ECE 493 **HW VC2 – Version 1.15 April 3, 2018** **Spring 2018**

Univ. of Illinois **Due Tue, April 10** **Prof. Allen**

Topic of this homework: Line, surface and volume integrals,
Grad: $\nabla()$, Div: $\nabla \cdot ()$, Curl: $\nabla \times ()$,
Laplacian, Helmholtz theorem,
Green's, Stokes and Divergence Thms

Deliverables: Show your work. Numerical results are not sufficient, expect when specifically requested.

There are two differents methods of defining vectors: a) $\overrightarrow{Z} = [1,\ 2,\ 3]^T$, where as before, the script T is transpose of the row-vector to a column, and b) $\mathbf{Z} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$. They represent exactly the same thing, and the rules of computation are identical. An arrow over the top, or the bold font, both indicate a column vector. You must adapt to these two equivalent representations.

Several of problems are adapted from (Greenberg, 1988; Allen, 2017).

1 Scalar fields and ∇:

1. Let $T(x, y)$ be a scalar temperature field in 2 dimensions. Describe or sketch the iso-temperature contours at T=10, 20, 30 degrees.

   (a) $T = x^2 + y^2$

   (b) $T = 10(x - y)$

2. Compute the gradient of $T = x^2 + y^2$.

3. Given some scalar field $\phi(x, y, z)$, find the rate of change, in the direction of maximum change. Hint: Take the gradient.

4. Define and then describe the meaning of the *directional derivative*

$$\frac{\partial T}{\partial n}\hat{\mathbf{n}} \tag{1}$$

   in the direction of unit vector $\hat{\mathbf{n}}$.

2 Vector fields

Let $\mathbf{v}(x, y, z) = x\hat{\mathbf{x}} - y\hat{\mathbf{y}} \in \mathbb{R}^3$.

1. Describe (or sketch) $\mathbf{v}(\mathbf{x})$.

2. Find the curves of constant scalar speed $s(\mathbf{x})$ of $\mathbf{v}(\mathbf{x})$, defined as the norm of the velocity $s(\mathbf{x}) = ||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

3 Divergence

1. Find the divergence of the following vector fields:

   (a) $\mathbf{v}(\mathbf{x}) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + \log(z)\hat{\mathbf{z}}$

   (b) $\mathbf{v}(\mathbf{x}) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + \log(z)\hat{\mathbf{z}}$ evaluated at $\mathbf{x} = [1, 2, 3]^T$.

   (c) $\mathbf{v}(\mathbf{x}, t) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + \log(z)\hat{\mathbf{z}} + t(\hat{\mathbf{x}} + \hat{\mathbf{y}})$

2. Gauss' Law

   Given a closed surface $\mathcal{S}$, the area is denoted as $|\mathcal{S}|$, and the volume enclosed as $||\mathcal{S}||$ (see the class notes, Lecture 39, Sect. 1.5.10).

   (a) In words, explain:

$$\iiint_{||\mathcal{S}||} \nabla\cdot\boldsymbol{D}\, ||d\mathcal{S}|| = \iint_{|\mathcal{S}|} \boldsymbol{D}\cdot\hat{\boldsymbol{n}}\, d|\mathcal{S}|. \tag{1}$$

   (b) In words, explain:

$$\nabla\cdot\boldsymbol{D} = \lim_{|\mathcal{S}|\to 0} \left\{ \frac{\int_{\mathcal{S}} \boldsymbol{D}\cdot\hat{\mathbf{n}}\; d|\mathcal{S}|}{|\mathcal{S}|} \right\} = \rho(x, y, z). \tag{2}$$

   (c) If $\phi(x, y) \equiv x + y$, find the unit vector pointing perpendicular ($\perp$) to plane $\phi = 1$

   (d) Explain the relationship between $\int_a^b F' dx = F(x)|_a^b = F(b) - F(a)$ and the Divergence Thm.

3. Assuming some vector potential field $\mathbf{w}(\mathbf{x})$, evaluate $I = \int_{|\mathcal{S}|} \hat{\mathbf{n}}\cdot\nabla\times\mathbf{w}(\mathbf{x})\, d|\mathcal{S}|$. Explain your reasoning. Hint: DoC/CoG.

