ECE 493	CV4 − Version 1.40 February 6, 2018	Spring 2018
Univ. of Illinois	Due Tue Feb 13	Prof. Allen

Topic of this homework: Convergence of transforms, ROC, Filter symmetry (continuous vs discrete time; pole-zero placement).

Deliverable: Show your work.

1 Convergence

In class I explained the use of the Cauchy residue theorem in computing the inverse Laplace transform. This required using the sign of $\Re st = \sigma_o t$ to determine how to close the integral as the limits go to ∞. The curve must be closed to use this theorem. The exercises below extend this to include non-causal signals. Such cases are not in agreement with physical reality, but the mathematics doesn't know about physical requirements, and it allows non-causal signals. These cases are called also the *two-sided* Laplace transform. I am not sure how useful they are, but you need to know about this possibility as you will see it referred to in the literature. If you need a refresher on how to decide where to close the contour at ∞, the discussion is on pages 114-115.

To do:

1. If $F(s) = 1/(1+s)$ and the ROC is $\sigma_0 > -1$, find the inverse Laplace transform ($\mathcal{L}^{-1}$).
2. If $F(s) = 1/(1+s)$ and the ROC is $\sigma_0 < -1$, find the inverse Laplace transform.
3. If $F(s) = 1/(s-1)$ and the ROC is $\sigma_0 > 1$, find the inverse Laplace transform.
4. If $F(s) = 1/(s-1)$ and the ROC is $\sigma_0 < 1$, find the inverse Laplace transform.
5. Find the inverse Laplace transform of

$$F(s) = \frac{1}{(s-1)(s+1)} \tag{1}$$

   (a) if the ROC is between the two poles.
      i. if the ROC is to the right of $\sigma = 1$.
      ii. if the ROC is to the left of $\sigma = -1$.

2 Scalar products

Review of scalar products: As explained on pages 73-74, and in more detail on pages 185-186, the scalar (i.e., dot) product between two vectors is the sum over the product of their respective components. The form of the scalar product depends on the nature of the vectors. For example, given vectors $x = [a, b, c]$ and $y = [\alpha, \beta, \gamma]$, the scalar product is $x \cdot y = a\alpha + b\beta + c\gamma$. When the vectors are functions, the scalar product is an integral $\int f(t)g(t)dt$ over t. The most common possibilities are summarized in Table 1. The complexity of the FTs is mostly in the implementation of the scalar product, which explicitly depends on the domain support. When the signals are continuous functions, the scalar product is an integral, and when the domain is discrete in time, the scalar product is a sum. Thus the complexity depends on the type of signal being transformed, and is in the details of the scalar product, which in turn depends on the signal support. The complexity is not inherent in the Fourier transform, which is a simple scalar product.

Name	Domain	scalar product	form	ON
Fourier T.	$-\infty < t \in \mathbb{R} < \infty$	$x(t)\cdot y(t)$	$\int_{-\infty}^{\infty} x(t)y(t)dt$	$e^{-j2\pi ft}$
"	$-\infty < f \in \mathbb{R} < \infty$	$X(f)\cdot Y(f)$	$\int_{-\infty}^{\infty} X(f)Y(f)\frac{d\omega}{2\pi}$	$e^{j2\pi ft}$
F. Series	$0 \le t \in \mathbb{R} \le T$	$x((t))\cdot y((t))$	$\frac{1}{T}\int_{t=0}^{T} x(t)y(t)dt$	$e^{-j2\pi f_k t}$
"	$-\infty < f_k = \frac{k}{T} \in \mathbb{N} < \infty$	$X_k \cdot Y_k$	$\sum_{k=-\infty}^{\infty} X_k Y_k$	$e^{j2\pi f_k t}$
DTFT	$-\infty < t_n < \infty$	$x_n \cdot y_n$	$\sum_{n=-\infty}^{\infty} x_n y_n$	$e^{-j2\pi t_n \Omega}$
"	$-\pi < \Omega < \pi$	$X((\Omega))\cdot Y((\Omega))$	$\int_{-\pi}^{\pi} X(e^{j\Omega})Y(e^{j\Omega})\frac{d\Omega}{2\pi}$	$e^{j2\pi t_n \Omega}$
DFT/FFT	$0 \le t_n = nT \le (N-1)T$	$x_n y_n$	$\sum_{n=0}^{N-1} x_n y_n$	$e^{-j2\pi t_n f_k}$
"	$0 \le f_k = \frac{k}{NT} \le \frac{(N-1)}{NT}$	$X_k Y_k$	$\frac{1}{N}\sum_{n=0}^{N-1} X_k Y_k$	$e^{j2\pi t_n f_k}$

