Analytic Series and its RoC:

Every polynomial of degree $N$ has coefficients related to its derivatives. There is an important distinction, between a polynomial of finite degree $N$ and of degree $N \to \infty$. When the degree of the polynomial is finite, the power series

$$P(x) = \sum_{0}^{\infty} a_n (x - x_o)^n,$$  \hspace{1cm} (1.32)

defines a single valued function $P(x)$ that is said to be analytic. As for the case of a polynomial, ($N \in \mathbb{N} < \infty$) the coefficients are precisely related to the derivatives. For the case where $N \to \infty$, the analytic function does not depend on the series to be convergent. For example

$$e^t u(t) = \sum_{0}^{\infty} \frac{1}{n!} t^n$$

with $t > 0$, diverges, yet the series exists and is a valid description of $e^t$ because it is entire. Note that it has a Laplace transform with an unstable pole at $s = 1$, where the Laplace transform is not analytic. For values of $x - x_o$ where the power series converges, namely when $|x - x_o| < 1$, $P(x)$ is said to be an analytic function in $x$, in the neighborhood of the expansion point $x_o$, within the region of convergence (RoC). For cases where the argument is complex ($x \in \mathbb{C}$), this is called the radius of convergence (RoC). We call the region $|x - x_o| > 1$ the region of divergence (RoD), and $|x - x_o| = 0$, the singular circle. Typically the function $P(s)$ defined by the series has a pole on the singular circle for $s \in \mathbb{C}$. Once may isolate such poles by moving the expansion point $s_o$ until the RoC approaches zero.

1.3.2 Taylor Series

For every analytic function $P(x)$ the coefficients $c_n$ are always determined by derivatives of $P(x)$, evaluated at $x = x_o$. The Taylor series coefficients are defined by

$$a_n = \frac{1}{n!} \left. \frac{d^n}{dx^n} P(x - x_o) \right|_{x = x_o}.$$  \hspace{1cm} (1.33)

The Taylor formula Eq. 1.32 states how to uniquely define the coefficients $a_n$. Without the Taylor series formula, we would have no way of determining $a_n$. The proof of the Taylor formula is transparent, simply by taking successive derivative of Eq. 1.32, and then evaluating the result at the expansion point. If $P(x)$ is analytic then this procedure will always work. If $P(x)$ fails to have a derivative of any order, then the function is not analytic and Eq. 1.32 does not valid for $P(x)$. For example if $P(x)$ has a pole at $x_o$ then it is not analytic at that point.

Exercise: Verify that $a_o$ and $a_1$ of Eq. 1.32 follow from Eq. 1.33.

The Taylor series plays an important role in mathematics, as the coefficients of the series uniquely determine any analytic function via its derivatives. The implications, and limitations of the power series representation are very specific. First, if the series fails to converge (i.e., outside the RoC), it is essentially meaningless. Second, the analytic function must be single valued. This follows from the fact that each term in Eq. 1.32 is single valued. Third, analytic functions are very “smooth,” since the may be differentiated an $\infty$ number of times, and the series still converges. There can be no jumps or kinks in such functions.
But these properties are both the curse and the blessing of the analytic function. On the positive side analytic functions are the ideal starting point for solving differential equations, which is exactly how they were used by Newton and others. Analytic functions are “smooth” in that they are infinitely differentiable, with coefficients given by Eq. 1.33. They are single valued, so there can be no ambiguity in their interpretation.

Two well known analytic functions are the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$  \hspace{1cm} (1.34)

and exponential series

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2} x^3 + \frac{1}{4 \cdot 3 \cdot 2} x^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$  \hspace{1cm} (1.35)

A third important series is the Brune Impedance functions, defined as the “rational” ratio of two polynomials, of degrees \(M\) and \(N\)

$$Z_{\text{Brune}} = \frac{P_N(s)}{P_M(s)} = \frac{s^N + a_1 s^{N-1} \ldots a_0}{s^M + b_1 s^{M-1} \ldots b_0},$$

where \(M = N \pm 1\) (i.e., \(N = M \pm 1\)). This fraction of polynomials is historically known as the “Pade approximation,” but more specifically is a Brune “pole-zero” impedance. The key priority of the Brune impedance is that its real part is non-negative for the right-hand \(s\) plane (RHP: \(\Re s = \sigma > 0\)).

Exercise: Verify that the coefficients of the above functions are given by Eq. 1.33.

Exercise: Find the RoC of the following by application of Eq. 1.33.

1. \(w(x) = \frac{1}{1-x^j}\). \textbf{Sol:} From a straightforward expansion we know the coefficients are

$$\frac{1}{1-x^j} = 1 + x^j + (x^j)^2 + (x^j)^3 \ldots = 1 + x^j - x^{2j} + -j x^{3j} \ldots .$$

Working this out using Eq. 1.33 by brute force is more work:

\begin{align*}
  c_0 &= w|_{x=0} = 1; c_1 = \frac{dw}{dx}|_{x=0} = -\frac{-j}{(1-x^j)^2} = j; c_2 = \frac{d^2 w}{dx^2}|_{x=0} = \frac{-2}{(1-x^j)^3} = -2; \\
  c_3 &= \frac{d^3 w}{dx^3}|_{x=0} = \frac{-6 j}{(1-x^j)^4} = -6 j;
\end{align*}

However if we take derivatives of the series expansion it is much easier, and one can even figure out the term for \(c_n\):

\begin{align*}
  c_0 &= 1; c_1 = \frac{d}{dx} \sum (j x^n)|_{x=0} = j; c_2 = \frac{d^2}{dx^2} \sum (j x^n)|_{x=0} = 2(j)^2; \\
  c_3 &= \frac{d^3}{dx^3} \sum (j x^n)|_{x=0} = 2 \cdot 3(j)^3 = -6 j;
\end{align*}

\ldots ,

\(c_n = j^n n!\).

2. \(w(x) = e^{x^j}\) \textbf{Sol:} \(c_n = \frac{j^n}{n!}\).
Region of convergence: Determining the RoC for a given analytic function is quite important, and may not always be obvious. In general the RoC is a circle having a radius, centered on the expansion point, out to the nearest pole. Thus when the expansion point is moved, the RoC changes, since the location of the pole is fixed.

