2.3 Problems AE-3

Topics of this homework:

Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.

Deliverables: Answers to problems

Two-port network analysis

Problem # *1: Perform an analysis of electrical two-port networks, shown in Fig. **??** (page **??**). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.*

The definition of the ABCD *transmission matrix* ($\mathcal{T}$) is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{AE-3.1}$$

The *impedance matrix*, where the determinant $\Delta_{\mathcal{T}} = AD - BC$, is given by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_{\mathcal{T}} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{AE-3.2}$$

– 1.1: Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.

Problem # *2: Consider a single circuit element with impedance $Z(s)$.*

– 2.1: What is the ABCD matrix for this element if it is in series?

– 2.2: What is the ABCD matrix for this element if it is in shunt?

Problem # *3: Find the ABCD matrix for each of the circuits of Fig. **??**.*

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency $s \in \mathbb{C}$, then (ii) substitute $s = 1\jmath$ and calculate the total transmission matrix at this single frequency.

– 3.1: Left circuit (let $R_1 = R_2 = 10$ kilo-ohms and $C = 10$ nano-farads)

– 3.2: Right circuit (use L and C values given in the figure), where the pressure P is analogous to the voltage V, and the velocity U is analogous to the current I.

– 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency $s = 1\jmath$ as in the previous part (feel free to use Matlab/Octave for your computation).

– 3.4: Right circuit: Repeat the analysis as in question 3.3.

Algebra

Problem # *4: Fundamental theorem of algebra (FTA).*

– *4.1: State the fundamental theorem of algebra (FTA).*

(13 pts) Algebra with complex variables

Problem # *5: (7 pts) Order and complex numbers:*
One can always say that $3 < 4$—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in $4 - 3 > 0$. Here we will explore how complex variables may be ordered. In the following define $\{x, y\} \in \mathbb{R}$ and complex variable $z = x + y\jmath \in \mathbb{C}$.

– *5.1: Explain the meaning of* $|z_1| > |z_2|$.

– *5.2: If* $x_1, x_2 \in \mathbb{R}$ *(are* real *numbers), define the meaning of* $x_1 > x_2$.

– *5.3: Explain the meaning of* $z_1 > z_2$.

– *5.4: (2 pts) What is the meaning of* $|z_1 + z_2| > 3$*?*

– *5.5: (2 pts) If time were complex, how might the world be different?*

Problem # *6: (1 pt) It is sometimes necessary to consider a function $w(z) = u + v\jmath$ in terms of the real functions $u(x, y)$ and $v(x, y)$ (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse $z(w) = x + y\jmath$, where $x(u, v)$ and $y(u, v)$ are real functions.*

– *6.1: (1 pts) Find* $u(x, y)$ *and* $v(x, y)$ *for* $w(z) = 1/z$.

Problem # *7: (5 pts) Find $u(x, y)$ and $v(x, y)$ for $w(z) = c^z$ with complex constant $c \in \mathbb{C}$ for questions 7.1, 7.2, and 7.3:*

– *7.1:* $c = e$

– *7.2:* $c = 1$ *(recall that* $1 = e^{\pm\jmath 2\pi k}$ *for* $k \in \mathbb{Z}$

– *7.3:* $c = \jmath$. Hint: $\jmath = e^{\jmath\pi/2 + \jmath 2\pi k}, \quad k \in \mathbb{Z}$.

– *7.4: (2 pts) What is* $\jmath^\jmath$*?*

Schwarz inequality

Problem # *8: The above figure shows three vectors for an arbitrary value of $\alpha \in \mathbb{R}$ and a specific value of $\alpha = \alpha^*$.*

– *8.1: Find the value of $\alpha \in \mathbb{R}$ such that the length (norm) of $\vec{E}$ (i.e., $||\vec{E}|| \geq 0$) is minimum. Show your derivation, not the answer ($\alpha = \alpha^*$).*

– *8.2: Find the formula for* $||\boldsymbol{E}(\alpha^*)||^2 \geq 0$. Hint: Substitute α^* into Eq. **??** (p. **??**) and show that this results in the Schwarz inequality

$$|\vec{U} \cdot \vec{V}| \leq ||\vec{U}||||\vec{V}||.$$

Problem # *9: Geometry and scaler products*

– *9.1: What is the geometrical meaning of the dot product of two vectors?*

– *9.2: Give the formula for the dot product of two vectors. Explain the meaning based on Fig.* **??** *(page* **??***).*

– *9.3: Write the formula for the dot product of two vectors* $\vec{U} \cdot \vec{V}$ *in* $\mathbb{R}^n$ *in polar form (e.g., assume the angle between the vectors is* θ*).*

– *9.4: How is the Schwarz inequality related to the Pythagorean theorem?*

– *9.5: Starting from* $||\boldsymbol{U} + \boldsymbol{V}||$*, derive the* triangle inequality

$$||\vec{U} + \vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||.$$

– *9.6: The* triangle inequality $||\vec{U}+\vec{V}|| \leq ||\vec{U}||+||\vec{V}||$ *is true for two and three dimensions: Does it hold for five-dimensional vectors?*

– *9.7: Show that the wedge product* $\vec{U} \wedge \vec{V} \perp \vec{U} \cdot \vec{V}$.

Probability

Problem # *10: Basic terminology of experiments*

– *10.1: What is the* mean *of a trial, and what is the average over all trials?*

– *10.2: What is the* expected value *of a random variable* X*?*

– *10.3: What is the* standard deviation *about the mean?*

– *10.4: What is the definition of* information *of a random variable?*

– *10.5: How do you combine events? Hint: If the event is the flip of a biased coin, the events are* $H = p$, $T = 1 - p$*, so the event is* $\{p, 1 - p\}$*. To solve the problem, you must find the probabilities of two independent events.*

– *10.6: What does the term* independent *mean in the context of question 10.5? Give an example.*

– *10.7: Define* odds.