ECE 493	**LA3 – Version 1.22 March 15, 2018**	**Spring 2018**
Univ. of Illinois	**Due Tue, Mar. 15**	**Prof. Allen**

Topic of this homework: Linear Algebra: Solutions of non-symmetric matrices (Tall and Fat), Singular value decomposition (SVD)

Deliverables: Show your work. Numerical results may not be sufficient, unless specifically requested.

1 Least-square solution of a non-square matrix

1.1 Tall (over-specified) systems

You are given the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{1}$$

1. Find the least squares (LS) solution of the *tall* (over-specified) system of equations $Ax = b$. Mentally note that leaving out the third equation (third row of A), which is very different than the first two, would result in a trivial solution of $x_1 = 1$ and $x_2 = 1$.

2. Justify that the inverse of the "tall" over-specified system of equations as being $(A^T A)^{-1} A^T$.

3. Find the LS solution, but unlike Eq. 1, the third (row) is close to the first row

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \epsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{2}$$

   where $|\epsilon| \ll 1$.

4. Starting from Eq. 1, find the eigen-values and eigen-vectors of the LS solution $A^T A$.

5. Summarize your conclusion about the impact of Eq. 2 on the LS solution, as a function of $|\epsilon \ll 1$.

1.2 Fat (under-specified) systems of equations

You are given the under-specified system

$$\begin{bmatrix} 1 & 2 & \epsilon \\ 2 & -1 & -\epsilon \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{3}$$

with $|\epsilon| \ll 1$.

1. Find the inverse of the *fat* under-specified system

2. Identify span(A) and null(A) for

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

   Define the operational definitions of these concepts.

3. Ignore this question. It was incomplete.

2 Singular Value Decomposition (SVD)

You are given

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}_{2\times 3}.$$

1. Find U, Σ and V such that $A = U\Sigma V^T$.

   Hint: If $\texttt{eig}(S)$ computes the matrix of eigenvectors of matrix S, then $U = \texttt{eig}(AA^T) \in \mathbb{R}^{2\times 2}$ and $V = \texttt{eig}(A^T A) \in \mathbb{R}^{3\times 3}$.

   Hint 2: $U^T U = I$ and $V^T V = I$.

   Hint 3: This problem was intended to be done using Matlab/Octave. For those of you that don't have access, for what ever reason, here are the eigen vectors of V:

   $$V = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{2} \\ -1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} = \texttt{eig}(A^T A)$$

   Note that the three are orthogonal.

2. Repeat calculation for A'

   $$A' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

   Hint: Take the transpose of the formula for A in terms of U, Σ, V.

3. Define the property of a unitary matrix.

4. Given that $A = U\Sigma V^T$, where U and V are orthogonal (real-unitary) and Σ is diagonal, show that U and V are the eigen-matrices of AA^T and $A^T A$, respectively.

5. What are the ranks of $A^T A$ and AA^T. Give a full explanation.

3 Operator symmetry

Each matrix (operator) A below has dimensions $m \times n$ (m rows and n columns). Define r as the smaller, and l as the larger, of m, n.

For each given matrix symmetry, provide the following information:

1. The *name* of the symmetry (e.g., Hermitian, unitary, self-adjoint, analytic, causal, etc.)

2. Definiteness: i.e., positive, semi-positive, negative, semi-negative definite

3. Eigen-vector properties (e.g., real, imaginary, complex, conjugate, zero, N, ON)

4. Eigen-value spectrum: i.e., discrete fixed set, infinite set, continuous infinite, etc.

3.1 Matrix Symmetry

Assume that A is $m \times n$, unless otherwise stated and $s = \sigma, \omega\jmath$.

1. $A = A^T$

2. $A = \overline{A}$

3. $A = -\overline{A}$

4. $S = A^T A$

5. $S = AA^T$

6. $A = A^\dagger$

7. $A^\dagger A$

8. $AA^\dagger$

9. Prove that the eigenvalues of $A^\dagger = A$, a Hermitian matrix, are real.

10. Prove that the eigenvectors of a symmetric matrix are orthogonal.

3.2 Complex matrix symmetry

1. Impedance matrix $Z(s) = R(s) + jX(s)$, for a network having two nodes is given as

$$\begin{bmatrix} V_1(\omega) \\ V_2(\omega) \end{bmatrix} = \begin{bmatrix} 1+s & 1/s \\ 1/s & 1+1/s \end{bmatrix} \begin{bmatrix} I_1(\omega) \\ I_2(\omega) \end{bmatrix} \tag{4}$$

   Is this matrix Hermitian?

2. Admittance matrix for the above impedance matrix $Y(s) = Z^{-1}(s)$ is given as

$$\begin{bmatrix} I_1(\omega) \\ I_2(\omega) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 1+1/s & -1/s \\ -1/s & 1+s \end{bmatrix} \begin{bmatrix} V_1(\omega) \\ V_2(\omega) \end{bmatrix} \tag{5}$$

   with $\Delta = (1+s)(1+1/s) - 1/s^2 = 2 + s + 1/s - 1/s^2$

Is this matrix Hermitian?

3.3 Continuous Symmetry

1. Given the following differential operator

$$A = \frac{d^2}{dt^2} + 2\frac{d}{dt} + 1.$$

   Find the eigen-values and -vectors. Hint: Take the Laplace Transform.