2.2 Problems AE-2

Topics of this homework:

Linear vs nonlinear systems of equations, Euclid's formula, Gaussian elimination, matrix permutations, Ohm's law, two-port networks,

   Deliverables: Answers to problems

(- pts) Gaussian elimination

 Problem # *1: (1 pts) Gaussian elimination*

   – *1.1: (1 pts) Find the inverse of*

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

   – *1.2: (1 pts) Verify that* $A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

 Problem # *2: (6 pts) Find the solution to the following 3×3 matrix equation $Ax = b$ by GE. Show your intermediate steps. You can check your work at each step using Octave/Matlab.*

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

   – *2.1: (2 pts) Show (i.e., verify) that the first GE matrix G_1, which zeros out all entries in the first column is given by*

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Identify the elementary row operations that this matrix performs.

   – *2.2 (2 pts) Find a second GE matrix, G_2, to put G_1A in upper triangular form. Identify the elementary row operations that this matrix performs.*

   – *2.3 (2 pts) Find a third GE matrix G_3 that scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.*

   – *2.4: (2 pts) Find the last GE matrix, G_4, which subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ($G_4G_3G_2G_1A = I$).*

– 2.5: *(2 pts) Solve for* $\{x_1, x_2, x_3\}^T$ *using the augmented matrix format* $G_4G_3G_2G_1\{A|b\}$ *(where* $\{A|b\}$ *is the augmented matrix). Note that if you've performed the preceding steps correctly,* $x = G_4G_3G_2G_1b$.

– 2.6: *(1 pt) Find the pivot matrix* G *that rescales the second row of the augmented matrix* $A|b$ *by 1/3.*

Two linear equations

Problem # *3 In this problem we transition from a general pair of equations*

$$f(x, y) = 0$$
$$g(x, y) = 0$$

to the important case of two linear equations

$$y = ax + b$$
$$y = \alpha x + \beta.$$

Note that to help keep track of the variables, roman coefficients (a, b) are used for the first equation and Greek (α, β) for the second.

– 3.1: *What does it mean, graphically, if these two linear equations have (1) a unique solution, (2) a nonunique solution, or (3) no solution?*

– 3.2: *Assuming the two equations have a unique solution, find the solution for* x *and* y.

– 3.3: *When will this solution fail to exist (for what conditions on* a, b, α, *and* β*)?*

– 3.4: *Write the equations as a* 2×2 *matrix equation of the form* $A\vec{x} = \vec{b}$, *where* $\vec{x} = \{x, y\}^T$.

– 3.5: *Find the inverse of the* 2×2 *matrix, and solve the matrix equation for* x *and* y.

– 3.6: *Discuss the properties of the determinant of the matrix* (Δ) *in terms of the slopes of the two equations* $(a$ *and* $\alpha)$.

Problem # *4: The application of* linear functional relationships *between* two variables

We use 2×2 matrices to describe two-port networks, as discussed in Sec. **??** (p. **??**). Transmission lines are a great example: Both voltage and current must be tracked as they travel along the line. The Figure below shows an example segment of a transmission line.
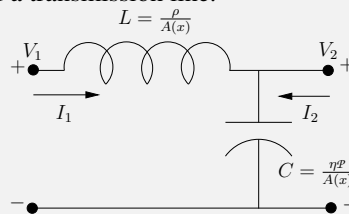

Fig. AE-2.1 *This figure shows a cell from an LC transmission line. The index* 1 *is at the input on the left and* 2 *represents the output, on the right.*

Suppose you are given the following pair of linear relationships between the input (source) variables V_1 and I_1 and the output (load) variables V_2 and I_2 of the transmission line:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

– 4.1: *Let the output (the load) be* $V_2 = 1$ *and* $I_2 = 2$ *(i.e.,* $V_2/I_2 = 1/2$ *{Ω}). Find the input voltage and current,* V_1 *and* I_1.

– 4.2: *Let the input (source) be* $V_1 = 1$ *and* $I_1 = 2$. *Find the output voltage and current,* V_2 *and* I_2.

Integer equations: applications and solutions

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

Problem # *5: (5 pts) A practical example of using a Diophantine equation:*

"A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?" - *Bachet de Bèziriac (1623)*[a].

Here, weighing is performed using a balance scale that has two pans, with weights on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weight in the other pan. Then the scale will be balanced. A solution to the four weights for Bachet's problem is $1 + 3 + 9 + 27 = 40$ pounds.

