

Homeworks

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Chapter 1

Number systems

1.1 Problems NS-1

Topic of this homework:

Introduction to Matlab/Octave (see the Matlab or Octave tutorial for help)

Deliverables: Report with charts and answers to questions.

Plotting complex quantities in Octave/Matlab

Problem # 1: Consider the functions $f(s) = s^2 + 6s + 25$ and $g(s) = s^2 + 6s + 5$.

– 1.1: Find the zeros of functions $f(s)$ and $g(s)$ using the command `roots()`.

Ans:

– 1.2: Show the roots of $f(s)$ as red circles and of $g(s)$ as blue plus signs.

The x -axis should display the real part of each root, and the y -axis should display the imaginary part. Use `hold on` and `grid on` when plotting the roots.

Ans:

– 1.3 Give your figure the title “Complex Roots of $f(s)$ and $g(s)$.” Label the x - and y -axes “Real Part” and “Imaginary Part.” Hint: Use `xlabel`, `ylabel`, `ylim([-10 10])`, and `xlim([-10 10])` to expand the axes.

Problem # 2: Consider the function $h(t) = e^{j2\pi ft}$ for $f = 5$ and $t = [0:0.01:2]$.

– 2.1: Use `subplot` to show the real and imaginary parts of $h(t)$.

Make two graphs in one figure. Label the x -axes “Time (s)” and the y -axes “Real Part” and “Imaginary Part.”

Ans:

– 2.2: Use `subplot` to plot the magnitude and phase parts of $h(t)$.

Use the command `angle` or `unwrap(angle())` to plot the phase. Label the x -axes “Time (s)” and the y -axes “Magnitude” and “Phase (radians).”

Ans:

Prime numbers, infinity**Problem # 3: Prime numbers**

– 3.1: Use the Matlab/Octave function `factor` to find the prime factors of 123, 248, 1767, and 999,999.

Ans:

– 3.2: Use the Matlab/Octave function `isprime` to determine whether 2, 3, and 4 are prime numbers. What does the function `isprime` return when a number is prime or not prime? Why?

Ans:

– 3.3: Use the Matlab/Octave function `primes` to generate prime numbers between 1 and 10^6 . Save them in a vector `x`. Plot this result using the command `hist(x)`.

Ans:

– 3.4: Now try `[n,bincenters] = hist(x)`. Use `length(n)` to find the number of bins.

Ans:

– 3.5: Set the number of bins to 100 by using an extra input argument to the function `hist`. Show the resulting figure, give it a title, and label the axes. Hint: `help hist` and `doc hist`.

Ans:

Problem # 4: Very large primes on Intel computers.

– 4.1: Find the largest prime number π_{\max} that can be stored on an Intel 64-bit computer; Hint: As explained in the Matlab/Octave command `help flintmax`.

The largest positive integer is 2^{53} ; however, the largest unsigned integer that can be factored is at least 2^{54} . Explain the logic of your answer. Hint: `help isprime()`.

Ans:

Problem # 5: We are interested in primes that are greater than π_{\max} .

– 5.1: How can you find them on an Intel computer (i.e., one using IEEE floating point)? Hint: Consider a sieve that contains only odd numbers, starting from 3 (not 2). Since every prime number greater than 2 is odd, there is no reason to check the even numbers. $n_{\text{odd}} \in \mathbb{N}/2$ contain all the primes other than 2.

Ans:

Special functions in Octave/Matlab and infinity (∞)**Problem # 6: Inf , NaN, and logarithms in Octave/Matlab.**

– 6.1: Try `1/0` and `0/0` in the Octave/Matlab command window. What are the results? What do these “numbers” mean in Octave/Matlab? **Ans:**

– 6.2: Try $\log(0)$, $\log_{10}(0)$, and $\log_2(0)$ in the command window.

In Matlab/Octave, the natural logarithm $\ln(\cdot)$ is computed using the function `log`. Functions \log_{10} and \log_2 are computed using `log10` and `log2`. [Ans:](#)

– 6.3: Try $\log(1)$ in the command window. What do you expect for $\log_{10}(1)$ and $\log_2(1)$? [Ans:](#)

– 6.4: Try $\log(-1)$ in the command window. What do you expect for $\log_{10}(-1)$ and $\log_2(-1)$? [Ans:](#)

– 6.5: Explain how Matlab/Octave arrives at the answer in problem 6.4. Hint: $-1 = e^{i\pi}$.

[Ans:](#)

– 6.6: Try $\log(\exp(j*\text{sqrt}(\text{pi})))$ (i.e., $\log e^{j\sqrt{\pi}}$) in the command window. What do you expect?

[Ans:](#)

– 6.7: What does “inverse” mean in this context? What is the inverse of $\ln f(x)$?

[Ans:](#)

– 6.8: What is a decibel? (Look up decibels on the internet.)

[Ans:](#)

Problem # 7: The following identity is interesting. Can you find a proof?

$$\begin{aligned}
 1 &= 1^2 \\
 1 + 3 &= 2^2 \\
 1 + 3 + 5 &= 3^2 \\
 1 + 3 + 5 + 7 &= 4^2 \\
 1 + 3 + 5 + 7 + 9 &= 5^2 \\
 &\vdots \\
 \sum_{n=0}^{N-1} 2n + 1 &= N^2.
 \end{aligned}$$

– 7.1: Can you find a proof?

[Ans:](#)

1.2 Problems NS-2

Topic of this homework:

Prime numbers, greatest common divisors, the continued fraction algorithm

Continued fractions

Problem # 1: Here we explore the continued fraction algorithm (CFA)

In its simplest form, the CFA starts with a real number, which we denote as $\alpha \in \mathbb{R}$. Let us work with an irrational real number, $\pi \in \mathbb{I}$, as an example because its CFA representation will be infinitely long. We can represent the CFA coefficients α as a vector of integers $n_k, k = 1, 2, \dots, \infty$:

$$\begin{aligned}\alpha &= [n_1; n_2, n_3, n_4, \dots] \\ &= n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \dots}}}\end{aligned}$$

The CFA is recursive, with three repeated steps per iteration. For example $\alpha_1 = \pi \approx 3.14159\dots$, $n_1 = 3$, $r_1 = \pi - 3$, and $\alpha_2 \equiv 1/r_1$.

$$\begin{aligned}\alpha_2 &= 1/0.1416 = 7.0625\dots \\ \alpha_1 &= n_1 + \frac{1}{\alpha_2} = n_1 + \frac{1}{n_2 + \frac{1}{\alpha_3}} = \dots\end{aligned}$$

In terms of a Matlab/Octave script,

```
alpha0 = pi;
K=10;
n=zeros(1,K); alpha=zeros(1,K);
alpha(1)=alpha0;

for k=2:K %k=1 to K
    n(k)=round(alpha(k-1));
    %n(k)=fix(alpha(k-1));
    alpha(k)= 1/(alpha(k-1)-n(k));
    %disp([fix(k), round(n(k)), alpha(k)]); pause(1)
end
disp([n; alpha]);
%Now compare this to matlab's rat() function
rat(alpha0,1e-20)
```

– 1.1: By hand find the first few values of n_k for $\alpha = e^\pi \approx 23.1407$.

Ans:

– 1.2: For the proceeding question, what is the error (remainder) when you truncate the continued fraction after n_1, \dots, n_3 ? Give the absolute value of the error and the percentage error relative to the original α .

Ans:

– 1.3: Use the Matlab/Octave program provided to find the first 10 values of n_k for $\alpha = e^\pi$, and verify your result using the Matlab/Octave command `rat()`.

Ans:

– 1.4: Discuss the similarities and differences between the Euclidean algorithm and the CFA.

Ans:

Greatest common divisors

Consider using the *Euclidean algorithm* to find the *greatest common divisor* (i.e., GCD; the largest common prime factor) of two numbers (Allen 2020, p. 42). This algorithm may be performed using one of two methods:

Method	Division	Subtraction
On each iteration...	$a_{i+1} = b_i$ $b_{i+1} = a_i - b_i \cdot \text{floor}(a_i/b_i)$	$a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$ $b_{i+1} = \min(a_i, b_i)$
Start with $i = 1$ and terminate when:	$b = 0$ (GCD = a)	$b = 0$ (GCD = a)

The division method (Matlab's floor function) (Eq. 2.1, Sec. 2.1.2, Ch. 2) is preferred because the subtraction method may require a huge number of iterations steps.

Problem # 2: Understanding the Euclidean algorithm (GCD)

– 2.1: Find the prime factors of $a = 85$ and $b = 15$.

Ans:

– 2.2: What is the greatest common prime factor of $a = 85$ and $b = 15$?

Ans:

– 2.3: By hand, perform the Euclidean algorithm for $a = 85$ and $b = 15$.

Ans:

– 2.4: By hand, perform the Euclidean algorithm for $a = 75$ and $b = 25$. Is the result a prime number?

Ans:

– 2.5: Consider the first step of the GCD division algorithm when $a < b$ (e.g., $a = 25$ and $b = 75$). What happens to a and b in the first step? Does it matter if you begin the algorithm with $a < b$ rather than $b < a$?

Ans:

– 2.6: Describe in your own words how the GCD algorithm works. Try the algorithm using numbers that have already been divided into factors (e.g., $a = 5 \cdot 3$ and $b = 7 \cdot 3$).

Ans:

– 2.7: Find the GCD of $2 \cdot \pi_{25}$ and $3 \cdot \pi_{25}$.

Ans:

Problem # 3: Coprimes

– 3.1: Define the term coprime.

Ans:

– 3.2: How can the Euclidean algorithm be used to identify coprimes?

Ans:

– 3.3: Give an important application of the Euclidean algorithm.

Ans:

– 3.4: Write a Matlab function, `function x = my_gcd(a,b)`, that uses the Euclidean algorithm to find the GCD of any two inputs a and b . Test your function on the (a, b) combinations from the previous problem. Include a printout (or hand-write) your algorithm to turn in.

Hints and advice:

- Don't give your variables the same names as Matlab functions! Since `gcd` is an existing Matlab/Octave function, if you use it as a variable or function name, you won't be able to use `gcd` to check your `gcd()` function. Try `clear all` to recover from this problem.
- Try using a "while" loop for this exercise (see Matlab documentation for help).
- You may need to use some temporary variables for a and b in order to perform the algorithm.

Ans:

Algebraic generalization of the GCD (Euclidean) algorithm

Problem # 4: In this problem we are looking for integer solutions $(m, n) \in \mathbb{Z}$ to the equations $ma + nb = \gcd(a, b)$ and $ma + nb = 0$ given positive integers $(a, b) \in \mathbb{Z}^+$.

Note that this requires that either m or n be negative. These solutions may be found using the Euclidean algorithm only if (a, b) are coprime ($a \perp b$). Note that integer (whole number) polynomial relations such as these are known as *Diophantine equations*. Such equations (e.g., $ma + nb = 0$) are linear Diophantine equations, possibly the simplest form of such relations.

Example: $\gcd(2, 3) = 1$: For $(a, b) = (2, 3)$, the result is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}}_{\substack{m \\ n}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus from the above equation we find the solution (m, n) to the integer equation

$$2m + 3n = \gcd(2, 3) = 1;$$

namely, $(m, n) = (-1, 1)$ (i.e., $-2 + 3 = 1$). There is also a second solution $(3, -2)$ (i.e., $3 \cdot 2 - 2 \cdot 3 = 0$) that represents the terminating condition. Thus these two solutions are a pair and the solution exists only if (a, b) are coprime ($a \perp b$).

