ECE 493 **CV3 − Version 1.86 January 30, 2018** **Spring 2018**

Univ. of Illinois **Due Tu, Feb 6, 2018** **Prof. Allen**

Topic of this homework: Riemann Sheets, branch cuts and multivalued functions; Filters; Positive definite functions;

1 2-port circuit networks

Background: An equivalent circuit for a battery is shown. The leakage current I_{leak} of the battery is determined by R_s, which decreases as the battery ages. R_s becomes a short (zero resistance) when the battery finally goes "dead." The open-circuit (unloaded) source voltage V_1 represents a chemical potential, which is assumed to be constant through out the batteries lifetime (Fig. 1).

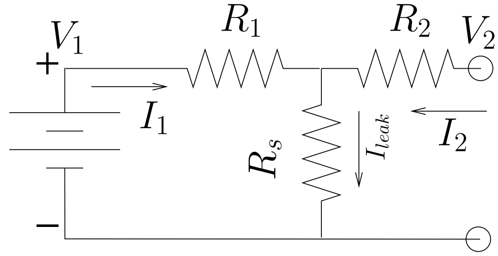


Fig. 1: Equivalent circuit for a battery having an unloaded voltage of V_1 (leakage current I_{leak}), an internal leakage resistance of R_s , and a source resistance of $R_1 + R_2$.

To Do:

1. Using 2-port 2x2 matrix theory (JBA Notes: Lec 16 (p. 79-82,193-196), write a set of relations between the input and output voltage and current, namely define the matrix elements of

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

   in terms of the impedances R_1, R_2, R_s.

   Once the equations are set up, set $R_{\text{leak}} = \infty$ ($I_{\text{leak}} = 0$), to simplify the analysis. For the simplified circuit ($R_s = \infty, I_{\text{leak=0}}$) find the input impedance when $I_2 = 0$, $V_2 = 0$ and $V_2 = R_{\text{load}} I_2$.

2. No load ($I_2 = 0$):

3. Shorted output ($V_2 = 0$):

4. Output loaded by R_{load}:

   The point is that solving systems of cascades of impedances is a straight forward procedure: 1) Set up the matrix equations; 2) simplify the equations; 3) apply the load conditions, and find the currents and voltages. With some help from Matlab/Octave, this can be reduced to a trivial analysis, taking only minutes to obtain the answer (once you have mastered the tools).

2 Brune Impedance

A Brune impedance is defined as the ratio of the force $F(s)$ over the flow $V(s)$, and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}. \tag{1}$$

It trivially follows that

$$D(s) = \prod_{k=1}^{K}(s - s_k) \qquad \text{and} \qquad c_k = \lim_{s \to s_k}(s - s_k)D(s) = \prod_{n'=1}^{K-1}(s - s_n),$$

where the prime on index n' means that $n = k$ is not included in the product.

There are several important theorems here, best summarized as *Brune's Theorem* on positive-real functions. But it goes beyond this since the impedance matrix and the transmission matrix are a rearrangement of the same matrix equation(see the Lecture notes for the details; for example, 2-port transfer functions, and their input impedance, have the same poles).

To do:

1. Find the Laplace transform ($\mathcal{L}$) of the three force relations in terms of the force $F(s)$ and the velocity V(s), along with the electrical equivalent impedance:
   (a) Hooke's Law $f(t) = Kx(t)$.
   (b) Dash-pot resistance $f(t) = Rv(t)$.
   (c) Newton's Law for Mass $f(t) = M dv(t)/dt$.
2. Take the Laplace transform ($\mathcal{L}$) of

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t). \tag{2}$$

   and find the total impedance $Z(s)$ of the mechanical circuit.
3. What are $N(s)$ and $D(s)$ (e.g. Eq. 1)?
4. Assume that $M = R = K = 1$, find the residue form of the admittance $Y(s) = 1/Z(s)$ (e.g. Eq. 1) in terms of the roots $s_\pm$. You may check your answer with the Matlab/Octave's `residue` command.
5. By applying the CRT, find the inverse Laplace transform ($\mathcal{L}^{-1}$). Use the residue form of the expression that you derived in the previous exercise.

2.1 Transfer functions

In this problem, we will look at the transfer function of a two-port network, shown in Fig. 1. We wish to model the dynamics of a freight-train having N such cars. The model of the train consists of masses connected by springs.

The velocity *transfer function* for this system is defined as the ratio of the output to the input velocity. Consider the engine on the left pulling the train at velocity V_1 and each car responding with a velocity of V_n. Then

$$H(s) = \frac{V_N(s)}{V_1(s)}$$

is the frequency domain ratio of the last car having velocity V_N to V_1, the velocity of the engine, at the left most spring (i.e., coupler).