---
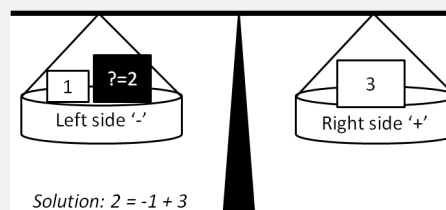
[a]Taken from **?**, p. 50

– 5.1: Show how the combination of 1-, 3-, 9-, and 27-pound weights can be used to weigh $1, 2, 3, \ldots, 8, 28,$ *and* 40 *pounds of milk (or something else, such as flour). Assuming that the milk is in the* left *pan, provide the position of the weights using a negative sign* $-$ *to indicate the left pan and a positive sign* $+$ *to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan,* $+1$*, will balance the scales.*

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]^T$. The pan assignments matrix should contain only the values -1 (left pan), $+1$ (right pan), and 0 (leave out). You can indicate these using $-$, $+$, and blanks.

Vector algebra in $\mathbb{R}^3$

Definitions of the scalar (also called a dot product) $\boldsymbol{A} \cdot \boldsymbol{B}$, cross $\boldsymbol{A} \times \boldsymbol{B}$ and triple product $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$, may be found in Appendix **??** (p. **??**), where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ in $\mathbb{R}^3 \subset \mathbb{C}^3$), as shown in Fig. **??**, p. **??**. A fourth "double-cross" (☿) vector product is:[a]

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \alpha_o \boldsymbol{B} - \beta_o \boldsymbol{C}.$$

where $\alpha_o = \boldsymbol{A} \cdot \boldsymbol{C}$ and $\beta_o = \boldsymbol{A} \cdot \boldsymbol{B}$ (Note: $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$).

---
[a]Greenberg p. 694, Eq. 8.

Problem # *6:* Scalar product A · B

– 6.1: If $\boldsymbol{A} = a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}} + a_z \hat{\mathbf{z}}$ *and* $\boldsymbol{B} = b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}} + b_z \hat{\mathbf{z}}$*, write out the definition of* A · B.

– 6.2: The dot product is often defined as $||\boldsymbol{A}|| \, ||\boldsymbol{B}|| \cos(\theta)$*, where* $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$ *and* θ *is the angle between* $\boldsymbol{A}, \boldsymbol{B}$*. If* $||\boldsymbol{A}|| = 1$*, describe how the dot product relates to the vector* B.

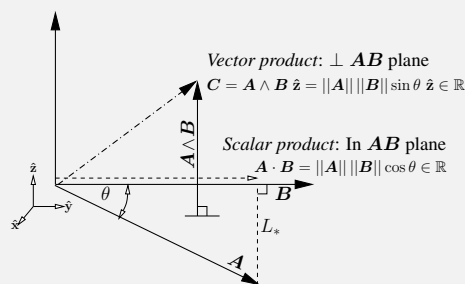## Problem # *7:* Vector (cross) product A × B

– 7.1: If $\boldsymbol{A} = a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}} + a_z \hat{\mathbf{z}}$ *and* $\boldsymbol{B} = b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}} + b_z \hat{\mathbf{z}}$*, write out the definition of* A × B.

– 7.2: Show that the cross product is equal to the area of the parallelogram formed by $\boldsymbol{A}, \boldsymbol{B}$*, namely* $||\boldsymbol{A}|| \, ||\boldsymbol{B}|| \sin(\theta)$*, where* $||\boldsymbol{A}|| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$ *and* θ *is the angle between* $\boldsymbol{A}$ *and* $\boldsymbol{B}$*.*

Problem # *8*: *Triple product* $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$
Let $\boldsymbol{A} = [a_1, a_2, a_3]^T$, $\boldsymbol{B} = [b_1, b_2, b_3]^T$, $\boldsymbol{C} = [c_1, c_2, c_3]^T$ be three vectors in $\mathbb{R}^3$.

– 8.1: Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Vector product: $\perp \boldsymbol{AB}$ plane
$\boldsymbol{C} = \boldsymbol{A} \wedge \boldsymbol{B}\,\hat{\boldsymbol{z}} = ||\boldsymbol{A}||\,||\boldsymbol{B}|| \sin\theta\,\hat{\boldsymbol{z}} \in \mathbb{R}$

Scalar product: In $\boldsymbol{AB}$ plane
$\boldsymbol{A} \cdot \boldsymbol{B} = ||\boldsymbol{A}||\,||\boldsymbol{B}|| \cos\theta \in \mathbb{R}$

This is figure is identical to Fig. **??** (p. **??**), Sec. **??**. Definitions of vectors A, B, C (vectors in $\mathbb{R}^3$) used in the definition of $\boldsymbol{A} \cdot \boldsymbol{B}$, $\boldsymbol{A} \times \boldsymbol{B}$ and $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$. There are two algebraic vector products, the *scalar (dot) product* $\boldsymbol{A} \cdot \boldsymbol{B} \in \mathbb{R}$ and the *vector (cross) product* $\boldsymbol{A} \times \boldsymbol{B} \in \mathbb{R}^3$. Note that the result of the dot product is a scalar, while the vector product yields a vector, which is $\perp$ to the plane containing $\boldsymbol{A}, \boldsymbol{B}$.