Table 1: *As summarized in this table of scalar products, each of the various types of Fourier transforms differ in their support in time and frequency. The transform types are* Fourier Transform, Fourier Series, Discrete time Fourier transform, Discrete time Fourier transform *and* Fast Fourier transform *(fast version of the DFT). The support then defines the inner product form. In this way all the various forms of Fourier transforms may be reduced to differences in the scalar product, as dictated by the support of the signals in time and frequency. In the above $t_n = nT$, $f_k = k/T$ represent discrete time and frequency samples, where T is one sample period. The signal period for the Fourier Series is T[s]. For the DFT the signal period is NT, where N is the length of the DFT. Typically this is taken to be a power of 2, such as N = 1024 samples. This is done to improve the speed of the transform. The term ON stands for* ortho-normal. *This column shows the signals that are used to when taking the transform. The signal is projected onto these vectors by the scalar product. (This table belongs in the book rather than in the assignment.)*

Norm of a signal: The scalar product of a vector with itself is a measure of its length, called the *norm* of that vector

$$||x(t)|| = \sqrt{x(t)\cdot x(t)} = \sqrt{\sum_k x_k^2(t)}.$$

Given a square integrable signal we define $||x(t)|| = \sqrt{\int x^2(t)dt}$ as the length (norm) of $x(t)$. Norm are variants on the Pythagorean theorem, for vectors.

Scalar product of two vectors: The scalar product between two vectors is an important because it quantifies how correlated the vectors are. It is also known as the *direction cosine*. As shown in Fig 1.13 (p. 74) the scalar product is a measure of how much one vector is pointing in the direction of the other vector. When expressed this way

$$x \cdot y = ||x||\,||y||\cos(\theta),$$

where $||x|| = \sqrt{x\cdot x}$ and $||y|| = \sqrt{y \cdot y}$ are the lengths of the two vectors. The above relation says that when the two vectors are in the same direction, the scalar product is simply the product of the lengths of the two vectors. When they are perpendicular, $\theta = 90°$, and the scalar product is zero. Namely the scalar product is a measure of how co-linear the two vectors are. The geometry of this is shown in Fig 1.13 on page 74 (and p. 186), with the derivation provided on that page.

2.1 Schwarz inequality

To do:

1. Derive the Schwarz inequality $|A \cdot B| \leq ||A|| \, ||B||$. Hint follow the derivation on page 74 of the Allen text. This is repeated in more detail on page 186.
2. Assuming that vector $U(\omega_k) = e^{-j\omega_k t_n}$ and $V(\omega_l) = e^{j\omega_l t_n}$, what must the scalar product be in time such that $U(\omega_k) \perp V(\omega_l)$?

2.2 The Fourier Transform is a scalar product

The Fourier transform is the scalar product of a signal $g(t)$ and the complex exponential $e^{-j\omega t}$, which form an ortho-normal set of functions

$$G(\omega) = g(t) \cdot e^{-j\omega t} = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

Thus the Fourier transform may be viewed as a scalar product of a function, say $g(t)$, and the set of orthogonal complex co-sinusoidal

$$e^{j\omega_k t} \cdot e^{-j\omega_l t} = \int e^{j(\omega_k - \omega_l)t} dt = \delta(k - l),$$

which is zero unless $k = l$, for which it is ∞. Thus the Fourier transform is a decomposition of the signal $g(t)$ into a sum of parts, each pointing in the direction of one of the complex exponential.