For the geometric series (Eq. 1.34), if the expansion point is taken as \( x_o = 1 \), the RoC is \( |x| < 1 \), since \( 1/(1 - x) \) has a pole at \( x = 1 \). We may move the expansion point by a linear transformation. For example, by replacing \( x \) with \( z + 3 \). Then the series becomes \( 1/((z + 3) - 1) = 1/(z + 2) \), so the RoC becomes 2, because in the \( z \) plane, the pole has moved to \( -2 \). A second important example is the function \( 1/(x^2 + 1) \), which has the same RoC as the geometric series, since it may be expressed in terms of its residue expansion (aka, partial fraction expansion)

\[
\frac{1}{x^2 + 1} = \frac{1}{(x + 1j)(x - 1j)} = \frac{1}{2j} \left( \frac{1}{x - 1j} - \frac{1}{x + 1j} \right).
\]

Each term has an RoC of \( |x| < |1j| = 1 \). In other words, it is the sum of two geometric series, with poles at \( \pm 1j \) which are not as obvious because the roots are complex, and conjugate. Once factored, it becomes clear what is going on.

The amplitude of each pole is called the residue. The residue for the pole at \( 1j \) is \( 1/2j \), a complex number, defined in Section 1.4.5 Eq. 1.94, p. 108.

Exercise: Verify the above expression is correct, and show that the residues are \( \pm 1/2j \). Sol: Cross-multiply and cancel, \( x \) cancels out and we are left with 1, as required.

Exercise: Find the residue of \( \frac{d}{dz} z^\pi \). Sol: Taking the derivative gives \( z^{\pi - 1} \) which has a pole at \( z = 0 \). Applying the formula for the residue (Eq. 1.94, p. 108) we find

\[
c^{-1} = \lim_{z \to 0} z z^{\pi - 1} = \lim_{z \to 0} z^\pi = 0.
\]

Thus the residue is zero.
The exponential series converges for every finite value of \( x \in \mathbb{R} \) (the RoC is the entire real line), thus the exponential is called an entire function.

Analytic functions:

Any function that has a Taylor series expansion is called an analytic function. Within the RoC, the series expansion defines a single valued function. Polynomials, \( 1/(1 - x) \) and \( e^x \) are examples of analytic functions that are real functions of their real argument \( x \). This is not the entire story. Because analytic functions are easily manipulated term by term, they may be used to find solutions of differential equations, since the derivatives of a series are uniquely determined within the RoC, due to Eq. 1.33.

Every analytic function has a corresponding differential equation, that is determined by the coefficients \( a_k \) of the analytic power series. An example is the exponential, which has the property that it is the eigen-function of the derivative operation

\[
\frac{d}{dx} e^{ax} = ae^{ax},
\]

which may be verified using Eq. 1.35. This relationship is a common definition of the exponential function, which is a very special, because it is the eigen-function of the derivative.
The complex analytic power series (i.e., complex analytic functions) may also be integrated, term by term, since
\[ \int f(x)dx = \sum \frac{a_k}{k+1} x^{k+1}. \] (1.36)

Newton took full advantage of this property of the analytic function and used the analytic series (Taylor series) to solve analytic problems, especially for working out integrals, allowing him to solve differential equations. To fully understand the theory of differential equations, one must master single valued analytic functions and their analytic power series.

**Single- vs. multi-valued functions:** Polynomials, and their \(\infty\)-degree extensions (analytic functions) are single valued: for each \(x\) there is a single value of \(P_N(x)\). The roles of the domain and codomain may be swapped, to obtain an *inverse function*, which is typically quite different in its properties compared to the function. For example \(y(x) = x^2 + 1\) has the inverse \(x = \pm\sqrt{y - 1}\), which is double valued, and complex when \(y < 1\). Periodic functions, such as \(y(x) = \sin(x)\) are even more "exotic," since \(x(y) = \arcsin(y) = \sin^{-1}(x)\) has an \(\infty\) number of \(x(y)\) values for each \(y\). This problem was first addressed in Riemann’s 1851 PhD thesis, while working with Gauss.

**Exercise:** Let \(y(x) = \sin(x)\). Then \(dy/dx = \cos(x)\). Show that \(dx/dy = -1/\sqrt{1-x^2}\). Hint: \(x(y) = \cos^{-1}(y) = \arccos(y)\).

**Exercise:** Let \(y(x) = \sin(x)\). Then \(dy/dx = \cos(x)\). Show that \(dx/dy = -1/\sqrt{1+y^2}\).

**Exercise:** Find the Taylor series coefficients of \(y = \sin(x)\) and \(x = \sin^{-1}(y)\).

**Complex analytic functions:** When the argument of an analytic function \(F(x)\) is complex, that is, \(x \in \mathbb{R}\) is replaced by \(s = \sigma + \omega j \in \mathbb{C}\) (recall that \(\mathbb{R} \subset \mathbb{C}\))

\[ F(s) = \sum_{n=0}^{\infty} c_n(s - \sigma_o)^n, \] (1.37)

with \(c_n \in \mathbb{C}\), that function is said to be a *complex analytic*.

For example, when the argument of the exponential becomes complex, it is periodic on the \(\omega\) axis, since

\[ e^{st} = e^{(\sigma + \omega j)t} = e^{\sigma t}e^{\omega jt} = e^{\sigma t} [\cos(\omega t) + j\sin(\omega t)]. \]

Taking the real part gives

\[ \Re \{e^{st}\} = e^{\sigma t} \frac{e^{\omega jt} + e^{-\omega jt}}{2} = e^{\sigma t} \cos(\omega t), \]

and \(\Im \{e^{st}\} = e^{\sigma t} \sin(\omega t)\). Once the argument is allowed to be complex, it becomes obvious that the exponential and circular functions are fundamentally related. This exposes the family of *entire circular functions* [i.e., \(e^s, \sin(s), \cos(s), \tan(s), \cosh(s), \sinh(s)\)] and their inverses [\(\ln(s), \arcsin(s), \arccos(s), \arctan(s), \cosh^{-1}(s), \sinh^{-1}(s)\)], first fully elucidated by Euler (c1750) (Stillwell, 2010, p. 315). Note that because a function, such as \(\sin(\omega t)\), is periodic, its inverse must be multi-valued. What is needed is some systematic way to account for this multi-valued property. That methodology was provided by Riemann 100 years later, in this 1851 PhD Thesis, supervised by Gauss, in the final years of Gauss’ life.