– 8.2: Describe why $|\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})|$ *is the volume of parallelepiped generated by* $\boldsymbol{A}, \boldsymbol{B}$, *and* $\boldsymbol{C}$.

– 8.3: Explain why three vectors A, B, C *are in one plane if and only if the triple product* $\mathrm{A} \cdot (\mathrm{B} \times \mathrm{C}) = \boldsymbol{0}$.

Problem # *9*: *Given two vectors* $\boldsymbol{A}, \boldsymbol{B}$ *in the* $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ *plane shown in Fig.* **??** *(same as* **??** *on page* **??***), with* $\boldsymbol{B} = \hat{\mathbf{y}}$ *(i.e.,* $||\boldsymbol{B}|| = 1$*).*

– 9.1: Show that $\boldsymbol{A}$ *may be split into two orthogonal parts, one in the direction of* $\boldsymbol{B}$ *and the other perpendicular* ($\perp$) *to* $\boldsymbol{B}$*. Hint: Express the vector products of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *(dot and cross) in polar coordinates* (**?***).*

$$\boldsymbol{A} = (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{B} + \boldsymbol{B} \times (\boldsymbol{A} \times \boldsymbol{B})$$
$$= \boldsymbol{A}_{\|} + \boldsymbol{A}_{\perp}.$$

Ohm's Law

In general, impedance is defined as the ratio of a force to a flow. For electrical circuits, the voltage is the force and the current is the flow. Ohm's law states that the voltage across and the current through a circuit element are related by the impedance of that element (which may be a function of frequency). For resistors, the voltage over the current is called the *resistance* and is a constant (e.g., the simplest case is $V/I = R$). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g., $V/I = Z(s)$, where s is the complex frequency $s \in \mathbb{C}$).

As shown in Table **??** (p. **??**), the impedance concept also holds in mechanics and acoustics. In mechanics, the force is equal to the mechanical force on an element (e.g., a mass, dashpot, or spring) and the flow is the velocity. In acoustics, the force is pressure and the flow is the volume velocity or particle velocity of air molecules.

Case	Force	Flow	Impedance	units
Electrical	voltage (V)	current (I)	Z	Ohms [Ω]
Mechanics	force (F)	velocity (V)	Z	Mechanical Ohms [Ω]
Acoustics	pressure (P)	particle velocity (U)	Z	Acoustic Ohms [Ω]
Thermal	temperature (T)	heat-flux (J)	Z	Thermal Ohms [Ω]

Problem # 10: *The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As the bulb lights up, the resistance of the metal filament increases.*
Ohm's law says that the current

$$\frac{V}{I} = R(T),$$

where T is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz]. Find the current when the light is first switched on.

Problem # 11: *(1 pts) The power in watts is the product of the force and the flow. What is the power of the lightbulb of Problem 10?*

Problem # 12: *(1 pts) State the impedance $Z(s)$ of each of the following circuit elements: (1) a resistor with resistance R, (2) an inductor with inductance L, and (3) a capacitor with capacitance C.*

Problem # 13: *Consider what happens at the triple point of water. As water freezes or thaws, the temperature remains constant at 0 (C°). Once all the water is frozen and more heat is removed, the temperature drops below 0°. As heat is added, water thaws but the temperature remains at 0°.*

– 13.1: Once all the ice has melted, what is the temperature as more heat is added?
Model the triple point using a Zener diode, a resistor, and a capacitor. A Zener diode holds the voltage constant independent of current. For the case of water's triple point, the voltage represents the temperature of water at the triple point, clamped at 0 [C°]. The current represents the heat flux. The latent heat of water at the triple point is 32 Cal/gm. Thus as the temperature rises from below freezing, the water is clamped at 0° once the triple point is reached. At that point, adding more heat flux has no effect on the temperature until all the ice melts. Once the ice has melted, the temperature again begins to rise until it hits the boiling point, where it again stays at 100° until all the water has evaporated.

Nonlinear (quadratic) to linear equations

In the following problems we deal with algebraic equations in more than one variable that are not linear equations. For example, the circle $x^2+y^2 = 1$ may be solved for $y(x) = \pm\sqrt{1-x^2}$. If we let $z_+ = x+y\jmath = x+\jmath\sqrt{1-x^2} = e^{\theta\jmath}$, we obtain the equation for half a circle $(y > 0)$. The entire circle is described by the magnitude of z as $|z|^2 = (x+y\jmath)(x-y\jmath) = 1$.

