3.2 Problems DE-2

3.2.1 Topics of this homework:

Integration of complex functions, Cauchy's theorem, integral formula, residue theorem, power series, Riemann sheets and branch cuts, inverse Laplace transforms, Quadratic forms.

Two fundamental theorems of calculus

Fundamental Theorem of Calculus (Leibniz):

According to the fundamental theorem of (real) calculus (FTC),

$$f(x) = f(a) + \int_a^x F(\xi)d\xi, \tag{DE-2.1}$$

where $x, a, \xi, F, f \in \mathbb{R}$. This is an indefinite integral (since the upper limit is unspecified). It follows that

$$\frac{df(x)}{dx} = \frac{d}{dx}\int_a^x F(x)dx = F(x).$$

This justifies also calling the indefinite integral the *antiderivative*.

For a closed interval $[a, b]$, the FTC is

$$\int_a^b F(x)dx = f(b) - f(a), \tag{DE-2.2}$$

thus the integral is independent of the path from $x = a$ to $x = b$.

Fundamental Theorem of Complex Calculus:

According to the fundamental theorem of complex calculus (FTCC),

$$f(z) = f(z_0) + \int_{z_0}^z F(\zeta)d\zeta, \tag{DE-2.3}$$

where $z_0, z, \zeta, f, F \in \mathbb{C}$. It follows that

$$\frac{df(z)}{dz} = \frac{d}{dz}\int_{z_0}^z F(\zeta)d\zeta = F(z). \tag{DE-2.4}$$

For a closed interval $[s, s_o]$, the FTCC is

$$\int_{s_o}^s F(\zeta)d\zeta = f(s) - f(s_o), \tag{DE-2.5}$$

thus the integral is independent of the path from $x = a$ to $x = b$.

Problem # *1*

– 1.1: (2 pts) Consider Equation DE-2.1. What is the condition on $F(x)$ for which this formula is true?

– 1.2: (2 pts) Consider Equation DE-2.3. What is the condition on $F(z)$ for which this formula is true?

– *1.3: Let* $F(z) = \sum_{k=0}^{\infty} c_k z^k$.

– *1.4: Let*

$$F(z) = \frac{\sum_{k=0}^{\infty} c_k z^k}{z - \jmath}.$$

Problem # *2: In the following problems, solve the integral*

$$I = \int_{\mathcal{C}} F(z)dz$$

for a given path $\mathcal{C} \in \mathbb{C}$.

– *2.1: Perform the following integrals* ($z = x + iy \in \mathbb{C}$):
$I = \int_0^{1+\jmath} z dz$

– *2.2:* $I = \int_0^{1+j} z dz$, *but this time make the path explicit: from* 0 *to* 1, *with* $y = 0$, *and then to* $y = 1$, *with* $x = 1$.

– *2.3: Discuss whether your results agree with Eq. DE-2.4?*

Problem # *3: Perform the following integrals on the closed path* $\mathcal{C}$, *which we define to be the unit circle. You should substitute* $z = e^{i\theta}$ *and* $dz = ie^{i\theta}d\theta$, *and integrate from* $\{-\pi, \pi\}$ *to go once around the unit circle.*

Discuss whether your results agree with Eq. DE-2.4?

– *3.1: (2 pts)* $\int_{\mathcal{C}} z dz$

– *3.2: (2 pts)* $\int_{\mathcal{C}} \frac{1}{z} dz$

– *3.3: (2 pts)* $\int_{\mathcal{C}} \frac{1}{z^2} dz$

– *3.4:* $I = \int_{\mathcal{C}} \frac{1}{(z+2\jmath)^2} dz$.

Recall that the path of integration is the unit circle, starting and ending at -1.

Problem # *4: FTCC and integration in the complex plane*
Let the function $F(z) = c^z$, where $c \in \mathbb{C}$ is given for each question. *Hint: Can you apply the FTCC?*

– *4.1: For the function* $f(z) = c^z$, *where* $c \in \mathbb{C}$ *is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that* $f(z)$ *is analytic for all* $z \in \mathbb{C}$.

– *4.2: Find the antiderivative of* $F(z)$.

