Lec 26: Multi-valued functions

In the field of mathematics there seems to have been a tug-of-war regarding the basic definition of the concept of function. The accepted definition today seems to be a single-valued mapping from the domain to the codomain (or range). This makes the discussion of multi-valued functions somewhat tedious. In 1851 Riemann (working with Gauss) seems to have resolved this problem for the natural set of multi-valued functions, by introducing the concept of the branch-cut and sheets.

Two simple examples multi-valued functions are the circle \( z^2 = x^2 + y^2 \) and \( w = \log(z) \). For example, assuming \( z \) is the radius of the circle, when solving for \( y(x) \) gives the multi-valued function

\[
y(x) = \pm \sqrt{z^2 - x^2}.
\]

If we accept the modern definition of a function, then \( y(x) \) is not a function, nor even two functions. For example what if \( x > z \)? Or worse, what if \( z = 2 \imath \) with \( |x| < 1 \). Riemann’s solution resolves all these difficulties (as best I know).

To proceed, we need definitions and classifications of the various types of complex singularities:

1. Poles of order 1 are called simple poles. Their amplitude called the residue (e.g. \( \alpha/s \) has residue \( \alpha \)). Simple poles are special (Eq. 1.97, p. 114)\(^{68}\) and play a key role in mathematical physics. Consider the function \( y(s) = \sqrt{s^2 - \alpha} \) with \( \alpha \in \mathbb{Z}, \mathbb{F}, \mathbb{R} \) and \( \mathbb{C} \).

2. When the numerator and denominator of a rational function have a common root (i.e., factor), that root is said to be removable.

3. A singularity that is not 1) removable, a 2) pole or 3) branch point, is called essential.

4. When the first derivative of a function \( Z(s) \) has a simple pole at \( s_o \), then \( s_o \) is said to be a branch point of \( Z(s) \) (e.g., \( d \ln(s^\alpha)/ds = \alpha/s \)). However the converse does not necessarily hold.

5. A complex function which is analytic, except for isolated poles, is called metamorphic.\(^{69}\)

Metamorphic functions can have any number of poles, even an infinite number.

More complex typologies are being researched today, and progress that is expected to accelerate due to modern computing technology.\(^{70}\) It is helpful to identify the physical meaning of these more complex surfaces, to guide us in their interpretation and possible applications.\(^{71}\)

Branch cuts: Up to this point we have only considered poles of order \( k \), of the form \( 1/s^k \), with \( k \in \mathbb{Z} \). The concept of a branch cut allows one to manipulate (and visualize) multi-valued functions, for which \( k \in \mathbb{F} \). This is done by breaking each region into a single valued sheets. The concepts of the branch cut, the sheets, and the extended plane, were first devised by Riemann, working with Gauss (1777-1855), and first described in his thesis of 1851. Of course it was these mathematical and geometrical constructions that provide the deep insight to complex analytic functions, supplementing the important earlier work of Cauchy (1789-1857) on the calculus of

\(^{68}\)https://en.wikipedia.org/wiki/Pole_(complex_analysis)

\(^{69}\)https://en.wikipedia.org/wiki/Meromorphic_function

\(^{70}\)https://www.maths.ox.ac.uk/about-us/departmental-art/theory

\(^{71}\)https://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201
complex analytic functions. For an intuitive discussion of Riemann sheets and branch cuts, see Boas (1987, Section 29, pp. 221-225).

Figure 1.23: Here we see the mapping for the square root function \( w(z) = \pm \sqrt{z} \) which has two single-valued sheets, corresponding to the two signs of the square root. The location of the branch cut may be moved by rotating the \( z \) coordinate system. For example, \( w(z) = \pm j \sqrt{z} \) and \( w(z) = \pm \sqrt{-z} \) have a different branch cuts, as may be easily verified using the Matlab/Octave commands \( j \text{zviz}(z) \) and \( \text{zviz}(-z) \). A function is analytic on the branch cut, since the cut may be moved. If a Taylor series is formed on the branch cut, it will describe the function on the two different sheets. Thus the complex analytic series (i.e., the Taylor formula, Eq. 1.80) does not depend on the location of a branch cut, as it only describes the function uniquely (as a single valued function), valid in its local region of convergence. This figure has been taken from Stillwell (2010, p. 303). A more comprehensive view presented in the text. The branch cut lies in the domain \( x \in \mathbb{C} \), not in the codomain \( w(x) \in \mathbb{C} \) (Kusse and Westwig, 2010). This becomes clear by studying how \( zviz.m \) works, and understanding its output. A example that shows this is Fig. 1.24, where the axes are \( s \) and \( \omega \), with the branch cut along the negative \( \sigma \) axis (\( \theta = \pi \)).