Given a complex analytic function of a complex variable, one must resort to the *extended complex plane, Riemann sheets and branch cuts*, as discussed in Section 1.3.9 (p. 80). The extended complex...
plane is a tool that extends the domain of complex analytic to include the point at infinity. This topic is critically important in engineering mathematics, and will be discussed in length in Sections 1.3.9-1.3.12 (pp. 80-87).

**Definition of the Taylor series of a complex analytic function:** However there is a fundamental problem, since we cannot formally define the Taylor series for the coefficients $c_k$, since we have not defined $dF(s)/ds$, the derivative with respect to the complex variable $s \in \mathbb{C}$. Thus simply substituting $s$ for $x$ in an analytic function is leaving a major hole in our understanding of the complex analytic function.

To gain a feeling of the nature of the problem, we make take derivatives of a function with respect to various variables. For example,

$$\frac{d}{dt} e^{st} = se^{st}.$$ 

Also

$$e^{\omega j} \frac{d}{d\sigma} e^{\sigma t} = \sigma e^{st},$$

and

$$e^{\sigma t} \frac{d}{d\omega j} e^{\omega j} = \omega je^{st}.$$ 

are straightforward.

It was the work of Cauchy (1814) (Fig. 1.12), who uncovered much deeper relationships within complex analytic functions (Sect. 1.3.10, p. 82) by defining differentiation and integration in the complex plane, leading to several fundamental theorems of complex calculus, including the *Fundamental theorem of complex integration*, and Cauchy’s formula. We shall explore this in and several fundamental theorems in Sect. 1.4.1 (p. 96).

There seems to be some disagreement as to the status of multi-valued functions: Are they functions, or is a function strictly single valued? If so, then we are missing out on a host of interesting possibilities, including all the inverses of nearly every complex analytic function. For example, the inverse of a complex analytic function is a complex analytic function (e.g., $e^s$ and $\log(s)$).

**Impact on Physics:** It seems likely, if not obvious, that the success of Newton was his ability to describe physics by the use of mathematics. He was inventing new mathematics at the same time as he was explaining new physics. The same might be said for Galileo. It seems likely that Newton was extending the successful techniques and results of Galileo (Galileo, 1638). Galileo died on Jan 8, 1642, and Newton was born Jan 4, 1643, just short of one year later. Certainly Newton was well aware of Galileo’s great success, and naturally would have been influenced by them.

The application of complex analytic functions to physics was dramatic, as may be seen in the six volumes on physics by Arnold Sommerfeld (1868-1951), and from the productivity of his many (36) students (e.g., Debye, Lenz, Ewald, Pauli, Guillemin, Bethe, Heisenberg and Seebach, to name a few), notable coworkers (i.e., Leon Brillouin) and others (i.e., John Bardeen), upon whom Sommerfeld had a strong influence. Sommerfeld is known for having many students who were awarded the Nobel Prize in Physics, yet he was not (the prize is not awarded in mathematics). Sommerfeld brought mathematical physics (the merging of physical and experimental principles with mathematics) to a new level with the use of complex integration of analytic functions to solve otherwise difficult problems, thus following the lead of Newton who used real integration of Taylor series to solve differential equations (Brillouin, 1960, Ch. 3 by Sommerfeld, A.).

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55. [https://www.aip.org/history-programs/niels-bohr-library/oral-histories/4661-1](https://www.aip.org/history-programs/niels-bohr-library/oral-histories/4661-1)
1.3.10 Lec 18: Complex analytic mappings (Domain-coloring)

One of the most difficult aspects of complex functions of a complex variable is understanding the mapping from \( z = x + yj \) to \( w(z) = u + vj \). For example, \( w(z) = \sin(x) \) is trivial when \( z = x + yj \) is real (i.e., \( y = 0 \)), because \( \sin(x) \) is real. Likewise for the \( \sin(x) \) because \( x, y, u \) and \( v \) are not well known \( x, y \) is not easily visualized. Thus when \( z \) functions, it is difficult to visualize. Fortunately with computer software today, \( z \) can be much more difficult to visualize. Examples you can render with \( zviz.m \) is trivial.

\[
\sin(yj) = \frac{e^{-y} - e^{y}}{2j} = j \sinh(y)
\]

is purely imaginary. But the general case, \( w(z) = \sin(z) \in \mathbb{C} \)

\[
\sin(zj) = \sin(xj - y) = j \sinh(z).
\]

is not easily visualized. Thus when \( u(x, y) \) and \( v(x, y) \) are not well known functions, \( w(z) \) are can be much more difficult to visualize. Fortunately with computer software today, this problem can be solved by adding color to the chart. A Matlab/Octave script \( zviz.m \) has been used to make the make the charts shown here.\(^{60}\) This tool is also known as Domain-coloring.\(^{61}\) Rather than plotting \( u(x, y) \) and \( v(x, y) \) separately, domain-coloring allows us to display the entire function on one chart. Note that for this visualization we see the complex polar form of \( w(s) = |w| e^{j \omega} \), rather than as the four dimensional Cartesian graph \( w(x + yj) = u(x, y) + v(x, y)j \).

Visualizing complex functions: The mapping from \( z = x + iy \) to \( w(z) = u(x, y) + iv(x, y) \) is a 2-2 = 4 dimensional graph. This is difficult to visualize, because for each point in the domain \( z \), we would like to represent both the magnitude and phase (or real and imaginary parts) of \( w(z) \). A good way to visualize these mappings is to use color (hue) to represent the phase and intensity (dark to light) to represent the magnitude. The Matlab program \( zviz.m \) has been provided to do this (see Lecture 17 on the class website).

