Topic of this homework: Convergence of transforms, ROC, Filter symmetry (continuous vs discrete time; pole-zero placement).
Deliverable: Show your work.

1 Convergence

1. If \( F(s) = \frac{1}{1 + s} \) and the ROC is \( \sigma_0 > -1 \), find the inverse Laplace transform \((L^{-1})\).

2. If \( F(s) = \frac{1}{1 + s} \) and the ROC is \( \sigma_0 < -1 \), find the inverse Laplace transform.

3. If \( F(s) = \frac{1}{s - 1} \) and the ROC is \( \sigma_0 > 1 \), find the inverse Laplace transform.

4. If \( F(s) = \frac{1}{s - 1} \) and the ROC is \( \sigma_0 < 1 \), find the inverse Laplace transform.

5. Find the inverse Laplace transform of
\[
F(s) = \frac{1}{(s - 1)(s + 1)}
\]  
(a) if the ROC is between the two poles.
   i. if the ROC is to the right of \( \sigma = 1 \).
   ii. if the ROC is to the left of \( \sigma = -1 \).

2 Fourier, Laplace and z Transforms

1. Write out the forward and reverse transform formula for the following:
   a. Fourier Transform?
   b. \( z \) transform?
   c. Fourier series?
   d. DFT?
   e. Laplace Transform?

2. Write out Parseval’s formula for the following:
   a. Fourier Transform?
   b. \( z \) transform?
   c. Fourier series?
   d. DFT?
   e. Laplace Transform?
   f. Derive this result for the case of the FT, starting from the basic definition of the FT transform and its inverse.
3 Filter Symmetries

Let $s = \sigma + i\omega$ be the Laplace (complex) frequency.

Let $z \equiv e^{sT}$ where $T$ is the “sample period” at which data is taken (every $T$ seconds). For example if $T = 22.7 = 1/44100$ seconds then the data is sampled at 44100 kHz. This is how a CD player works with high quality music. Thus the unit-time delay operator $z^{-1}$ as

$$z^{-1} \equiv e^{-sT} \leftrightarrow \delta(t - T).$$

Note that $z^{-1}$ is in the frequency domain and produces a delay of $T$ (1 “time sample” of delay).

3.1 Given the function:

$$F(s) = \frac{(s + 1)(s - 1)}{(s + 2)},$$

1. Find the minimum phase $M(s)$ and all-pass $A(s)$ parts.
2. Find the magnitude of $M(s)$
3. Find the phase of $M(s)$
4. Find the magnitude of $A(s)$
5. Find the phase of $A(s)$

3.2 $z$ transforms

3.2.1 Define $F(z)$ as

$$F(z) = -z^2 + z^{-1} + z^0 - z^{-2} - z^{-4}$$

1. Find the time function $f[n] \leftrightarrow F(z)$ (i.e., find the inverse $z$ transform)
2. State if this sequence is
   (a) causal/anti-causal
   (b) minimum phase
   (c) all-pass
   (d) FIR
   (e) IIR
   (f) passive or active
3. Find the causal part of this sequence ($c[n]$)
4. Is the causal part minimum phase?

3.2.2 Product of transforms:

1. Find the square of the $z$ transform of $U(z)$ (i.e., $U^2(z)$) given that

$$U(z) = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} \ldots$$
4 Riemann Zeta function, in the time domain

The zeta function \( \zeta(s) \) is defined by the series

\[
\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.
\]

(2)

Expressing this as exponentials base functions [as I suggested in class, you to *always* make this transformation (it must be a knee-jerk operation)]

\[
\zeta(s) \equiv \sum_{n=1}^{\infty} e^{-L_n s}
\]

where \( L_n \equiv \ln(n) \) is the natural log of integer \( n \). Now each integer may be uniquely written as the product of primes, thus \( n = \prod_k \pi_k^{\alpha_k} \) (e.g., \( 12 = 2^2 \cdot 3 \) with \( \alpha_2 = 2 \) and \( \alpha_3 = 1 \)).

1. Given this setup, form the inverse Laplace transform of the above expression, of each exponential factor, and express the result as a sum over convolutions, using the convolution relationship

\[
\delta(t - T_1 - T_2) = \delta(t - T_1) * \delta(t - T_2).
\]

4.1 Euler product formula

As was first published by Euler in 1737, one may recursively factor out the lead term leading to an infinite product. For example the first factor may be found from

\[
\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\right).
\]

(3)

Canceling the common (even) terms gives

\[
\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \cdots.
\]

This is equivalent to long-division by the factor on the left. Recursing this to the next level removes all factors of 3 (a second long-division):

\[
\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} + \cdots.
\]

Continuing this recursion knocks out all the prime-factors, one at a time, resulting in Euler’s famous product formula:

\[
\frac{1}{\zeta(s)} = \prod_{\pi_k \in \mathbb{P}} (1 - 1/\pi_k^s).
\]

(4)

This construction assures that the lead term has all of its factors removed, and thus is necessarily prime, named after the Greek mathematician who first proposed this procedure, in a alternate form and now called *Eratosthenes’ sieve*.¹

¹http://en.wikipedia.org/wiki/Euler_product_formula
To do:

1. Find the inverse Laplace transform of the Euler Product formula.

2. What is the RoC of each factor $\zeta_k(s)$ as a function of $k$.

3. Write a summary conclusion as to what these formulas mean.

4. Can you find anything about the $\mathcal{L}^{-1}$ transform of the Zeta function on the Web? If so, tell us about it.