– *4.3:* $c = 1/e = 1/2.7183, \ldots$ *where* $\mathcal{C}$ *is* $\zeta = 0 \to i \to z$

– *4.4:* $c = 2$, *where* $\mathcal{C}$ *is* $\zeta = 0 \to (1+i) \to z$

– *4.5:* $c = i$, *where the path* $\mathcal{C}$ *is an inward spiral described by* $z(t) = 0.99^t e^{i2\pi t}$ *for* $t = 0 \to t_0 \to \infty$

– *4.6:* $c = e^{t-\tau_0}$, *where* $\tau_0 > 0$ *is a real number and* $\mathcal{C}$ *is* $z = (1 - i\infty) \to (1 + i\infty)$. Hint: Do you recognize this integral? If you do not, please do not spend a lot of time trying to solve it via the "brute force" method.

3.2.2 Cauchy's theorems CT-1, CT-2, CT-3

There are three basic definitions related to Cauchy's integral formula. They are all related and can greatly simplify integration in the complex plane. When a function depends on a complex variable, we use uppercase notation, consistent with the engineering literature for the Laplace transform.

Problem # *5: Describe the relationships between the theorems:*

– *5.1: CT-1 and CT-2*

– *5.2: CT-1 and CT-3*

– *5.3: CT-2 and CT-3*

– *5.4: Consider the function with poles at* $z = \pm j$,

$$F(z) = \frac{1}{1+z^2} = \frac{1}{(z-j)(z+j)}.$$

Find the residue expansion.

Problem # *6: Apply Cauchy's theorems to solve the following integrals. State which theorem(s) you used and show your work.*

– *6.1:* $\oint_{\mathcal{C}} F(z)dz$, *where* $\mathcal{C}$ *is a circle centered at* $z = 0$ *with a radius of* $\frac{1}{2}$

– *6.2:* $\oint_{\mathcal{C}} F(z)dz$, *where* $\mathcal{C}$ *is a circle centered at* $z = j$ *with a radius of* 1

– *6.3:* $\oint_{\mathcal{C}} F(z)dz$, *where* $\mathcal{C}$ *is a circle centered at* $z = 0$ *with a radius of* 2

Integration of analytic functions

Problem # *7: In the following questions, you'll be asked to integrate* $F(s) = u(\sigma,\omega) + iv(\sigma,\omega)$ *around the contour* $\mathcal{C}$ *for complex* $s = \sigma + i\omega$,

$$\oint_{\mathcal{C}} F(s)ds. \tag{DE-2.6}$$

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations

– *7.1:* $F(s) = \sin(s)$

– *7.2: Given function* $F(s) = \frac{1}{s}$ *State where the function is and is not analytic.*

– *7.3: Explicitly evaluate the integral when* $\mathcal{C}$ *is the unit circle, defined as* $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

– *7.4: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.*

– *7.5:* $F(s) = \frac{1}{s^2}$ *State where the function is and is not analytic.*

– *7.6: Explicitly evaluate the integral when* $\mathcal{C}$ *is the unit circle, defined as* $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

– *7.7: What does your result imply about the residue of the second-order pole at* $s = 0$*?*

– *7.8:* $F(s) = e^{st}$*: State where the function is and is not analytic.*

– *7.9: Explicitly evaluate the integral when* $\mathcal{C}$ *is the square* $(\sigma, \omega) = (1, 1) \to (-1, 1) \to (-1, -1) \to (1, -1) \to (1, 1)$.

– *7.10: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.*

– *7.11:* $F(s) = \frac{1}{s+2}$*: State where the function is and is not analytic.*

– *7.12: Let* $\mathcal{C}$ *be the unit circle, defined as* $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$*. Evaluate the integral using Cauchy's theorem and/or the residue theorem.*

– *7.13: Let* $\mathcal{C}$ *be a circle of radius 3, defined as* $s = 3e^{i\theta}$, $0 \leq \theta \leq 2\pi$*. Evaluate the integral using Cauchy's theorem and/or the residue theorem.*

– *7.14:* $F(s) = \frac{1}{2\pi i} \frac{e^{st}}{(s+4)}$ *State where the function is and is not analytic.*

– *7.15: Let* $\mathcal{C}$ *be a circle of radius 3, defined as* $s = 3e^{i\theta}$, $0 \leq \theta \leq 2\pi$*. Evaluate the integral using Cauchy's theorem and/or the residue theorem.*

– *7.16: Let* $\mathcal{C}$ *contain the entire left half* s *plane. Evaluate the integral using Cauchy's theorem and/or the residue theorem. Do you recognize this integral?*

– *7.17:* $F(s) = \pm \frac{1}{\sqrt{s}}$ *(e.g.,* $F^2 = \frac{1}{s}$*) State where the function is and is not analytic.*