Figure 1.24: Colorized plots of two sheets of \( w(s) = \pm \sqrt{s} \) with the branch cut at \( \theta = \pi \). **Left:** Color map reference plane \( (z = s) \). **Center:** \( e^{\pi j} \sqrt{s} = -\sqrt{s} \). This sheet was generated by rotating \( w \) by 180°. This should be compared to the right panel of Fig. 1.25 which shows \(-\sqrt{-s}\), and to the upper sheet of Fig. 1.23. Note how the panel on the right of Fig. 1.25 matches the right half of \( s \) (purple = -90°, yellow/green = +90°) while the middle panel above comes from the left side of \( s \) (green to purple). The center sheet is green at -180°, and purple at +180°, which matches the right sheet at \( \pm 180° \) respectively. (i.e., \( e^{\pi j} \sqrt{s} \)). **Right:** the lower sheet of Fig. 1.23.

**Square root function:** The branch cut is a line that separates the various single valued parts of a multi-valued function. For example, in Fig. 1.23 we see the single valued function \( w(z) = s^2 \) (left), and on the right, its inverse, the double-valued mapping of \( s(w) = \pm \sqrt{w} \).
The multi-valued nature of the square root is best understood by working with the function in polar coordinates. Let

\[ s_k = re^{\theta j}e^{2\pi kj}, \]

where \( k \) is the sheet-index, and

\[ w = \rho e^{\psi j} = \sqrt{r}e^{\theta/2}e^{\pi k}, \]

where \( r, \rho \in \mathbb{R} \) are the magnitudes and \( \theta, \psi \in \mathbb{R} \) are the angles. The domain-coloring program \( \text{zviz.m} \) assumes that the angles go from \(-\pi < \theta < \pi\), with \( \theta = 0 \) being a light red and \( \pm \pi \) a blue color. This color map is shown in the left panel of Fig. 1.24. The first Riemann sheet \((k = 0)\) for \( s_k \) is define for \(-\pi < \theta < \pi\).

The second sheet \((k = 1)\) picks up at \( \theta = \pi \) and continues on to \( \pi + 2\pi = 3\pi \). The first sheet maps the angle of \( w \) (i.e., \( \phi = \angle w = \theta/2 \)) from \(-\pi/2 < \phi < \pi/2 \) \((w = \sqrt{r}e^{\theta/2})\). This corresponds to \( u = \Re\{w(s)\} > 0 \). The second sheet maps \( \pi/2 < \psi < 3\pi/2 \) (i.e., \( 90^\circ \) to \( 270^\circ \)), which is \( \Re\{w\} = u < 0 \). In summary, twice around the \( s \) plane is once around the \( w(s) \) plane, because the angle is half due to the \( \sqrt{s} \). This then describes the multi-valued nature of the square root function.

This concept of analytic inverses becomes important only when the function is multi-valued. For example, if \( w(s) = s^2 \) then \( s(w) = \pm \sqrt{w} \) is multivalued. Riemann dealt with such extensions with the concept of a branch-cut with multiple sheets, labeled by a sheet number. Each sheet describes an analytic function (Taylor series), that converges within some RoC, having a radius out to the nearest pole. This Riemann’s branch cut and sheets explicitly deal with the need to define unique single valued inverses of multi-valued functions. Since the square root function has two overlapping regions, corresponding to the \( \pm \) due to the radical, there must be two connected regions, sort of like mathematical Siamese-twins, distinct, yet the same.

