ECE 493 **HW VC1 – Version 1.08 April 2, 2018** **Spring 2018**

Univ. of Illinois **Due Tu, Apr 3** **Prof. Allen**

Topic of this homework: By lecture:
L1: Vector identities: cross and triple product;
Scalar and vector fields;
Implicit function thm

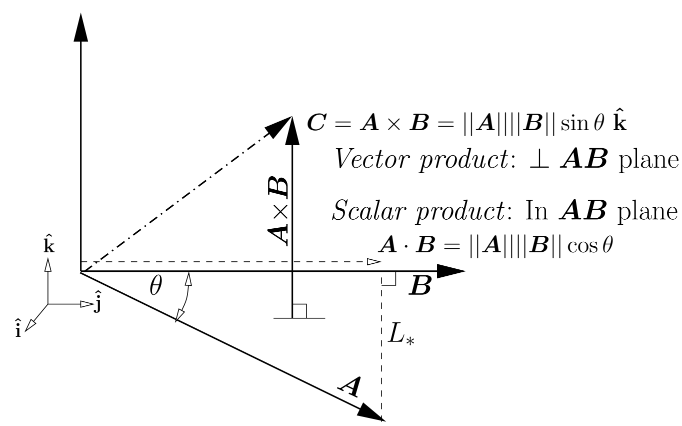
Deliverables: Show your work. Numerical results are not sufficient, expect when specifically requested.

General definitions: $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors in the x, y, z directions, that satisfy

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1, \quad \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = 0, \tag{1}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = 0, \quad \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}, \text{etc.} \tag{2}$$

as defined in the figure:



There are two differents methods of defining vectors: a) $\overrightarrow{Z} = [1,\ 2,\ 3]^T$, where as before, the script T is transpose of the row-vector to a column, and b) $\mathbf{Z} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. They represent exactly the same thing, and the rules of computation are identical. An arrow over the top, or the bold font, both indicate a column vector. You must adapt to these two equivalent representations.

Several of these problems are adapted from (Greenberg, 1988, p. 695).

1 Condition for vector independence

1. Apply the Gram Schmidt (GS) procedure to the vectors $\vec{A} = [2, -1]^T$ and $\vec{B} = [0, 1]^T$.

2. What is the condition that two vectors $\vec{A}$ and $\vec{B}$ are independent? Hint: Look up the Schwarz inequality.

2 Cross product $\vec{A} \times \vec{B}$

1. If $\vec{A} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$ and $\vec{B} = b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}}$, write out the definition of $\vec{A} \times \vec{B}$:

2. Derive the formula for $\vec{C} = \vec{A} \times \vec{B}$. Explain your logic. Hint: Define a matrix $\vec{Q} = [\vec{C}\vec{A}\vec{B}]^T$, where $\vec{A}$ and $\vec{B}$ are independent $(0 \le |\vec{A} \cdot \vec{B}| < ||\vec{A}||\,||\vec{B}||)$, $\vec{C} \perp \vec{A}$ $(\vec{C} \cdot \vec{A} = 0)$ and $\vec{C} \perp \vec{B}$ $(\vec{C} \cdot \vec{B} = 0)$.

3. Show that $||\vec{C}||^2 + |\vec{A} \cdot \vec{B}|^2 = ||\vec{A}||^2\vec{B}||^2$.

4. Can θ be complex?

3 Triple product

Let $\vec{A} = [a_1, a_2, a_3]^T$, $\vec{B} = [b_1, b_2, b_3]^T$, $\vec{C} = [c_1, c_2, c_3]^T$ be three vectors in $\mathbb{R}^3$.

1. Show that three vectors $\vec{A}$,$\vec{B}$,$\vec{C}$ are in one plane if and only if their triple product is 0.

2. If $\vec{B}$ is replaced with the gradient operator ∇, write out $A \cdot \nabla \times C$ as the determinant of a matrix, but replacing $\vec{B}$ with

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}.$$

   The vector product of $\nabla \times \vec{C}(\mathbf{x})$ is called the *curl* of $\vec{C}$.

3. What are the implications if the determinant of $A \cdot \nabla \times C$ being zero?

4 Span and vector cross products

As shown in the figure above, assume $\vec{A}, \vec{B}, \vec{C}, \in \mathbb{R}^3$ with $\vec{C} = \vec{A} \times \vec{B}$.

1. What is the span$\{\vec{C}\}$?

2. Given a new vector $\vec{Q} \in \mathbb{R}^3$ (not shown), what is the span$\{\vec{Q} \times \vec{C}\}$? Hint consider your answer to the previous problem.

3. Given the span of $\vec{Q} \times (\vec{A} \times \vec{B})$, explain the following identity (Hint: Recall GS):

$$\vec{Q} \times (\vec{A} \times \vec{B}) = (\vec{Q} \cdot \vec{B})\vec{A} - (\vec{Q} \cdot \vec{A})\vec{B}. \tag{3}$$

4. What is the name of the above identity?

5. Are the brackets () necessary in identity (4)? Namely is

$$\vec{Q} \times (\vec{A} \times \vec{B}) = (\vec{Q} \times \vec{A}) \times \vec{B}? \tag{4}$$

   Either explain, or give a numerical example to verify.

6. Assume $\vec{A} = [1,1,1]^T$, $\vec{B} = [1,2,1]^T$ and $\vec{Q} = [0,0,1]^T$. Verify the above identity (3).

7. Repeat the calculation by first computing the RHS of Eq. 4.

8. Prove that if $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$, that they are *linearly dependent* if and only if

$$\vec{x} \cdot \vec{y} \times \vec{z} = 0.$$

5 Vector fields

Let $\mathbf{R}(x,y,x) \equiv x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \sim [x(t), y(t), z(t)]^T$

1. What is $\mathbf{R}(x,y,z) \cdot \mathbf{R}(a,b,c)$?

2. What is $\frac{d}{dt}\left(\mathbf{R}(x,y,z) \cdot \mathbf{R}(a,b,c)\right)$?

3. Write out the formula for the mean temperature $\overline{T}(\mathbf{x})$, give a temperature field $T(\mathbf{x})$, defined over a unit cube (between x, y, z, each between $[0,1]$).

4. As for an electrical current, the heat *current* $\vec{J}(\mathbf{x}) = -\nabla T(\mathbf{x})$ is given by the gradient of a potential. Write out the formula for the mean flux $\vec{J}(\mathbf{x})$ over the unit cube V_o.

6 Implicit function method

1. Using the *Implicit function* method of (Greenberg, 1988, pp. 642-649), find the Taylor series of $y(x)$ about $x_0, y_0 = (-2, 1)$, given

$$x[\cos(\pi y) + 1] + x^3 y + 8 = 0. \tag{5}$$

   Show your work. HINT: Verify your answer at (Greenberg, 1988, p. 1299).

References

Greenberg, M. D. (**1988**), *Advanced Engineering Mathematics* (Prentice Hall, Upper Saddle River, NJ, 07458), URL `http://jontalle.web.engr.illinois.edu/uploads/493/Greenberg.pdf`.