To use the program in Matlab/Octave, use the syntax \( zviz <\text{function of } z> \) (for example, type \( zviz z.\hat{z}^2 \)). Note the period between \( z \) and \( \hat{z}^2 \). This will render a ‘domain coloring’ (aka colorized) version of the function. Examples you can render with \( zviz \) are given in the comments at the top of the \( zviz.m \) program. A good example for testing is \( zviz z - sqrt(j) \), which should show a dark spot (a zero) at \((1 + 1j)/\sqrt{2} = 0.707(1 + 1j)\).

Example: Figure 1.18 shows a colorized plot of \( w(z) = \sin(\pi(z - i)/2) \) resulting from the Matlab/Octave command \( zviz \ sin(pi*(z-i)/2) \). The abscissa (horizontal axis) is the real \( x \) axis and the ordinate (vertical axis) is the complex \( iy \) axis. The graph is offset along the ordinate axis by

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\(^{60}\)http://jontalle.web.engr.illinois.edu/uploads/298/zviz.zip

\(^{61}\)This is also called ‘domain coloring’: https://en.wikipedia.org/wiki/Domain_coloring
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Figure 1.19: This domain–color map allows one to visualize complex mappings by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase). The white and black lines are the iso-real and iso-imaginary contours of the mapping. LEFT: This figure shows the domain–color map for the complex mapping from the $z = x + iy$ plane to the $w(z) = u + jv = e^{x+iy}$ plane, which goes to zero as $x \to -\infty$, causing the domain–color map to become dark for $x < -2$. The white and black lines are always perpendicular because $e^z$ is complex analytic everywhere. RIGHT: This shows the principal value of the inverse function $w(x, y) + v((x, y))j = \log(x + yj)$, which has a zero (dark) at $z = 1$, since there $\log(1) = 0$ (the imaginary part is zero). Note the branch cut from $x = 0$ to $x = -\infty j$. On branches other than the one shown, there is are no zeros, since the phase ($\angle z = 2\pi n$) is not zero.

1i, since the argument $z - i$ causes a shift of the sine function by 1 in the positive imaginary direction. The visible zeros of $w(z)$ appear as dark regions at $(-2, 1), (0, 1), (2, 1)$. As a function of $x$, $w(x + 1j)$ oscillates between red (phase is zero degrees), meaning the function is positive and real, and sea-green (phase is $180^\circ$), meaning the function is negative and real.

Along the vertical axis, the function is either a $\cosh(y)$ or $\sinh(y)$, depending on $x$. The intensity becomes lighter as $|w|$ increases, and white as $w \to \infty$, the intensity becomes darker as $|w|$ decreases, and black as $w \to 0$.

Mathematicians typically use the more abstract (i.e., non–physical) notation $w(z)$, where $w = u + vj$ and $z = x + yj$. Engineers think in terms of a physical complex impedance $Z(s) = R(s) + X(s)j$, having resistance $R(s)$ and reactance $X(s)$ [Ohms], as function of the complex Laplace radian frequency $s = \sigma + \omega j$ [rad], as used, for example, with the Laplace transform (Sect. 1.3.12, p. 87).

In Fig. 1.17 we use both notations, with $Z(s) = s$ on the left and $w(z) = z - \sqrt{j}$ on the right, where we show this color code as a 2x2 dimensional domain-coloring graph. Intensity (dark to light) represents the magnitude of the function, while hue (color) represents the phase, where (see Fig. 1.17) red is $0^\circ$, sea-green is $90^\circ$, blue-green is $135^\circ$, blue is $180^\circ$, and violet is $-90^\circ$ (or $270^\circ$). The function $w = s$ has a dark spot (a zero) at $s = 0$, and becomes brighter away from the origin. On the right is $w(z) = z - \sqrt{j}$, which shifts the zero to $z = \sqrt{j}$. Thus domain–coloring gives the full picture of the complex analytic function mappings $w(x, y) = u(x, y) + v(x, y)j$ in colorized polar coordinates.

Two additional examples are given in Fig. 1.19 to help you interpret the two complex mappings.

In the right panel note the zero for $\ln(w) = \ln |w| + \omega j$ at $w = 1$. The root of the log function is $\log(w_r) = 0$ is $w_r = 1, \phi = 0$, since $\log(1) = 0$. More generally, the log of $w = |w|e^{\phi j}$ is $s = \ln |w| + \phi j$. Thus $s(w)$ can be zero only when the angle of $w$ is zero.
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Gibbs ringing (it oscillates around the step, hence does not converge at the jump).\(^\text{62}\) The LT does not exhibit Gibbs ringing.

8. The FT is not always analytic in \(\omega\), as in this example of the step function. The step function cannot be expanded in a Taylor series about \(\omega = 0\), because \(e^{\delta(\omega)}\) is not analytic in \(\omega\).

9. The Fourier \(\delta\) function is denoted \(\tilde{\delta}(t)\), to differentiate it from the Laplace delta function \(\delta(t)\). They differ because the step functions differ, due to the convergence problem described above.

10. One may define

\[
\tilde{u}(t) = \int_{-\infty}^{t} \tilde{\delta}(t) dt,
\]

and define the somewhat questionable notation

\[
\tilde{\delta}(t) \equiv \frac{d}{dt} \tilde{u}(t),
\]

since the Fourier step function is not analytic.

11. The rec\((t)\) function is defined as

\[
\text{rec}(t) = \frac{\tilde{u}(t) - \tilde{u}(t - T_o)}{T_o} = \begin{cases} 
0 & \text{if } t > 0 \\
1/T_o & 0 < t < T_o \\
0 & \text{if } t < 0 
\end{cases}
\]

It follows that \(\tilde{\delta}(t) = \lim_{T_o \to 0}\). Like \(\tilde{\delta}(t)\), the rec\((t)\) has unit area.

1.3.12  Lec 20: Systems: Laplace transforms

When dealing with engineering problems it is convenient to separate the signals we use from the systems that process them. We do this by treating signals, such as a music signal, differently from a system, such as a filter. In general signals may start and end at any time. The concept of causality has no mathematical meaning in signal space. Systems, on the other hand, obey very rigid rules (to assure that they remain physical). These physical restrictions are described in terms of the Network Postulates, which are first discussed in Sect. 1.3.13, and then in greater detail in Sect. 3.5.1.