Branch cuts emanate from branch points, singularities that can have fractional order, for example \( 1/\sqrt{s} \), and terminate at either another singularity having the same fractional order, which may be at \( \infty \). For example, suppose that in the neighborhood of the pole, at \( s_o \), the function is

\[ f(s) = \frac{w(s)}{(s - s_o)^k}, \]

where \( w, s, K, k \in \mathbb{C} \). When \( k = 1 \), \( s_o = \sigma_o + \omega_o j \) is the pole location in the \( s \) plane and \( w(s_o) \) is the residue. When \( k \) is a complex (or real) constant, it defines the order of the pole.\(^{72}\)

When \( k \in \mathbb{F} \) there must be a branch cut, of order \( k \). For example, if \( k = 1/2 \), the singularity (branch cut) is of order \( 1/2 \), and there are two Riemann sheets, as shown in Fig. 1.23. An important example is the Bessel function

\[ \delta(t) + \frac{1}{t} J_1(t)u(t) \leftrightarrow \sqrt{s^2 + 1}, \]

which is related to the solution to the wave equation in two-dimensional cylindrical coordinates (Table 1.5, p. 94). Bessel functions are the solutions to guided acoustic waves in round pipes, or surface waves on the earth (seismic waves), or waves on the surface of a pond.

\(^{72}\)We shall refer to the order of a pole and the degree of a differential equation.
It is important to understand that the function is analytic on the branch cut, but not at the branch point. One is free to move the branch cut (at will). It does not need to be on a line, it could be cut in almost any connected manner, such as a spiral. The only rule is that it must start and stop at the matching branch points, or at $\infty$, which must have the same order.

There are a limited number of possibilities for the order, $k \in \mathbb{Z}$ or $\in \mathbb{F}$. If the order is drawn from $\mathbb{R} \notin \mathbb{F}$, the pole cannot not have a residue. According to the definition of the residue, $k \in \mathbb{F}$ will not give a residue. But there remains open the possibility of generalizing the concept of the Riemann Integral Theorem, to include $k \in \mathbb{F}$. This seems unlikely however.

If the singularity had an irrational order ($k \in \mathbb{I}$), the branch cut has the same “irrational order.” Accordingly there would be an infinite number of Riemann sheets, as in the case of the log function. An example is $k = \pi$, for which

$$F(s) = \frac{1}{s^\pi} = e^{-\log(s^\pi)} = e^{-\pi \log(s)} = e^{-\pi \log(\rho)} e^{-\pi \theta \phi},$$

where the domain is expressed in polar coordinates $s = \rho e^{\phi \theta}$. When $k \in \mathbb{F}$ is may be arbitrarily close to 1 (e.g., two very close primes, such as 881 and 883, primes 152 and 153), the branch cut could be very subtle (it could even be unnoticed), but it would have a significant impact on the nature of the function, and of course, on the inverse Laplace transform.

**Multivalued functions:** The two basic functions we review, to answer the questions about multi-valued functions and branch cuts, are $w(s) = s^2$ and $w(s) = e^s$, along with their inverse functions $w(s) = \sqrt{s}$ and $w(s) = \log(s)$. For uniformity we shall refer to the complex abscissa ($s = \sigma + \omega \phi$) and the complex ordinate $w(s) = u + v \phi$. When the complex abscissa and domain are swapped, by taking the inverse of a function, multi-valued functions are a common consequence. For example, $f(t) = \sin(t)$ is single valued, and analytic in $t$, thus has a Taylor series. The inverse function $t(f)$ is not so fortunate as it is multivalued.

The modern terminology is the domain and range, or alternatively the co-domain.\footnote{The best way to create confusion is the rename things. The confusion grows geometrically with each renaming. I}

\footnote{since there are no even primes, the minimum difference is 2. Out of $10^6$ primes 5 have a spacing of 80, and the distribution is linear on a log scale.)}
Log function: Next we discuss the multi-valued nature of the log function. In this case there are an infinite number of Riemann sheets, not well captured by Fig. 1.19 (p. 86), which only displays the principal sheet. However if we look at the formula for the log function, the nature is easily discerned. The abscissa \( s \) may be defined as multi-valued since

\[
s_k = r e^{2\pi k j} e^{\theta j},
\]

Here we have extended the angle of \( s \) by \( 2\pi k \), where \( k \) is the sheet index \( \in \mathbb{Z} \). Taking the log

\[
\log(s) = \log(r) + (\theta + 2\pi k) j.
\]

When \( k = 0 \) we have the principal value sheet, which is zero when \( s = 1 \). For any other value of \( k \) \( w(s) \neq 0 \), even when \( r = 1 \), since the angle is not zero, except for the \( k = 0 \) sheet.

1.4.5 Lec 27: Three Cauchy Integral Theorems

Cauchy’s theorems for integration in the complex plane

There are three basic deﬁnitions related to Cauchy’s integral formula. They are closely related, and can greatly simplify integration in the complex plane.