Subtraction method: This method is more complicated than the division algorithm because at each stage we must check whether $a < b$. Define

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where Q sets $a_{i+1} = a_i - b_i$ and $b_{i+1} = b_i$ assuming $a_i > b_i$, and S is a swap matrix that swaps a_i and b_i if $a_i < b_i$. Using these matrices, we implement the algorithm by assigning

$$\begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = Q \begin{bmatrix} a_i \\ b_i \end{bmatrix} \text{ for } a_i > b_i, \quad \begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} = QS \begin{bmatrix} a_i \\ b_i \end{bmatrix} \text{ for } a_i < b_i.$$

The result of this method is a cascade of Q and S matrices. For $(a, b) = (2, 3)$, the result is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_Q \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_S \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_Q \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_S \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}_{\substack{m \\ n}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus we find two solutions (m, n) to the integer equation $2m + 3n = \gcd(2, 3) = 1$.

– 4.1: By inspection, find at least one integer pair (m, n) that satisfies $12m + 15n = 3$.

Ans:

– 4.2: Using matrix methods for the Euclidean algorithm, find integer pairs (m, n) that satisfy $12m + 15n = 3$ and $12m + 15n = 0$. Show your work!!!

Ans:

– 4.3: Does the equation $12m + 15n = 1$ have integer solutions for n and m ? Why or why not?

Ans:

Problem # 5: Matrix approach:

It can be difficult to keep track of the a 's and b 's when the algorithm has many steps. We need an alternative way to run the Euclidean algorithm using matrix algebra. Matrix methods provide a more transparent approach to the operations on (a, b) . Thus the Euclidean algorithm can be classified in terms of standard matrix operations. Write out the indirect matrix approach discussed at the end of Sec. ?? (Eq. ??).

[Ans:](#)

Prime numbers**Problem # 6: Every integer may be written as a product of primes.**

– 6.1: Write the numbers 1,000,000, 1,000,004, and 999,999 in the form $N = \prod_k \pi_k^{\beta_k}$. Hint: Use Matlab/Octave to find the prime factors.

[Ans:](#)

– 6.2: Give a generalized formula for the natural logarithm of a number $\ln(N)$ in terms of its primes π_k and their multiplicities β_k . Express your answer as a sum of terms.

[Ans:](#)

Problem # 7: Using the computer

– 7.1: Explain why the following brief Matlab/Octave program returns the prime numbers π_k between 1 and 100.

```
n=2:100;
k = isprime(n);
n(k)
```

[Ans:](#)

– 7.2: How many primes are there between 2 and $N = 100$?

[Ans:](#)

Problem # 8: Prime numbers may be identified using a sieve.

– 8.1: By hand, complete the sieve of Eratosthenes for $n = 1, \dots, 49$. Start by writing out a table of the integers 1-50, as 5 rows of 10 numbers. Starting with the first prime, $p_k = 2$, $k = 1$, circle it and cross out all multiples (e.g., $2\pi_k = 2$, $3\pi_k = 6, \dots$, $24 * \pi_2 = 48$). Then repeat for the second, third, and higher primes π_2 . When done, only the circled primes should remain. Be sure you look up the definition of a prime.

[Ans:](#)

– 8.2: What is the largest number you need to consider before only primes remain? Look up the definition of the Matlab/Octave floor function (e.g., $\lfloor \pi \rfloor = 3$).

[Ans:](#)

– 8.3: Generalize: For $n = 1, \dots, N$, what is the largest number you need to consider before only the primes remain?

[Ans:](#)

– 8.4: Write each of these numbers as a product of primes: 22, 30, 34, 43, 44, 48, 49. [Ans:](#)

– 8.5: Find the largest prime $\pi_k \leq 100$. Do not use Matlab/Octave other than to check your answer. Hint: Write the numbers starting with 100 and count backward: 100, 99, 98, 97, Cross off the even numbers, leaving 99, 97, 95, Pull out a factor (only one is necessary to show that it is not prime).

[Ans:](#)

– 8.6: Find the largest prime $\pi_k \leq 1000$. Do not use Matlab/Octave other than to check your answer.

[Ans:](#)

– 8.7: Explain why $\pi_k^{-s} = e^{-s \ln \pi_k}$.

[Ans:](#)

Problem # 9: CFA of ratios of large primes

– 9.1: (4pts) Expand $23/7$ as a continued fraction. Express your answer in bracket notation (e.g., $\pi = [3., 7, 16, \dots]$). Show your work. [Ans:](#)

– 9.2: Starting from the primes below 10^6 , form the CFA of π_j/π_k with $j = 78498$ and $k < j$.

[Ans:](#)

– 9.3: Look at other ratios of prime numbers and look for a pattern in the CFA of the ratios of large primes. What is the most obvious conclusion? [Ans:](#)

– 9.4: **(1pts)** Try the Matlab/Octave functions `rats(23/7)`, `rats(3.2857)`, and `rats(3.2856)`. What can you conclude?

[Ans:](#)

– 9.5: **(2pts)** Can $\sqrt{2}$ be represented as a finite continued fraction? Why or why not?

[Ans:](#)

– 9.6: **(2pts)** What is the CFA for $\sqrt{2} - 1$?

Hint: $\sqrt{2} + 1 = \frac{1}{\sqrt{2} - 1} = [2; 2, 2, 2, \dots].$

[Ans:](#)

– 9.7: Show that

$$\frac{1}{1 - \sqrt{a}} = a^{\frac{11}{2}} + a^{\frac{9}{2}} + a^{\frac{7}{2}} + a^{\frac{5}{2}} + a^{\frac{3}{2}} + \sqrt{a} + a^5 + a^4 + a^3 + a^2 + a + 1 = 1 - a^6$$

```
syms a,b
b= taylor(1/(1-sqrt(a)))
simplify((1-sqrt(a))*b) = 1-a^6
```

Use symbolic analysis to show this, then explain. [Ans:](#)

1.3 Problems NS-3

Topic of this homework: Pythagorean triplets, Pell's equation, Fibonacci sequence

Pythagorean triplets

Problem # 1: Euclid's formula for the Pythagorean triplets a, b, c is $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.

– 1.1: What condition(s) must hold for p and q such that a, b , and c are always positive and nonzero?

Ans:

– 1.2: Solve for p and q in terms of a, b , and c .

Ans:

Problem # 2: The ancient Babylonians (ca. 2000 BCE) cryptically recorded (a, c) pairs of numbers on a clay tablet, archeologically denoted Plimpton-322 (see ??).

– 2.1: Find p and q for the first five pairs of a and c shown here from Plimpton-322.

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97

Find a formula for a in terms of p and q .

Ans:

– 2.2: Based on Euclid's formula, show that $c > (a, b)$.

Ans:

– 2.3: What happens when $c = a$?

Ans:

– 2.4: Is $b + c$ a perfect square? Discuss.

Ans:

Pell's equation:

Problem # 3: Pell's equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{I}$. We seek integer solutions of

$$x^2 - Ny^2 = 1.$$

As shown in Sec. ??, the solutions x_n, y_n for the case of $N = 2$ are given by the linear 2×2 matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1_J \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1_J = \sqrt{-1} = e^{j\pi/2}$. It follows that the general solution to Pell's equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{j\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

To calculate solutions to Pell's equation using the matrix equation above, we must calculate

$$A^n = e^{j\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n = e^{j\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

which becomes tedious for $n > 2$.

– 3.1: Find the companion matrix and thus the matrix A that has the same eigenvalues as Pell's equation. Hint: Use Matlab's function `[E, Lambda] = eig(A)` to check your results!

Ans:

– 3.2: Solutions to Pell's equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell's equation is relevant to $\sqrt{2}$.

Ans:

– 3.3: Find the first three values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell's equation for $N = 2$. **Ans:** By hand, find the eigenvalues λ_{\pm} of the 2×2 Pell's equation matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Ans:

– 3.4: By hand, show that the matrix of eigenvectors, E , is

$$E = [\vec{e}_+ \quad \vec{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}.$$

Ans:

– 3.5: Using the eigenvalues and eigenvectors you found for A , verify that

$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

Ans:

– 3.6: Once you have diagonalized A , use your results for E and Λ to solve for the $n = 10$ solution $(x_{10}, y_{10})^T$ to Pell's equation with $N = 2$.

Ans:

The Fibonacci sequence

The Fibonacci sequence is famous in mathematics and has been observed to play a role in the mathematics of genetics. Let x_n represent the Fibonacci sequence,

$$x_{n+1} = x_n + x_{n-1}, \quad (\text{NS-3.1})$$

where the current input sample x_n is equal to the sum of the previous two inputs. This is a “discrete time” recurrence relationship. To solve for x_n , we require some initial conditions. In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \dots\}$ for $n = 0, 1, 2, 3, \dots$.

Equation NS-3.1 is equivalent to the 2×2 matrix equations

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \quad (\text{NS-3.2})$$

Problem # 4: Here we seek the general formula for x_n . Like Pell’s equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast x_n as a 2×2 matrix relationship and then proceed, as we did for the Pell case.

– 4.1: Show that the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$ may be generated by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (\text{NS-3.3})$$

Ans:

– 4.2: What is the relationship between y_n and x_n ?

Ans:

– 4.3: Write a Matlab/Octave program to compute x_n using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is x_{40} ? Note: Consider using the eigenanalysis of A , described by Eq. ?? of the text.

Ans:

– 4.4: Using the eigenanalysis of the matrix A (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence

$$x_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]. \quad (\text{NS-3.4})$$

– 4.5: What are the eigenvalues λ_{\pm} of the matrix A ?

Ans:

– 4.6: How is the formula for x_n related to these eigenvalues? Hint: Find the eigenvectors.

Ans:

– 4.7: What happens to each of the two terms

$$\left[\left(1 \pm \sqrt{5} \right) / 2 \right]^{n+1} ?$$

Ans:

– 4.8: What happens to the ratio x_{n+1}/x_n ?

Ans:

Problem # 5: Replace the Fibonacci sequence with

$$x_n = \frac{x_{n-1} + x_{n-2}}{2},$$

such that the value x_n is the average of the previous two values in the sequence.

– 5.1: What matrix A is used to calculate this sequence?

Ans:

– 5.2: Modify your computer program to calculate the new sequence x_n . What happens as $n \rightarrow \infty$?

Ans:

– 5.3: What are the eigenvalues of your new A ? How do they relate to the behavior of x_n as $n \rightarrow \infty$? Hint: You can expect the closed-form expression for x_n to be similar to Eq. NS-3.4.

Ans:

Problem # 6: Consider the expression

$$\sum_1^N f_n^2 = f_N f_{N+1}.$$

– 6.1: Find a formula for f_n that satisfies this relationship. Hint: It holds for only the Fibonacci recursion formula.