Definition of the Laplace transform: The forward and inverse Laplace transforms are

\[
F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt \\
f(t) = \frac{1}{2\pi j} \int_{\sigma_o - \infty j}^{\sigma_o + \infty j} F(s) e^{st} ds
\]

(1.68)  (1.69)

1. Time \(t \in \mathbb{R}\) [s] and Laplace frequency [rad] are defined as \(s = \sigma + \omega j \in \mathbb{C}\).

2. Given a Laplace transform (LT) pair \(f(t) \leftrightarrow F(s)\), in the engineering literature, the time domain is always lower case \([f(t)]\) and causal (i.e., \(f(t < 0) = 0\)) and the frequency domain is upper-case [e.g. \(F(s)\)]. Maxwell’s venerable equations are the unfortunate exception to this otherwise universal rule.

\(^{62}\)https://en.wikipedia.org/wiki/Gibbs_phenomenon
3. When taking the forward transform \((t \to s)\), the sign of the exponential is negative. This is necessary to assure that the integral converges when the integrand \(f(t) \to \infty\) as \(t \to \infty\). For example, if \(f(t) = e^t u(t)\) (i.e., without the negative \(\sigma\) exponent), the integral does not converge.

4. The target time function \(f(t < 0) = 0\) (i.e., it must be causal). The time limits are \(0^- < t < \infty\). Thus the integral must start from slightly below \(t = 0\) to integrate over a delta functions at \(t = 0\). For example if \(f(t) = \delta(t)\), the integral must include both sides of the impulse. If you wish to include non-causal functions such as \(\delta(t + 1)\) it is necessary to extend the lower time limit. In such cases simply set the lower limit of the integral to \(-\infty\), and let the integrand \((f(t))\) determine the limits.

5. The limits on the integrals of the forward transform are \(t : (0^-, \infty) \in \mathbb{R}\), and the reverse LTs are \([\sigma_o - \infty j, \sigma_o + \infty j] \in \mathbb{C}\). These limits will be further discussed in Section 1.4.9 (p. 112).

6. When taking the inverse Laplace transform, the normalization factor of \(1/2\pi j\) is required to cancel the \(2\pi j\) in the differential \(ds\) of the integral.

7. The frequency for the LT must be is complex, and in general \(F(s)\) is complex analytic for \(\sigma > \sigma_o\). It follows that the real and imaginary parts of \(\bar{F}(s)\) are related. Given \(\Re\{F(s)\}\) it is possible to find \(\Im\{F(s)\}\) (Boas, 1987). More on this in Section 1.3.12 (p. 87).

8. To take the inverse Laplace transform, we must learn how to integrate in the complex \(s\) plane. This will be explained in Sections 1.4.5-1.4.9 (p. 107-112).

9. The Laplace step function is defined as

\[
    u(t) = \int_{-\infty}^{t} \delta(t) \, dt = \begin{cases} 
        1 & \text{if } t > 0 \\
        \text{NaN} & \text{if } t = 0 \\
        0 & \text{if } t < 0 
    \end{cases}
\]

Alternatively one could define \(\delta(t) = du(t)/dt\).

10. It is easily shown that \(u(t) \leftrightarrow 1/s\) by direct integration

\[
    F(s) = \int_0^{\infty} u(t) e^{-st} \, dt = -\frac{e^{-st}}{s} \bigg|_0^\infty = \frac{1}{s}.
\]

With the LT step \((u(t))\) there is no Gibbs ringing effect.

11. In many physical applications, the Laplace transform takes the form of a ratio of two polynomials. In such case the roots of the numerator polynomial are call the zeros while the roots of the denominator polynomial are called the poles. For example the LT of \(u(t) \leftrightarrow 1/s\) has a pole at \(s = 0\), which represents integration, since

\[
    u(t) \ast f(t) = \int_{-\infty}^{t} f(\tau) \, d\tau \leftrightarrow \frac{F(s)}{s}.
\]

12. The LT is quite different from the FT in terms of its analytic properties. For example, the step function \(u(t) \leftrightarrow 1/s\) is complex analytic everywhere, except at \(s = 0\). The FT of \(1 \leftrightarrow 2\pi \delta(\omega)\) is not analytic anywhere.
13. Dilated step function

\[ u(at) \leftrightarrow \int_{-\infty}^{\infty} u(at)e^{-st}\,dt = \frac{1}{a} \int_{-\infty}^{\infty} u(\tau)e^{-(s/a)\tau}\,d\tau = \frac{a}{|a|} \frac{1}{s} = \pm \frac{1}{s}, \]

where we have made the change of variables \( \tau = at \). The only effect that \( a \) has on \( u(at) \) is the sign of \( t \), since \( u(t) = u(2t) \). However \( u(-t) \neq u(t) \), since \( u(t) \cdot u(-t) = 0 \), and \( u(t) + u(-t) = 1 \), except at \( t = 0 \), where it is not defined.

Once complex integration in the complex plane has been defined (Section 1.4.2, p. 97), we can justify the definition of the inverse LT (Eq. 1.68).

Example: Here \( s = \sigma + j\omega \) is the complex Laplace frequency in [radians] and \( t \) is time in [seconds].

As discussed in Section 1.4.8 (p. 110), we must use the Cauchy Residue Theorem (CRT), requiring closure of the contour \( C \) at \( \omega j \rightarrow \pm j\infty \)

\[ \int_{C} = \int_{\gamma} + \int_{C_{\infty}} \]

where the path represented by \( \gamma \) is a semicircle of infinite radius. For a causal, ‘stable’ (e.g. doesn’t “blow up” in time) signal, all of the poles of \( F(s) \) must be inside of the Laplace contour, in the left-half \( s \)-plane.

Hooke’s Law for a spring states that the force \( f(t) \) is proportional to the displacement \( x(t) \), i.e., \( f(t) = Kx(t) \). The formula for a dash-pot is \( f(t) = Rv(t) \), and Newton’s famous formula for mass is \( f(t) = d[Mv(t)]/dt \), which for constant \( M \) is \( f(t) = Mdv/dt \).