– *7.18: This function is multivalued. How many Riemann sheets do you need in the domain* (s) *and the range* (f) *to fully represent this function? Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.*

– *7.19: Explicitly evaluate the integral* $\int_{\mathcal{C}} \frac{1}{\sqrt{z}} dz$ *when* $\mathcal{C}$ *is the unit circle, defined as* $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$*. Is this contour closed? State why or why not.*

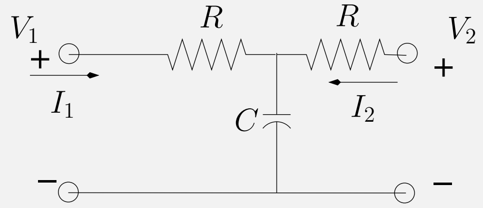
– *7.20: Explicitly evaluate the integral* $\int_{\mathcal{C}} \frac{1}{\sqrt{z}} dz$ *when* $\mathcal{C}$ *is* twice *around the unit circle, defined as* $s = e^{i\theta}$, $0 \leq \theta \leq 4\pi$*. Is this contour closed? State why or why not. Hint: Note that* $\sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi} e^{i\theta}} = e^{i\pi} \sqrt{e^{i\theta}} = -1\sqrt{e^{i\theta}}$.

– 7.21: What does your result imply about the residue of the (twice-around $\frac{1}{2}$ order) pole at $s = 0$?

– 7.22: Show that the residue is zero. Hint: Apply the definition of the residue.

3.2.3 Laplace transform applications

Problem # *8: A two-port network application for the Laplace transform*



This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$. It is convenient to define the dimensionless ratio $s/s_c = RCs$ in terms of a time constant $\tau = RC$ and cutoff frequency $s_c = 1/\tau$.

3.2.4 Computer exercises with Matlab/Octave

Problem # *9: With the help of a computer*

Now we look at a few important concepts using Matlab/Octave's `syms` commands or Wolfram Alpha's symbolic math toolbox.[1]

For example, to find the Taylor series expansion about $s = 0$ of

$$F(s) = -\log(1 - s),$$

we first consider the derivative and its Taylor series (about $s = 0$)

$$F'(s) = \frac{1}{1 - s} = \sum_{n=0}^{\infty} s^n.$$

Then, we integrate this series term by term:

$$F(s) = -\log(1 - s) = \int^s F'(s)ds = \sum_{n=0}^{\infty} \frac{s^n}{n}.$$

Alternatively we can use Matlab/Octave commands:

```
syms s
taylor(-log(1-s),'order',7)
```

– 9.1: Use Octave's `taylor(-log(1-s))` to the seventh order, as in the example above. Try the above Matlab/Octave commands. Give the first seven terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above).

– 9.2: What is the inverse Laplace transform of this series? Consider the series term by term.

[1] https://www.wolframalpha.com/

– 9.3: The function $1/\sqrt{z}$ has a branch point at $z = 0$; thus it is singular there. Can you apply Cauchy's integral theorem when integrating around the unit circle?

– 9.4: This Matlab/Octave code computes $\int_0^{4\pi} \frac{dz}{\sqrt{z}}$ using Matlab's/Octave's symbolic analysis package. Run the following script:

```
syms  z
I=int(1/sqrt(z))
J = int(1/sqrt(z),exp(-j*pi),exp(j*pi))
eval(J)
```

What answers do you get for I and J?

– 9.5: Modify this code to integrate $f(z) = 1/z^2$ once *around the unit circle. What answers do you get for I and J?*

– 9.6: Bessel functions can describe waves in a cylindrical geometry.
The Bessel function has a Laplace transform with a branch cut

$$J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}.$$

Draw a hand sketch showing the nature of the branch cut. Hint: Use `zviz`.

Problem # *10: Matlab/Octave exercises*

– *10.1: Try the following Matlab/Octave commands, and then comment on your findings.*

```
%Take the inverse LT of 1/sqrt(1+s^2)
syms s
I=ilaplace(1/(sqrt((1+s^2))));
disp(I)

%Find the Taylor series of the LT
T = taylor(1/sqrt(1+s^2),10); disp(T);
```

```
%Verify this
syms t
J=laplace(besselj(0,t));
disp(J);

%plot the Bessel function
t=0:0.1:10*pi;
b=besselj(0,t);
plot(t/pi,b);
grid on;
```

– *10.2: When did Friedrich Bessel live?*

– *10.3: What did he use Bessel functions for?*

Problem # *11: Colorized plots of analytic functions. Use `zviz` for each of the following.*

– *11.1: Describe the plot generated by `zviz S=Z`.*

– *11.2: Describe the plot generated by* `zviz` $1./sqrt(1+S.^2)$.