1. Cauchy’s (Integral) Theorem:

\[
\oint_C F(s) ds = 0, \tag{1.95}
\]

if and only if \( F(s) \) is complex-analytic inside of a simple closed curve \( C \) (Boas, 1987, p. 45),(Stillwell, 2010, 319). The FTCC (Eq. 1.79) says that the integral only depends on the end points if \( F(s) \) is complex-analytic. By closing the path (contour \( C \)) the end points are the same, thus the integral must be zero, as long as \( F(s) \) is complex analytic.

2. Cauchy’s Integral Formula:

\[
\frac{1}{2\pi j} \oint_C \frac{F(s)}{s-s_0} ds = \begin{cases} F(s_0), & s_0 \in C \text{ (inside)} \\ 0, & s_0 \notin C \text{ (outside)} \end{cases} \tag{1.96}
\]

Here \( F(s) \) is required to be analytic everywhere within (and on) the contour \( C \) (Greenberg, 1988, p. 1200),(Boas, 1987, p. 51), (Stillwell, 2010, p. 220). The value \( F(s_0) \in \mathbb{C} \) is the residue of the pole \( s_o \) of \( F(s)/(s-s_o) \).

3. The (Cauchy) Residue Theorem (Greenberg, 1988, p. 1241), (Boas, 1987, p. 73)

\[
\oint_C F(s) ds = 2\pi j \sum_{k=1}^{K} c_k = \sum_{k=1}^{K} \oint_{s-s_k} \frac{F(s)}{s-s_k} ds, \tag{1.97}
\]

where the residues \( c_k \in \mathbb{C} \), corresponding to the \( k \)th poles of \( f(s) \), enclosed by the contour \( C \). By the use of Cauchy’s integral formula, the last form of the residue theorem is equivalent to the residue form. \(^{75}\)

\(^{75}\)This theorem is the same as a 2D version of Stokes thm (citations).
How to calculate the residue: The case of first order poles, while special, is important, since the Brune impedance only allows simple poles and zeros, increasing the utility of this special case. The residues for simple poles are $F(s_k)$, which is complex analytic in the neighborhood of the pole, but not at the pole.

Consider the function $f(s) = F(s)/(s - s_k)$, where we have factored $f(s)$ to isolate the first-order pole at $s = s_k$, with $F(s)$ analytic at $s_k$. Then the residue of the poles at $c_k = F(s_k)$. This coefficient is computed by removing the singularity, by placing a zero at the pole frequency, and taking the limit as $s \to s_k$, namely

$$c_k = \lim_{s \to s_k} [(s - s_k)F(s)]$$  \hspace{1cm} (1.98)


When the pole is an $N$th order, the procedure is much more complicated, and requires taking $N - 1$ derivatives of $f(s)$, followed by the limit process (Greenberg, 1988, p. 1242). Higher order poles are rarely encountered, thus it is good to know that this formula exists, but perhaps it is not worth the effort to memorize it.

Summary and examples: These three theorems, all attributed to Cauchy, collectively are related to the fundamental theorems of calculus. Because the names of the three theorems are so similar, they are easily confused.

1. In general it makes no sense to integrate through a pole, thus the poles (or other singularities) must not lie on $C$.

2. The Cauchy integral theorem (Eq. 1.95), follows trivially from the fundamental theorem of complex calculus (Eq. 1.79, p. 102), since if the integral is independent of the path, and the path returns to the starting point, the closed integral must be zero. Thus Eq. 1.95 holds when $F(s)$ is complex analytic within $C$.

3. Since the real and imaginary parts of every complex analytic function obey Laplace’s equation (Eq. 1.85, p. 104), it follows that every closed integral over a Laplace field, i.e., one defined by Laplace’s equation, must be zero. In fact this is the property of a conservative system, corresponding to many physical systems. If a closed box has fixed potentials on the walls, with any distribution whatsoever, and a point charge (i.e., an electron) is placed in the box, then a force equal to $F = qE$ is required to move that charge, thus work is done. However if the point is returned to its starting location, the net work done is zero.

4. Work is done in charging a capacitor, and energy is stored. However when the capacitor is discharged, all of the energy is returned to the load.

5. Soap bubbles and rubber sheets on a wire frame, obey Laplace’s equation.

6. These are all cases where the fields are Laplacian, thus closed line integrals must be zero. Laplacian fields are commonly observed, because they are so basic.