The equation of motion for the mechanical oscillator in Fig. 1.20 is given by Newton’s second law; the sum of the forces must balance to zero

\[ M \frac{d^2x(t)}{dt^2} + R \frac{dx(t)}{dt} + Kx(t) = f(t). \]  

(1.70)

These three constants, the mass \( M \), resistance \( R \) and stiffness \( K \) are all real and positive. The dynamical variables are the driving force \( f(t) \leftrightarrow F(s) \), the position of the mass \( x(t) \leftrightarrow X(s) \) and its velocity \( v(t) \leftrightarrow V(s) \), with \( v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s) \).

Newton’s second law (c1650) is the mechanical equivalent of Kirchhoff’s (c1850) voltage law (KCL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., \( F = qE \)).

Equation 1.70 may be re-expressed in terms of impedances, the ratio of the force to velocity, once it is transformed into the Laplace frequency domain.

The key idea that every impedance must be complex analytic and \( \geq 0 \) for \( \sigma > 0 \) was first proposed by Otto Brune in his PhD at MIT, supervised by a student of Arnold Sommerfeld, Ernst Guilliman, an
Table 1.5: The following table provides a brief table of Laplace Transforms of \( f(t), \delta(t), \ u(t), \text{rect}(t), T_0, p, e, \in \mathbb{R} \) and \( F(s), G(s), s, a \in \mathbb{C} \). Given a Laplace transform (LT) pair \( f(t) \leftrightarrow F(s) \), the frequency domain will always be upper-case [e.g. \( F(s) \)] and the time domain lower case [\( f(t) \)] and causal (i.e., \( f(t < 0) = 0 \)). An extended table of transforms is given in Table 3.1 on page 189.

| \( f(t) \leftrightarrow F(s) \) \( t \in \mathbb{R} \); \( s, F(s) \in \mathbb{C} \) | Name                                      |
|-------------------------------------------------------------------------------------------------|
| \( \delta(t) \leftrightarrow 1 \) \( t \in \mathbb{R} \) \( \delta(\tfrac{t}{a}) \leftrightarrow \tfrac{1}{|a|} \) \( a \neq 0 \) | Dirac \( \delta(t - T_0) \leftrightarrow e^{-st_0} \) \( \delta(t - T_0) \ast f(t) \leftrightarrow F(s)e^{-st_0} \) delayed Dirac \( \sum_{n=0}^{\infty} \delta(t - nT_0) = \frac{1}{1 - \delta(t - T_0)} \leftrightarrow \frac{1}{1 - e^{-st_0}} = \sum_{n=0}^{\infty} e^{-snT_0} \) one-sided impulse train |
| \( u(t) \leftrightarrow \frac{1}{s} \) \( u(-t) \leftrightarrow -\frac{1}{s} \) \( u(at) \leftrightarrow \frac{a}{s} \) \( a \in \mathbb{C}, a \neq 0 \) | step \( e^{at}u(t) \leftrightarrow \frac{1}{s + a} \) \( e^{\pm at}u(t) \leftrightarrow \frac{1}{s \pm a}, \ a \in \mathbb{C} \) damped step \( e^{\pm at}u(t) \leftrightarrow \frac{1}{s \pm a}, \ a \in \mathbb{C} \) modulated step |
| \( \cos(at)u(t) \leftrightarrow \frac{1}{2} \left( \frac{1}{s - a} + \frac{1}{s + a} \right) \) \( \sin(at)u(t) \leftrightarrow \frac{1}{2j} \left( \frac{1}{s - a} - \frac{1}{s + a} \right) \) \( a \in \mathbb{R} \) | \( \cos \) \( \sin \) “damped” sin |
| \( u(t - T_0) \leftrightarrow \frac{1}{s} e^{-st_0} \) \( T_0 \in \mathbb{R}^+ \) | time delay |
| \( \text{rect}(t) = \frac{1}{T_o} [u(t) - u(t - T_o)] \leftrightarrow \frac{1}{T_o} \left( 1 - e^{-st_0} \right) \) | rect-pulse |
| \( u(t) \ast u(t) = tu(t) \leftrightarrow 1/s^2 \) \( u(t) \ast u(t) \ast u(t) = \frac{1}{2} t^2 u(t) \leftrightarrow 1/s^3 \) | ramp \( t^p u(t) \leftrightarrow \frac{\Gamma(p + 1)}{sp^{p + 1}} \) \( p \in \mathbb{R} \geq 0 \) double ramp
MIT ECE professor, who played a major role in the development of circuit theory. Brune’s primary (non-MIT) advisor was Cauer, who was also well trained in 19th century German mathematics. 63

Summary: While the definitions of the FT ($\mathcal{F}T$) and LT ($\mathcal{L}T$) transforms appear superficially similar, they are not. The key difference is that the time response of the Laplace transform is causal, leading to a complex analytic frequency responses. The frequency response of the Fourier transform is real, thus typically not analytic. These are not superficial differences. The concept of symmetry is helpful in understanding the many different types of time-frequency transforms. Two fundamental types of symmetry are causality and periodicity.

The Fourier transform $\mathcal{F}T$ characterizes the steady-state response while the Laplace transform $\mathcal{L}T$ characterizes both the transient and steady-state response. Given a system response $H(s) \leftrightarrow h(t)$ with input $x(t)$, the output is

$$y(t) = h(t) \ast x(t) \leftrightarrow Y(\omega) = H(s)\bigg|_{s=j\omega} X(\omega).$$

