2.3 Problems AE-3

Topics of this homework:

Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.

Deliverables: Answers to problems

Two-port network analysis

Problem # 1: Perform an analysis of electrical two-port networks, shown in Fig. ?? (page ??). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

The definition of the ABCD transmission matrix \( T \) is

\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.
\] (AE-3.1)

The impedance matrix, where the determinant \( \Delta_T = AD - BC \), is given by

\[
\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A & \Delta_T \\ 1 & D \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}.
\] (AE-3.2)

— 1.1: Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.

Problem # 2: Consider a single circuit element with impedance \( Z(s) \).

— 2.1: What is the ABCD matrix for this element if it is in series?

— 2.2: What is the ABCD matrix for this element if it is in shunt?

Problem # 3: Find the ABCD matrix for each of the circuits of Fig. ??.

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency \( s \in \mathbb{C} \), then (ii) substitute \( s = 1/j \) and calculate the total transmission matrix at this single frequency.

— 3.1: Left circuit (let \( R_1 = R_2 = 10 \) kilo-ohms and \( C = 10 \) nano-farads)

— 3.2: Right circuit (use \( L \) and \( C \) values given in the figure), where the pressure \( P \) is analogous to the voltage \( V \), and the velocity \( U \) is analogous to the current \( I \).

— 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency \( s = 1/j \) as in the previous part (feel free to use Matlab/Octave for your computation).

— 3.4: Right circuit: Repeat the analysis as in question 3.3.
Algebra

Problem # 4: Fundamental theorem of algebra (FTA).

– 4.1: State the fundamental theorem of algebra (FTA).

(13 pts) Algebra with complex variables

Problem # 5: (7 pts) Order and complex numbers:
One can always say that 3 < 4—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in 4 − 3 > 0. Here we will explore how complex variables may be ordered. In the following define \( \{x, y\} \in \mathbb{R} \) and complex variable \( z = x + yj \in \mathbb{C} \).

– 5.1: Explain the meaning of \(|z_1| > |z_2|\).

– 5.2: If \( x_1, x_2 \in \mathbb{R} \) (are real numbers), define the meaning of \( x_1 > x_2 \).

– 5.3: Explain the meaning of \( z_1 > z_2 \).

– 5.4: (2 pts) What is the meaning of \(|z_1 + z_2| > 3|\)?

– 5.5: (2 pts) If time were complex, how might the world be different?

Problem # 6: (1 pt) It is sometimes necessary to consider a function \( w(z) = u + vj \) in terms of the real functions \( u(x, y) \) and \( v(x, y) \) (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse \( z(w) = x + yj \), where \( x(u, v) \) and \( y(u, v) \) are real functions.

– 6.1: (1 pts) Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = 1/z \).

Problem # 7: (5 pts) Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = e^z \) with complex constant \( c \in \mathbb{C} \) for questions 7.1, 7.2, and 7.3:

– 7.1: \( c = e \)

– 7.2: \( c = 1 \) (recall that \( 1 = e^{\pm 2\pi k} \) for \( k \in \mathbb{Z} \)

– 7.3: \( c = j \). Hint: \( j = e^{\pi/2 + 2\pi k}, \quad k \in \mathbb{Z} \).

– 7.4: (2 pts) What is \( j^j \) ?

Schwarz inequality

Problem # 8: The above figure shows three vectors for an arbitrary value of \( \alpha \in \mathbb{R} \) and a specific value of \( \alpha = \alpha^* \).

– 8.1: Find the value of \( \alpha \in \mathbb{R} \) such that the length (norm) of \( \vec{F} \) (i.e., \( ||\vec{F}|| \geq 0 \)) is minimum. Show your derivation, not the answer (\( \alpha = \alpha^* \)).
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– 8.2: Find the formula for $||E(\alpha^*)||^2 \geq 0$. Hint: Substitute $\alpha^*$ into Eq. ?? (p. ??) and show that this results in the Schwarz inequality

$$||U \cdot V|| \leq ||U|| ||V||.$$

**Problem # 9: Geometry and scalar products**

– 9.1: What is the geometrical meaning of the dot product of two vectors?

– 9.2: Give the formula for the dot product of two vectors. Explain the meaning based on Fig. ?? (page ??).

– 9.3: Write the formula for the dot product of two vectors $\vec{U} \cdot \vec{V}$ in $\mathbb{R}^n$ in polar form (e.g., assume the angle between the vectors is $\theta$).

– 9.4: How is the Schwarz inequality related to the Pythagorean theorem?

– 9.5: Starting from $||U + V||$, derive the triangle inequality

$$||U + V|| \leq ||U|| + ||V||.$$

– 9.6: The triangle inequality $||U + V|| \leq ||U|| + ||V||$ is true for two and three dimensions: Does it hold for five-dimensional vectors?

– 9.7: Show that the wedge product $\vec{U} \wedge \vec{V} \perp \vec{U} \cdot \vec{V}$.

**Probability**

**Problem # 10: Basic terminology of experiments**

– 10.1: What is the mean of a trial, and what is the average over all trials?

– 10.2: What is the expected value of a random variable $X$?

– 10.3: What is the standard deviation about the mean?

– 10.4: What is the definition of information of a random variable?

– 10.5: How do you combine events? Hint: If the event is the flip of a biased coin, the events are $H = p$, $T = 1 − p$, so the event is $\{p, 1 − p\}$. To solve the problem, you must find the probabilities of two independent events.

– 10.6: What does the term independent mean in the context of question 10.5? Give an example.

– 10.7: Define odds.