Ans:

CFA as a matrix recursion

Problem # 7: The CFA may be written as a matrix recursion. For this we adopt a special notation, unlike other matrix notations,^a with $k \in \mathbb{N}$:

$$\begin{bmatrix} n \\ x \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & \lfloor x_k \rfloor \\ 0 & \frac{1}{x_k - \lfloor x_k \rfloor} \end{bmatrix} \begin{bmatrix} n \\ x \end{bmatrix}_k.$$

This equation says that $n_{k+1} = \lfloor x_k \rfloor$ and $x_{k+1} = 1/(x_k - \lfloor x_k \rfloor)$. It does *not* mean that $n_{k+1} = \lfloor x_k \rfloor x_k$, as would be implied by standard matrix notation. The lower equation says that $r_k = x_k - \lfloor x_k \rfloor$ is the *remainder*—namely, $x_k = \lfloor x - k \rfloor + r_k$ (Octave/Matlab's `rem(x, floor(x))` function), also known as `mod(x, y)`.

– 7.1: Start with $n_0 = 0 \in \mathbb{N}$, $x_0 \in \mathbb{I}$, $n_1 = \lfloor x_0 \rfloor \in \mathbb{N}$, $r_1 = x - \lfloor x \rfloor \in \mathbb{I}$, and $x_1 = 1/r_1 \in \mathbb{I}$, $r_n \neq 0$. For $k = 1$ this generates on the left the next CFA parameter $n_2 = \lfloor x_1 \rfloor$ and $x_2 = 1/r_2 = 1/(x_0 - \lfloor x_0 \rfloor)$ from n_0 and x_0 . Find $[n, x]_{k+1}^T$ for $k = 2, 3, 4, 5$.

Ans:

^aThis notation is highly nonstandard due to the nonlinear operations. The matrix elements are *derived* from the vector rather than multiplying them. These calculation may be done with the help of Matlab/Octave.

Chapter 2

Algebraic Equations

2.1 Problems AE-1

Topics of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton's root finding method, Riemann zeta function. Deliverables: Answers to problems

Note: The term analytic is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

Problem # 1: A polynomial of degree N is defined as

$$P_N(x) = a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N.$$

– 1.1: How many coefficients a_n does a polynomial of degree N have?

Ans:

– 1.2: How many roots does $P_N(x)$ have?

Ans:

Problem # 2: The fundamental theorem of algebra (FTA)

– 2.1: State and then explain the FTA.

Ans:

– 2.2: Using the FTA, prove your answer to question 1.2. Hint: Apply the FTA to prove how many roots a polynomial $P_N(x)$ of order N has.

Ans:

Problem # 3: Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$ and the related function $F(x) = 1/P_2(x)$. What are the roots (e.g., zeros) x_{\pm} of $P_2(x)$? Hint: Complete the square on the polynomial $P_2(x) = 1 + x^2$ of degree 2, and find the roots.

Ans:

Problem # 4: $F(x)$ may be expressed as $(A, B, x_{\pm} \in \mathbb{C})$

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \quad (\text{AE-1.1})$$

where x_{\pm} are the roots (zeros) of $P_2(x)$, which become the *poles* of $F(x)$; A and B are the *residues*. The expression for $F(x)$ is sometimes called a *partial fraction expansion* or *residue expansion*, and it appears in many engineering applications.

– 4.1: Find $A, B \in \mathbb{C}$ in terms of the roots x_{\pm} of $P_2(x)$.

Ans:

– 4.2: Verify your answers for A and B by showing that this expression for $F(x)$ is indeed equal to $1/P_2(x)$.

Ans:

– 4.3: Give the values of the poles and zeros of $P_2(x)$.

Ans:

– 4.4: Give the values of the poles and zeros of $F(x) = 1/P_2(x)$.

Ans:

2.1.1 Analytic functions

Overview: Analytic functions are defined by infinite (power) series. The function $f(x)$ is said to be *analytic* at any value of constant $x = x_o$, where there exists a convergent power series

$$P(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

such that $P(x_o) = f(x_o)$. The point $x = x_o$ is called the *expansion point*. The region around x_o such that $|x - x_o| < 1$ is called the *radius of convergence*, or region of convergence (RoC). The local power series for $f(x)$ about $x = x_o$ is defined by the Taylor series:

$$\begin{aligned} f(x) &\approx f(x_o) + \left. \frac{df}{dx} \right|_{x=x_o} (x - x_o) + \frac{1}{2!} \left. \frac{d^2 f}{dx^2} \right|_{x=x_o} (x - x_o)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n}{dx^n} f(x) \right|_{x=x_o} (x - x_o)^n. \end{aligned}$$

Two classic examples are the geometric series¹ where $a_n = 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad (\text{AE-1.2})$$

and the exponential function where $a_n = 1/n!$, Eq. ?? (p. ??). The coefficients for both series may be derived from the Taylor formula.

Problem # 5: The geometric series

– 5.1: What is the region of convergence (RoC) for the power series Eq. AE-1.2 of $1/(1-x)$ given above—for example, where does the power series $P(x)$ converge to the function value $f(x)$? State your answer as a condition on x . Hint: What happens to the power series when $x > 1$?

Ans:

¹The geometric series is *not* defined as the function $1/(1-x)$, it is defined as the series $1 + x + x^2 + x^3 + \cdots$, such that the ratio of consecutive terms is x .

– 5.2: In terms of the pole, what is the RoC for the geometric series in Eq. AE-1.2?

Ans:

– 5.3: How does the RoC relate to the location of the pole of $1/(1-x)$?

Ans:

– 5.4: Where are the zeros, if any, in Eq. AE-1.2?

Ans:

– 5.5: Assuming x is in the RoC, prove that the geometric series correctly represents $1/(1-x)$ by multiplying both sides of Eq. AE-1.2 by $(1-x)$.

Ans:

Problem # 6: Use the geometric series to study the degree N polynomial. It is very important to note that all the coefficients c_n of this polynomial are 1.

$$P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^N x^n. \quad (\text{AE-1.3})$$

– 6.1: Prove that

$$P_N(x) = \frac{1 - x^{N+1}}{1 - x}. \quad (\text{AE-1.4})$$

Ans:

– 6.2: What is the RoC for Eq. AE-1.3?

Ans:

– 6.3: What is the RoC for Eq. AE-1.4?

Ans:

– 6.4: How many poles does $P_N(x)$ (Eq. AE-1.3) have? Where are they?

Ans:

– 6.5: How many zeros does $P_N(x)$ (Eq. AE-1.4) have? State where are they in the complex plane.

Ans:

– 6.6: Explain why Eqs. AE-1.3 and AE-1.4 have different numbers of poles and zeros.

Ans:

– 6.7: Is the function $1/(1-x)$ analytic outside of the RoC?

Ans:

– 6.8: *Extra credit. Evaluate $P_N(x)$ at $x = 0$ and $x = 0.9$ for the case of $N = 100$, and compare the result to that from Matlab.*

```
%sum the geometric series and P_100(0.9)
clear all;close all;format long
N=100; x=0.9; S=0;
for n=0:N
    S=S+x^n
end
P100=(1-x^(N+1))/(1-x);
disp(sprintf('S= %g, P100= %g, error= %g',S,P100, S-P100))
```

[Ans:](#)

Problem # 7: The exponential series

– 7.1: *What is the RoC for the exponential series Eq. ???*

[Ans:](#)

– 7.2: *Let $x = j$ in Eq. ??, and write out the series expansion of e^x in terms of its real and imaginary parts.*

[Ans:](#)

– 7.3: *Let $x = j\theta$ in Eq. ??, and write out the series expansion of e^x in terms of its real and imaginary parts. How does your result relate to Euler's identity ($e^{j\theta} = \cos(\theta) + j\sin(\theta)$)?*

[Ans:](#)

(5 pts) Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Note that the inverse is not to be confused with the *reciprocal*. The function $(x, y \in \mathbb{C})$

$$y(x) = \frac{1}{1-x},$$

has a pole at $x = 1$ and a zero at $x = \infty$. Its inverse

$$x(y) = \frac{y-1}{y} = 1 - \frac{1}{y}$$

has a zero at $y = 1$ and a pole at $y = 0$, with $x(\infty) = 1$. The reciprocal of $y(x)$ is $1/y(x) = (1-x)$, which is very different from $x(y)$.

Problem # 8(2 pts) Consider the exponential function $z(x) = e^x$ ($x, z \in \mathbb{C}$).

– 8.1: *Find the inverse $x(z)$.*

[Ans:](#)

– 8.2: *Where are the poles and zeros of $x(z)$?*

[Ans:](#)

Problem # 9: (3 pts) Composition.

– 9.1: If $y(s) = 1/(1-s)$ and $z(s) = e^s$, compose these two functions to obtain $(y \circ z)(s)$. Give the expression for $(y \circ z)(s) = y(z(s))$. [Ans:](#)

– 9.2: Where are the poles and zeros of $(y \circ z)(s)$?

[Ans:](#)

– 9.3: Where (for what condition on s) is $(y \circ z)(s)$ analytic?

[Ans:](#)

Eigen-analysis**Problem # 10: (4 pts) The vectorized eigen-equation for a matrix A is**

$$AE = E\Lambda. \quad (\text{AE-1.5})$$

– 10.1: (4pt) Provide a formula for A^3 in terms of the eigenvector E and eigenvalue Λ matrices.

[Ans:](#)

– 10.2: (4 pts) Find the eigenvalues of the matrix, and find the roots, by completing the square, where $a, b, c, d \in \mathbb{C}$, and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

[Ans:](#)

(4 pts) Convolution

Multiplying two short or simple polynomials is not demanding. However, if the polynomials have many terms, it can become tedious. For example, multiplying two 10th-degree polynomials is not something one would like to do every day.

An alternative is a method called *convolution*. The inverse of convolution is called *deconvolution*, which is equivalent to long-division of polynomials, also known as factoring polynomials (Sec. 3.4.1, p. 109-111). Newton's method is a reliable and accurate algorithm to extract roots from polynomials using term by term deconvolution. When the roots are well approximated by fractional numbers, the method is accurate to within computational accuracy. For example, if the root is $\pi \approx \hat{\pi}_{19} \equiv 817696623/260280919 \in \mathbb{F}$, as given by `rats(pi, 19)`. $\hat{\pi}_{19}$ is the 64 bit machine's internal representation of π since $\pi - \hat{\pi}_{19} = 0$ (See text Fig. 2.6, p. 48).

Problem # 11: (4 pts) Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Manually evaluate the following convolutions. Show your work!

– 11.1: (2 pts) Multiplying two polynomials is the same as convolving their coefficients.

Given

$$\begin{aligned} f(x) &= x^3 + 3x^2 + 3x + 1 \leftrightarrow [1; 3, 3, 1] \\ g(x) &= x^3 + 2x^2 + x + 2 \leftrightarrow [1; 2, 1, 2]. \end{aligned}$$

show that

$$f(x)g(x) = x^6 + 5x^5 + 10x^4 + 12x^3 + 11x^2 + 7x + 2 \leftrightarrow [1; 3, 3, 1] \star [1; 2, 1, 2].$$

[Ans:](#)

Here is how you put [] inside the latex's

`begin{question}[: bracketed text]`

```
\begin{question}
[: (1 pts)
  \(\char91 1;-1 \char93 \star \char91 0; 1, 2, 4, 7, 0\char93\)
]
\end{question}
```

as used in the following question:

– 11.2: (1 pts) $[1; -1] \star [0; 1, 2, 4, 7, 0]$

Ans: $\cdot = [0, 1, 1, 2, 3, -7, 0, \dots]$.