Problem # *14: Give the curve defined by the equation:*

$$x^2 + xy + y^2 = 1$$

– 14.1: Find the function $y(x)$.

– 14.2: Using Matlab/Octave, plot $y(x)$ and describe the graph.

– 14.3: What is the name of this curve?

– 14.4: Find the solution (in x, p, and q) to these equations:

$$x + y = p$$
$$xy = q.$$

– 14.5: Find an equation that is linear in y starting from equations that are quadratic (second-degree) in the two unknowns x and y:

$$x^2 + xy + y^2 = 1 \tag{AE-2.1}$$
$$4x^2 + 3xy + 2y^2 = 3. \tag{AE-2.2}$$

– 14.6: Compose the following two quadratic equations and describe the results.

$$x^2 + xy + y^2 = 1$$
$$2x^2 + xy \quad = 1$$

chapter[Euclid's formula derivation] Euclid's formula derivation

Nonlinear intersection in analytic geometry

Euclid's formula for Pythagorean triplets (Eq. **??**, p. **??**) can be derived by intersecting a circle and a secant line. Consider the nonlinear equation of a unit circle having radius 1, centered at $(x, y) = (0, 0)$,
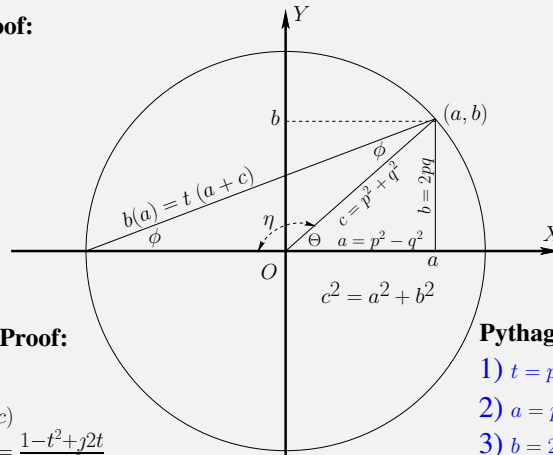
$$x^2 + y^2 = 1,$$

and the secant line through $(-1, 0)$,
$$y = t(x + 1),$$

a linear equation having slope t and intercept $x = -1$. If the slope $0 < t < 1$, the line intersects the circle at a second point (a, b) in the positive x, y quadrant. The goal is to find $a, b \in \mathbb{N}$ and then show that $c^2 = a^2 + b^2$. Since the construction gives a right triangle with short sides $a, b \in \mathbb{N}$, then it follows that $c \in \mathbb{N}$.

Euclidean Proof:

1) $2\phi + \eta = \pi$

2) $\eta + \Theta = \pi$

3) $\therefore \phi = \Theta/2$

Diophantus's Proof:

1) $c^2 = a^2 + b^2$

2) $b(a) = t(a + c)$

3) $\zeta(t) \equiv a + \jmath b = \frac{1 - t^2 + \jmath 2t}{1 + t^2}$

4) $\zeta = |c|e^{i\theta} = |c|\frac{1+it}{1-it} = |c|(\cos(\theta) + i\sin(\theta))$

Pythagorean triplets:

1) $t = p/q \in \mathbb{Q}$

2) $a = p^2 - q^2$

3) $b = 2pq$

4) $c = p^2 + q^2$

*Derivation of Euclid's formula for the Pythagorean triplets (PT) $[a, b, c]$, based on a composition of a line, having a rational slope $t = p/q \in \mathbb{F}$, and a circle $c^2 = a^2 + b^2$, $[a, b, c] \in \mathbb{N}$. This analysis is attributed to Diophantus (Di·o·phan′·tus) (250 CE), and today such equations are called Diophantine (Di·o·phan′·tine) equations. PTs have applications in architecture and scheduling, and many other practical problems. Most interesting is their relation to Rydberg's formula for the eigenstates of the hydrogen atom (Appendix **??**).*

Problem # 15: *Derive Euclid's formula*

  – 15.1: Draw the circle and the line, given a positive slope $0 < t < 1$.

Problem # 16: *Substitute $y = t(x + 1)$ (the line equation) into the equation for the circle, and solve for $x(t)$.*
Hint: Because the line intersects the circle at two points, you will get two solutions for x. One of these solutions is the trivial solution $x = -1$.

  – 16.1: Substitute the $x(t)$ you found back into the line equation, and solve for $y(t)$.

  – 16.2: Let $t = q/p$ be a rational number, where p and q are integers. Find $x(p, q)$ and $y(p, q)$.

  – 16.3: Substitute $x(p, q)$ and $y(p, q)$ into the equation for the circle, and show how Euclid's formula for the Pythagorean triples is generated.

For full points you must show that you understand the argument. Explain the meaning of the comment "magic happens" when t^4 cancels.