There are three basic types of signal support, continuous time or frequency, discrete time or frequency and periodic time or frequency. When one domain (e.g., time) is periodic then the transform domain (i.e., frequency) must be discrete. This follows from the fact that periodic signals have harmonics (discrete frequencies). The DFT case is both periodic and discrete in both time and frequency. The DTFT is periodic in frequency and discrete in time. The Fourier series is periodic in time and discrete in frequency.

To do:

1. Write out the forward and reverse transform formula for the following:
   (a) Fourier Transform? Hint: in this case the scalar product is the integral over all time t and the ortho-normal set of functions is $e^{-j2\pi t}$.
   (b) Fourier series? In this case we assume the signal is periodic with period T such that $f(t) = f(t + T)$. What is the appropriate definition of the inner product in this case? Hint integrate over one period.
   (c) Discrete Fourier transform (DFT)? In this case the signal $g(t_n)$ is both discrete in time and periodic with a period of N samples.
   Find the natural scalar product given this signal.

3 Model of a transmission line

When waves travel on a *transmission line* (TL) they obey the wave equation (read Class Notes, Section 1.5.9, The uniform horn, page 141-143). A uniform transmission line supports bi-directional wave propagation, with a finite velocity of propagation (Notes Eq. 1.84, p. 103). A TL, having a delay of $T = L/c$, has a response to an impulse at its input that is simply a delay $\delta(t-1) \leftrightarrow e^{-sL/c}$, where L is the line length and c is the wave velocity. This represents the one-way delay of a plane-wave. Once the impulse reaches the end ($x = L$) it is reflected, and the pulse travels back to the input, where round trip delay is $2L/c$, as describe by Eq. 1.144, p. 142 of the notes.

The formula for the *input impedance* of such a system, is defined as the voltage over the current (or pressure over velocity), and is $Z_{\text{in}} = r_o \tanh(s2L/c)$ or $r_o \cosh(s2L/c)$, depending on the reflection conditions.

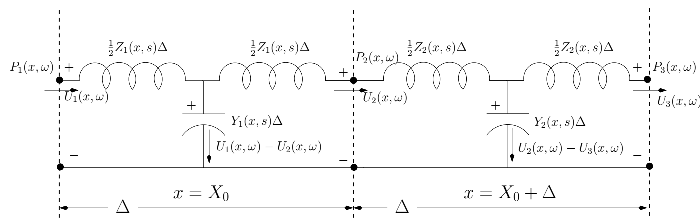


Figure 1: *A transmission line (TL) can be made from a cascade of inductors and capacitors. Each section introduces a delay, with the cascade making the delay longer. If an impulse of voltage (or current) is introduced at the input, it will appear at the output delayed, in proportion to the number of cells. Shown are two cells each of length Δ. Thus the delay is $T = 2\Delta/c$ where $c = 1/\sqrt{LC}$ is the speed of the wave, L is the size of the inductor and C is the size of the capacitor. As the wave travels from the input, the characteristic resistance of the TL is $r_o = V/I = \sqrt{L/C}$. When the wave hits the end, it is reflected, depending on the boundary impedance. If the impedance is zero, then the voltage will be zero, and if the impedance is infinite, the current must be zero.*

To do:

1. Find the formula for the impedance $Z(s)$ of the TL of length L, in terms of the reflectance $\Gamma(s)$ Hint:
$$Z(s, x = 0) = \frac{V}{I} = \frac{V_+ + V_-}{I_+ - I_-}$$
with $r_o = V_+/V_-$ and the input impedance at $x = 0$ (Eq. 1.143, p. 142).

2. Assume the TL is shorted at the end ($V = 0$) so that $\Gamma_o = -1$. Find the formula for the input impedance.

3. Find the formula for the reflectance $\Gamma(s, x)$ in terms of the impedance $Z(s, x)$ (see Eq. 1.142, p. 143).

4 Filter classes

Operators[1] are causal linear *filters*, that modify an input *signal*, transforming it to an output signal. Operators include *transfer functions* (the ratio of two voltages or currents) and *impedances* (the ratio of a voltage over a current). Below we define the Laplace (complex) radian frequency $s = \sigma + i\omega$, where $\omega = 2\pi f$, where f is the frequency in Hz. Operators are functions of s and signal are functions of f [Hz], or equivalently $\omega = 2\pi f$ [rad].