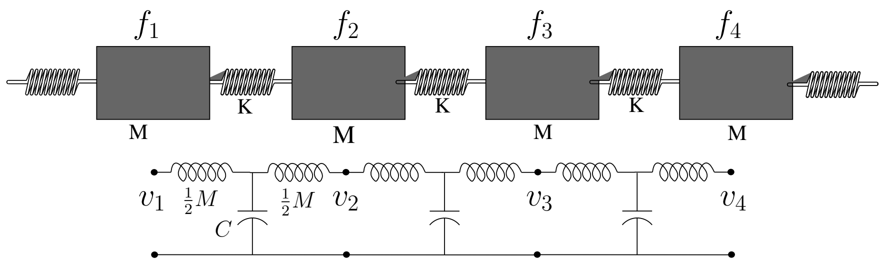


Figure 1: *Depiction of a train consisting of cars, treated as a mass M and linkages, treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $v_n(t)$.*

To do: Use the ABCD method to find the matrix representation of Fig. 1. Consistent with the figure, break the model into cells each consisting of three elements: a series inductor representing half the mass ($L = M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$). Each cell is symmetric, making the model a cascade of identical cells.

At each node define the force $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$ at junction n.

1. Write the ABCD matrix $\boldsymbol{T}$ for a single cell ($N = 2$), composed of series mass $M/2$, shunt compliance C and series mass $M/2$, that relates the input node 1 to node 2 where

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} F_2(\omega) \\ -V_2(\omega) \end{bmatrix}$$

   Note that here the mechanical force F is analogous to electrical voltage, and the mechanical velocity V is analogous to electrical current.

2. Assuming that $F_2 = 0$, find the velocity transfer function $H(s) \equiv V_2/V_1$. From the results of the $\boldsymbol{T}$ matrix you determined above, find

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

3. Find $h_{21}(t)$, the inverse Laplace transform of $H_{21}(s)$.

4. What is the input impedance $Z_2 = F_2/V_2$ if $F_3 = -r_0 V_3$?

5. Simplify the expression for Z_2 by making 3 assumptions:

   (a) Formula for the characteristic resistance: $r_0 = \sqrt{M/C}$
   (b) Load impedance is the characteristic resistance: $F_3 = -r_0 V_3$ (i.e., V_3 cancels),
   (c) Low frequency approximation: $s\sqrt{MC} < 1$

6. State the ABCD matrix relationship between the first and Nth node, in terms of of the cell matrix.

7. What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$? Hint: Use an eigen matrix diagonalization as described in the next section, since

$$T^N = E\Gamma E^{-1} E\Lambda E^{-1} E\Lambda E^{-1} \cdots = E \begin{bmatrix} \lambda^N & \\ 0 & \lambda^N \end{bmatrix} E^{-1} \cdots$$

2.2 Eigen analysis of $\boldsymbol{T}$

Given an ABCD transmission matrix $\boldsymbol{T}$

$$\boldsymbol{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the eigenvalues are and vectors are given in Appendix C of the Notes (p. 245), repeated below. Note that the determinant Δ of every $\boldsymbol{T}$ is always $+1$ for reciprocal systems, and -1 for anti-reciprocal systems.

Eigenvalues:

$$\begin{bmatrix} \lambda_+ \\ \lambda_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (A+D) - \sqrt{(A-D)^2 + 4BC} \\ (A+D) + \sqrt{(A-D)^2 + 4BC} \end{bmatrix}$$

Reversable systems have the symmetry $A = D$ ($|\boldsymbol{T}| = A^2 - BC = 1$), and this simplifies to $\lambda_\pm = A \mp \sqrt{BC}$, and the eigenmatrix is

$$\Lambda = \begin{bmatrix} A - \sqrt{BC} & 0 \\ 0 & A + \sqrt{BC} \end{bmatrix}$$

Eigenvectors: The eigenvectors are

$$[\boldsymbol{E}_\pm] = \begin{bmatrix} \frac{1}{2C} \left[(A-D) \mp \sqrt{(A-D)^2 + 4BC} \right] \\ 1 \end{bmatrix}$$

When $A = D$ The eigenvectors simplifying to

$$[\boldsymbol{E}_\pm] = \begin{bmatrix} \mp\sqrt{\frac{B}{C}} \\ 1 \end{bmatrix}$$

defining the Eigenmatrix and it inverse:

$$\boldsymbol{E} = \begin{bmatrix} -\sqrt{\frac{B}{C}} & +\sqrt{\frac{B}{C}} \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{E}^{-1} = \frac{1}{2} \begin{bmatrix} -\sqrt{\frac{C}{B}} & 1 \\ +\sqrt{\frac{C}{B}} & 1 \end{bmatrix}$$

