Chapter 3

Differential equations

3.1 Problems DE-1

3.1.1 Topics of this homework:

Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

3.1.2 Complex Power Series

Problem # *1: In each case derive (e.g., using Taylor's formula) the power series of $w(s)$ about $s = 0$ and give the RoC of your series. If the power series doesn't exist, state why!* Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.

– *1.1:* $1/(1-s)$

– *1.2:* $1/(1-s^2)$

– *1.3:* $1/(1+s^2)$.

– *1.4:* $1/s$

– *1.5:* $1/(1-|s|^2)$

Problem # *2: Consider the function $w(s) = 1/s$*

– *2.1: Expand this function as a power series about $s = 1$.* Hint: Let $1/s = 1/(1-1+s) = 1/(1-(1-s))$.

– *2.2: What is the RoC?*

– *2.3: Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$.*

– *2.4: What is the RoC?*

– *2.5: What is the residue of the pole?*

Problem # *3: Consider the function* $w(s) = 1/(2-s)$

– *3.1: Expand* $w(s)$ *as a power series in* $s^{-1} = 1/s$. *State the RoC as a condition on* $|s^{-1}|$. Hint: Multiply top and bottom by s^{-1}.

– *3.2: Find the inverse function* $s(w)$. *Where are the poles and zeros of* $s(w)$, *and where is it analytic?*

Problem # *4:Summing the series*

The Taylor series of functions have more than one region of convergence.

– *4.1: Given some function* $f(x)$, *if* $a = 0.1$, *what is the value of*

$$f(a) = 1 + a + a^2 + a^3 + \cdots ?$$

Show your work.

– *4.2: Let* $a = 10$. *What is the value of*

$$f(a) = 1 + a + a^2 + a^3 + \cdots ?$$

3.1.3 Cauchy-Riemann Equations

Problem # *5: For this problem* $\jmath = \sqrt{-1}$, $s = \sigma + \omega\jmath$, *and* $F(s) = u(\sigma,\omega) + \jmath v(\sigma,\omega)$. *According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of* $F(s)$ *is defined as*

$$\frac{dF}{ds} = \frac{d}{ds}\left[u(\sigma,\omega) + \jmath v(\sigma,\omega)\right]. \tag{DE-1.1}$$

If the integral is independent of the path, then the derivative must also be independent of the direction:

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \jmath\omega}. \tag{DE-1.2}$$

The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma,\omega)}{\partial \sigma} = \frac{\partial v(\sigma,\omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma,\omega)}{\partial \omega} = -\frac{\partial v(\sigma,\omega)}{\partial \sigma}$$

may be used to show where Equation DE-1.2 holds.

– *5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.*

– *5.2: Merge the CR equations to show that* u *and* v *obey Laplace's equations.*

$$\nabla^2 u(\sigma,\omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma,\omega) = 0.$$

– *5.3: What can you conclude?*

Problem # *6: Apply the CR equations to the following functions. State for which values of* $s = \sigma + i\omega$ *the CR conditions do or do not hold (e.g., where the function* $F(s)$ *is or is not analytic).* Hint: Review where CR-1 and CR-2 hold.

– *6.1:* $F(s) = e^s$

– *6.2:* $F(s) = 1/s$

3.1.4 Branch cuts and Riemann sheets

Problem # *7: Consider the function $w^2(z) = z$. This function can also be written as $w_\pm(z) = \sqrt{z_\pm}$. Assume $z = re^{\phi \jmath}$ and $w(z) = \rho e^{\theta \jmath} = \sqrt{r} e^{\phi \jmath/2}$.*

– *7.1: How many Riemann sheets do you need in the* domain *(z) and the* range *(w) to fully represent this function as single-valued?*

– *7.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.*

– *7.3: Use* `zviz.m` *to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.*

– *7.4: Where does* `zviz.m` *place the branch cut for this function?*

– *7.5: Must the branch cut necessarily be in this location?*

Problem # *8: Consider the function $w(z) = \log(z)$. As in Problem 7, let $z = re^{\phi \jmath}$ and $w(z) = \rho e^{\theta \jmath}$.*

– *8.1: Describe with a sketch and then discuss the* branch cut *for $f(z)$.*

– *8.2: What is the inverse of the function $z(f)$? Does this function have a branch cut? If so, where is it?*

– *8.3: Using* `zviz.m`*, show that*

$$\tan^{-1}(z) = -\frac{\jmath}{2} \log \frac{\jmath - z}{\jmath + z}. \qquad \text{(DE-1.3)}$$

In Fig. **??** (p. **??**) these two functions are shown to be identical.

– *8.4: Algebraically justify Eq. DE-1.3.* Hint: Let $w(z) = \tan^{-1}(z)$ and $z(w) = \tan w = \sin w/\cos w$; then solve for e^{wj}.

3.1.5 A Cauer synthesis of any Brune impedance

Problem # *9: One may synthesize a transmission line (ladder network) from a positive real impedance $Z(s)$ by using the continued fraction method. To obtain the series and shunt impedance values, we can use a residue expansion. Here we shall explore this method.*

– *9.1: Starting from the Brune impedance $Z(s) = \frac{1}{s+1}$, find the impedance network as a ladder network.*

– 9.2: Use a residue expansion in place of the CFA floor function (Sec. **??**, *p.* **??***) for polynomial expansions. Find the residue expansion of* $H(s) = s^2/(s+1)$ *and express it as a ladder network.*

– 9.3: Discuss how the series impedance $Z(s,x)$ *and shunt admittance* $Y(s,x)$ *determine the wave velocity* $\kappa(s,x)$ *and the characteristic impedance* $z_o(s,x)$ *when (1)* $Z(s)$ *and* $Y(s)$ *are both independent of* x*; (2)* $Y(s)$ *is independent of* x*,* $Z(s,x)$ *depends on* x*; (3)* $Z(s)$ *is independent of* x*,* $Y(s,x)$ *depends on* x*; and (4) both* $Y(s,x), Z(s,x)$ *depend on* x*.*