Topic of this homework: Linear Algebra (Schwartz inequality for n dimensional space. Eigenvalues and eigenvectors)

Deliverables: Show your work. Numerical results are not sufficient expect when specifically requested.

1 Permutation operations

Determine the sequence of $P$ operations (as I discussed in class) that renders each matrix zero below the main diagonal. Hint check the determinant

1. $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

2. Try to find the Det by trivial means.

$$ A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2.01 \\ 3 & 3 & 3.01 \end{bmatrix} $$

Since the last two rows are identical, the matrix has no inverse, consistent with $det(A) = 0$.

3. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

2 Eigenvalues and eigenvectors

Write the Eigen matrix $E = [e_1 e_2 \cdots e_n]$

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

3 Inequalities

Prove the Schwartz inequality by following the argument below.

1. Use the argument based on the proof on pages 424-426 in the text book. Let $V, U, E(a)$ be vectors in $R^n$ and $a$ is a real scalar. Taking the norm of the error $E(a)$ gives $||E(a)|| = ||V - aU||$. Find the minimum of $E$ with respect to $a$. As shown in the book, use this to prove the Schwartz inequality (13) page 424.
2. Give a diagram showing what is happening when you minimize $E(a)$ with respect to $a$. Provide a diagram showing the minimum value of $E(a^*)$ where $a^*$ is the value such that the gradient of $E$ with respect to $a$ is zero.

1. Prove the triangular inequality $||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$. All of these are to be done for $\mathbb{R}^n$ where $n$ is greater that 3. In other words, don’t assume that $n = 2$ or 3.

2. What is the name of the triangular inequality when $n = 2$? Who first showed this?

4  Gram-Schmidt

Apply Gram-Schmidt method to vectors $v_1 = (1,1,0,1)^T$, $v_2 = (1,-2,0,0)^T$ and $v_3 = (1,0,-1,2)^T$ to get three orthogonal (or orthonormal) vectors. As shown in class (Glob and vanLoan, p. 230):

$$e_K = v_K - \sum_{k=1}^{K-1} (v_K \cdot \hat{e}_k)\hat{e}_k$$

(1)

where $\hat{e}_k = e_k/||e_k||$.

5  Vector sums

Given $A\vec{x} = \vec{b}$, expand this as a sum of vectors of the form $x_1A_1 + x_2A_2 + \cdots = \vec{b}$. Give the formula for $A_1, A_2$, etc. What is $x_1$ here? What is $\vec{b}$?

6  Quadratic forms

- Expand the following as a quadratic form

$$f(x_1, x_2, x_3) = x^T \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix} x$$

- Now expand this in the form: $f = \lambda_1\tilde{x}_1^2 + \lambda_2\tilde{x}_2^2 + \lambda_3\tilde{x}_3^3$

- Is the matrix positive definite? (Page 592) Explain.

- Is this equivalent to completing the squares? Explain.