2.3 Problems AE-3

Topics of this homework:
Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.

Deliverables: Answers to problems

Two-port network analysis

Problem # 1: Perform an analysis of electrical two-port networks, shown in Fig. 3.8 (page 105). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

The definition of the ABCD transmission matrix \((T)\) is
\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix}
= \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}.
\]
\[\text{(AE-3.1)}\]

The impedance matrix, where the determinant \(\Delta_T = AD - BC\), is given by
\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= \frac{1}{C}
\begin{bmatrix}
A & \Delta_T \\
1 & D
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\]
\[\text{(AE-3.2)}\]

- 1.1: Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.

Ans:

Problem # 2: Consider a single circuit element with impedance \(Z(s)\).

- 2.1: What is the ABCD matrix for this element if it is in series?

Ans:

- 2.2: What is the ABCD matrix for this element if it is in shunt?

Ans:
Problem # 3: Find the $ABCD$ matrix for each of the circuits of Fig. 3.8.
For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency $s \in \mathbb{C}$, then (ii) substitute $s = 1j$ and calculate the total transmission matrix at this single frequency.

- 3.1: Left circuit (let $R_1 = R_2 = 10$ kilo-ohms and $C = 10$ nano-farads)

- 3.2: Right circuit (use $L$ and $C$ values given in the figure), where the pressure $P$ is analogous to the voltage $V$, and the velocity $U$ is analogous to the current $I$.

- 3.3: Convert both transmission ($ABCD$) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency $s = 1j$ as in the previous part (feel free to use Matlab/Octave for your computation).

- 3.4: Right circuit: Repeat the analysis as in question 3.3.

Algebra

Problem # 4: Fundamental theorem of algebra (FTA).

- 4.1: State the fundamental theorem of algebra (FTA).

Ans:

(13 pts) Algebra with complex variables

Problem # 5: (7 pts) Order and complex numbers:
One can always say that $3 < 4$—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in $4 - 3 > 0$. Here we will explore how complex variables may be ordered. In the following define $\{x, y\} \in \mathbb{R}$ and complex variable $z = x + yj \in \mathbb{C}$.

- 5.1: Explain the meaning of $|z_1| > |z_2|$.

Ans:

- 5.2: If $x_1, x_2 \in \mathbb{R}$ (are real numbers), define the meaning of $x_1 > x_2$.

Ans:
2.3. PROBLEMS AE-3

– 5.3: Explain the meaning of $z_1 > z_2$.

**Ans:**

– 5.4: (2 pts) What is the meaning of $|z_1 + z_2| > 3$?

**Ans:**

– 5.5: (2 pts) If time were complex, how might the world be different?

**Problem # 6: (1 pt)** It is frequently necessary to consider a function $w(z) = u + vj$ in terms of the real functions $u(x, y)$ and $v(x, y)$ (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse $z(w) = x + yj$, where $x(u, v)$ and $y(u, v)$ are real functions.

– 6.1: (1 pts) Find $u(x, y)$ and $v(x, y)$ for $w(z) = 1/z$.

**Ans:**

**Problem # 7: (5 pts)** Find $u(x, y)$ and $v(x, y)$ for $w(z) = c^2$ with complex constant $c \in \mathbb{C}$ for questions 7.1, 7.2, and 7.3:

– 7.1: $c = e$

**Ans:**
Figure 2.2: This figure shows how to derive the Schwarz inequality, by finding the value of \( \alpha = \alpha^* \) corresponding to \( \min_{\alpha} |E(\alpha)| \). It is identical to Fig. 3.5 on page 89.

- 7.2: \( c = 1 \) (recall that \( 1 = e^{\pm j2\pi k} \) for \( k \in \mathbb{Z} \))

\textbf{Ans:}

- 7.3: \( c = j \). Hint: \( j = e^{j\pi/2 + j\pi k}, \quad k \in \mathbb{Z} \).

\textbf{Ans:}

- 7.4: (2 pts) What is \( j^j \)?

\textbf{Schwarz inequality}

\textbf{Problem # 8:} The above figure shows three vectors for an arbitrary value of \( \alpha \in \mathbb{R} \) and a specific value of \( \alpha = \alpha^* \).

- 8.1: Find the value of \( \alpha \in \mathbb{R} \) such that the length (norm) of \( \vec{E} \) (i.e., \( ||\vec{E}|| \geq 0 \)) is minimum. Show your derivation, not the answer (\( \alpha = \alpha^* \)).

\textbf{Ans:}
2.3. PROBLEMS AE-3

– 8.2: Find the formula for \( ||E(\alpha^*)||^2 \geq 0 \). Hint: Substitute \( \alpha^* \) into Eq. 3.5.9 (p. 90) and show that this results in the Schwarz inequality

\[ |\vec{U} \cdot \vec{V}| \leq ||\vec{U}||||\vec{V}|. \]

Ans:

Problem # 9: Geometry and scaler products

– 9.1: What is the geometrical meaning of the dot product of two vectors?
Ans:

– 9.2: Give the formula for the dot product of two vectors. Explain the meaning based on Fig. 3.4 (page 85).
Ans:

– 9.3: Write the formula for the dot product of two vectors \( \vec{U} \cdot \vec{V} \) in \( \mathbb{R}^n \) in polar form (e.g., assume the angle between the vectors is \( \theta \)).
Ans:

– 9.4: How is the Schwarz inequality related to the Pythagorean theorem?
Ans:
– 9.5: Starting from $||\mathbf{U} + \mathbf{V}||$, derive the triangle inequality

$$||\mathbf{U} + \mathbf{V}|| \leq ||\mathbf{U}|| + ||\mathbf{V}||.$$  

– 9.6: The triangle inequality $||\mathbf{U} + \mathbf{V}|| \leq ||\mathbf{U}|| + ||\mathbf{V}||$ is true for two and three dimensions: Does it hold for five-dimensional vectors?

– 9.7: Show that the wedge product $\mathbf{U} \wedge \mathbf{V} \perp \mathbf{U} \cdot \mathbf{V}$.

**Ans:**

**Probability**

**Problem # 10: Basic terminology of experiments**

– 10.1: What is the mean of a trial, and what is the average over all trials?

– 10.2: What is the expected value of a random variable $X$?

– 10.3: What is the standard deviation about the mean?

– 10.4: What is the definition of information of a random variable?

– 10.5: How do you combine events? Hint: If the event is the flip of a biased coin, the events are $H = p$, $T = 1 - p$, so the event is $\{p, 1 - p\}$. To solve the problem, you must find the probabilities of two independent events.

– 10.6: What does the term independent mean in the context of question 10.5? Give an example.

– 10.7: Define odds.