**ECE 493** **BV2 – Version 1.06 April 24, 2018** **Spring 2018**

**Univ. of Illinois** **Due Thu May 1** **Prof. Allen**

**Topic of this homework:** Special equations and functions of Physics, Sturm-Liouville Theory, Fourier and Laplace Transforms, Transmission lines.

Deliverable: Show your work.

**Overview:** Sturm-Louville equations form a class of differential equations used to solve partial differential equations. They are always scalar (ordinary) differential and typically follow after separation of variables has been applied, to simplify the wave equation. They always have the form

$$[p(x)y'(x)]' + [q(x) + \lambda^2 w(x)]y(x) = 0, \tag{1}$$

where $y'(x)$ is the derivative with respect to the independent variable $x$. The physical meaning of the functions $p(x), q(x), w(x)$ are not identified, but $\lambda$ plays the role as an eigen value. The function $p(x)$ plays a pivotal role, typically is real, and cannot be zero. Thus it is taken to be positive ($p(x) > 0$). The other functions have special constraints place on them, such as $q(x) \neq 0$ and must be real. When solving real world problems, it may be possible to identify the physical meaning of $p(x)$ and $q(x)$, and in such cases, it seems likely that the solution may be physically insightful.

Expanding the left most term gives

$$[p(x)y'(x)]' = p(x)y''(x) + p'(x)y'(x)$$

Note that $p'/p = \partial_x \log p$.

Generalize impedance boundary conditions are specified as

$$\begin{aligned} \alpha y(a) &+ \beta y'(a) = 0 \\ \gamma y(b) &+ \delta y'(b) = 0. \end{aligned}$$

When $y(x)$ is constrained at the boundary $x = a$, $\beta = 0$, and when the slope $y'(a)$ is constrained, $\alpha = 0$. Linear combinations of $y$ and $y'$ are allowed.[1]

# 1 PDE Equations from Physics

**To do:**

1. For the equations (a), (b):

   i) Identify all the differential equation parameters $[p(x), q(x), w(x), \lambda^2]$, as well as the boundary conditions.

   ii) Solve for the eigenvalues. If the characteristic equation is too difficult to solve analytically, state that, and proceed with the rest of the problem, assuming that the eigenvalues $\lambda_n$'s may be determined (e.g., by numerical means). Clearly state the sign of $\lambda^2$, thus if $\lambda$ is zero0, real, or complex.

   iii) Find the unnormalized eigen-functions.

   iv) Work out the eigenfunction expansion of the given function $f(x)$.

   (a) $y'' + \lambda^2 y = 0$, $y(0) = 0$, $y'(L) = 0$, $f(x) = 100$.

   (b) $y'' + \lambda^2 y = 0$, $y(0) = 0$, $y(L) = 0$, and[2]

$$f(x) = \begin{cases} 1 & 0 \le x \le L/2 \\ 0 & L/2 \le x \le L \end{cases}$$

[1] Greenberg p. 887 Eq. (1a)

[2] Greenberg, solution 17.7(a), p. 1304.

2. Find $y(x)$ given the first order equation

$$a(x)y'(x) + b(x)y = f(x).$$

Hint: Consider the homogeneous equation, isolate $y$ and its derivatives from $a(x), b(x)$.

3. Show that the equation

$$A(x)y'' + B(x)y' + C(x)y + \lambda^2 D(x)y = 0 \tag{2}$$

may be transformed into Sturm-Liouville (SL) form (Eq. 1) if and only if $p(x) = A(x)$ and $B(x) = A'(x)$ (Greenberg, 1988, p. 66).

Note: In general $By'/A$ will not be a perfect differential. However if we multiply $A(x)$ and $B(x)$ by a scale function $\sigma(x)$, called the integration factor (Greenberg, 1988, p. 66), it may be possible to make it a perfect differential.

4. *Sturm-Liouville* equations may be obtained using *integration by parts*, following the multiplication by an "integration factor" $\sigma(r)$. State the conditions for the integration factor to exist.

5. Assuming it exists, rewrite Eq. 2 (above) in SL notation (i.e., find $p(x), u(x)$, etc.).

6. Derive the *integration factor* that will transform Eq. 2 into the SL form (i.e., Eq. 1).

7. Use the previous result to *recast*

$$y'' + 2y' + xy + \lambda^2 x^2 y = 0 \tag{3}$$

in SL form.

## 2 *Webster horn equation* (WHEN)

The Webster horn equation

$$\frac{1}{A(x)}\frac{d}{dx}A(x)\frac{d\mathcal{P}}{dx} = -\frac{s}{c_o}^2\mathcal{P}(x,\omega) \leftrightarrow \frac{1}{c^2}\frac{\partial^2 \varrho(x,t)}{\partial t^2}. \tag{4}$$

was first proposed in by (Webster, 1919). In the mathematical literature, equations of this form are known as *Sturm-Liouville* (SL) problems. Sturm-Liouville equations may be obtained from the 3D wave equation, either by seperation of variables or the application of Gauss' law, both of which transform them into a "1-dimensional" (i.e., single range variable) equivalent horn equation (Allen, 2017, Lec 37).

Salmon (1946) transformed Eq. 4 to its equivalent Schrödinger wave equation, having constant coefficients. In this problem we study the Horn equation.

