ECE 493 | **BV1 – Version 1.83 April 17, 2018** | **Spring 2018**

Univ. of Illinois | **Due Tu, Apr 24** | **Prof. Allen**

Topic of this homework: Partial differential equations: Wave, diffusion, Poisson; Solution methods: separation of variables, Sturm-Liouville BV Theory Transmission lines. Special functions.

Deliverables: Show your work.

Assuming there is no angular dependence only axial variation, the formula for the Laplacian in N dimensions is

$$\nabla_r^2 T(\mathbf{x}) \equiv \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial T(\mathbf{x})}{\partial r}\right). \tag{1}$$

For example, in $N = 1$ dimension, $\nabla_x^2 T = \partial^2 T(x)/\partial x^2$, whereas in spherical coordinates ($N = 3$)

$$\nabla_r^2 T(\mathbf{x}) \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T(\mathbf{x})}{\partial r}\right) \tag{2}$$

1 Laplace' equation

Rectangular coordinate: The Laplacian is the sum of two $N = 1$ independent terms

$$\nabla_{xy} T(\mathbf{x}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T(x, y) = 0. \tag{3}$$

The RHS is zero because the only sources of heat are at the walls.

1. Assuming *separation of variables* $T(x, y) = X(x)Y(y)$, we rewrite the equation as

   $$\nabla_{xy} T(\mathbf{x}) = Y(y)\frac{\partial^2}{\partial x^2}X(x) + X(x)\frac{\partial^2}{\partial y^2}Y(y) = 0.$$

   Dividing by XY

   $$\frac{1}{X}\frac{\partial^2}{\partial x^2}X(x) = k^2 = -\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y(y),$$

   factors the equation into the two terms,

   $$\frac{\partial^2}{\partial x^2}X(x) = k^2 X(x) \qquad \text{and} \qquad \frac{\partial^2}{\partial y^2}Y(y) = -k^2 Y(y), \tag{4}$$

   each of which are equal to the *separation constant*: k^2. Once we find X, Y, $T(x, y) = X(x)Y(y)$.

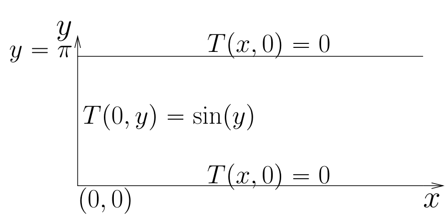

Figure 1: *Two grounded parallel conductors at $y = 0$ and $y = \pi$. At $x = 0$ the temperature is $T(0, y) = \sin(y)$. The problem is to find $T(x, y)$ between the conductors.*

To Do: A 2 dimensional box (see Fig. 1) having coordinates (x, y) and ∞ long in z direction, is grounded along the $x = 0$ and $x = \pi$ surfaces. Two grounded parallel conductors at $y = 0$ and $y = \pi$. At $x = 0$ the temperature is $T(0, y) = \sin(y)$. The problem is to use Eq. 3 to find $T(x, y)$.

(a) What are X, Y when $k = 0$?

(b) What is the separation constant for the case of Fig. 1?

(c) Given k, what are $X(x)$ and $Y(y)$ as defined by Eqs. 4?

(d) What is the temperature as $x \to \infty$?

(e) Summarize the four boundary conditions (BC).

(f) Write out the final solution for $T(x, y)$, that agrees with the BCs.

(g) Verify the solution satisfies Laplace's equation and the BCs.

(h) Sketch the solution.

2. Discuss qualitatively what would happen to $T(x, y)$ if the $x = 0$ BC where changed to $T(0, y) = \sin 10\pi y$?

Cylindrical coordinates: $(N = 2)$. For detailed examples on separation of variables in cylindrical coordinates, see Greenberg (1988, p. 1077).

3. We are seeking the temperature $T(r, z)$ for a cylinder of radius $d = 1$ meter and infinitely long, having temperature $T(r = 1, z) = 0$ on the boundary?

(a) As before, assume separation of variables $T(r, z) = R(r)Z(z)$, and find the separation constant for Laplace's equation

(b) Note the choice of signs is critically important. Why is the signs of k^2 important?

(a) Find $Z(z)$ and $R(r)$ for cylindrical coordinates $(N = 2)$.

(b) Show that the equation and its solution are

$$\nabla_r^2 R(r) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + k^2 R(r) = 0 \quad \text{thus} \quad R(r) = C J_0(kr). \tag{5}$$

(c) Find the first root of $J_0(r_0) = 0$. Hint: Using either Matlab, Octave or Python,[1] plot $J_0(r)$ to locate the first zero crossing.

(d) Why do we need to know the zero of $J_0(r)$?

(e) Write out the solution for $T(r, z)$.

2 Poisson equation.

One of Maxwell's equation is $\nabla \cdot \vec{D} = q(x, y, z)$ where q is the charge distribution. A second relation is $\vec{D} = \epsilon \vec{E}$.