4 Curl and Stokes Thm

The curl may be defined as the limit of the integral

$$\nabla\times\boldsymbol{H} \equiv \lim_{|\mathcal{S}|\to 0} \left\{ \frac{\int_{\mathcal{S}} \hat{\boldsymbol{n}} \times \boldsymbol{H}\; d|\mathcal{S}|}{|\mathcal{S}|} \right\}. \tag{1}$$

Stokes' theorem (law) says

$$\iint_{\mathcal{S}} (\nabla\times\boldsymbol{H})\cdot\hat{\boldsymbol{n}}\; d|d\mathcal{S}| = \oint_{\mathcal{B}} \boldsymbol{H}\cdot d\boldsymbol{l}. \tag{2}$$

1. Expand the formula for the curl in rectangular coordinates, about the top row.

2. If $\mathbf{v}(x, y, z) = \nabla(1/x + 1/y + 1/z)$, find $\nabla \times \mathbf{v}(x, y, z)$.

3. One of Maxwell's Equations, under steady state conditions, is $\nabla \times \mathbf{H} = \mathbf{J}$, where $\mathbf{H}$ [amps/m] is the magnetic field strength and $\mathbf{J}$ [amps/m^2] is the current density. Apply Stokes' Thm to this equation, and explain the result.

4. Determine when $\nabla\times\boldsymbol{V}(\mathbf{x})$ is independent of $\boldsymbol{V}(\mathbf{x})$ Hint: Consider the definition of the triple product $\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C}$.

5. Prove CoG: $\nabla\times\nabla\phi(\mathbf{x}) = 0$

6. Prove DoC: $\nabla\cdot\nabla\times\boldsymbol{V} = 0$

5 Laplacian

1. If $\vec{v}(x,y,z) = \nabla(1/x + 1/y + 1/z)$, then what is $\nabla\cdot\vec{v}(x,y,z)$? What does this mean?

2. For $u = xe^y$ evaluate

$$\nabla^2 u = \nabla^2 xe^y$$

   and

$$\nabla\times\nabla u =$$

3. For $\vec{v} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, evaluate

$$\nabla\cdot(\nabla\times\vec{v}) =$$

   and

$$\nabla\times(\nabla\times\vec{v}) = \nabla(\nabla\cdot\vec{v}) - \boldsymbol{\nabla}^2\vec{v} = .$$

6 Compression-Rotation decomposition Thm.

Helmholtz decomposition theorem: Every vector field [e.g., $\vec{v}(x,y,z)$] may be defined as the sum of the gradient of a *scalar potential* $\Phi(x,y,z)$, and the curl of a *vector potential* $\vec{A}(x,y,z)$:

$$\vec{v}(x,y,z) \equiv \nabla\Phi(x,y,z) + \nabla\times\vec{A}(x,y,z).$$

Given the two vector identities CoG $\nabla\times\nabla\Phi = 0$ and DoC $\nabla\cdot\nabla\times\vec{A} = 0$ what can you say about Φ and $\vec{A}$?

1. First take the (a) divergence and second (b) the curl of $\vec{v}(x.,y,z)$. Give an example from Maxwell's equations (or fluid flow) of the meaning of each result.

2. When is a field *irrotational*?

3. Classify $\mathbf{v}(x,y,z) = x\hat{\mathbf{x}} - y\hat{\mathbf{y}} \in \mathbb{R}^3$ as (in)compressible and (ir)rotational by computing $\nabla\cdot\mathbf{v}(\mathbf{x})$ and $\nabla\times\mathbf{v}(\mathbf{x})$ and finding its potential $\Phi(\mathbf{x})$ (i.e., $\mathbf{v}(\mathbf{x}) = \nabla\boldsymbol{\Phi}$).

4. When is a field *incompressible*?

6.1 Field classification:

Provide your reasoning. Hint: look at Fig. 1.33 of Lec. 40, Sect. 1.5.11.

1. Classify the following fields as (in)compressible or (ir)rotational?

   (a) $\vec{v}(x,y,z) = \nabla(3x^3 + y\cos(xy))$

   (b) $\vec{v}(x,y,z) = xy\hat{\mathbf{x}} - z\hat{\mathbf{y}} + f(z)\hat{\mathbf{z}}$

2. Define the purely rotational (i.e., incompressable) field, and provide your reasoning.

References

Allen, J. B. (**2017**), *An invitation to mathematical physics, and its history* (unpublished), URL http://jontalle.web.engr.illinois.edu/uploads/298/anInvitation.pdf.

Greenberg, M. D. (**1988**), *Advanced Engineering Mathematics* (Prentice Hall, Upper Saddle River, NJ, 07458), URL http://jontalle.web.engr.illinois.edu/uploads/493/Greenberg.pdf.