– 11.3: (1 pts) $[1; 2, 1] \star [1; -1]$

Ans:

Newton's root-finding method

Problem # 12: (2 pts) Use Newton's iteration to find the roots of the polynomial

$$P_3(x) = 1 - x^3.$$

– 12.1: Draw a graph describing the first step of the iteration starting with $x_0 = (1/2, 0)$.

Ans:

– 12.2: Calculate x_1 and x_2 . What number is the algorithm approaching?

Ans:

– 12.3: Here is an Octave/Matlab script for the $P_2(x)$ case. Modify it to find $P_3(x)$:

```
x(1)=1/2; %x(1)=0.9; %x(1)=-10
y(1)=x(1);
for n=2:10
x(n) = x(n-1) + (1-x(n-1)^2)/(2*x(n-1));
y(n) = (1+y(n-1)^2)/(2*x(n-1));
end
semilogy(abs(x)-1); hold on
semilogy(abs(7)-1,'or'); hold off
```

Ans:

– 12.4: (1 pts) For $n = 4$, what is the absolute difference between the root and the estimate, $|x_r - x_4|$?

Ans:

– 12.5: Does Newton's method work for $P_2(x) = 1 + x^2$? If so, why? Hint: The poles and zeros are exactly known!

[Ans:](#)

– 12.6: What if we let $x_0 = (1 + j)/2$ for the case of $P_2(x) = 1 + x^2$?

[Ans:](#)

Problem # 13: In this problem we consider the case of fractional roots, and take advantage of this fact during the iteration. Given that the roots are integers, composed of primes, we may uniquely identify the primes by factoring the numerator and denominator of the rational approximation of the root.

The method is:

1. Start the Newton iteration

$$s_{n+1} = s_n - \frac{M(s_n)}{M'(s_n)}$$

2. Apply the CFA to the next output $\text{rats}(s_{n+1})$
3. Factor the Num and Dem of the CFA
4. Terminate when the factors converge

Using this method, show that we can find either the best possible fractional approximation to the roots (or even the exact roots, when the answer is within machine accuracy).

– 13.1: Find the roots of a Monic having coefficients $m_k \in \mathbb{F}$.

Let

$$M_3(x) = (x - 254/17)(x - 2047/13)(x - 17/13)$$

In this case the root vector R becomes

$$R = [14.9412, 157.4615, 1.3077].$$

Verify that $\text{rats}(M)$ returns the rational set of roots. [Ans:](#)

2.2 Problems AE-2

Topics of this homework:

Linear vs nonlinear systems of equations, Euclid's formula, Gaussian elimination, matrix permutations, Ohm's law, two-port networks,

Deliverables: Answers to problems

(- pts) Gaussian elimination

Problem # 1: (1 pts) Gaussian elimination

– 1.1: (1 pts) Find the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Ans:

– 1.2: (1 pts) Verify that $A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Ans:

Problem # 2: (6 pts) Find the solution to the following 3×3 matrix equation $Ax = b$ by GE. Show your intermediate steps. You can check your work at each step using Octave/Matlab.

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

– 2.1: (2 pts) Show (i.e., verify) that the first GE matrix G_1 , which zeros out all entries in the first column is given by

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Identify the elementary row operations that this matrix performs. Ans:

– 2.2 (2 pts) Find a second GE matrix, G_2 , to put G_1A in upper triangular form. Identify the elementary row operations that this matrix performs.

Ans:

– 2.3 (2 pts) Find a third GE matrix G_3 that scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.

Ans:

– 2.4: (2 pts) Find the last GE matrix, G_4 , which subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ($G_4G_3G_2G_1A = I$).

Ans:

– 2.5: (2 pts) Solve for $\{x_1, x_2, x_3\}^T$ using the augmented matrix format $G_4G_3G_2G_1\{A|b\}$ (where $\{A|b\}$ is the augmented matrix). Note that if you've performed the preceding steps correctly, $x = G_4G_3G_2G_1b$.

Ans:

– 2.6: (1 pt) Find the pivot matrix G that rescales the second row of the augmented matrix $A|b$ by $1/3$.

Ans:

Two linear equations

Problem # 3 *In this problem we transition from a general pair of equations*

$$f(x, y) = 0$$

$$g(x, y) = 0$$

to the important case of two linear equations

$$y = ax + b$$

$$y = \alpha x + \beta.$$

Note that to help keep track of the variables, roman coefficients (a, b) are used for the first equation and Greek (α, β) for the second.

– 3.1: *What does it mean, graphically, if these two linear equations have (1) a unique solution, (2) a nonunique solution, or (3) no solution?*

Ans:

– 3.2: *Assuming the two equations have a unique solution, find the solution for x and y .*

Ans:

– 3.3: *When will this solution fail to exist (for what conditions on a, b, α , and β)?*

Ans:

– 3.4: *Write the equations as a 2×2 matrix equation of the form $A\vec{x} = \vec{b}$, where $\vec{x} = \{x, y\}^T$.*

Ans:

– 3.5: *Find the inverse of the 2×2 matrix, and solve the matrix equation for x and y .*

Ans:

– 3.6: *Discuss the properties of the determinant of the matrix (Δ) in terms of the slopes of the two equations (a and α).*

Ans:

Problem # 4: The application of linear functional relationships between two variables

We use 2×2 matrices to describe two-port networks, as discussed in Sec. ?? (p. ??). Transmission lines are a great example: Both voltage and current must be tracked as they travel along the line. The Figure below shows an example segment of a transmission line.

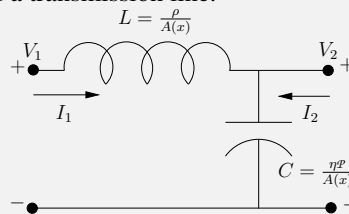


Fig. AE-2.1 This figure shows a cell from an LC transmission line. The index 1 is at the input on the left and 2 represents the output, on the right.

Suppose you are given the following pair of linear relationships between the input (source) variables V_1 and I_1 and the output (load) variables V_2 and I_2 of the transmission line:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} j & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

– 4.1: Let the output (the load) be $V_2 = 1$ and $I_2 = 2$ (i.e., $V_2/I_2 = 1/2 \{\Omega\}$). Find the input voltage and current, V_1 and I_1 .

Ans:

– 4.2: Let the input (source) be $V_1 = 1$ and $I_1 = 2$. Find the output voltage and current, V_2 and I_2 .

Ans:

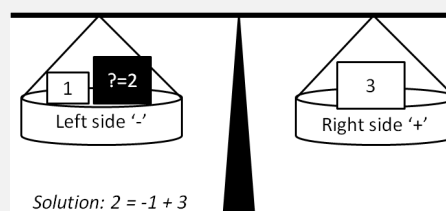
Integer equations: applications and solutions

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

Problem # 5: (5 pts) A practical example of using a Diophantine equation:

“A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?” - *Bachet de Béziriac (1623)^a*.

Here, weighing is performed using a balance scale that has two pans, with weights on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weight in the other pan. Then the scale will be balanced. A solution to the four weights for Bachet’s problem is $1 + 3 + 9 + 27 = 40$ pounds.



^aTaken from ?, p. 50

– 5.1: Show how the combination of 1-, 3-, 9-, and 27-pound weights can be used to weigh 1, 2, 3, ..., 8, 28, and 40 pounds of milk (or something else, such as flour). Assuming that the milk is in the left pan, provide the position of the weights using a negative sign – to indicate the left pan and a positive sign + to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan, +1, will balance the scales.

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]^T$. The pan assignments matrix should contain only the values –1 (left pan), +1 (right pan), and 0 (leave out). You can indicate these using –, +, and blanks.

[Ans:](#)

Vector algebra in \mathbb{R}^3

Definitions of the scalar (also called a dot product) $A \cdot B$, cross $A \times B$ and triple product $A \cdot (B \times C)$, may be found in Appendix ?? (p. ??), where A, B, C in $\mathbb{R}^3 \subset \mathbb{C}^3$, as shown in Fig. ??, p. ??. A fourth “double-cross” (\otimes) vector product is:^a

$$A \times (B \times C) = \alpha_o B - \beta_o C.$$

where $\alpha_o = A \cdot C$ and $\beta_o = A \cdot B$ (Note: $A \times (B \times C) \neq (A \times B) \times C$).

^aGreenberg p. 694, Eq. 8.

Problem # 6: Scalar product $A \cdot B$

– 6.1: If $A = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$ and $B = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}$, write out the definition of $A \cdot B$.

[Ans:](#)

– 6.2: The dot product is often defined as $\|A\| \|B\| \cos(\theta)$, where $\|A\| = \sqrt{A \cdot A}$ and θ is the angle between A, B . If $\|A\| = 1$, describe how the dot product relates to the vector B .

[Ans:](#)

Problem # 7: Vector (cross) product $A \times B$

– 7.1: If $A = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$ and $B = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}$, write out the definition of $A \times B$.

[Ans:](#)

– 7.2: Show that the cross product is equal to the area of the parallelogram formed by A, B , namely $\|A\| \|B\| \sin(\theta)$, where $\|A\| = \sqrt{A \cdot A}$ and θ is the angle between A and B .

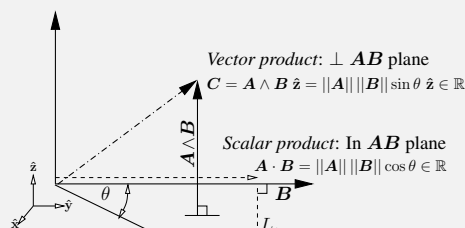
[Ans:](#)

Problem # 8: Triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

Let $\mathbf{A} = [a_1, a_2, a_3]^T$, $\mathbf{B} = [b_1, b_2, b_3]^T$, $\mathbf{C} = [c_1, c_2, c_3]^T$ be three vectors in \mathbb{R}^3 .

– 8.1: Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Ans:



This figure is identical to Fig. ?? (p. ??), Sec. ??. Definitions of vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ (vectors in \mathbb{R}^3) used in the definition of $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{A} \times \mathbf{B}$ and $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. There are two algebraic vector products, the *scalar (dot) product* $\mathbf{A} \cdot \mathbf{B} \in \mathbb{R}$ and the *vector (cross) product* $\mathbf{A} \times \mathbf{B} \in \mathbb{R}^3$. Note that the result of the dot product is a scalar, while the vector product yields a vector, which is \perp to the plane containing \mathbf{A}, \mathbf{B} .

– 8.2: Describe why $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$ is the volume of parallelepiped generated by \mathbf{A}, \mathbf{B} , and \mathbf{C} .

Ans:

– 8.3: Explain why three vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are in one plane if and only if the triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$.

Ans:

Problem # 9: Given two vectors \mathbf{A}, \mathbf{B} in the \hat{x}, \hat{y} plane shown in Fig. ?? (same as ?? on page ??), with $\mathbf{B} = \hat{y}$ (i.e., $\|\mathbf{B}\| = 1$).

