ECE 493 **CV5 – Version 1.07 February 12, 2018** **Spring 2018**

Univ. of Illinois **Due Thur, Feb 20, 2018** **Prof. Allen**

Topic of this homework: Harder Laplace transforms
Deliverable: Show your work.

1 Inverse of analytic functions

1. Start from the definition of $\tan(s) = \sin(s)/\cos(s)$, derive its "inverse" (page 1199, 1135)

$$\tan^{-1}(s) = \frac{1}{2i}\ln\left(\frac{i-s}{i+s}\right). \tag{1}$$

   (a) Step 1: Since we are looking for $z = \tan^{-1}(s)$, show that

$$s = \tan(z) = \frac{\sin(z)}{\cos(z)} = -i\,\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}. \tag{2}$$

   (b) Step 2: Solve this expression for e^{iz} in terms of s, take the natural log, thus obtain Eq. 1.

2. Starting from the geometric series $1/(1-s) = \sum s^n$,

   (a) show that

$$\ln\left(\frac{1}{1-s}\right) = \sum_{n=1}^{\infty}\frac{s^n}{n}$$

   (b) What is the ROC?

3. In class I demonstrated that the multiplication of two complex numbers can give different answer if you work in polar vs rectangular coordinates. This example show the same, but gives some insight into what is going on.

   Suppose we have a cascade of two delay lines, The first has delay T_1 and the second T_2. Thus the time response for each of these systems is $h_1(t) = \delta(t-T_1)$ and $h_2(t) = \delta(t-T_2)$.

   (a) Working in the time domain, find the system response $h_s(t)$ of the cascade.
   (b) Working in the frequency domain, find the system response of the cascade.
   (c) Working in the frequency domain, discuss the phase due to the delay.

1. In a handout on the website[1] a method is described for finding the conjugate part of an analytic function. For example, if $f(z) = u(x,y) + v(x,y)\jmath$ and you are given $u(x,y)$, find $v(x,y)$. Alternatively, given $v(x,y)$, find $u(x,y)$.

   Boas' method is called **RULE B**:

   > If u is harmonic in a neighborhood of a point $z_0 = x_0+iy_0$, then in this neighborhood we have, up to an additive imaginary constant,
   >
   > $$f(z) = 2u\left(\frac{z+\bar{z}_0}{2}, \frac{z-\bar{z}_0}{2i}\right) - u(x_0,y_0).$$
   >
   > Important note: We may take $z_0 = 0$ if u is harmonic in a neighborhood of 0, simplifying the method for an important general case.

[1] http://jontalle.web.engr.illinois.edu/uploads/493/Boas159-163.pdf

To do: If $u(x, y) = x^2 - y^2 + 1$ find $v(x, y)$ using Rule B.

2 One-sided-periodic functions

1. Define the somewhat weird notation:
$$f(t))_T \equiv \sum_{n=0}^{\infty} f(t - nT).$$
Find an expression for $f(t))_T$ in terms of the Laplace transform ($\mathcal{L}$) of $f(t)$. Hint: Write this as a convolution, then take the $\mathcal{L}$ of the convolution. This relationship is discussed in the notes near Eq. (3.36) and in Figs. 3.5 and 3.6 (p. 202).

2. Describe the poles of $f(t))_T$ in terms of the poles of $f(t)$.

3. What is the ROC for this example?

3 Generalized transmission and impedance Matrices

As discussed in the notes in Sect. 1.3.8 (pp. 80-81), there are a number of important closely related concepts in mathematics. This list includes transmission lines (TLs), impedance matricies, positive definite matricies, conservation of energy, and eigen-functions, eigen-values and reflectance functions (aka, Smith charts). Here we shall restrict oursleves to 2x2 transmission matrices.