7. We have presented the impedance as the primary example of a complex analytic function. Physically, every impedance has an associated stored energy, and every system having stored energy has an associated impedance. This impedance is usually defined in the frequency $s$
domain, as a force over a flow (i.e., voltage over current). The power $P(t)$ is defined as the force times the flow and the energy $\mathcal{E}(t)$ as the time integral of the power

$$\mathcal{E}(t) = \int_{-\infty}^{t} P(t) dt,$$

which is similar to Eq. 1.76 (p. 101) [see Section 3.2.1, Eq. 3.10 (p. 189)]. In summary, impedance and power and energy are all fundamentally related.

### 1.4.6 Lec 28: Cauchy Integral Formula & Residue Theorem

The Cauchy integral formula (Eq. 1.96) is an important extension of the Cauchy integral theorem (Eq. 1.95) in that a pole has been explicitly injected into the integrand at $s = s_0$. If the pole location is outside of the curve $C$, the result of the integral is zero, in keeping with Eq. 1.95. However, when the pole is inside of $C$, the integrand is no longer complex analytic, and a new result follows. By a manipulation of the contour in Eq. 1.96, the pole can be isolated with a circle around the pole, and then taking the limit, the radius goes to zero.

For the related Cauchy residue theorem (Eq. 1.97) the same result holds, except it is assumed that there are $K$ simple poles in the function $F(s)$. This requires the repeated application of Eq. 1.96, $K$ times, so it represents a minor extension of Eq. 1.96. The function $F(s)$ may be written as $f(s)/P_K(s)$, where $f(s)$ is analytic in $C$ and $P_K(s)$ is a polynomial of degree $K$, with all of its roots $s_k \in C$.

**Non-integral degree singularities:** The key point is that this theorem applies when $n \in \mathbb{I}$, including fractionals $n \in \mathbb{F}$, but for these cases the residue is always zero, since by definition the residue is the amplitude of the $1/s$ term (Boas, 1987, p. 73).

**Examples:**

1. The function $1/\sqrt{s}$ has a zero residue (Hint: apply the definition of the residue Eq. 1.98).
2. When $n \in \mathbb{F}$, the residue is, by definition, zero.
3. When $n \in \mathbb{I}$, the residue is, by definition, zero.
4. When $n = 1$, the residue is non-zero.

This point is equally important when defining the inverse Laplace transform. When integrating over $\omega \in \mathbb{R}$, the value passes through all possible exponents, including all rational and irrational numbers. The only value of $\omega$ that has a residue, are those at the poles of the integrand.

### 1.4.7 Lec 29: Inverse Laplace transform & Cauchy residue theorem

The inverse Laplace transform Eq. 1.71 transforms a function of complex frequency $F(s)$ and returns a causal function of time $f(t)$

$$f(t) \leftrightarrow F(s),$$

where $f(t) = 0$ for $t < 0$. Examples are provided in Table 1.5 (p. 94). We next discuss the details of finding the inverse transform by use of the Cauchy residue theorem, and how the causal requirement $f(t < 0) = 0$ comes about.
The integrand of the inverse transform is $F(s)e^{st}$ and the limits of integration are $-\sigma_o \pm \omega j$. To find the inverse we must close the curve, at infinity, and show that the integral at $\omega j \to \infty$. There are two ways to close these limits, to the right $\sigma > 0$ (RHP), and to the left $\sigma < 0$ (LHP), but there needs to be some logical reason for this choice. That logic is the sign of $t$. For the integral to converge the term $e^{st}$ must go to zero as $\omega \to \infty$. In terms of the real and imaginary parts of $s = \sigma + \omega j$, the exponential may be rewritten as $e^{\sigma t}e^{\omega t}$. Note that both $t$ and $\omega$ go to $\infty$. So it is the interaction between these two limits that determines how we pick the closure, RHP vs. LHP.

Case for causality ($t < 0$): Let us first consider negative time, including $t \to -\infty$. If we were to close $C$ in the left half plane ($\sigma < 0$), then the product $\sigma t$ is positive ($\sigma < 0$, $t < 0$, thus $\sigma t > 0$). In this case as $\omega \to \infty$, the closure integral $|s| \to \infty$ will diverge. Thus we may not close in the LHP for negative time. If we close in the RHP $\sigma > 0$ then the product $\sigma t < 0$ and $e^{st}$ will go to zero as $\omega \to \infty$. This then justifies closing the contour, allowing for the use the Cauchy theorems.