– 9.1: Show that \mathbf{A} may be split into two orthogonal parts, one in the direction of \mathbf{B} and the other perpendicular (\perp) to \mathbf{B} . Hint: Express the vector products of \mathbf{A} and \mathbf{B} (dot and cross) in polar coordinates (?).

$$\begin{aligned} \mathbf{A} &= (\mathbf{A} \cdot \mathbf{B})\mathbf{B} + \mathbf{B} \times (\mathbf{A} \times \mathbf{B}) \\ &= \mathbf{A}_{\parallel} + \mathbf{A}_{\perp}. \end{aligned}$$

Ans:

Ohm's Law

In general, impedance is defined as the ratio of a force to a flow. For electrical circuits, the voltage is the force and the current is the flow. Ohm's law states that the voltage across and the current through a circuit element are related by the impedance of that element (which may be a function of frequency). For resistors, the voltage over the current is called the *resistance* and is a constant (e.g., the simplest case is $V/I = R$). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g., $V/I = Z(s)$, where s is the complex frequency $s \in \mathbb{C}$).

As shown in Table ?? (p. ??), the impedance concept also holds in mechanics and acoustics. In mechanics, the force is equal to the mechanical force on an element (e.g., a mass, dashpot, or spring) and the flow is the velocity. In acoustics, the force is pressure and the flow is the volume velocity or particle velocity of air molecules.

Case	Force	Flow	Impedance	units
Electrical	voltage (V)	current (I)	Z	Ohms [Ω]
Mechanics	force (F)	velocity (V)	Z	Mechanical Ohms [Ω]
Acoustics	pressure (P)	particle velocity (U)	Z	Acoustic Ohms [Ω]
Thermal	temperature (T)	heat-flux (J)	Z	Thermal Ohms [Ω]

Problem # 10: *The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As the bulb lights up, the resistance of the metal filament increases.*

Ohm's law says that the current

$$\frac{V}{I} = R(T),$$

where T is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz]. Find the current when the light is first switched on. **Ans:**

Problem # 11: (1 pts) *The power in watts is the product of the force and the flow. What is the power of the lightbulb of Problem 10?*

Ans:

Problem # 12: (1 pts) *State the impedance $Z(s)$ of each of the following circuit elements: (1) a resistor with resistance R , (2) an inductor with inductance L , and (3) a capacitor with capacitance C .*

Ans:

Problem # 13: Consider what happens at the triple point of water. As water freezes or thaws, the temperature remains constant at 0°C . Once all the water is frozen and more heat is removed, the temperature drops below 0° . As heat is added, water thaws but the temperature remains at 0° .

– 13.1: Once all the ice has melted, what is the temperature as more heat is added?

Model the triple point using a Zener diode, a resistor, and a capacitor. A Zener diode holds the voltage constant independent of current. For the case of water's triple point, the voltage represents the temperature of water at the triple point, clamped at 0°C . The current represents the heat flux. The latent heat of water at the triple point is 32 Cal/gm . Thus as the temperature rises from below freezing, the water is clamped at 0° once the triple point is reached. At that point, adding more heat flux has no effect on the temperature until all the ice melts. Once the ice has melted, the temperature again begins to rise until it hits the boiling point, where it again stays at 100° until all the water has evaporated.

Ans:

Nonlinear (quadratic) to linear equations

In the following problems we deal with algebraic equations in more than one variable that are not linear equations. For example, the circle $x^2 + y^2 = 1$ may be solved for $y(x) = \pm\sqrt{1 - x^2}$. If we let $z_+ = x + jy = x + j\sqrt{1 - x^2} = e^{j\theta}$, we obtain the equation for half a circle ($y > 0$). The entire circle is described by the magnitude of z as $|z|^2 = (x + jy)(x - jy) = 1$.

Problem # 14: Give the curve defined by the equation:

$$x^2 + xy + y^2 = 1$$

– 14.1: Find the function $y(x)$.

Ans:

– 14.2: Using Matlab/Octave, plot $y(x)$ and describe the graph.

Ans:

– 14.3: What is the name of this curve?

Ans:

– 14.4: Find the solution (in x , p , and q) to these equations:

$$x + y = p$$

$$xy = q.$$

Ans:

– 14.5: Find an equation that is linear in y starting from equations that are quadratic (second-degree) in the two unknowns x and y :

$$x^2 + xy + y^2 = 1 \quad (\text{AE-2.1})$$

$$4x^2 + 3xy + 2y^2 = 3. \quad (\text{AE-2.2})$$

Ans:

– 14.6: Compose the following two quadratic equations and describe the results.

$$\begin{aligned}x^2 + xy + y^2 &= 1 \\ 2x^2 + xy &= 1\end{aligned}$$

Ans:

chapter[Euclid's formula derivation] Euclid's formula derivation

Nonlinear intersection in analytic geometry

Euclid's formula for Pythagorean triplets (Eq. ??, p. ??) can be derived by intersecting a circle and a secant line. Consider the nonlinear equation of a unit circle having radius 1, centered at $(x, y) = (0, 0)$,

$$x^2 + y^2 = 1,$$

and the secant line through $(-1, 0)$,

$$y = t(x + 1),$$

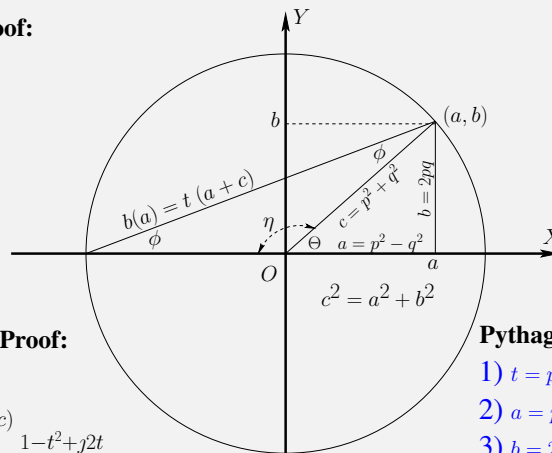
a linear equation having slope t and intercept $x = -1$. If the slope $0 < t < 1$, the line intersects the circle at a second point (a, b) in the positive x, y quadrant. The goal is to find $a, b \in \mathbb{N}$ and then show that $c^2 = a^2 + b^2$. Since the construction gives a right triangle with short sides $a, b \in \mathbb{N}$, then it follows that $c \in \mathbb{N}$.

Euclidean Proof:

- 1) $2\phi + \eta = \pi$
- 2) $\eta + \Theta = \pi$
- 3) $\therefore \phi = \Theta/2$

Diophantus's Proof:

- 1) $c^2 = a^2 + b^2$
- 2) $b(a) = t(a + c)$
- 3) $\zeta(t) \equiv a + jb = \frac{1 - t^2 + j2t}{1 + t^2}$
- 4) $\zeta = |c|e^{i\theta} = |c|\frac{1 + jt}{1 - jt} = |c|(\cos(\theta) + i\sin(\theta))$



Pythagorean triplets:

- 1) $t = p/q \in \mathbb{Q}$
- 2) $a = p^2 - q^2$
- 3) $b = 2pq$
- 4) $c = p^2 + q^2$

Derivation of Euclid's formula for the Pythagorean triplets (PT) $[a, b, c]$, based on a composition of a line, having a rational slope $t = p/q \in \mathbb{F}$, and a circle $c^2 = a^2 + b^2$, $[a, b, c] \in \mathbb{N}$. This analysis is attributed to Diophantus (Di-o-phan'-tus) (250 CE), and today such equations are called Diophantine (Di-o-phan'-tine) equations. PTs have applications in architecture and scheduling, and many other practical problems. Most interesting is their relation to Rydberg's formula for the eigenstates of the hydrogen atom (Appendix ??).

Problem # 15: Derive Euclid's formula

– 15.1: Draw the circle and the line, given a positive slope $0 < t < 1$.

Ans:

Problem # 16: *Substitute $y = t(x + 1)$ (the line equation) into the equation for the circle, and solve for $x(t)$.*

Hint: Because the line intersects the circle at two points, you will get two solutions for x . One of these solutions is the trivial solution $x = -1$. [Ans:](#)

– 16.1: *Substitute the $x(t)$ you found back into the line equation, and solve for $y(t)$.*
[Ans:](#)

– 16.2: *Let $t = q/p$ be a rational number, where p and q are integers. Find $x(p, q)$ and $y(p, q)$.*
[Ans:](#)

– 16.3: *Substitute $x(p, q)$ and $y(p, q)$ into the equation for the circle, and show how Euclid's formula for the Pythagorean triples is generated.*
[Ans:](#)

For full points you must show that you understand the argument. Explain the meaning of the comment “magic happens” when t^4 cancels.

2.3 Problems AE-3

Topics of this homework:

Visualizing complex functions, bilinear/Möbius transformation, Riemann sphere.

Deliverables: Answers to problems

Two-port network analysis

Problem # 1: Perform an analysis of electrical two-port networks, shown in Fig. ?? (page ??). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

The definition of the ABCD transmission matrix (\mathcal{T}) is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \quad (\text{AE-3.1})$$

The impedance matrix, where the determinant $\Delta_{\mathcal{T}} = AD - BC$, is given by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_{\mathcal{T}} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \quad (\text{AE-3.2})$$

– 1.1: Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1). Show your work.

Ans:

Problem # 2: Consider a single circuit element with impedance $Z(s)$.

– 2.1: What is the ABCD matrix for this element if it is in series?

Ans:

– 2.2: What is the ABCD matrix for this element if it is in shunt?

Ans:

Problem # 3: Find the ABCD matrix for each of the circuits of Fig. ??.

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency $s \in \mathbb{C}$, then (ii) substitute $s = 1j$ and calculate the total transmission matrix at this single frequency.

– 3.1: Left circuit (let $R_1 = R_2 = 10$ kilo-ohms and $C = 10$ nano-farads)

Ans:

– 3.2: Right circuit (use L and C values given in the figure), where the pressure P is analogous to the voltage V , and the velocity U is analogous to the current I .

Ans:

– 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency $s = 1j$ as in the previous part (feel free to use Matlab/Octave for your computation).

Ans:

– 3.4: Right circuit: Repeat the analysis as in question 3.3.

Ans:

Algebra**Problem # 4: Fundamental theorem of algebra (FTA).**

– 4.1: State the fundamental theorem of algebra (FTA).

Ans:

(13 pts) Algebra with complex variables**Problem # 5: (7 pts) Order and complex numbers:**

One can always say that $3 < 4$ —namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in $4 - 3 > 0$. Here we will explore how complex variables may be ordered. In the following define $\{x, y\} \in \mathbb{R}$ and complex variable $z = x + yj \in \mathbb{C}$.

– 5.1: Explain the meaning of $|z_1| > |z_2|$.

Ans:

– 5.2: If $x_1, x_2 \in \mathbb{R}$ (are real numbers), define the meaning of $x_1 > x_2$.

Ans:

– 5.3: Explain the meaning of $z_1 > z_2$.

Ans:

– 5.4: (2 pts) What is the meaning of $|z_1 + z_2| > 3$?

Ans:

– 5.5: (2 pts) If time were complex, how might the world be different?

Ans:

Problem # 6: (1 pt) It is sometimes necessary to consider a function $w(z) = u + vj$ in terms of the real functions $u(x, y)$ and $v(x, y)$ (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse $z(w) = x + yj$, where $x(u, v)$ and $y(u, v)$ are real functions.