Operators vs. signals: The signature of an *operator* is that it is linear and *causal* (zero for negative time), and has a *Laplace transform*, as functions of complex (Laplace) frequency s. Operators operate on *signals*. The signature of a signal is that it is typically not causal, and is characterized by its Fourier transform, expressed as a function of a real frequency f or ω. Every physical real-world filter is causal. When doing image processing one may work with two-sided filters, since causality is a time-domain construct, which does not apply in spatial coordinates.

There are many types of filters, and each given a unique name, as described next.

[1] The word is a formal (mathematical) term for what engineers would call a filter. An operator is more general concept than a simple filter, but here we treat them as equivalent concepts.

Definitions: The following discussion is built around the 10 postulates (P1-P10) described on pages 95-97.

1. All causal filters (P1) we shall deal with are *real time* (P4) responses. Specifically let $a(t) \in \mathbb{R}$ be a filter's *impulse response*. This requires that the poles and zeros come in conjugate pairs. For example, if there is a pole (or zero) at $s_+ = -1 + \omega\jmath$ then there must be a matching pole (or zero) at $s_- = s_+^* = -1 - \omega\jmath$. If the pole is real (the imaginary part is zero), there is no conjugate pole. For example a single resistor-capacitor (RC) filter has a single real pole. It is easily verified that this condition forces the time response to be real.

2. A *causal* filter $h(t) \leftrightarrow H(s)$ is one that is zero for negative time. It necessarily has a Laplace transform, which are characterized by their *poles* and *zeros*, or poles and *residues*.

3. An *finite impulse response* (FIR) filter has finite duration, namely if $f(t)$ is FIR, then it is zero for $t < 0$ (causal) and for $t > T$, where T is positive constant (time). FIR filters only have *zeros* (they do not have poles).

4. An *Infinite impulse response* (IIR) filter is one that is non-zero in magnitude as $t \to \infty$, namely $|a(t)| > 0$, other than a isolated times (the real roots of $a(t)$). IIR filters typically have poles and zeros.

5. A *Minimum Phase* filter $m(t) \leftrightarrow M(\omega)$ is a causal filter having the smallest phase (i.e., $\angle M(\omega)$) of any filter with magnitude $|M(\omega)|$, on the $i\omega$ axis. A minimum phase filter also satisfies the very special condition

$$\aleph(t) \leftrightarrow N(s) \equiv \frac{1}{M(s)},$$

namely the inverse of $m(t)$ is causal. Thus $m(t) \star \aleph(t) = \delta(t)$ where $\star$ represents convolution. Since $M(s)$ has a causal inverse $N(s)$, the poles of $N(s)$ (i.e., the zeros of $M(s)$), are also in the LHP. Minimum phase filters are common: Every impedance is minimum phase.

6. An *allpass* filter modifies the *phase* but not the *magnitude* of a signal; namely if $a(t)$ is a causal impulse response of a causal allpass filter, having Laplace Transform $A(s) \equiv |A(s)|e^{i\phi(\omega)}$, then $|A(\omega)| = 1$. It follows that the phase $\phi(\omega)$ characterizes an allpass filter. One may make an allpass filter by placing stable poles in the left half plane, and placing a zero in the right half plane symmetrically across from the poles. For example, if a pole is at $s_p = -1 + j\omega$, then there must be a zero at $s_z = 1 + j\omega$.

   The *group delay* is defined as

$$\tau_g(\omega) \equiv -\frac{\partial \phi(\omega)}{\partial \omega},$$

   Since

$$\phi(\omega) - \phi(0) = \int_0^\omega \tau_g(\omega) d\omega,$$

   $\tau_g(\omega)$ may be used as the definition of any allpass system.