### Table 1.6: Functional relationships between Laplace Transforms.

<table>
<thead>
<tr>
<th>Functional properties</th>
<th>Convolution</th>
<th>Scaling</th>
<th>Damped</th>
<th>Damped and Delayed</th>
<th>Reverse Time</th>
<th>Time-Reversed &amp; Damped</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t) \ast g(t) = \int_{t=0}^{t} f(t-\tau)g(\tau)d\tau \leftrightarrow F(s)G(s)$</td>
<td>convolution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u(t) \ast f(t) = \int_{0^-}^{t} f(t)dt \leftrightarrow \frac{F(s)}{s}$</td>
<td>convolution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(at)u(at) \leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right)$, $a \in \mathbb{R} \neq 0$</td>
<td>scaling</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(t)e^{-at}u(t) \leftrightarrow F(s+a)$</td>
<td>damped</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(t-T)e^{-a(t-T)}u(t-T) \leftrightarrow e^{-sT}F(s+a)$</td>
<td>damped and delayed</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(-t)u(-t) \leftrightarrow F(-s)$</td>
<td>reverse time</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(-t)e^{-at}u(-t) \leftrightarrow F(a-s)$</td>
<td>time-reversed &amp; damped</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{df(t)}{dt} = \delta'(t) \ast f(t) \leftrightarrow sF(s)$</td>
<td>deriv</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{\sin(t)u(t)}{t} \leftrightarrow \tan^{-1}(1/s)$</td>
<td>half-sync</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

63 It must be noted that Prof. ‘Mac’ Van Valkenburg from the University of IL., was arguably more influential in circuit theory, during the same period. Mac’s book are certainly more accessible, but perhaps less widely cited.
1.4.1 Lec 23: Fundamental theorems of calculus

History of the fundamental theorem of calculus: It some sense, the story of calculus begins with the fundamental theorem of calculus (FTC), also known generically as Leibniz’s formula. The simplest integral is the length of a line \( L = \int_0^L dx \). If we label a point on a line as \( x = 0 \) and wish to measure the distance to any other point \( x \), we form the line integral between the two points. If the line is straight, this integral is simply the Euclidean length given by the difference between the two ends (Eq. 1.46, p. 71).

If \( F(x) \) describes a height above the line, then the area under \( F(x) \) is

\[
f(x) - f(0) = \int_{x=0}^{x} F(\chi) d\chi,
\]

(1.72)

where \( \chi \) is a dummy variable of integration. Thus the area under \( F(x) \) (\( F(x), x \in \mathbb{R} \)) only depends on the difference of the area evaluated at the end points. This makes intuitive sense, by viewing \( f(x) \) as the area of a graph of \( F(x) \).

This property, of the area as an integral over an interval, only depending on the end points, has important consequences in physics in terms of conservation of energy, making generalizations important.

Fundamental theorems of real calculus:

If \( F(x) \) is analytic (Eq. 1.32, p. 59), then

\[
F(x) = \frac{d}{dx} f(x),
\]

(1.73)

which is known as the fundamental theorem of (real) calculus (FTC). Thus Eq. 1.73 may be viewed as an anti-derivative, or exact differential. This is easily shown by evaluating

\[
\frac{df(x)}{dx} = \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta} = F(x),
\]

starting from Eq. 1.72. If \( F(x) \) is not analytic then the limit may not exist, so this seems like a necessary condition. There are many important variations on this very basic theorem (see Sect. 1.4). For example, the limits could depend on time. Also when taking Fourier and Laplace transforms, the integrand depends on both time and frequency via an exponential “kernel” function \( e^{\pm st} \).

Case of a complex path The fundamental theorem of complex calculus (FTCC) states that for any complex analytic function \( F(s) \in \mathbb{C} \) with \( s = \sigma + \omega j \in \mathbb{C} \)

\[
f(s) - f(s_0) = \int_{s_0}^{s} F(\zeta) d\zeta.
\]

(1.74)

(Greenberg, 1988, p. 1197). If we compare this to Eq. 1.73, they differ in that the path of the integral is complex, meaning that the integral is over \( s \in \mathbb{C} \), rather than a real integral over \( x \in \mathbb{R} \). The fundamental theorems of complex calculus (FTCC) states that the integral only depends on the end points, since

\[
F(s) = \frac{d}{ds} f(s).
\]

(1.75)

Comparing Eq. 1.73 (FTC) with Eq. 1.75 (FTCC), it would appear that this can only be true if \( F(s) \in \mathbb{C} \) is complex analytic, meaning it must have a Taylor series in powers of \( s \in \mathbb{C} \).
Complex analytic functions: The definition of a complex analytic function $F(s)$ of $s \in \mathbb{C}$ is that the function may be expanded in a Taylor series (Eq. 1.37, p. 62) about an expansion point $s_0 \in \mathbb{C}$. This definition follows the same logic as the FTC. Thus we need a definition for the coefficients $c_n \in \mathbb{C}$, which most naturally follow from Taylor’s formula:

$$c_n = \frac{1}{n!} \frac{d^n}{ds^n} F(s) \bigg|_{s=s_0}.$$ \hfill (1.76)

The requirement that $F(s)$ have a Taylor series naturally follows by taking derivatives with respect to $s$ at $s_0$. The problem is that both integration and differentiation of functions of complex Laplace frequency $s = \sigma + \omega j$ have not yet been defined.

Thus the question is: “What does it mean to take the derivative of a function $F(s) \in \mathbb{C}$, $s = \sigma + \omega j \in \mathbb{C}$, with respect to $s$, where $s$ defines a plane rather than a real line?” We learned how to form the derivative on the real line. Can the same derivative concept be extended to the complex plane?

The answer is affirmative. The question may be resolved by applying the rules of the real derivative when defining the derivative in the complex plane. However for the complex case, there is an issue regarding direction. Given any analytic function $F(s)$, is the partial derivative with respect to $\sigma$ different from the partial derivative with respect to $\omega j$? For complex analytic functions, the FTCC states that the integral is independent of the path in the $s$ plane. Based on the chain rule, the derivative must also be independent of direction at $s_0$. This directly follows from the FTCC. If the integral of a function of a complex variable is to be independent of the path, the derivative of a function with respect to a complex variable must be independent of the direction. This follows from Taylor’s formula, Eq. 1.76 for the coefficients of the complex analytic formula. The FTC defines the area as an integral over a real differential ($dx \in \mathbb{R}$), while the FTCC relates an integral over a complex function $F(s) \in \mathbb{C}$, along a complex interval (i.e., path) ($ds \in \mathbb{C}$). For the FTC the area under the curve only depends on the end points of the anti-derivative $f(x)$. But what is the meaning of an “area” along a complex path? The Cauchy-Riemann conditions provide the answer.