– 6.1: (1 pts) Find $u(x, y)$ and $v(x, y)$ for $w(z) = 1/z$.

Ans:

Problem # 7: (5 pts) Find $u(x, y)$ and $v(x, y)$ for $w(z) = c^z$ with complex constant $c \in \mathbb{C}$ for questions 7.1, 7.2, and 7.3:

– 7.1: $c = e$

Ans:

– 7.2: $c = 1$ (recall that $1 = e^{\pm j2\pi k}$ for $k \in \mathbb{Z}$)

Ans:

– 7.3: $c = j$. Hint: $j = e^{j\pi/2 + j2\pi k}$, $k \in \mathbb{Z}$.

Ans:

– 7.4: (2 pts) What is j^j ?

Ans:

Schwarz inequality

Problem # 8: *The above figure shows three vectors for an arbitrary value of $\alpha \in \mathbb{R}$ and a specific value of $\alpha = \alpha^*$.*

– 8.1: *Find the value of $\alpha \in \mathbb{R}$ such that the length (norm) of \vec{E} (i.e., $\|\vec{E}\| \geq 0$) is minimum. Show your derivation, not the answer ($\alpha = \alpha^*$).*

Ans:

– 8.2: *Find the formula for $\|\vec{E}(\alpha^*)\|^2 \geq 0$. Hint: Substitute α^* into Eq. ?? (p. ??) and show that this results in the Schwarz inequality*

$$|\vec{U} \cdot \vec{V}| \leq \|\vec{U}\| \|\vec{V}\|.$$

Ans:

Problem # 9: *Geometry and scalar products*

– 9.1: *What is the geometrical meaning of the dot product of two vectors?*

Ans:

– 9.2: *Give the formula for the dot product of two vectors. Explain the meaning based on Fig. ?? (page ??).*

Ans:

– 9.3: *Write the formula for the dot product of two vectors $\vec{U} \cdot \vec{V}$ in \mathbb{R}^n in polar form (e.g., assume the angle between the vectors is θ).*

Ans:

– 9.4: *How is the Schwarz inequality related to the Pythagorean theorem?*

Ans:

– 9.5: *Starting from $\|\vec{U} + \vec{V}\|$, derive the triangle inequality*

$$\|\vec{U} + \vec{V}\| \leq \|\vec{U}\| + \|\vec{V}\|.$$

Ans:

– 9.6: *The triangle inequality $\|\vec{U} + \vec{V}\| \leq \|\vec{U}\| + \|\vec{V}\|$ is true for two and three dimensions: Does it hold for five-dimensional vectors?*

Ans:

– 9.7: *Show that the wedge product $\vec{U} \wedge \vec{V} \perp \vec{U} \cdot \vec{V}$.*

Ans:

Probability

Problem # 10: Basic terminology of experiments

- 10.1: What is the mean of a trial, and what is the average over all trials?

Ans:

- 10.2: What is the expected value of a random variable X ?

Ans:

- 10.3: What is the standard deviation about the mean?

Ans:

- 10.4: What is the definition of information of a random variable?

Ans:

– 10.5: How do you combine events? Hint: If the event is the flip of a biased coin, the events are $H = p$, $T = 1 - p$, so the event is $\{p, 1 - p\}$. To solve the problem, you must find the probabilities of two independent events.

Ans:

– 10.6: What does the term independent mean in the context of question 10.5? Give an example.

Ans:

- 10.7: Define odds.

Ans:

Chapter 3

Differential equations

3.1 Problems DE-1

3.1.1 Topics of this homework:

Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

3.1.2 Complex Power Series

Problem # 1: *In each case derive (e.g., using Taylor's formula) the power series of $w(s)$ about $s = 0$ and give the RoC of your series. If the power series doesn't exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.*

– 1.1: $1/(1 - s)$

Ans:

– 1.2: $1/(1 - s^2)$

Ans:

– 1.3: $1/(1 + s^2)$.

Ans:

– 1.4: $1/s$

Ans:

– 1.5: $1/(1 - |s|^2)$

Ans:

Problem # 2: *Consider the function $w(s) = 1/s$*

– 2.1: *Expand this function as a power series about $s = 1$. Hint: Let $1/s = 1/(1 - (1 - s)) = 1/(1 - (1 - s))$.*

Ans:

– 2.2: What is the RoC?

Ans:

– 2.3: Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$.

Ans:

– 2.4: What is the RoC?

Ans:

– 2.5: What is the residue of the pole?

Ans:

Problem # 3: Consider the function $w(s) = 1/(2 - s)$

– 3.1: Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the RoC as a condition on $|s^{-1}|$.

Hint: Multiply top and bottom by s^{-1} .

Ans:

– 3.2: Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic?

Ans:

Problem # 4: Summing the series

The Taylor series of functions have more than one region of convergence.

– 4.1: Given some function $f(x)$, if $a = 0.1$, what is the value of

$$f(a) = 1 + a + a^2 + a^3 + \dots?$$

Show your work. **Ans:**

– 4.2: Let $a = 10$. What is the value of

$$f(a) = 1 + a + a^2 + a^3 + \dots?$$

Ans:

3.1.3 Cauchy-Riemann Equations

Problem # 5: For this problem $j = \sqrt{-1}$, $s = \sigma + \omega j$, and $F(s) = u(\sigma, \omega) + jv(\sigma, \omega)$. According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of $F(s)$ is defined as

$$\frac{dF}{ds} = \frac{d}{ds} [u(\sigma, \omega) + jv(\sigma, \omega)]. \quad (\text{DE-1.1})$$

If the integral is independent of the path, then the derivative must also be independent of the direction:

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial j\omega}. \quad (\text{DE-1.2})$$

The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}$$

may be used to show where Equation DE-1.2 holds.

– 5.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.

Ans:

– 5.2: Merge the CR equations to show that u and v obey Laplace's equations.

$$\nabla^2 u(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma, \omega) = 0.$$

Ans:

– 5.3: What can you conclude?

Ans:

Problem # 6: Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g., where the function $F(s)$ is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

– 6.1: $F(s) = e^s$

Ans:

– 6.2: $F(s) = 1/s$

Ans:

3.1.4 Branch cuts and Riemann sheets

Problem # 7: Consider the function $w^2(z) = z$. This function can also be written as $w_{\pm}(z) = \sqrt{z}_{\pm}$. Assume $z = re^{i\phi}$ and $w(z) = \rho e^{i\theta} = \sqrt{r}e^{i\phi/2}$.

– 7.1: How many Riemann sheets do you need in the domain (z) and the range (w) to fully represent this function as single-valued?

Ans:

– 7.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

Ans:

– 7.3: Use `zviz.m` to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.

Ans:

– 7.4: Where does `zviz.m` place the branch cut for this function?

Ans:

– 7.5: Must the branch cut necessarily be in this location?

Ans:

Problem # 8: Consider the function $w(z) = \log(z)$. As in Problem 7, let $z = re^{i\phi}$ and $w(z) = \rho e^{i\theta}$.

- 8.1: Describe with a sketch and then discuss the branch cut for $f(z)$.

Ans:

- 8.2: What is the inverse of the function $z(f)$? Does this function have a branch cut? If so, where is it?

Ans:

- 8.3: Using `zviz.m`, show that

$$\tan^{-1}(z) = -\frac{j}{2} \log \frac{j-z}{j+z}. \quad (\text{DE-1.3})$$

In Fig. ?? (p. ??) these two functions are shown to be identical.

Ans:

- 8.4: Algebraically justify Eq. DE-1.3. Hint: Let $w(z) = \tan^{-1}(z)$ and $z(w) = \tan w = \sin w / \cos w$; then solve for e^{wj} .

Ans:

3.1.5 A Cauer synthesis of any Brune impedance

Problem # 9: One may synthesize a transmission line (ladder network) from a positive real impedance $Z(s)$ by using the continued fraction method. To obtain the series and shunt impedance values, we can use a residue expansion. Here we shall explore this method.

- 9.1: Starting from the Brune impedance $Z(s) = \frac{1}{s+1}$, find the impedance network as a ladder network.

Ans:

- 9.2: Use a residue expansion in place of the CFA floor function (Sec. ??, p. ??) for polynomial expansions. Find the residue expansion of $H(s) = s^2/(s+1)$ and express it as a ladder network.

Ans:

- 9.3: Discuss how the series impedance $Z(s, x)$ and shunt admittance $Y(s, x)$ determine the wave velocity $\kappa(s, x)$ and the characteristic impedance $z_o(s, x)$ when (1) $Z(s)$ and $Y(s)$ are both independent of x ; (2) $Y(s)$ is independent of x , $Z(s, x)$ depends on x ; (3) $Z(s)$ is independent of x , $Y(s, x)$ depends on x ; and (4) both $Y(s, x)$, $Z(s, x)$ depend on x .

Ans:

3.2 Problems DE-2

3.2.1 Topics of this homework:

Integration of complex functions, Cauchy's theorem, integral formula, residue theorem, power series, Riemann sheets and branch cuts, inverse Laplace transforms, Quadratic forms.

Two fundamental theorems of calculus

Fundamental Theorem of Calculus (Leibniz):

According to the fundamental theorem of (real) calculus (FTC),

$$f(x) = f(a) + \int_a^x F(\xi) d\xi, \quad (\text{DE-2.1})$$

where $x, a, \xi, F, f \in \mathbb{R}$. This is an indefinite integral (since the upper limit is unspecified). It follows that

$$\frac{df(x)}{dx} = \frac{d}{dx} \int_a^x F(x) dx = F(x).$$

This justifies also calling the indefinite integral the *antiderivative*.

For a closed interval $[a, b]$, the FTC is

$$\int_a^b F(x) dx = f(b) - f(a), \quad (\text{DE-2.2})$$

thus the integral is independent of the path from $x = a$ to $x = b$.

Fundamental Theorem of Complex Calculus:

According to the fundamental theorem of complex calculus (FTCC),

$$f(z) = f(z_0) + \int_{z_0}^z F(\zeta) d\zeta, \quad (\text{DE-2.3})$$

where $z_0, z, \zeta, f, F \in \mathbb{C}$. It follows that

$$\frac{df(z)}{dz} = \frac{d}{dz} \int_{z_0}^z F(\zeta) d\zeta = F(z). \quad (\text{DE-2.4})$$

For a closed interval $[s, s_o]$, the FTCC is

$$\int_{s_o}^s F(\zeta) d\zeta = f(s) - f(s_o), \quad (\text{DE-2.5})$$

thus the integral is independent of the path from $x = a$ to $x = b$.

Problem # 1

– 1.1: (2 pts) Consider Equation DE-2.1. What is the condition on $F(x)$ for which this formula is true?

Ans:

– 1.2: (2 pts) Consider Equation DE-2.3. What is the condition on $F(z)$ for which this formula is true?

Ans:

$$- 1.3: \text{ Let } F(z) = \sum_{k=0}^{\infty} c_k z^k.$$

Ans:

– 1.4: Let

$$F(z) = \frac{\sum_{k=0}^{\infty} c_k z^k}{z - j}.$$

Ans:

Problem # 2: In the following problems, solve the integral

$$I = \int_C F(z) dz$$

for a given path $C \in \mathbb{C}$.