If $F(s)$ is analytic in the RHP, the FTCC applies, and the resulting $f(t)$ must be zero, and the inverse Laplace transform must be causal. This argument holds for any $F(s)$ that is analytic in the RHP ($\sigma > 0$).

Case of unstable poles: An important but subtle point arises: If $F(s)$ has a pole in the RHP, then the above argument still applies if we pick $\sigma_o$ to be to the right of the RHP pole. This means that the inverse transform may still be applied to unstable poles (those in the RHP). This explains the need for the $\sigma_o$ in the limits. If $F(s)$ has no RHP poles, then $\sigma_o = 0$ is adequate, and this factor may be ignored.

Case for zero time ($t = 0$): When time is zero, the integral does not, in general, converge, leaving $f(t)$ undefined. This is most clear in the case of the step function $u(t) \leftrightarrow 1/s$, where the integral may not be closed, because the convergence factor $e^{st} = 1$ is lost for $t = 0$.

1.4.8 Lec 30: Inverse Laplace transform ($t > 0$)

Case of $t > 0$: Next we investigate the convergence of the integral for positive time $t > 0$. In this case we must close the integral in the LHP ($\sigma < 0$) for convergence, so that $st < 0$ ($\sigma \leq 0$ and $t > 0$). When there are poles on the $\omega j = 0$ axis, $\sigma_o > 0$ assures convergence by keeping the on-axis poles inside the contour. At this point the Cauchy residue theorem (Eq. 1.97) is relevant. If we restrict ourselves to simple poles (as required for a Brune impedance), the residue theorem may be directly applied.

The most simple example is the step function, for which $F(s) = 1/s$ and thus

$$u(t) = \int_{\text{LHP}} \frac{e^{st}}{s} \frac{ds}{2\pi j} \leftrightarrow \frac{1}{s},$$

which is a direct application of the Cauchy Residue theorem, Eq. 1.97 (p. 114). The forward transform of $u(t)$ is straight forward, as discussed in Section 1.3.12 (p. 91). This is true of most if not all of the elementary forward Laplace transforms. In these cases, causality is built into the integral by the limits, so is not a result, as it must be in the inverse transform. An interesting problem is proving that $u(t)$ is not defined at $t = 0$.
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\[ u + jv = \text{besselj}(0, \pi(x+jy)) \]

Figure 1.27: Colorized plot of the of \( 1/(s^2 + 1) \leftrightarrow J_0(t)u(t) \).

The inverse Laplace transform of \( F(s) = 1/(s + 1) \) has a residue of 1 at \( s = -1 \), thus that is the only contribution to the integral. More demanding cases are Laplace transform pairs

\[ \frac{1}{\sqrt{t}}u(t) \leftrightarrow \sqrt{\frac{\pi}{s}} \quad \text{and} \quad J_0(t)u(t) \leftrightarrow \frac{1}{s^2 + 1} = \frac{1}{(s+j)(s-j)}. \]

Many of these are easily proved in the forward direction, but are much more difficult in the inverse direction, due to the properties at \( t = 0 \), unless of course, the residue theorem (Eq. 1.97, p. 114) is invoked. The last \( \mathcal{L}\mathcal{T} \)-pair helps us understand the basic nature of the Bessel functions \( J_0(z) \), and \( H^{(1)}_0(z^2) \), with a branch cut along the negative axis (see Fig. 3.3, p. 199).

**Some open questions:** Without the use of the CRT (Eq. 1.97) it is difficult to see how to evaluate the inverse Laplace transform of \( 1/s \) directly. For example, how does one show that the above integral is zero for negative time (or that it is 1 for positive time)? The CRT neatly solves this difficult problem, by the convergence of the integral for negative and positive time. Clearly the continuity of the integral at \( \omega \to \infty \) plays an important role. Perhaps the Riemann sphere plays a role in this, that has not yet been explored.