### 1.4.2 Lec 24: Cauchy-Riemann conditions

For the integral of $Z(s) = R(\sigma, \omega) + X(\sigma, \omega)j$ to be independent of the path, the derivative of $Z(s)$ must be independent of the direction of the derivative. As we show next, this leads to a pair of equations known as the Cauchy-Riemann conditions. This is an important generalization of Eq. 1.1, p. 18 which goes from real integration ($x \in \mathbb{R}$) to complex integration ($s \in \mathbb{C}$), based on lengths, thus on area.

To define

$$\frac{d}{ds} Z(s) = \frac{d}{ds} [R(\sigma, \omega) + jX(\sigma, \omega)]$$

take partial derivatives of $Z(s)$ with respect to $\sigma$ and $j\omega$, and equate them:

$$\frac{\partial Z}{\partial \sigma} = \frac{\partial R}{\partial \sigma} + j \frac{\partial X}{\partial \sigma} \quad \equiv \quad \frac{\partial Z}{j\partial \omega} = \frac{\partial R}{\partial j\omega} + j \frac{\partial X}{\partial j\omega}.$$ 

This says that a horizontal derivative, with respect to $\sigma$, is equivalent to a vertical derivative, with respect to $j\omega$. Taking the real and imaginary parts gives the Cauchy Riemann conditions

$$\text{CR-1:} \quad \frac{\partial R(\sigma, \omega)}{\partial \sigma} = j \frac{\partial X(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \text{CR-2:} \quad \frac{\partial R(\sigma, \omega)}{\partial \omega} = -j \frac{\partial X(\sigma, \omega)}{\partial \sigma}. \hfill (1.77)$$

The $j$ cancels in CR-1, but introduces a $j^2 = -1$ in CR-2. These equations are known as the Cauchy-Riemann (CR) conditions. They may also be written in polar coordinates ($s = r e^{\theta j}$) as

$$\frac{\partial R}{\partial r} = \frac{1}{r} \frac{\partial X}{\partial \theta} \quad \text{and} \quad \frac{\partial X}{\partial r} = -\frac{1}{r} \frac{\partial R}{\partial \theta}.$$
If you are wondering what would happen if we took a derivative at 45 degrees, then we only need to multiply the function by \( e^{i\pi/4} \). But that will not change the derivative. Thus we may take the derivative in any direction by multiplying by \( e^{i\phi} \), and the CR conditions will not change.

The CR conditions are necessary conditions that the integral of \( Z(s) \), and thus its derivative, be independent of the path, expressed in terms of conditions on the real and imaginary parts of \( Z \). This is a very strong condition on \( Z(s) \), which follows assuming that \( Z(s) \) may be approximated by a Taylor series in \( s \)

\[
Z(s) = Z_0 + Z_1 s + \frac{1}{2} Z_2 s^2 + \cdots,
\]

where \( Z_n \in \mathbb{C} \) are complex constants given by the Taylor series formula (Eq. 1.76, p. 97).

Every function that may be expressed as a Taylor series about a point \( s \) is said to be complex analytic at that point. This series, which is single valued, is said to converge within a radius of convergence (RoC). This highly restrictive condition has significant physical consequences. As an important example, every impedance function \( Z(s) \) obeys the CR conditions over large regions of the \( s \) plane, including the entire right half plane (RHP) (\( \sigma > 0 \)). This condition is summarized by the Brune condition \( \Re\{Z(\sigma > 0)\} \geq 0 \) (Eq. 1.122, Section 1.4.3). When the CR condition is generalized to volume integrals, it is called Green’s theorem, used heavily in the solution of boundary value problems in engineering and physics. (Kusse and Westwig, 2010) The last chapter of this course shall depend heavily on these concepts.

We may merge these equations into a pair of second order equations by taking a second round of partials. Specifically, eliminating the real part \( R(\sigma, \omega) \) of Eq. 1.77 gives

\[
\frac{\partial^2 R(\sigma, \omega)}{\partial \sigma \partial \omega} = \frac{\partial^2 X(\sigma, \omega)}{\partial^2 \omega} = -\frac{\partial^2 X(\sigma, \omega)}{\partial^2 \sigma},
\]

which may be compactly written as \( \nabla^2 X(\sigma, \omega) = 0 \). Eliminating the imaginary part gives

\[
\frac{\partial^2 X(\sigma, \omega)}{\partial \omega \partial \sigma} = \frac{\partial^2 R(\sigma, \omega)}{\partial^2 \sigma} = -\frac{\partial^2 R(\sigma, \omega)}{\partial^2 \omega},
\]

which may be written as \( \nabla^2 R(\sigma, \omega) = 0 \).

In summary, for a function \( Z(s) \) to be complex analytic, the derivative \( dZ/ds \) must be independent of direction (path), which requires that the real and imaginary parts of the function obey Laplace’s equation, i.e.,

\[
\text{CR-3: } \nabla^2 R(\sigma, \omega) = 0 \quad \text{and} \quad \text{CR-4: } \nabla^2 X(\sigma, \omega) = 0.
\]

The CR equations are easier to work with because they are first order, but the physical intuition is best understood by noting fact 1) the derivative of a complex analytic function is independent of its direction, and fact 2) the real and imaginary parts of the function each obey Laplace’s equation. Such relationships are known as harmonic functions.\(^{65}\)

As we shall see in the next few lectures, complex analytic functions must be smooth since every analytic function may be differentiated an infinite number of times, within the RoC. The magnitude must attain its maximum and minimum on the boundary. For example, when you stretch a rubber sheet over a jagged frame, the height of the rubber sheet obeys Laplace’s equation. Nowhere can the height of the sheet rise above or below its value at the boundary.

Harmonic functions define Conservative fields, which means that energy (like a volume or area) is conserved. The work done in moving a mass from \( a \) to \( b \) in such a field is conserved. If you return the mass from \( b \) back to \( a \), the energy is retrieved, and zero net work has been done.

\(^{65}\)When the function is the ratio of two polynomials, as in the cases of the Brune impedance, they are also related to Möbius transformations, also know as bi-harmonic operators.