ECE 493 **CV2 – Version 1.29 (January 21, 2018)** **Spring 2018**

Univ. of Illinois **Due: Tu Jan 23, 2018** **Prof. Allen**

Topic of this homework: Analytic functions: Integration of analytic functions; Cauchy integral formula; Riemann Sheets and Branch cuts; Region of Convergence; inverse Laplace transforms;

Deliverable: Show your work.

For this homework $i = \sqrt{-1}$.

1 Ordering complex numbers

One can always say that 3<4, namely that real numbers have *order*. We will explore if complex numbers have order. Let $z = x + iy$ be a complex number.

1. Can you define a meaning to $|z_1| > |z_2|$?
2. If z and w are *real* numbers, define the meaning of $z > w$.
3. If z and w are complex numbers, define the meaning of $z > w$.
4. How about $|z_1 + z_2| > 3$?

2 Analytic functions

State the regions where the following functions are analytic (Note: I'm *not* asking you to apply the CR conditions, just state the region. Remember that the analytic function has a power series that converges in the region of convergence (ROC). Thus an analytic function can be differentiated any number of times. Try to expand the function is a power series, and then look for the ROC. Consider also the expansion of $df(z)/dz$.

1. $f(z) = z^2$
2. $f(z) = 1/z$
3. $f(z) = \ln(\sqrt{z})$
4. $f(z) = \sqrt{1 - z^2}$
5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n = 1$ (independent of n). Find $f'(z)$ and state the region where $f(z)$ and $f'(z)$ are analytic.

3 Integration of Analytic (and non Analytic) functions

State where the function is and is not analytic. Integrate $w = f(z) = u(x, y) + iv(x, y)$ over curve $C \in z = x + iy$, as given.

1. $f(z) = z$, C on the *unit circle* defined as $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.
2. $f(z) = \sin(z)$ on the unit circle.
3. $f(z) = 1/z$ on the unit circle.
4. $f(z) = 1/(2 - z)$ on the unit circle.

5. $f(z) = 1/\sqrt{z}$ on the unit circle.

6. $f(z) = 1/\sqrt{z}$ twice around the unit circle ($0 \le \theta \le 4\pi$). This function has a branch cut, can you apply the Cauchy theorem?

7. $f(z) = 1/z^2$ on the unit circle.

4 Taylor Series

1. Explain the difference between $1/.5$, $1/(1-.5)$, $1/z$, $1/(1-z)$

2. Express $1/(1-z)$ as a power series in z. What is the ROC?

3. Express $1/(1-z^2)$ as a power series in z. What is the ROC?

4. Express $1/(1-z)^2$ as a power series in z. What is the ROC?

5. Express $1/z$ as a power (Laurent) series in z, and give the ROC.

6. Express $1/(1-|z|^2)$ as a power series in z. What is the ROC? (Hint: This is not analytic! State why?)

7. Express $1/(2-z)$ as a power series in $1/z$. What is the ROC?

8. Express the inverse of $1/(2-z)$ as a power series in z. What is the ROC?

9. Why are poles and zeros of a function important?

10. If $a = 0.1$ what is the value of

$$x = 1 + a + a^2 + a^3 \cdots ?$$

    Show your work.

11. If $a = 10$ what is the value of

$$x = 1 + a + a^2 + a^3 \cdots ?$$

5 Cauchy integral formula

1. Integrate the following:

   (a) $\int_C z dz$ with $C : z = e^{i\theta}$ for $\theta = [-\pi, \pi]$.

2. If $w = u + iv$ and $z = x + iy$, find $u(x, y)$ and $v(x, y)$ for $w = c^z$ with complex constant $c \in \mathbb{C}$:

   (a) $c = e$

   (b) $c = 1$

   (c) $c = \sqrt{2i}$

6 CR conditions

For the following problem: $i = \sqrt{-1}$, $s = \sigma + i\omega$, $s = re^{i\theta}$, with $r \equiv |s| = \sqrt{\sigma^2 + \omega^2}$, $\theta \equiv \angle s$, and $f(s) = u(\sigma, \omega) + iv(\sigma, \omega)$. Show that the CR conditions for $f(s)$ may also be expressed in the following coordinate systems:

1. Rectangular:
$$\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial i\omega}, \tag{1}$$

2. Polar:
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{2}$$