

## Chapter 3

# Differential equations

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### 3.1 Problems DE-1

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#### 3.1.1 Topics of this homework:

Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

#### 3.1.2 Complex Power Series

**Problem # 1:** *In each case derive (e.g., using Taylor's formula) the power series of  $w(s)$  about  $s = 0$  and give the RoC of your series. If the power series doesn't exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at  $s = 0$ .*

– 1.1:  $1/(1 - s)$

Ans:

– 1.2:  $1/(1 - s^2)$

Ans:

– 1.3:  $1/(1 + s^2)$ .

Ans:

– 1.4:  $1/s$

Ans:

– 1.5:  $1/(1 - |s|^2)$

Ans:

**Problem # 2:** *Consider the function  $w(s) = 1/s$*

– 2.1: *Expand this function as a power series about  $s = 1$ . Hint: Let  $1/s = 1/(1 - (1 - s)) = 1/(1 - (1 - s))$ .*

Ans:

– 2.2: What is the RoC?

**Ans:**

– 2.3: Expand  $w(s) = 1/s$  as a power series in  $s^{-1} = 1/s$  about  $s^{-1} = 1$ .

**Ans:**

– 2.4: What is the RoC?

**Ans:**

– 2.5: What is the residue of the pole?

**Ans:**

**Problem # 3:** Consider the function  $w(s) = 1/(2 - s)$

– 3.1: Expand  $w(s)$  as a power series in  $s^{-1} = 1/s$ . State the RoC as a condition on  $|s^{-1}|$ .

Hint: Multiply top and bottom by  $s^{-1}$ .

**Ans:**

– 3.2: Find the inverse function  $s(w)$ . Where are the poles and zeros of  $s(w)$ , and where is it analytic?

**Ans:**

**Problem # 4: Summing the series**

The Taylor series of functions have more than one region of convergence.

– 4.1: Given some function  $f(x)$ , if  $a = 0.1$ , what is the value of

$$f(a) = 1 + a + a^2 + a^3 + \dots?$$

Show your work. **Ans:**

– 4.2: Let  $a = 10$ . What is the value of

$$f(a) = 1 + a + a^2 + a^3 + \dots?$$

**Ans:**

### 3.1.3 Cauchy-Riemann Equations

**Problem # 5:** For this problem  $j = \sqrt{-1}$ ,  $s = \sigma + \omega j$ , and  $F(s) = u(\sigma, \omega) + jv(\sigma, \omega)$ . According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of  $F(s)$  is defined as

$$\frac{dF}{ds} = \frac{d}{ds} [u(\sigma, \omega) + jv(\sigma, \omega)]. \quad (\text{DE-1.1})$$

If the integral is independent of the path, then the derivative must also be independent of the direction:

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial j\omega}. \quad (\text{DE-1.2})$$

The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}$$

may be used to show where Equation DE-1.2 holds.

– 5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.

**Ans:**

– 5.2: Merge the CR equations to show that  $u$  and  $v$  obey Laplace's equations.

$$\nabla^2 u(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma, \omega) = 0.$$

**Ans:**

– 5.3: What can you conclude?

**Ans:**

**Problem # 6:** Apply the CR equations to the following functions. State for which values of  $s = \sigma + i\omega$  the CR conditions do or do not hold (e.g., where the function  $F(s)$  is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

– 6.1:  $F(s) = e^s$

**Ans:**

– 6.2:  $F(s) = 1/s$

**Ans:**

### 3.1.4 Branch cuts and Riemann sheets

**Problem # 7:** Consider the function  $w^2(z) = z$ . This function can also be written as  $w_{\pm}(z) = \sqrt{z_{\pm}}$ . Assume  $z = re^{i\phi}$  and  $w(z) = \rho e^{i\theta} = \sqrt{r}e^{i\phi/2}$ .

– 7.1: How many Riemann sheets do you need in the domain ( $z$ ) and the range ( $w$ ) to fully represent this function as single-valued?

**Ans:**

– 7.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

**Ans:**

– 7.3: Use `zviz.m` to plot the positive and negative square roots  $+\sqrt{z}$  and  $-\sqrt{z}$ . Describe what you see.

**Ans:**

– 7.4: Where does `zviz.m` place the branch cut for this function?

**Ans:**

– 7.5: Must the branch cut necessarily be in this location?

**Ans:**

**Problem # 8:** Consider the function  $w(z) = \log(z)$ . As in Problem 7, let  $z = re^{i\phi}$  and  $w(z) = \rho e^{i\theta}$ .

- 8.1: Describe with a sketch and then discuss the branch cut for  $f(z)$ .

Ans:

- 8.2: What is the inverse of the function  $z(f)$ ? Does this function have a branch cut? If so, where is it?

Ans:

- 8.3: Using `zviz.m`, show that

$$\tan^{-1}(z) = -\frac{j}{2} \log \frac{j-z}{j+z}. \quad (\text{DE-1.3})$$

In Fig. ?? (p. ??) these two functions are shown to be identical.

Ans:

- 8.4: Algebraically justify Eq. DE-1.3. Hint: Let  $w(z) = \tan^{-1}(z)$  and  $z(w) = \tan w = \sin w / \cos w$ ; then solve for  $e^{wj}$ .

Ans:

### 3.1.5 A Cauer synthesis of any Brune impedance

**Problem # 9:** One may synthesize a transmission line (ladder network) from a positive real impedance  $Z(s)$  by using the continued fraction method. To obtain the series and shunt impedance values, we can use a residue expansion. Here we shall explore this method.

- 9.1: Starting from the Brune impedance  $Z(s) = \frac{1}{s+1}$ , find the impedance network as a ladder network.

Ans:

- 9.2: Use a residue expansion in place of the CFA floor function (Sec. ??, p. ??) for polynomial expansions. Find the residue expansion of  $H(s) = s^2/(s+1)$  and express it as a ladder network.

Ans:

- 9.3: Discuss how the series impedance  $Z(s, x)$  and shunt admittance  $Y(s, x)$  determine the wave velocity  $\kappa(s, x)$  and the characteristic impedance  $z_o(s, x)$  when (1)  $Z(s)$  and  $Y(s)$  are both independent of  $x$ ; (2)  $Y(s)$  is independent of  $x$ ,  $Z(s, x)$  depends on  $x$ ; (3)  $Z(s)$  is independent of  $x$ ,  $Y(s, x)$  depends on  $x$ ; and (4) both  $Y(s, x)$ ,  $Z(s, x)$  depend on  $x$ .

Ans: