3.3 Problems DE-3

3.3.1 Topics of this homework: Brune impedance

lattice transmission line analysis

3.3.2 Brune Impedance

**Problem # 1: Residue form**

A Brune impedance is defined as the ratio of the force $F(s)$ to the flow $V(s)$ and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)} \quad \text{(DE-3.1)}$$

with

$$D(s) = \prod_{k=1}^{K} (s - s_k) \quad \text{and} \quad c_k = \lim_{s \to s_k} (s - s_k)D(s) = \prod_{n'=1}^{K-1} (s - s_n).$$

The prime on the index $n'$ means that $n = k$ is not included in the product.

- **1.1:** Find the Laplace transform ($LT$) of a (1) spring, (2) dashpot, and (3) mass. Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: (1) Hooke’s law $f(t) = Kx(t)$, (2) dashpot resistance $f(t) = Rv(t)$, and (3) Newton’s law for mass $f(t) = Mdv(t)/dt$.

- **1.2:** Take the Laplace transform ($LT$) of Eq. DE-3.2 and find the total impedance $Z(s)$ of the mechanical circuit.

$$M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + Kx(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s). \quad \text{(DE-3.2)}$$

- **1.3:** What are $N(s)$ and $D(s)$ (see Eq. DE-3.1)?

- **1.4:** Assume that $M = R = K = 1$ and find the residue form of the admittance $Y(s) = 1/Z(s)$ (see Eq. DE-3.1) in terms of the roots $s_{\pm}$. Hint: Check your answer with Octave’s/Matlab’s residue command.

- **1.5:** By applying Eq. ?? (page ??), find the inverse Laplace transform ($LT^{-1}$). Use the residue form of the expression that you derived in question 1.4.
Depiction of a train consisting of cars treated as masses $M$ and linkages treated as springs of stiffness $K$ or compliance $C = 1/K$.

Below it is the electrical equivalent circuit for comparison. The masses are modeled as inductors and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is $\Delta$ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL. This method, called a Cauer synthesis, is based on the ABCD transmission line method of Sec. ?? (p. ??).

### 3.3.3 Transmission-line analysis

**Problem # 2: (14 pts) Train-mission-line** We wish to model the dynamics of a freight train that has $N$ such cars and study the velocity transfer function under various load conditions.

As shown in Fig. 3.3.2, the train model consists of masses connected by springs.

**Problem # 3: Transfer functions**

Use the ABCD method (see the discussion in Appendix ??, p. ??) to find the matrix representation of the system of Fig. 3.3.2. Define the force on the $n$th train car $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$. Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($\mathcal{A} = \mathcal{D}$) identical cell matrices $T(s)$.

– 3.1: Find the elements of the ABCD matrix $T$ for the single cell that relate the input node 1 to output node 2

$$
\begin{bmatrix} F \\ V \end{bmatrix}_1 = \sigma \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2.
$$ (DE-3.3)
- 3.2: Express each element of $\mathcal{T}(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi j f$, $s_c = 2\pi j f_c$). The Nyquist wavelength sampling condition is $\lambda_c > 2\Delta$. It says the critical wavelength $\lambda_c > 2\Delta$. Namely it is defined in terms the minimum number of cells $2\Delta$, per minimum wavelength $\lambda_c$.

The Nyquist wavelength sampling theorem says that there are at least two cars per wavelength.

Proof: From the figure, the distance between cars $\Delta = c_o T_o [m]$, where

$$c_o = \frac{1}{\sqrt{MC}} [m/s].$$

The cutoff frequency obeys $f_c \lambda_c = c_o$. The Nyquist critical wavelength is $\lambda_c = c_o/f_c > 2\Delta$. Therefore the Nyquist sampling condition is

$$f < f_c \equiv \frac{c_o}{\lambda_c} = \frac{c_o}{2\Delta} = \frac{1}{2\Delta \sqrt{MC}} \text{ [rad/sec]}.$$

Finally, $s_c = \gamma 2\pi f_c$.

- 3.3: Use the property of the Nyquist sampling frequency $\omega < \omega_c$ (Eq. DE-3.4) to remove higher order powers of frequency

$$1 + \left(\frac{s}{s_c}\right)^2 \approx 1$$

to determine a band-limited approximation of $\mathcal{T}(s)$.

Problem # 4: (4 pts) Now consider the cascade of $N$ such $\mathcal{T}(s)$ matrices and perform an eigenanalysis.

- 4.1: (4 pts) Find the eigenvalues and eigenvectors of $\mathcal{T}(s)$ as functions of $s/s_c$.

Problem # 5: (14 pts) Find the velocity transferfunction $H_{12}(s) = \frac{V_2}{V_1}|_{F_2=0}$.

- 5.1: (3 pts) Assuming that $N = 2$ and $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv \frac{V_2}{V_1}$. From the results of the $\mathcal{T}$ matrix, find

$$H_{21}(s) = \frac{V_2}{V_1}|_{F_2=0}$$

Express $H_{12}$ in terms of a residue expansion.

- 5.2: (2 pts) Find $h_{21}(t) \leftrightarrow H_{21}(s)$.

- 5.3: (2 pts) What is the input impedance $Z_2 = F_2/V_2$, assuming $F_3 = -r_0 V_3$?
- **5.4: (5 pts)** Simplify the expression for $Z_2$ as follows:

1. Assuming the characteristic impedance $r_0 = \sqrt{M/C}$.
2. Terminate the system in $r_0$: $F_2 = -r_0V_2$ (i.e., $-V_2$ cancels).
3. Assume higher-order frequency terms are less than 1 ($|s/s_c| < 1$).
4. Let the number of cells $N \to \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteristic impedance $r_0$, the input impedance $Z_1(s) = r_0$. Thus, when we simplify the expression for $T(s)$, it should be equal to $r_0$. Show that this is true for this setup.

- **5.5: (1 pts)** State the ABCD matrix relationship between the first and $N$th nodes in terms of the cell matrix. Write out the transfer function for one cell, $H_{21}$.

- **5.6: (1 pts)** What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?