

1 2-port circuit networks

Background: An equivalent circuit for a battery is shown. The leakage current $I_{\text{leak}}$ of the battery is determined by $R_s$, which decreases as the battery ages. $R_s$ becomes a short (zero resistance) when the battery finally goes "dead." The open-circuit (unloaded) source voltage $V_1$ represents a chemical potential, which is assumed to be constant throughout the batteries lifetime (Fig. 1).

Fig. 1: Equivalent circuit for a battery having an unloaded voltage of $V_1$ (leakage current $I_{\text{leak}}$), an internal leakage resistance of $R_s$, and a source resistance of $R_1 + R_2$.

To Do:

1. Using 2-port 2x2 matrix theory (JBA Notes: Lec 16 (p. 79-82,193-196), write a set of relations between the input and output voltage and current, namely define the matrix elements of

$$
\begin{bmatrix}
V_1 \\
I_1 \\
\end{bmatrix} = \begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2 \\
\end{bmatrix}
$$

in terms of the impedances $R_1, R_2, R_s$.

Once the equations are set up, set $R_{\text{leak}} = \infty$ ($I_{\text{leak}} = 0$), to simplify the analysis. For the simplified circuit ($R_s = \infty, I_{\text{leak}}=0$) find the input impedance when $I_2 = 0$, $V_2 = 0$ and $V_2 = R_{\text{load}}I_2$.

2. No load ($I_2 = 0$):

3. Shorted output ($V_2 = 0$):

4. Output loaded by $R_{\text{load}}$:

The point is that solving systems of cascades of impedances is a straightforward procedure: 1) Set up the matrix equations; 2) simplify the equations; 3) apply the load conditions, and find the currents and voltages. With some help from Matlab/Octave, this can be reduced to a trivial analysis, taking only minutes to obtain the answer (once you have mastered the tools).
2 Brune Impedance

A Brune impedance is defined as the ratio of the force $F(s)$ over the flow $V(s)$, and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}. \tag{1}$$

It trivially follows that

$$D(s) = \prod_{k=1}^{K} (s - s_k) \quad \text{and} \quad c_k = \lim_{s \to s_k} (s - s_k)D(s) = \prod_{n'=1}^{K-1} (s - s_n),$$

where the prime on index $n'$ means that $n = k$ is not included in the product.

There are several important theorems here, best summarized as Brune’s Theorem on positive-real functions. But it goes beyond this since the impedance matrix and the transmission matrix are a rearrangement of the same matrix equation (see the Lecture notes for the details; for example, 2-port transfer functions, and their input impedance, have the same poles).

To do:

1. Find the Laplace transform ($L$) of the three force relations in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance:

   (a) Hooke’s Law $f(t) = K x(t)$.
   (b) Dash-pot resistance $f(t) = R v(t)$.
   (c) Newton’s Law for Mass $f(t) = M \frac{dv}{dt}$.

2. Take the Laplace transform ($L$) of

   $$M \frac{d^2 x(t)}{dt^2} + R \frac{dx(t)}{dt} + K x(t) = f(t). \tag{2}$$

   and find the total impedance $Z(s)$ of the mechanical circuit.

3. What are $N(s)$ and $D(s)$ (e.g. Eq. 1)?

4. Assume that $M = R = K = 1$, find the residue form of the admittance $Y(s) = 1/Z(s)$ (e.g. Eq. 1) in terms of the roots $s_\pm$. You may check your answer with the Matlab/Octave’s residue command.

5. By applying the CRT, find the inverse Laplace transform ($L^{-1}$). Use the residue form of the expression that you derived in the previous exercise.

2.1 Transfer functions

In this problem, we will look at the transfer function of a two-port network, shown in Fig. 1. We wish to model the dynamics of a freight-train having $N$ such cars. The model of the train consists of masses connected by springs.

The velocity transfer function for this system is defined as the ratio of the output to the input velocity. Consider the engine on the left pulling the train at velocity $V_1$ and each car responding with a velocity of $V_n$. Then

$$H(s) = \frac{V_N(s)}{V_1(s)}$$

is the frequency domain ratio of the last car having velocity $V_N$ to $V_1$, the velocity of the engine, at the left most spring (i.e., coupler).
To do: Use the ABCD method to find the matrix representation of Fig. 1. Consistent with the figure, break the model into cells each consisting of three elements: a series inductor representing half the mass \((L = M/2)\), a shunt capacitor representing the spring \((C = 1/K)\), and another series inductor representing half the mass \((L = M/2)\). Each cell is symmetric, making the model a cascade of identical cells.

At each node define the force \(f_n(t) \leftrightarrow F_n(\omega)\) and the velocity \(v_n(t) \leftrightarrow V_n(\omega)\) at junction \(n\).

1. Write the ABCD matrix \(T\) for a single cell \((N = 2)\), composed of series mass \(M/2\), shunt compliance \(C\) and series mass \(M/2\), that relates the input node 1 to node 2 where

\[
\begin{bmatrix}
  F_1 \\
  V_1 
\end{bmatrix} = T \begin{bmatrix}
  F_2(\omega) \\
  -V_2(\omega)
\end{bmatrix}
\]

Note that here the mechanical force \(F\) is analogous to electrical voltage, and the mechanical velocity \(V\) is analogous to electrical current.

