2.3 Problems AE-3

Topics of this homework:

Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.

Deliverables: Answers to problems

Two-port network analysis

Problem # 1: Perform an analysis of electrical two-port networks, shown in Fig. 3.8 (page 106). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

The definition of the ABCD transmission matrix \((\mathcal{T})\) is

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}.
\] (AE-3.1)

The impedance matrix, where the determinant \(\Delta_T = AD - BC\), is given by

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \frac{1}{C}
\begin{bmatrix}
A & \Delta_T \\
1 & D
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\] (AE-3.2)

\[1.1:\] Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.

Ans:

Problem # 2: Consider a single circuit element with impedance \(Z(s)\).

\[2.1:\] What is the ABCD matrix for this element if it is in series?

Ans:

\[2.2:\] What is the ABCD matrix for this element if it is in shunt?

Ans:

Problem # 3: Find the ABCD matrix for each of the circuits of Fig. 3.8.

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency \(s \in \mathbb{C}\), then (ii) substitute \(s = 1/j\) and calculate the total transmission matrix at this single frequency.
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– 3.1: Left circuit (let \( R_1 = R_2 = 10 \) kilo-ohms and \( C = 10 \) nano-farads)

Ans:

– 3.2: Right circuit (use \( L \) and \( C \) values given in the figure), where the pressure \( P \) is analogous to the voltage \( V \), and the velocity \( U \) is analogous to the current \( I \).

Ans:

– 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency \( s = 1 \) as in the previous part (feel free to use Matlab/Octave for your computation).

Ans:

– 3.4: Right circuit: Repeat the analysis as in question 3.3.

Ans:

Algebra

Problem # 4: Fundamental theorem of algebra (FTA).

– 4.1: State the fundamental theorem of algebra (FTA).

Ans:

Algebra with complex variables

Problem # 5: Order and complex numbers:
One can always say that \( 3 < 4 \)—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in \( 4 - 3 > 0 \). Here we will explore how complex variables may be ordered. Define the complex variable \( z = x + y_1 \in \mathbb{C} \).
5.1: Explain the meaning of $|z_1| > |z_2|$.

Ans:

5.2: If $x_1, x_2 \in \mathbb{R}$ (are real numbers), define the meaning of $x_1 > x_2$. Hint: Take the difference.

Ans:

5.3: Explain the meaning of $z_1 > z_2$.

Ans:

5.4: If time were complex, how might the world be different?

Ans:

Problem # 6: It is sometimes necessary to consider a function $w(z) = u + vz$ in terms of the real functions $u(x, y)$ and $v(x, y)$ (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse $z(w) = x + yj$, where $x(u, v)$ and $y(u, v)$ are real functions.

6.1: Find $u(x, y)$ and $v(x, y)$ for $w(z) = 1/z$.

Ans:

Problem # 7: Find $u(x, y)$ and $v(x, y)$ for $w(z) = c^z$ with complex constant $c \in \mathbb{C}$ for questions 7.1, 7.2, and 7.3:

7.1: $c = e$

Ans:
- 7.2: $c = 1$ (recall that $1 = e^{j2\pi k}$ for $k = 0, 1, 2, \ldots$)

**Ans:**

- 7.3: $c = j$. Hint: $j = e^{j\pi/2 + j2\pi m}$, $m \in \mathbb{Z}$.

**Ans:**

- 7.4: Find $u(x, y)$ for $w(z) = \sqrt{z}$.

**Ans:**

**Problem #8: Convolution of an impedance $z(t)$ and its inverse $y(t)$:**

In the frequency domain a Brune impedance is defined as the ratio of a numerator polynomial $N(s)$ to a denominator polynomial $D(s)$.

- 8.1: Consider a Brune impedance defined by the ratio of numerator and denominator polynomials, $Z(s) = N(s)/D(s)$. Since the admittance $Y(s)$ is defined as the reciprocal of the impedance, the product must be 1. If $z(t) \leftrightarrow Z(s)$ and $y(t) \leftrightarrow Y(s)$, it follows that $z(t) \ast y(t) = \delta(t)$. What property must $n(t) \leftrightarrow N(s)$ and $d(t) \leftrightarrow D(s)$ obey for this to be true?

**Ans:**

- 8.2: The definition of a minimum phase function is that it must have a causal inverse. Show that every impedance is minimum phase.

**Ans:**

**Schwarz inequality**

**Problem #9:** The above figure shows three vectors for an arbitrary value of $\alpha \in \mathbb{R}$ and a specific value of $\alpha = \alpha^*$. 


- 9.1: Find the value of $\alpha \in \mathbb{R}$ such that the length (norm) of $E$ (i.e., $||E|| \geq 0$) is minimum. Show your derivation, not the answer ($\alpha = \alpha^*$).

Ans:

- 9.2: Find the formula for $||E(\alpha^*)||^2 \geq 0$. Hint: Substitute $\alpha^*$ into Eq. 3.5.9 (p. 92) and show that this results in the Schwarz inequality $|U \cdot V| \leq ||U|| ||V||$.

Ans:

Problem # 10: Geometry and scaler products

- 10.1: What is the geometrical meaning of the dot product of two vectors?
Ans: }
– 10.3: Write the formula for the dot product of two vectors $\vec{U} \cdot \vec{V}$ in $\mathbb{R}^n$ in polar form (e.g., assume the angle between the vectors is $\theta$).

\textbf{Ans:}

– 10.4: How is the Schwarz inequality related to the Pythagorean theorem?

\textbf{Ans:}

– 10.5: Starting from $||U + V||$, derive the triangle inequality

\[ ||U + V|| \leq ||U|| + ||V||. \]

\textbf{Ans:}

– 10.6: The triangle inequality $||U + V|| \leq ||U|| + ||V||$ is true for two and three dimensions: Does it hold for five-dimensional vectors?

\textbf{Ans:}

– 10.7: Show that the wedge product $\vec{U} \wedge \vec{V} \perp \vec{U} \cdot \vec{V}$.

\textbf{Ans:}

\textbf{Probability}

\textbf{Problem # 11: Basic terminology of experiments}
11.1: What is the mean of a trial, and what is the average over all trials?

\textbf{Ans:}

11.2: What is the expected value of a random variable \( X \)?

\textbf{Ans:}

11.3: What is the standard deviation about the mean?

\textbf{Ans:}

11.4: What is the definition of information of a random variable?

\textbf{Ans:}

11.5: How do you combine events? Hint: If the event is the flip of a biased coin, the events are \( H = p, \ T = 1 - p \), so the event is \( \{ p, 1 - p \} \). To solve the problem, you must find the probabilities of two independent events.

\textbf{Ans:}

11.6: What does the term independent mean in the context of question 11.5? Give an example.

\textbf{Ans:}

11.7: Define odds.

\textbf{Ans:}