7. A *positive real* (PR) filter $z(t) \leftrightarrow Z(s) = R(s) + iX(s)$ is both minimum phase, and has a positive (i.e., non negative) real part in the right half s plane, namely

$$R(\sigma > 0) \geq 0$$

   that is, for $\sigma > 0$ $\Re Z > 0$. Every "Brune" impedance (those made up of inductors, capacitors and resistors) is PR. Since impedance is used in the definition of power, it represents a *positive definite* operator. Note differentiation and integration are operators, corresponding to inductors and capacitors. More generally if one convolves a current $i(t)$ with an impedance, a voltage results (i.e, $v(t) = z(t) \star i(t)$). Since instantaneous power is the voltage across an impedance times the current through it $\mathcal{P}(t) \equiv v(t)i(t)$. The *time average power* is the time average of $\mathcal{P}(t)$, $\overline{\mathcal{P}(t)} \equiv \int_t \mathcal{P}(t)dt$.

To do: Prove each of the following:

1. The convolution relation $h(t) \star g(t) \leftrightarrow H(\omega)G(s)$ where $H(s) \leftrightarrow h(t)$ and $G(s) \leftrightarrow g(t)$. Hint: Write out the definition for $h(t) \star g(t)$ and then substitute the definitions of the FT of $f(t)$ and $g(t)$.
   (a) Start by writing out the formula for the convolution of $h(t)$ and $g(t)$, denoted $h(t) \star g(t)$. Note that $g(t) = 0$ for $t < 0$, i.e., $g(t)$ is causal, as noted on page 925 of Greenberg.
   (b) Then show that the inverse transform of a product of filters is a convolution.
   (c) The point of this example is that one function is of ω while the other a function of s. How does this impact the convolution?
2. Is an allpass filter minimum phase? Explain.
3. Prove that $\delta(t - 5)$ is allpass.
4. Is $\delta(t + 5)$ allpass?
5. Is $e^{-t}u(t)$ Allpass or minimum phase? Justify your answer.
6. When is $F(s) = \frac{s-a}{s+b}$ allpass?
7. When is $F(s) = \frac{s-a}{s+b}$ minimum phase?
8. Does $F(s) = \frac{s+i}{s-i}$ have a real impulse response?
9. In the continuous time domain, a pure delay by T [s] may be written as $\delta(t - T) \leftrightarrow e^{-i\omega T}$. Find the expression for z^{-N}, a pure delay of N samples in the discrete time domain.
10. Where are the poles and zeros for
    (a) a Stable filter
    (b) an Allpass filter
    (c) a Minimum phase filter
    (d) an Impedance (PR function)

5 Filter Symmetries

Let $s = \sigma + i\omega$ be the Laplace (complex) frequency. Let $z \equiv e^{sT}$ where T is the "sample period" at which data is taken (every T seconds). For example if $T = 22.7 = 1/44100$ seconds then the data is sampled at 44100 kHz. This is how a CD player works with high quality music. Thus the *unit-time delay operator* z^{-1} as

$$z^{-1} \equiv e^{-sT} \leftrightarrow \delta(t - T).$$

Note that z^{-1} is in the frequency domain and produces a delay of T (1 "time sample" of delay).

To do:

5.1 Analog (continuous in time) Filters

Given the function:

$$F(s) = \frac{(s+1)(s-1)}{(s+2)},$$

1. Find the minimum phase $M(s)$ and all-pass $A(s)$ parts.
2. Find the magnitude of $M(s)$
3. Find the phase of $M(s)$
4. Find the magnitude of $A(s)$
5. Find the phase of $A(s)$

5.2 Digital (discrete in time) filters

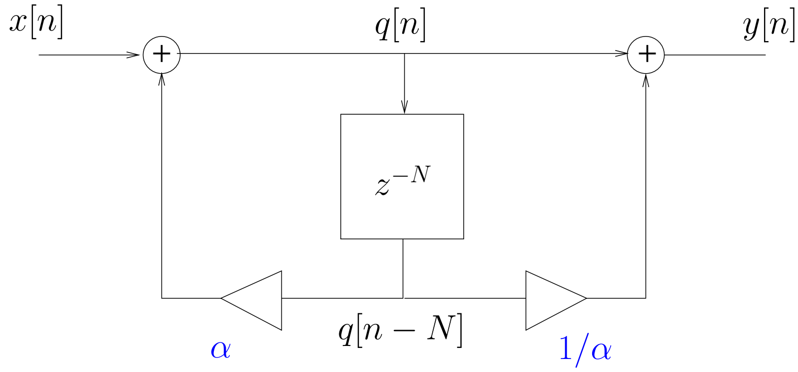
$$F(z) = -z^2 + z^{-1} + z^0 - z^{-2} - z^{-4}$$

