1 Ordering complex numbers

One can always say that $3 < 4$, namely that real numbers have order. We will explore if complex numbers have order. Let $z = x + iy$ be a complex number.

1. Can you define a meaning to $|z_1| > |z_2|$?
2. If $z$ and $w$ are real numbers, define the meaning of $z > w$.
3. If $z$ and $w$ are complex numbers, define the meaning of $z > w$.
4. How about $|z_1 + z_2| > 3$?

2 Analytic functions

State the regions where the following functions are analytic (Note: I’m not asking you to apply the CR conditions, just state the region. Remember that the analytic function has a power series that converges in the region of convergence (ROC). Thus an analytic function can be differentiated any number of times. Try to expand the function is a power series, and then look for the ROC. Consider also the expansion of $df(z)/dz$.

1. $f(z) = z^2$
2. $f(z) = 1/z$
3. $f(z) = \ln(\sqrt{z})$
4. $f(z) = \sqrt{1 - z^2}$
5. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n = 1$ (independent of $n$). Find $f'(z)$ and state the region where $f(z)$ and $f'(z)$ are analytic.

3 Integration of Analytic (and non Analytic) functions

State where the function is and is not analytic. Integrate $w = f(z) = u(x, y) + iv(x, y)$ over curve $C : z = x + iy$, as given.

1. $f(z) = z$, $C$ on the unit circle defined as $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.
2. $f(z) = \sin(z)$ on the unit circle.
3. $f(z) = 1/z$ on the unit circle.
4. $f(z) = 1/(2 - z)$ on the unit circle.
5. \( f(z) = 1/\sqrt{z} \) on the unit circle.

6. \( f(z) = 1/\sqrt{z} \) twice around the unit circle \((0 \leq \theta \leq 4\pi)\). This function has a branch cut, can you apply the Cauchy theorem?

7. \( f(z) = 1/z^2 \) on the unit circle.

4 Taylor Series

1. Explain the difference between \( 1/.5 \), \( 1/(1 -.5) \), \( 1/z \), \( 1/(1 - z) \)

2. Express \( 1/(1 - z) \) as a power series in \( z \). What is the ROC?

3. Express \( 1/(1 - z^2) \) as a power series in \( z \). What is the ROC?

4. Express \( 1/(1 - z)^2 \) as a power series in \( z \). What is the ROC?

5. Express \( 1/z \) as a power (Laurent) series in \( z \), and give the ROC.

6. Express \( 1/(1 - |z|^2) \) as a power series in \( z \). What is the ROC? (Hint: This is not analytic! State why?)

7. Express \( 1/(2 - z) \) as a power series in \( 1/z \). What is the ROC?

8. Express the inverse of \( 1/(2 - z) \) as a power series in \( z \). What is the ROC?

9. Why are poles and zeros of a function important?

10. If \( a = 0.1 \) what is the value of

\[
x = 1 + a + a^2 + a^3 \ldots?
\]

Show your work.

11. If \( a = 10 \) what is the value of

\[
x = 1 + a + a^2 + a^3 \ldots?
\]

5 Cauchy integral formula

1. Integrate the following:
   (a) \( \int_C \, z \, dz \) with \( C : z = e^{i\theta} \) for \( \theta = [-\pi, \pi] \).

2. If \( w = u + iv \) and \( z = x + iy \), find \( u(x, y) \) and \( v(x, y) \) for \( w = c^2 \) with complex constant \( c \in \mathbb{C} \):
   (a) \( c = e \)
   (b) \( c = 1 \)
   (c) \( c = \sqrt{2i} \)
6 CR conditions

For the following problem: $i = \sqrt{-1}, s = \sigma + i\omega, s = re^{i\theta}$, with $r \equiv |s| = \sqrt{\sigma^2 + \omega^2}$, $\theta \equiv \angle s$, and $f(s) = u(\sigma, \omega) + iv(\sigma, \omega)$. Show that the CR conditions for $f(s)$ may also be expressed in the following coordinate systems:

1. Rectangular:

$$\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial i\omega}, \quad (1)$$

2. Polar:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}. \quad (2)$$