3 Riemann Sheets and Branch cuts

1. What is a Riemann sheet (define it, and detail its properties)
2. Describe the Riemann sheet of $G(z) = \ln(z)$.
3. Describe with a figure and then discuss the *branch cut* for $f(z) = \log(z)$.
4. Describe the branch cut for $f(z) = e^z$.
5. Find $f(z)$ and $f'(z)$ at $z_0 = 0$ and $z_1 = e^{i\pi/2} = i$ for

   $$f(z) = \sqrt[3]{1 - z^2}$$

   and describe the branch-cuts required to make this function single valued.
6. If $w \equiv F(s) = 1 + s^2$

(a) What is $s = G(w) \equiv F^{-1}(w)$.

(b) Map out the range of $s = G(w)$ for the two domains of w. Explain.

(c) Discuss reasonable places to place the branch cut(s)?

7. Starting with the formula for the impedance of a transmission line (TL):

$$z(s) = \tan(s)$$

show that

$$\tan^{-1}(z) = -\frac{i}{2} \log \frac{i - z}{i + z}. \tag{3}$$

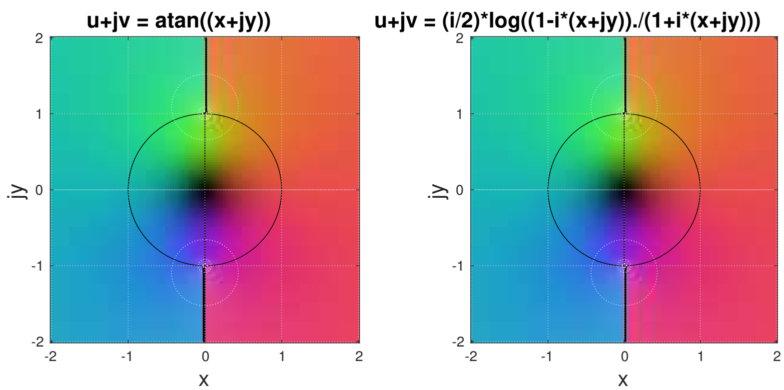
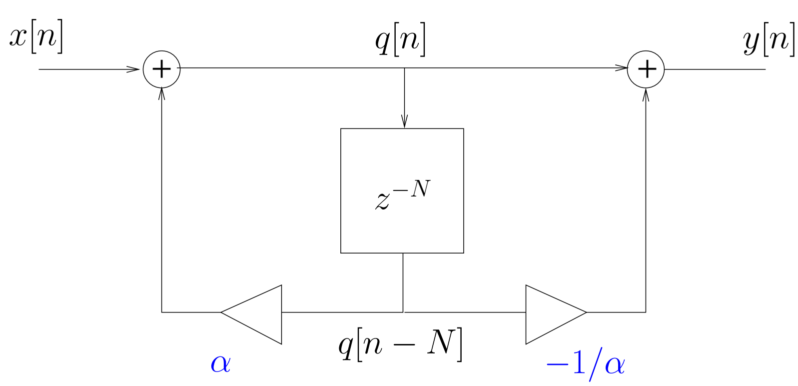


Figure 2: *This figure compares the colorized plots of $w = tan^{-1}(z)$ and $w = \frac{i}{2}\ln(1 - iz)/(1 + iz)$, to verify they are the same function.*

4 Positive definite operators

Structure of a *bilinear filter*, composed from a bounded allpass filter z^{-N} (Figure 4):



$$q[n] = \alpha q[n - N] + x[n]$$
$$y[n] = q[n] - (1/\alpha)q[n - N]$$

Fig. 4 Digital filter with N sample delay, feed-forward and feed-back taps.

1. Assuming that $N = 1$, compute the first 5 samples of the impulse response of the filter depicted in Fig. 4. Determine the transfer function, and find its poles and zeros. Namely fill out a table of values for x, q, $q[n - N]$ and y, for $n = 0, 1, ..5$.

2. If $N = 10$, find the frequency relationship between $y[n]$ and $x[n]$. This may be done by either a Fourier Transform or a z Transform. Hint: To write a formula, formulate the sum around the + junction, I.E. the z transform of

$$q[n] = \alpha q[n-N] + x[n]$$

is

$$Q(z) = \alpha z^{-N} Q(z) + X(z),$$

thus

$$H(z) \equiv \frac{Q(z)}{X(z)} = \frac{1}{1-\alpha z^{-N}} = 1 + \alpha z^{-1} + \alpha^2 z^{-2} \cdots .$$

3. What are the poles and zeros of $H(z)$?

4. Is the impulse response of the operator FIR or IIR?

5. What is the real part of the transfer function $H(z)$.