[1]or your hand calculator

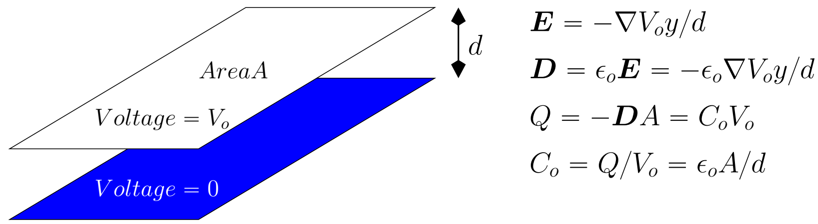

Figure 2: *Capacitor made from two conduction plates having areas A, separated by a distance d.*

To do:

1. Find $\vec{D}(x,y,z)$ if the charge consists of two sheets of two sheets having area 1 [cm^2] = 10^{-4} [m^2], $d = \mu$-meter (10^{-6} [m]) apart, with a voltage of 1 volt between the two plates. Find the capacitance [Fd].
2. What is ϵ? Give its value along with SI units.
3. Compute C_o
4. Find $\nabla \cdot \boldsymbol{D}$.
5. Assuming that $\nabla \times \vec{E} = 0$, rewrite these relations to derive the *Poisson Equation*

$$\nabla^2 \Phi(x,y,z) = -q(x,y,z)/\epsilon_o.$$

3 Wave equation

For detailed examples on separation of variables of the wave equation, see (Greenberg, 1988, p. 1077).

1. Show that d'Alembert's solution, is in fact a solution to the wave equation:

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p(x,t)}{\partial t^2}$$

   where $p(x,t) = f(t-x/c)+g(t+x/c)$ and $f(\xi)$ and $f(\xi)$ are any functions, where $f'(\xi) \equiv \partial f(\xi)/\partial \xi$.

2. Solution to the wave equation in spherical coordinates:
   (a) Write down the wave equation in spherical coordinates.
   (b) Show that the following is true (Hint, expand both sides):

$$\nabla_\rho^2 R(\rho) \equiv \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho}R(\rho) = \frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}\rho R(\rho) \tag{6}$$

   (c) Using the results of Eq. 6, show that the solution to the spherical wave equation is

$$\nabla_\rho^2 p(\rho,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p(\rho,t) \qquad \text{thus:} \quad R(\rho,t) = \frac{f(t-\rho/c)}{\rho} + \frac{g(t+\rho/c)}{\rho} \tag{7}$$

   (d) With $f(\xi) = \sin(\xi)U(\xi)$ and $g(\xi) = e^\xi U(\xi)$, where $U(\cdot)$ is the Heaviside step function, write down the solutions to the spherical wave equation.
   (e) Sketch the above solutions at several different times, as a function of ρ.

(f) What happens when the inbound wave reaches the center at $\rho = 0$?

3. **Webster horn equation:**

   (a) Rewrite the Webster horn equation [Class notes; Olson, p. 101 (1947); Morse, p. 269, (1948)]

$$\frac{\partial}{\partial x}\begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix} = -\begin{bmatrix} 0 & \frac{\rho_0}{A(x)} \\ \frac{A(x)}{\eta_0 P_0} & 0 \end{bmatrix} \frac{\partial}{\partial t}\begin{bmatrix} p(x,t) \\ \nu(x,t) \end{bmatrix} \tag{8}$$

   as a 2d order differential equation in the pressure $p(x,t)$. Here $\nu(x,t) = A(x)u(x,t)$ is the *volume velocity*, more formally defined as the integral over the normal component of the particle velocity $u(x,t)$, over the cross-sectional area $A(x)$ of the tube, or

$$\nu \equiv \int_S \mathbf{u} \cdot d\mathbf{A}.$$

   (b) Compare the Webster horn equation to the Spherical wave equation in ρ. That is let $A(\rho) = A_0\rho^2$ be the area of the surface of a sphere of radius ρ.

   (c) Compare the Webster horn equation to the cylindrical wave equation in r, namely let set $A(r) = A_0 r$.

4 Special functions:

In Matlab or Octave, use the symbolic Taylor series command for each function and show that:

1. Use the defining series for the Bessel function to obtain $J_{1/2} = \sqrt{\frac{2}{\pi x}}\sin x$ given the general definition of $J_\nu(x)$:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k},$$

2. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$.

3. $J_{-n}(x) = (-1)^n J_n(x)$

5 Symmetry:

Find the symmetric, anti-symmetric, real and imaginary parts of

1. e^{ix}.

2. $e^{i(x-4)}$

3. 1

4. $U(t)$, the step function 0 for $t < 0$, 1 for $t > 0$, undefined at $t = 0$.

5. $1/t$

6. $\text{sgn}(t) \equiv \frac{|t|}{t}$

References

Greenberg, M. D. (**1988**), *Advanced Engineering Mathematics* (Prentice Hall, Upper Saddle River, NJ, 07458), URL `http://jontalle.web.engr.illinois.edu/uploads/493/Greenberg.pdf`.