**To do:**

1. Transform Eq. 4 by assuming an effective pressure $E(x) = \mathcal{P}(x,\omega)/\sqrt{A(x)}$ (Salmon, 1946, Eq. 2.2), resulting in Salmon's Eq. 2.3[3]

$$\frac{d^2E}{dx^2} + \left(\frac{\omega^2}{c_o^2} - \frac{1}{\sqrt{A(x)}}\frac{d^2\sqrt{A(x)}}{dx^2}\right)E = 0. \tag{5}$$

This transformation assumes the energy $A(x)|\mathcal{P}|^2$ is independent of position $x$, where the effective radius is $R(x) \equiv \sqrt{A(x)/\pi}$ (Morse, 1948, p. 270).

2. Show that the equation is equivalent to the formula (Pierce, 1981, page 360, (7-8.5a))

$$\left[\frac{\partial^2}{\partial x^2} - m^2 - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\right]A(x)^{\frac{1}{2}}\varrho(x,t) \leftrightarrow \left[\frac{\partial^2}{\partial x^2} - m^2 + \frac{\omega^2}{c_o^2}\right]A(x)^{\frac{1}{2}}\mathcal{P}(x,\omega) = 0$$

where $m$ is the *horn flair parameter*

$$-m^2 = \frac{1}{4A^2}\left[(A')^2 - 2AA''\right].$$

[3]Salmon has an error in this equation, fixed here, as indicated in red.

## 3 Classification

In terms of a table having 5 columns, labeled with the abbreviations: L/NL, TI/TV, P/A, C/NC, Re/Clx:

L/NL : linear(L)/nonlinear(NL)

TI/TV : time-invariant(TI)/time varying(TV)

P/A : passive(P)/active(A)

C/NC : causal(C)/non-causal(NC)

Re/Cx : real(Re)/complex(Cx)

1. Along the rows of the table, classify the following *systems*:

   (a) Resistor $v(t) = r(t)i(t)$

   (b) inductor $v(t) = L\frac{di(t)}{dt}$

   (c) Capacitor $i(t) = C\frac{dv(t)}{dt}$

   (d) Switch connected to a battery

   $$v(t) \equiv \begin{cases} 0 & t \leq 0 \\ V_0 & t > 0. \end{cases}$$

   Explain what happens at $t = 0$.

   (e) A diode (or transistor)

   (f) 1 meter of transmission line, terminated in a capacitor

   (g) $v(t) = e^{-i\omega t}i(t)$

   (h) $F(\omega) = e^{i\omega T}G(\omega)$

   (i) $p(x,t) = \int_{\xi=-\infty}^{\infty} h(\xi - x)v(\xi, t)d\xi$

## 4 Step Functions and the Fourier and Laplace Transforms

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The unit step function is defined as:

$$u(t) = \begin{cases} 0 & t < 0 \\ \text{undefined} & t = 0 \\ 1 & t > 0 \end{cases} \quad \leftrightarrow \quad \frac{1}{s}$$

and the Fourier Step Function $\tilde{u}(t)$

$$\tilde{u}(t) = \frac{1 + sgn(t)}{2} = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & t = 0 \\ 1 & t > 0 \end{cases} \quad \leftrightarrow \quad \pi\delta(\omega) + \frac{1}{j\omega}$$

Convolution is defined as

$$f(t) \star g(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\, d\tau \quad \leftrightarrow \quad F(\omega)G(\omega) \,\text{or}\, F(s)G(s).$$

1. Show that $g(t) \equiv u(t) \star u(t) = tu(t)$
2. Find $\tilde{g}(t) \equiv \tilde{u}(t) \star \tilde{u}(t)$.
3. Show that $f(t) \equiv u(t) \star \tilde{u}(t) = tu(t)$.

4. Repeat (1-3) in the frequency domain. That is, use the convolution formula properties to find these three quantities (e.g.: $g(t) \leftrightarrow 1/s^2 \leftrightarrow t\,u(t)$).

5. Explain why (2) fails to work in the frequency domain.

6. Compute $sgn(t) \star 1(t)$ in the frequency domain.

7. Compute $1(t) \star 1(t)$ in the frequency domain.

8. Compute $sgn(t) \star sgn(t)$ in the frequency domain.

9. Summarize what you have learned about the LT and FT from this exercise.

# 5 Fourier Methods

1. Describe each function below using the notation

$$G((t))_T \equiv g(t) = g(t+T) \tag{6}$$

to describe a function that is periodic with period $T$

(a) $g(t) = \cos(t)$

(b) $g(t) = \sin(2\pi t)$

(c) $g(t, \omega) = e^{-i2\omega t}$

(d) $g_n(t, T) = e^{-i2\pi nt/T}$

(e) $g(t) = e^t$

2. In terms of an integral over $f(t)$, find the formula for the Fourier Series coefficients $F_k$ of $f((t))_T = \sum_n f(t+nT)$.

3. If $f(t) \leftrightarrow F(\omega)$ define a Fourier transform pair, if we define $f((t))_T = \sum_n f(t+nT)$, then prove the following:

$$\sum_{n=-\infty}^{\infty} f(t+nT) \leftrightarrow \omega_0 \sum_{k=-\infty}^{\infty} F(\omega_k)\, \delta(\omega - \omega_k)\,. \tag{7}$$

where $\omega_0 = \frac{2\pi}{T}$ and $\omega_k = k\omega_0$.

4. If $f(t)$ is real, what symmetry property will its Fourier transform $F(\omega)$ have?

5. If $f((t))_T$ is real, what symmetry property will its Fourier series coefficients $F_k$ have?

6. Can the function $F_k$ [i.e., the Fourier coefficients of $f((t))_T$] be a real function of integer $k$? Explain.

# References


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