2. Assuming that \(F_2 = 0\), find the velocity transfer function \(H(s) \equiv V_2/V_1\). From the results of the \(T\) matrix you determined above, find

\[
H_{21}(s) = \left. \frac{V_2}{V_1} \right|_{F_2=0}
\]

3. Find \(h_{21}(t)\), the inverse Laplace transform of \(H_{21}(s)\).

4. What is the input impedance \(Z_2 = F_2/V_2\) if \(F_3 = -r_0V_3\)?

5. Simplify the expression for \(Z_2\) by making 3 assumptions:

   (a) Formula for the characteristic resistance: \(r_0 = \sqrt{M/C}\)
   (b) Load impedance is the characteristic resistance: \(F_3 = -r_0V_3\) (i.e., \(V_3\) cancels),
   (c) Low frequency approximation: \(s\sqrt{MC} < 1\)

6. State the ABCD matrix relationship between the first and \(N\)th node, in terms of of the cell matrix.

7. What is the velocity transfer function \(H_{N1} = \frac{V_N}{V_1}\)? Hint: Use an eigen matrix diagonalization as described in the next section, since

\[
T^N = EE^{-1}E\Lambda E^{-1}E\Lambda E^{-1} \cdots = E \begin{bmatrix}
  \lambda^N \\
  0 
\end{bmatrix} \Lambda^{-1} \cdots
\]
2.2 Eigen analysis of $T$

Given an ABCD transmission matrix $T$

$$A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the eigenvalues are and vectors are given in Appendix C of the Notes (p. 245), repeated below. Note that the determinant $\Delta$ of every $T$ is always $+1$ for reciprocal systems, and $-1$ for anti-reciprocal systems.

**Eigenvalues:**

$$\frac{\lambda_+}{\lambda_-} = \frac{1}{2} \left( \frac{(A + D) - \sqrt{(A - D)^2 + 4BC}}{(A + D) + \sqrt{(A - D)^2 + 4BC}} \right)$$

Reversible systems have the symmetry $A = D$ ($|T| = A^2 - BC = 1$), and this simplifies to $\lambda_\pm = A \mp \sqrt{BC}$, and the eigenmatrix is

$$\Lambda = \begin{bmatrix} A - \sqrt{BC} & 0 \\ 0 & A + \sqrt{BC} \end{bmatrix}$$

**Eigenvectors:** The eigenvectors are

$$[E_\pm] = \frac{1}{2\pi} \begin{bmatrix} (A - D) \mp \sqrt{(A - D)^2 + 4BC} \\ 1 \end{bmatrix}$$

When $A = D$ The eigenvectors simplifying to

$$[E_\pm] = \begin{bmatrix} \mp \sqrt{\frac{B}{C}} \\ 1 \end{bmatrix}$$

defining the Eigenmatrix and it inverse:

$$E = \begin{bmatrix} -\sqrt{\frac{B}{C}} & +\sqrt{\frac{B}{C}} \\ 1 & 1 \end{bmatrix}, \quad E^{-1} = \frac{1}{2} \begin{bmatrix} -\sqrt{\frac{B}{C}} & 1 \\ +\sqrt{\frac{B}{C}} & 1 \end{bmatrix}$$

3 Riemann Sheets and Branch cuts

1. What is a Riemann sheet (define it, and detail its properties)

2. Describe the Riemann sheet of $G(z) = \ln(z)$.

3. Describe with a figure and then discuss the branch cut for $f(z) = \log(z)$.

4. Describe the branch cut for $f(z) = e^z$.

5. Find $f(z)$ and $f'(z)$ at $z_0 = 0$ and $z_1 = e^{i\pi/2} = i$ for

$$f(z) = \frac{3}{\sqrt{1 - z^2}}$$

and describe the branch-cuts required to make this function single valued.

6. If $w \equiv F(s) = 1 + s^2$
(a) What is $s = G(w) \equiv F^{-1}(w)$.
(b) Map out the range of $s = G(w)$ for the two domains of $w$. Explain.
(c) Discuss reasonable places to place the branch cut(s)?

7. Starting with the formula for the impedance of a transmission line (TL):

$$z(s) = \tan(s)$$

show that

$$\tan^{-1}(z) = -i \frac{1}{2} \log \frac{i - z}{i + z}. \quad (3)$$

![Figure 2: This figure compares the colorized plots of $w = \tan^{-1}(z)$ and $w = \frac{1}{2} \ln(1 - iz)/(1 + iz)$, to verify they are the same function.]

4 Positive definite operators

Structure of a bilinear filter, composed from a bounded allpass filter $z^{-N}$ (Figure 4):

$$q[n] = \alpha q[n - N] + x[n]$$

$$y[n] = q[n] - \frac{1}{\alpha} q[n - N]$$

Fig. 4 Digital filter with $N$ sample delay, feed-forward and feed-back taps.

1. Assuming that $N = 1$, compute the first 5 samples of the impulse response of the filter depicted in Fig. 4. Determine the transfer function, and find its poles and zeros. Namely fill out a table of values for $x, q, q[n - N]$ and $y$, for $n = 0, 1, ..5$. 
2. If $N = 10$, find the frequency relationship between $y[n]$ and $x[n]$. This may be done by either a Fourier Transform or a z Transform. Hint: To write a formula, formulate the sum around the $+$ junction, I.E. the z transform of

$$q[n] = \alpha q[n - N] + x[n]$$

is

$$Q(z) = \alpha z^{-N}Q(z) + X(z),$$

thus

$$H(z) = \frac{Q(z)}{X(z)} = \frac{1}{1 - \alpha z^{-N}} = 1 + \alpha z^{-1} + \alpha^2 z^{-2} \cdots .$$

3. What are the poles and zeros of $H(z)$?

4. Is the impulse response of the operator FIR or IIR?

5. What is the real part of the transfer function $H(z)$. 