1.4.9 Lec 31: Properties of the LT (e.g., Linearity, convolution, time-shift, modulation, etc.)

As shown in the table of Laplace transforms, there are integral (i.e., integration, not integer) relationships, or properties, that are helpful to identify. The first of these is a definition not a property:76

\[ f(t) \leftrightarrow F(s). \]

When taking the LT, the time response is given in lower case (e.g., \( f(t) \)) and the frequency domain transform is denoted in upper case (e.g., \( F(s) \)). It is required, but not always explicitly specified, that \( f(t < 0) = 0 \), that is, the time function must be causal (P1: Section 1.3.13).

76Put this notional property in Appendix A.
**Linearity:** A key property so basic that it almost is forgotten, is the linearity property of the LT. These properties are summarized as P2 of Section 1.3.13, 95).

**Convolution property:** One of the most basic and useful properties is that the product of two LTs in frequency, results in convolution in time

\[ f(t) \ast g(t) = \int_0^t f(\tau)g(t - \tau)\, d\tau \leftrightarrow F(s)G(s), \]

where we use the \( \ast \) operator as a shorthand for the convolution of two time functions.

A key application of convolution is filtering, which takes on many forms. The most basic filter is the moving average, the moving sum of data samples, normalized by the number of samples. Such a filter has very poor performance. It also introduces a delay of half the length of the average, which may, or may not constitute a problem, depending on the application. Another important example is a low-pass filter, that removes high frequency noise, or a notch filter that removes line-noise (i.e., 60 [Hz] in the US, and its 2d and 3d harmonics, 120 and 180 [Hz]). Such noise is typically a result of poor grounding and ground loops. It is better to solve the problem at its root, than to remove it with a notch filter. Still, filters are very important in engineering.

By taking the LT of the convolution we can derive this relationship

\[
\int_0^\infty [f(t) \ast g(t)]e^{-st}\, dt = \int_0^{\infty} \int_0^t f(\tau)g(t - \tau)d\tau e^{-st}\, dt \\
= \int_0^t f(\tau) \left( \int_0^{\infty} g(t - \tau)e^{-st}\, dt \right) d\tau \\
= \int_0^t f(\tau) \left( e^{-st} \int_0^{\infty} g(t')e^{-s't}\, dt' \right) d\tau \\
= G(s) \int_0^t f(\tau)e^{-s\tau}\, d\tau \\
= G(s)F(s)
\]

We first encountered this relationship in Section 1.3.4 (p. 70)) in the context of multiplying polynomials, which was the same as convolving their coefficients. Hopefully the parallel is obvious. In the case of polynomials, the convolution was discrete in the coefficients, and here it is continuous in time. But the relationships are the same.

**Time-shift propriety:** When a function is time-shifted by time \( T_o \), the LT is modified by \( e^{sT_o} \), leading to the propriety

\[ f(t - T_o) \leftrightarrow e^{-sT_o} F(s). \]

This is easily shown by applying the definition of the LT to a delayed time function.

**Time derivative:** The key to the eigen-function analysis provided by the LT, is the transformation of a time derivative on a time function, that is

\[ \frac{d}{dt}f(t) \leftrightarrow sF(s). \]

Here \( s \) is the eigen-value corresponding to the time derivative of \( e^{st} \). Given the definition of the derivative of \( e^{st} \) with respect to time, this definition seems trivial. Yet that definition was not
obvious to Euler. It needed to be extended to the space of complex analytic function $e^{st}$, which did not happen until at least Riemann (1851).

*Given a differential equation of order $K$, the LT results in a polynomial in $s$, of degree $K$.

It follows that this LT property is the corner-stone of why the LT is so important to scalar differential equations, as it was to the early analysis of Pell’s equation and the Fibonacci sequence, as presented in earlier chapters. This property was first uncovered by Euler. It is not clear if he fully appreciated its significance, but by the time of his death, it certainly would have been clear to him. Who first coined the term *eigen-value* and *eigen-function*? The word *eigen* is a German word meaning *of one*. It seem likely that this term became popular somewhere between the 19th and 20th century.

**Initial and final value theorems:** There are much more subtle relations between $f(t)$ and $F(s)$ that characterize $f(0^+)$ and $f(t \to \infty)$. While these properties can be very important in certain application, they are beyond the scope of the present treatment. These relate to so-called *initial value theorems*. If the system under investigation has potential energy at $t = 0$, then the voltage (velocity) need not be zero for negative time. An example is a charged capacitor or a moving mass. These are important situations, but better explored in a more in depth treatment.

1.4.10  **Lec 32: ’Lecture:’ Review for Exam III (Evening Exam)**