– 2.1: Perform the following integrals ($z = x + iy \in \mathbb{C}$):

$$I = \int_0^{1+j} z dz$$

Ans:

– 2.2: $I = \int_0^{1+j} z dz$, but this time make the path explicit: from 0 to 1, with $y = 0$, and then to $y = 1$, with $x = 1$.

Ans:

– 2.3: Discuss whether your results agree with Eq. DE-2.4?

Ans:

Problem # 3: Perform the following integrals on the closed path C , which we define to be the unit circle. You should substitute $z = e^{i\theta}$ and $dz = ie^{i\theta} d\theta$, and integrate from $\{-\pi, \pi\}$ to go once around the unit circle.

Discuss whether your results agree with Eq. DE-2.4?

– 3.1: (2 pts) $\int_C z dz$

Ans:

– 3.2: (2 pts) $\int_C \frac{1}{z} dz$

Ans:

– 3.3: (2 pts) $\int_C \frac{1}{z^2} dz$

Ans:

– 3.4: $I = \int_C \frac{1}{(z+2j)^2} dz.$

Recall that the path of integration is the unit circle, starting and ending at -1.

Ans:

Problem # 4: FTCC and integration in the complex plane

Let the function $F(z) = c^z$, where $c \in \mathbb{C}$ is given for each question. *Hint: Can you apply the FTCC?*

– 4.1: For the function $f(z) = c^z$, where $c \in \mathbb{C}$ is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that $f(z)$ is analytic for all $z \in \mathbb{C}$.

Ans:

– 4.2: Find the antiderivative of $F(z)$.

Ans:

– 4.3: $c = 1/e = 1/2.7183, \dots$ where C is $\zeta = 0 \rightarrow i \rightarrow z$

Ans:

– 4.4: $c = 2$, where C is $\zeta = 0 \rightarrow (1+i) \rightarrow z$

Ans:

– 4.5: $c = i$, where the path C is an inward spiral described by $z(t) = 0.99^t e^{i2\pi t}$ for $t = 0 \rightarrow t_0 \rightarrow \infty$

Ans:

– 4.6: $c = e^{t-\tau_0}$, where $\tau_0 > 0$ is a real number and C is $z = (1-i\infty) \rightarrow (1+i\infty)$. *Hint: Do you recognize this integral? If you do not, please do not spend a lot of time trying to solve it via the “brute force” method.*

Ans:

3.2.2 Cauchy’s theorems CT-1, CT-2, CT-3

There are three basic definitions related to Cauchy’s integral formula. They are all related and can greatly simplify integration in the complex plane. When a function depends on a complex variable, we use uppercase notation, consistent with the engineering literature for the Laplace transform.

Problem # 5: Describe the relationships between the theorems:

– 5.1: CT-1 and CT-2

Ans:

– 5.2: CT-1 and CT-3

Ans:

– 5.3: CT-2 and CT-3

Ans:

– 5.4: Consider the function with poles at $z = \pm j$,

$$F(z) = \frac{1}{1+z^2} = \frac{1}{(z-j)(z+j)}.$$

Find the residue expansion.

Ans:

Problem # 6: Apply Cauchy's theorems to solve the following integrals. State which theorem(s) you used and show your work.

– 6.1: $\oint_C F(z)dz$, where C is a circle centered at $z = 0$ with a radius of $\frac{1}{2}$

Ans:

– 6.2: $\oint_C F(z)dz$, where C is a circle centered at $z = j$ with a radius of 1

Ans:

– 6.3: $\oint_C F(z)dz$, where C is a circle centered at $z = 0$ with a radius of 2

Ans:

Integration of analytic functions

Problem # 7: In the following questions, you'll be asked to integrate $F(s) = u(\sigma, \omega) + iv(\sigma, \omega)$ around the contour C for complex $s = \sigma + i\omega$,

$$\oint_C F(s)ds. \quad (\text{DE-2.6})$$

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations

– 7.1: $F(s) = \sin(s)$

Ans:

– 7.2: Given function $F(s) = \frac{1}{s}$ State where the function is and is not analytic.

Ans:

– 7.3: Explicitly evaluate the integral when C is the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

Ans:

– 7.4: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.

Ans:

– 7.5: $F(s) = \frac{1}{s^2}$ State where the function is and is not analytic.

Ans:

– 7.6: Explicitly evaluate the integral when C is the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

Ans:

– 7.7: What does your result imply about the residue of the second-order pole at $s = 0$?

Ans:

– 7.8: $F(s) = e^{st}$: State where the function is and is not analytic.

Ans:

– 7.9: Explicitly evaluate the integral when C is the square $(\sigma, \omega) = (1, 1) \rightarrow (-1, 1) \rightarrow (-1, -1) \rightarrow (1, -1) \rightarrow (1, 1)$.

Ans:

– 7.10: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.

Ans:

– 7.11: $F(s) = \frac{1}{s+2}$: State where the function is and is not analytic.

Ans:

– 7.12: Let C be the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Ans:

– 7.13: Let C be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Ans:

– 7.14: $F(s) = \frac{1}{2\pi i} \frac{e^{st}}{(s+4)}$ State where the function is and is not analytic.

Ans:

– 7.15: Let C be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Ans:

– 7.16: Let C contain the entire left half s plane. Evaluate the integral using Cauchy's theorem and/or the residue theorem. Do you recognize this integral?

Ans:

– 7.17: $F(s) = \pm \frac{1}{\sqrt{s}}$ (e.g., $F^2 = \frac{1}{s}$) State where the function is and is not analytic.

Ans:

– 7.18: This function is multivalued. How many Riemann sheets do you need in the domain (s) and the range (f) to fully represent this function? Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

Ans:

– 7.19: Explicitly evaluate the integral $\int_C \frac{1}{\sqrt{z}} dz$ when C is the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Is this contour closed? State why or why not.

Ans:

– 7.20: Explicitly evaluate the integral $\int_C \frac{1}{\sqrt{z}} dz$ when C is twice around the unit circle, defined as $s = e^{i\theta}$, $0 \leq \theta \leq 4\pi$. Is this contour closed? State why or why not. Hint: Note that $\sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi} e^{i\theta}} = e^{i\pi} \sqrt{e^{i\theta}} = -1 \sqrt{e^{i\theta}}$.

Ans:

– 7.21: What does your result imply about the residue of the (twice-around $\frac{1}{2}$ order) pole at $s = 0$?

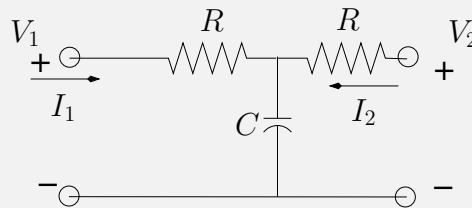
Ans:

– 7.22: Show that the residue is zero. Hint: Apply the definition of the residue.

Ans:

3.2.3 Laplace transform applications

Problem # 8: A two-port network application for the Laplace transform



This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$. It is convenient to define the dimensionless ratio $s/s_c = RCs$ in terms of a time constant $\tau = RC$ and cutoff frequency $s_c = 1/\tau$.

3.2.4 Computer exercises with Matlab/Octave

Problem # 9: With the help of a computer

Now we look at a few important concepts using Matlab/Octave's `syms` commands or Wolfram Alpha's symbolic math toolbox.¹

For example, to find the Taylor series expansion about $s = 0$ of

$$F(s) = -\log(1 - s),$$

we first consider the derivative and its Taylor series (about $s = 0$)

$$F'(s) = \frac{1}{1-s} = \sum_{n=0}^{\infty} s^n.$$

Then, we integrate this series term by term:

$$F(s) = -\log(1 - s) = \int^s F'(s) ds = \sum_{n=0}^{\infty} \frac{s^{n+1}}{n+1}.$$

Alternatively we can use Matlab/Octave commands:

¹<https://www.wolframalpha.com/>

```
syms s
taylor(-log(1-s), 'order', 7)
```

– 9.1: Use Octave's `taylor(-log(1-s))` to the seventh order; as in the example above. Try the above Matlab/Octave commands. Give the first seven terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above).

[Ans:](#)

– 9.2: What is the inverse Laplace transform of this series? Consider the series term by term.

[Ans:](#)

– 9.3: The function $1/\sqrt{z}$ has a branch point at $z = 0$; thus it is singular there. Can you apply Cauchy's integral theorem when integrating around the unit circle?

[Ans:](#)

– 9.4: This Matlab/Octave code computes $\int_0^{4\pi} \frac{dz}{\sqrt{z}}$ using Matlab's/Octave's symbolic analysis package. Run the following script:

```
syms z
I=int(1/sqrt(z))
J = int(1/sqrt(z), exp(-j*pi), exp(j*pi))
eval(J)
```

What answers do you get for I and J ?

[Ans:](#)

– 9.5: Modify this code to integrate $f(z) = 1/z^2$ once around the unit circle. What answers do you get for I and J ?

[Ans:](#)

– 9.6: Bessel functions can describe waves in a cylindrical geometry. The Bessel function has a Laplace transform with a branch cut

$$J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}.$$

Draw a hand sketch showing the nature of the branch cut. Hint: Use `zviz`. [Ans:](#)

Problem # 10: Matlab/Octave exercises

– 10.1: Try the following Matlab/Octave commands, and then comment on your findings.

```
%Take the inverse LT of 1/sqrt(1+s^2)
syms s
I=ilaplace(1/(sqrt((1+s^2))));
disp(I)
```

[Ans:](#)

```
%Find the Taylor series of the LT
T = taylor(1/sqrt(1+s^2),10); disp(T);
```

[Ans:](#)

```
%Verify this
syms t
J=laplace(besselj(0,t));
disp(J);
```

[Ans:](#)

```
%plot the Bessel function
t=0:0.1:10*pi;
b=besselj(0,t);
plot(t/pi,b);
grid on;
```

[Ans:](#)

– 10.2: When did Friedrich Bessel live?

[Ans:](#)

– 10.3: What did he use Bessel functions for?

[Ans:](#)

Problem # 11: Colorized plots of analytic functions. Use `zviz` for each of the following.

– 11.1: Describe the plot generated by `zviz S=Z`.

[Ans:](#)

– 11.2: Describe the plot generated by `zviz 1./sqrt(1+S.^2)`.

[Ans:](#)

– 11.3: Describe the plot generated by `zviz 1./sqrt(1-S.^2)`. Is this function a Brune impedance (i.e., does this function obey

[Ans:](#)

– 11.4: `zviz 1./(1+sqrt(S))`

[Ans:](#)

3.2.5 Inverse of Riemann $\zeta(s)$ function**Problem # 12: Inverse zeta function (This problem is for extra credit).**

– 12.1: Find the $\mathcal{L}T^{-1}$ of one factor of the Riemann zeta function $\zeta_p(s)$, where $\zeta_p(s) \leftrightarrow z_p(t)$. Describe your results in words. Hint: Consider the geometric series representation

$$\zeta_p(s) = \frac{1}{1 - e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p}, \quad (\text{DE-2.7})$$

for which you can look up the $\mathcal{L}T^{-1}$ of each term.

[Ans:](#)

Problem # 13: Inverse transform of products:

The time-domain version of Eq. DE-2.7 may be written as the convolution of all the $z_k(t)$ factors:

$$z(t) \equiv z_2(t) \star z_3(t) \star z_5(t) \star z_7(t) \star \cdots \star z_p(t) \star \cdots, \quad (\text{DE-2.8})$$

where \star represents time convolution.