1. Find the time function $f[n] \leftrightarrow F(z)$ (i.e., find the inverse z transform)
2. State if this sequence is
   (a) causal/anti-causal
   (b) minimum phase
   (c) all-pass
   (d) FIR
   (e) IIR
   (f) Passive or active
3. Find the causal part of this sequence ($c[n]$)
4. Is the causal part minimum phase?
5. Find the square of the z transform of $U(z)$ (i.e., $U^2(z)$) given that

$$U(z) = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} \cdots.$$

6 All-pass filter

An allpass discrete-time filter is shown in Fig. 2). Allpass filters only change the phase of the input signal $x[n]$ but not the magnitude spectrum.



$$q[n] = \alpha q[n-N] + x[n]$$
$$y[n] = q[n] + (1/\alpha)q[n-N]$$

Figure 2: Digital filter with N sample delay, feed-forward and feed-back taps.

1. Assuming that $N = 1$ (unit delay), compute the first 5 samples of the impulse response of the filter depicted in Fig. 2. Determine the transfer function, and find its poles and zeros. Namely fill out a table of values for $x[n]$, $q[n]$, and $y[n]$, for $n = 0, 1, ..5$.

   Taking the Z transform ($N = 1$) gives
   $$(1 - \alpha z^{-1})Q(z) = X(z),$$
   thus the transfer function $H(z) = Q/X = 1/(1 - \alpha z^{-1}) = z/(z - \alpha)$, which has a zero at $z = 0$ and a pole at $\alpha = 0.9$. The transfer function $Y(z)/Q(z)$ has the z transform
   $$Y(z) = (1 - z^{-1}/\alpha)Q(z) = \frac{(z - \alpha^{-1})}{z}Q(z).$$
   Thus
   $$Y/X = \frac{z}{z-\alpha}\frac{z-\alpha^{-1}}{z} = \frac{z-\alpha^{-1}}{z-\alpha} = \frac{z-1.1}{z-0.9}.$$

2. If $N = 8$, find the frequency relationship between $y[n]$ and $x[n]$. This may be done by either a Fourier Transform or a z Transform. Hint: To write a formula, formulate the sum around the + junction, I.E. the z transform of
   $$q[n] = \alpha q[n-N] + x[n]$$
   is
   $$Q(z) = \alpha z^{-N}Q(z) + X(z),$$
   thus
   $$H(z) \equiv \frac{Q(z)}{X(z)} = \frac{1}{1-\alpha z^{-N}} = 1 + \alpha z^{-10} + \alpha^2 z^{-20} \cdots .$$

3. What are the poles and zeros of $H(z)$?

4. Is the impulse response of the operator FIR or IIR?

5. What is the real part of the transfer function $H(z)$.

Matlab/Octave program to analyze allpass filter. The support files are on the website at http://jontalle.web.engr.illinois.edu/uploads/493/M/

```
alpha=0.5;
Z=[1 0 0 0 0 0 0 0 1/alpha]; %Zeros N=8
P=[1 0 0 0 0 0 0 0 alpha];   %Poles N=8
x=zeros(1024,1); x(1)=1;
q=filter(1,P,x); %filter impulse
y = filter(Z,P,x); %filter impulse

fig(1);
pltfast(fast(q));
ylabel('Q(\omega)')
xlabel('frequency')
grid on;

fig(2)
pltfast(fast(y));
ylabel('Y(\omega)')
xlabel('frequency')
grid on;

fig(3)
plot(roots(Z),'o'); hold on
plot(roots(P),'xr');
title('Poles & zeros')

R=exp(j*2*pi*(0:.01:1));
plot(R,'k:'); grid on
axis([-1.2 1.2 -1.2 1.2]); axis square

fig(4)
Y=fast(y);
plot((0:512/513,unwrap(angle(Y)),'-b');hold on
title('Unwrapped phase of allpass filter')
xlabel('frequency');ylabel('angle Y');
grid on;
```