ECE 493 **CV1 − Version 1.41 (January 4, 2018)** **Spring 2018**

Univ. of Illinois **Due Tu, Jan 23, 2018** **Prof. Allen**

Topic of this homework: Analytic functions of a complex variable;
Deliverable: Please, show your work.
For the following the unit step function is defined as:

$$u(t) = \begin{cases} 0 & t < 0 \\ \text{undefined} & t = 0 \\ 1 & t > 0 \end{cases}$$

1 Complex Arithmetic

This may come as a shock, but multiplying two complex numbers is not as straight forward as it may seem. The following is meant to be an enlightening example:

1. Let $a = i$ and $b = -1$. What is $c = a * b$?
2. Let $a = e^{i\pi/2}$ and $b = e^{i\pi}$. What is $c = a * b$? (Hint: $c \neq -i$. Use polar coordinates.)
3. Let $a = e^{i\pi/2}$ and $b = e^{-i\pi}$. What is $c = a * b$?
4. The Laplace transform of a delay of T [s] is e^{-sT} where the Laplace frequency $s = \sigma + \omega \jmath$. Consider a system that first delays the signal by T_1 [sec], and then the output of that system is feed into a second system having delay T_2 [s].
   (a) What is the total delay?
   (b) What is the Laplace transform of the cascaded system?
   (c) What is the phase of the cascaded system?
   (d) Justify your answer.

2 Complex Algebra

Find numerical values in the form $a + ib$ for the following:

1. $x^2 + 1 = 0$
2. $x^3 + 8 = 0$
3. i^i (Show your work, as always!)
4. What is the frequency, in [Hz], of $a^t u(t)$, given a constant $a \in \mathbb{C}$? Here

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

   is called the Heaviside step function.

3 Complex functions

Domain: $s \equiv \sigma + i\omega$, Range: $Z(s) \equiv R(s) + iX(s)$.

The *Domain* (e.g., s) and *Range* (e.g., $Z(s)$) are described in the text on page 1114.

In engineering terms think of $Z(s) = X + iY$ as an *impedance* having a real part (*resistance*) X, and an imaginary part (*reactance*) Y.

Make two axes, one for the $s = \sigma + i\omega$ plane and a second for the $Z(s) = X(s) + iY(s)$ plane. Label the two sets of axes: On the left (s), the horizontal axis (abscissa) is labeled σ, while the vertical axis (ordinate) is $i\omega$. For the $Z(s)$ axis (on the right), the abscissa is labeled X and the ordinate axis is iY.

Plot the *Range* $Z(s)$ in terms of the specified *Domain* in s. In some cases, it may help you to label a few points in the s domain, then label corresponding points in the $Z(s)$ domain.

1. Domain: $s = \sigma$, Range: $Z(s) = 1 + s$.
2. Domain: $s = i\omega$, Range: $Z(s) = 1 + s$.
3. Domain: $s = i\omega$, Range: $H(s) = 1 + s^2$.
4. Reverse the range and domain. Thus the Domain is $H(s) = 1 + s^2$ while the Range s. Plot the range ($s = ?$) for domain Real$\{H(s)\}$.

4 Harmonic functions

1. Show that if $F(s) = e^s$ that the real and imaginary parts obey the Cauchy-Riemann conditions.
2. If $F(s) = s/(1 + s)$, where are the Cauchy-Riemann conditions valid, or not valid? Explain.
3. If $F(s) = \log(s)$, where are the Cauchy-Riemann conditions valid, or not? Explain.
4. If $F(s) = \sqrt{1 + s^2}$, where are the Cauchy-Riemann conditions valid, or not? Explain.

5 Laplace Transforms

1. Find the Laplace transform of 1, $df(t)/dt$, $\int_{-\infty}^{t} \delta(t)dt$, and $\int_{-\infty}^{t} u(t)dt$. Assume $f(t) \leftrightarrow F(s)$.
2. If $f(t) = 1/\sqrt{\pi t}$ has a Laplace transform $F(s) = 1/\sqrt{s}$. In engineering shorthand

$$\frac{1}{\sqrt{\pi t}} \leftrightarrow \frac{1}{\sqrt{s}}.$$

   (a) What is the inverse Laplace transform of $g(t) \leftrightarrow \sqrt{s}$? This is called a *semi-inductor* and $1/\sqrt{s}$ is called a semi capacitor. This term appears in the *skin effect*[1] at the surface of a conductor. It also appears in fluid mechanics in the viscous boundary layer.[2] There are many types of boundary layers.[3]

   (b) What is $f(-1)$?

[1] https://en.wikipedia.org/wiki/Skin_effect
[2] http://www.thermopedia.com/content/595/
[3] https://en.wikipedia.org/wiki/Boundary_layer