Transmission matrices: We start from a general transmission matrix, as defined in the class notes (p. 79-80)
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_1(s) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_2(s) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & Z_N(s) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_N \\ -I_N \end{bmatrix} = \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix} \begin{bmatrix} V_N \\ -I_N \end{bmatrix}. \tag{1}$$
Each 2x2 matrix follows from Ohm's law, thus forms a linearized expansion of the system, since each element is a linear impedance, defined is the ratio of the voltage over the current (or the force over the velocity), as discussed in Fig. 1.15 (p. 81). From the above it is clear that the $\boldsymbol{T}(s)$ matrix
$$\boldsymbol{T}(s) = \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix}$$
has determinant $\Delta_T = \mathcal{A}(s)\mathcal{D}(s) - \mathcal{B}(s)\mathcal{C}(s) = 1$. The elements of $\boldsymbol{T}(s)$ $(\mathcal{A}(s), \mathcal{B}(s), \mathcal{C}(s), \mathcal{D}(s))$, are complex analytic causal functions of complex frequency $s = \sigma + \omega\jmath$ (i.e., they have inverse Laplace transforms). These four functions are defined in Eq. (1.62) (p. 80) of the class notes. The 2x2 transmission matrix $\boldsymbol{T}(s)$ generalizes to cascaded systems of transmission lines, which includes a large class of problems in engineering, as discussed in Fig. 1.15 (p. 81). For a specific example look at Sect. 1.3.3.3 (pp. 193-196).

The functions $Z_k(s)$ and $Y_k(s)$ are series impedances and shunt admittances. Their form can be quite general, including ratios of polynomials of s. For example $Z_1(s)$ and $Y_2(s)$ can be the parallel combination of an inductor and capacitor.

The impedance matrix: The *impedance matrix* corresponding to $\boldsymbol{T}(s)$ is (Notes, p. 81-82)
$$\begin{bmatrix} V_1 \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{1,1}(s) & Z_{1,2}(s) \\ Z_{2,1}(s) & Z_{2,2}(s) \end{bmatrix} \begin{bmatrix} I_1 \\ I_N. \end{bmatrix}$$
The relations between $Z_{i,j}$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ is given is given in Eq. (1.63) (p. 80). We denote the impedance matrix as
$$\mathcal{Z}(s) = \begin{bmatrix} Z_{1,1}(s) & Z_{1,2}(s) \\ Z_{2,1}(s) & Z_{2,2}(s) \end{bmatrix}. \tag{2}$$

Brune impedance $Z(s)$: $Z_{1,1}(s)$ is an example of a complex analytic Brune impedance, requiring that $\Re Z(\sigma \geq 0) \geq 0$. This relation, first proposed by Brune in 1931, says that the real part (resistance) of every Brune impedance is non-negative in the entire right half s plane ($\sigma > 0$). This condition follows from the requirement that every impedance has a positive real part (non-negative in the RHP). Conservation of energy follows from this condition. All impedance matricies are symmetric ($Z_{1,2} = Z_{2,1}$), and the roots are (simple no double roots). This last property has never been formally proven, but no counter examples are known.

Reflection function $\Gamma(s)$: The Brune impedance $Z_{1,1}(s)$ looking into a transmission line (TL) is related to the TL's reflectance $\Gamma(s)$ via a *Smith chart*, according to the relation

$$Z_{1,1}(s) = r_o \frac{1+\Gamma(s)}{1-\Gamma(s)},$$

where r_o is the characteristic impedance of the TL and $\Gamma(s) = e^{-\kappa(s)2L}$. The function $\kappa(s) = s/c$ is called the TL's *complex propagation function*.

Positive definite (PD) matricies: A 2x2 positive definite matrix $\mathcal{Z}(s)$ has the properity that $I \cdot \mathcal{Z}(s) I \geq 0$ for all vectors I. PD matricies are commonly used in mathematics.

Symmetric matricies have positive eigen-values: The eigen-values of real symmetric matricies are positive, as long as the eigen-values are distinct (no double roots).

To do:

1. Starting from Eq. 1, show that $\Delta_{\boldsymbol{T}} = 1$.
2. Derive the relation (above) between the input impedance $Z_{1,1}$ and the reflection coefficient $\Gamma(s)$.
3. Given that $\Delta_{\boldsymbol{T}}(s) = 1$, show that the impedance matrix (Eq. 2) is symmetric (i.e., $Z_{1,2} = Z_{2,1}$).
4. Find an expression for the *Thévenin voltage* in terms of the elements of the $\boldsymbol{T}$ matrix. Hint: The Thévenin voltage is defined as the open circuit voltage V_2 when current I_2 is zero.
5. Find the expression for the *Thévenin impedance* of the transmission line, in terms of the elements of the $\boldsymbol{T}(s)$ matrix.
6. What is the definition of a *self adjoint* matrix?
7. Is an impedance matrix self-ajoint?
8. What is the take-home message? Is a transmission line physical, yet not self-adjoint?