– *11.3: Describe the plot generated by* `zviz` $1./sqrt(1-S.^2)$. *Is this function a Brune impedance (i.e., does this function obey*

– *11.4:* `zviz` $1./(1+sqrt(S))$

3.2.5 Inverse of Riemann $\zeta(s)$ function

Problem # *12: Inverse zeta function (This problem is for extra credit).*

– *12.1: Find the $\mathcal{LT}^{-1}$ of one factor of the Riemann zeta function $\zeta_p(s)$, where $\zeta_p(s) \leftrightarrow z_p(t)$. Describe your results in words. Hint: Consider the geometric series representation*

$$\zeta_p(s) = \frac{1}{1-e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p}, \tag{DE-2.7}$$

for which you can look up the $\mathcal{LT}^{-1}$ of each term.

Problem # *13: Inverse transform of products:*
The time-domain version of Eq. DE-2.7 may be written as the convolution of all the $z_k(t)$ factors:

$$z(t) \equiv z_2(t) \star z_3(t) \star z_5(t) \star z_7(t) \star \cdots \star z_p(t) \star \cdots, \tag{DE-2.8}$$

where $\star$ represents time convolution.

Explain what this means in physical terms. Start with two terms (e.g., $z_1(t) \star z_2$). Hint: The input admittance of this cascade may be interpreted as the analytic continuation of $\zeta(s)$ by defining a cascade of eigenfunctions with eigenvalues derived from the primes. For a discussion of this idea see Sec. **??** and **??**.

Physical interpretation: Such functions may be generated in the time domain, as shown in Fig. **??**, using a feedback delay of T_p seconds, described by the two equations in the Fig. **??** with a unity feedback gain $\alpha = -1$. Taking the Laplace transform of the system equation, we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $\zeta_p(s)$, which is an all-pole function, since

$$Q(s) = e^{-sT_n}Q(s) + V(s), \text{ or } \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}. \quad \text{(DE-2.9)}$$

Closing the feed-forward path gives a second transfer function $Y(s) = I(s)/V(s)$—namely,

$$Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}. \quad \text{(DE-2.10)}$$

If we take $i(t)$ as the current and $v(t)$ as the voltage at the input to the transmission line, then $y_p(t) \leftrightarrow \zeta_p(s)$ represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the $j\omega$ axis. By a slight modification, $\zeta_p(s)$ may alternatively be written as

$$Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j\tan(sT_p/2). \quad \text{(DE-2.11)}$$

Every impedance $Z(s)$ has a corresponding *reflectance* function given by a Möbius transformation, which may be read off of Eq. DE-2.11 as

$$\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}, \quad \text{(DE-2.12)}$$

since impedance is also related to the round-trip delay T_p on the line. The inverse Laplace transform of $\Gamma(s)$ is the round-trip delay T_p on the line

$$\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}. \quad \text{(DE-2.13)}$$

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations that are not obvious in the frequency domain (See Sec. **??**). Transforming to the time domain is a form of analytic continuation of $\zeta(s)$ that depends on the assumption that $Z^{eta}(t) \leftrightarrow \zeta(s)$ is one-sided in time (causal).

3.2.6 Quadratic forms

A matrix that has positive eigenvalues is said to be positive-definite. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy, since the power is the voltage times the current. Given an impedance matrix

$$\mathbf{V} = \mathcal{Z}\mathbf{I},$$

the power $\mathcal{P}$ is

$$\mathcal{P} = \mathbf{I}\cdot\mathbf{V} = \mathbf{I}\cdot\mathcal{Z}\mathbf{I},$$

which must be positive-definite for the system to obey conservation of energy.

Problem # *14: In this problem, consider the* 2×2 *impedance matrix*

$$\mathcal{Z} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

– 14.1: Solve for the power $\mathcal{P}(i_1, i_2)$ *by multiplying out this matrix equation (which is a* quadratic form*):*

$$\mathcal{P}(i_1, i_2) = \mathbf{I}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{I}.$$

– 14.2: Is the impedance matrix positive-definite? Show your work by finding the eigenvalues of the matrix $\mathbf{Z}$*.*

– 14.3: Should an impedance matrix always be positive-definite? Explain.