Explain what this means in physical terms. Start with two terms (e.g., $z_1(t) \star z_2$). Hint: The input admittance of this cascade may be interpreted as the analytic continuation of $\zeta(s)$ by defining a cascade of eigenfunctions with eigenvalues derived from the primes. For a discussion of this idea see Sec. ?? and ??.

Ans:

Physical interpretation: Such functions may be generated in the time domain, as shown in Fig. ??, using a feedback delay of T_p seconds, described by the two equations in the Fig. ?? with a unity feedback gain $\alpha = -1$. Taking the Laplace transform of the system equation, we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $\zeta_p(s)$, which is an all-pole function, since

$$Q(s) = e^{-sT_p} Q(s) + V(s), \text{ or } \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}. \quad (\text{DE-2.9})$$

Closing the feed-forward path gives a second transfer function $Y(s) = I(s)/V(s)$ —namely,

$$Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}. \quad (\text{DE-2.10})$$

If we take $i(t)$ as the current and $v(t)$ as the voltage at the input to the transmission line, then $y_p(t) \leftrightarrow \zeta_p(s)$ represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the $j\omega$ axis. By a slight modification, $\zeta_p(s)$ may alternatively be written as

$$Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j \tan(sT_p/2). \quad (\text{DE-2.11})$$

Every impedance $Z(s)$ has a corresponding *reflectance* function given by a Möbius transformation, which may be read off of Eq. DE-2.11 as

$$\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}, \quad (\text{DE-2.12})$$

since impedance is also related to the round-trip delay T_p on the line. The inverse Laplace transform of $\Gamma(s)$ is the round-trip delay T_p on the line

$$\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}. \quad (\text{DE-2.13})$$

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations that are not obvious in the frequency domain (See Sec. ??). Transforming to the time domain is a form of analytic continuation of $\zeta(s)$ that depends on the assumption that $Z^{eta}(t) \leftrightarrow \zeta(s)$ is one-sided in time (causal).

3.2.6 Quadratic forms

A matrix that has positive eigenvalues is said to be positive-definite. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy, since the power is the voltage times the current. Given an impedance matrix

$$\mathbf{V} = \mathbf{Z}\mathbf{I},$$

the power \mathcal{P} is

$$\mathcal{P} = \mathbf{I} \cdot \mathbf{V} = \mathbf{I} \cdot \mathbf{Z}\mathbf{I},$$

which must be positive-definite for the system to obey conservation of energy.

Problem # 14: In this problem, consider the 2×2 impedance matrix

$$\mathbf{Z} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

– 14.1: Solve for the power $\mathcal{P}(i_1, i_2)$ by multiplying out this matrix equation (which is a quadratic form):

$$\mathcal{P}(i_1, i_2) = \mathbf{I}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{I}.$$

Ans:

– 14.2: Is the impedance matrix positive-definite? Show your work by finding the eigenvalues of the matrix \mathbf{Z} .

Ans:

– 14.3: Should an impedance matrix always be positive-definite? Explain.

Ans:

3.3 Problems DE-3

3.3.1 Topics of this homework: Brune impedance

lattice transmission line analysis

3.3.2 Brune Impedance

Problem # 1: Residue form

A Brune impedance is defined as the ratio of the force $F(s)$ to the flow $V(s)$ and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^K \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)} \quad (\text{DE-3.1})$$

with

$$D(s) = \prod_{k=1}^K (s - s_k) \quad \text{and} \quad c_k = \lim_{s \rightarrow s_k} (s - s_k) D(s) = \prod_{n'=1}^{K-1} (s - s_{n'}).$$

The prime on the index n' means that $n = k$ is not included in the product.

– 1.1: Find the Laplace transform (\mathcal{LT}) of a (1) spring, (2) dashpot, and (3) mass.

Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: (1) Hooke's law $f(t) = Kx(t)$, (2) dashpot resistance $f(t) = Rv(t)$, and (3) Newton's law for mass $f(t) = Mdv(t)/dt$. [Ans:](#)

– 1.2: Take the Laplace transform (\mathcal{LT}) of Eq. DE-3.2 and find the total impedance $Z(s)$ of the mechanical circuit.

$$M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + Kx(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s). \quad (\text{DE-3.2})$$

[Ans:](#)

– 1.3: What are $N(s)$ and $D(s)$ (see Eq. DE-3.1)?

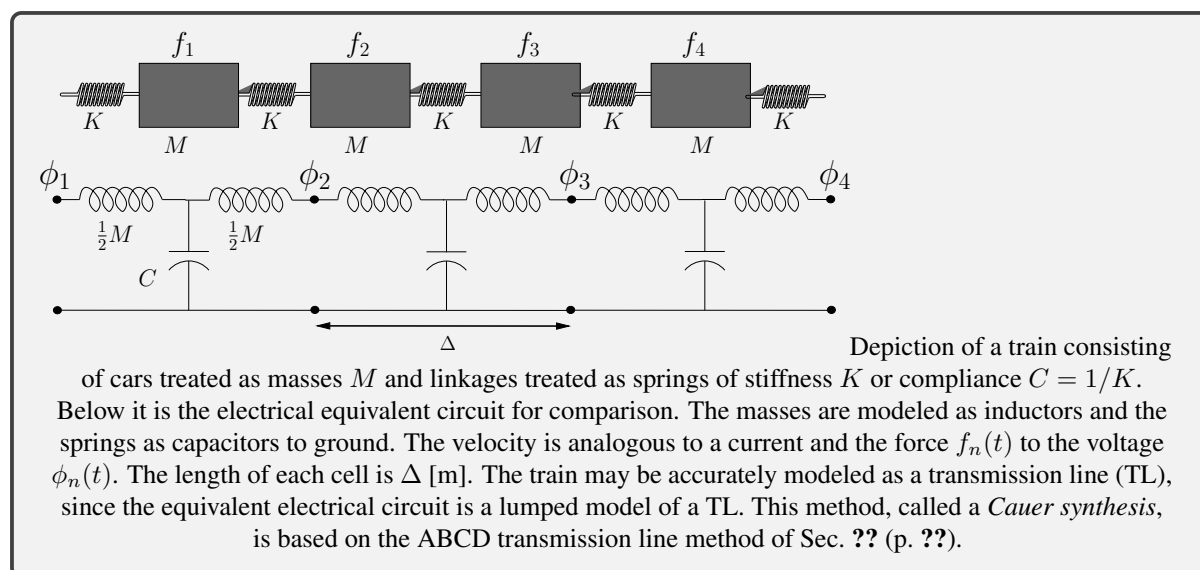
[Ans:](#)

– 1.4: Assume that $M = R = K = 1$ and find the residue form of the admittance $Y(s) = 1/Z(s)$ (see Eq. DE-3.1) in terms of the roots s_{\pm} . Hint: Check your answer with Octave's/Matlab's `residue` command.

[Ans:](#)

– 1.5: By applying Eq. ?? (page ??), find the inverse Laplace transform (\mathcal{LT}^{-1}). Use the residue form of the expression that you derived in question 1.4.

[Ans:](#)



3.3.3 Transmission-line analysis

Problem # 2: (14 pts) **Train-mission-line** We wish to model the dynamics of a freight train that has N such cars and study the velocity transfer function under various load conditions.

As shown in Fig. 3.3.2, the train model consists of masses connected by springs.

Problem # 3: Transfer functions

Use the ABCD method (see the discussion in Appendix ??, p. ??) to find the matrix representation of the system of Fig. 3.3.2. Define the force on the n th train car $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$. Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($\mathcal{A} = \mathcal{D}$) identical cell matrices $\mathcal{T}(s)$.

– 3.1: Find the elements of the ABCD matrix \mathcal{T} for the single cell that relate the input node 1 to output node 2

$$\begin{bmatrix} F \\ V \end{bmatrix}_1 = \mathcal{T} \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2. \quad (\text{DE-3.3})$$

Ans:

– 3.2: Express each element of $\mathcal{T}(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi j f$, $s_c = 2\pi j f_c$). The Nyquist wavelength sampling condition is $\lambda_c > 2\Delta$. It says the critical wavelength $\lambda_c > 2\Delta$.^a It says the critical wavelength $\lambda_c > 2\Delta$. Namely it is defined in terms the minimum number of cells 2Δ , per minimum wavelength λ_c .

The Nyquist wavelength sampling theorem says that there are at least two cars per wavelength.

Proof: From the figure, the distance between cars $\Delta = c_o T_o$ [m], where

$$c_o = \frac{1}{\sqrt{MC}} \quad [\text{m/s}].$$

The cutoff frequency obeys $f_c \lambda_c = c_o$. The Nyquist critical wavelength is $\lambda_c = c_o / f_c > 2\Delta$. Therefore the Nyquist sampling condition is

$$f < f_c \equiv \frac{c_o}{\lambda_c} = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}} \quad [\text{rad/sec}]. \quad (\text{DE-3.4})$$

Finally, $s_c = j2\pi f_c$.

Ans:

^aThe history of this relation has been traced back to 1841, as discussed by (? , Chap. I,II, Eq. 4.7).

– 3.3: Use the property of the Nyquist sampling frequency $\omega < \omega_c$ (Eq. DE-3.4) to remove higher order powers of frequency

$$1 + \left(\frac{s}{s_c} \right)^2 \approx 1 \quad (\text{DE-3.5})$$

to determine a band-limited approximation of $\mathcal{T}(s)$.

Ans:

Problem # 4: (4 pts) Now consider the cascade of N such $\mathcal{T}(s)$ matrices and perform an eigenanalysis.

– 4.1: (4 pts) Find the eigenvalues and eigenvectors of $\mathcal{T}(s)$ as functions of s/s_c .

Ans:

Problem # 5: (14 pts) Find the velocity transferfunction $H_{12}(s) = V_2/V_1|_{F_2=0}$.

– 5.1: (3 pts) Assuming that $N = 2$ and $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the \mathcal{T} matrix, find

$$H_{21}(s) = \left. \frac{V_2}{V_1} \right|_{F_2=0}$$

Express H_{12} in terms of a residue expansion.

Ans:

– 5.2: (2 pts) Find $h_{21}(t) \leftrightarrow H_{21}(s)$.

Ans:

– 5.3: (2 pts) What is the input impedance $Z_2 = F_2/V_2$, assuming $F_3 = -r_0V_3$?

[Ans:](#)

– 5.4: (5 pts) Simplify the expression for Z_2 as follows:

1. Assuming the characteristic impedance $r_0 = \sqrt{M/C}$,
2. terminate the system in r_0 : $F_2 = -r_0V_2$ (i.e., $-V_2$ cancels).
3. Assume higher-order frequency terms are less than 1 ($|s/s_c| < 1$).
4. Let the number of cells $N \rightarrow \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteristic impedance r_0 , the input impedance $Z_1(s) = r_0$. Thus, when we simplify the expression for $\mathcal{T}(s)$, it should be equal to r_0 . Show that this is true for this setup.

[Ans:](#)

– 5.5: (1 pts) State the ABCD matrix relationship between the first and N th nodes in terms of the cell matrix. Write out the transfer function for one cell, H_{21} .

[Ans:](#)

– 5.6: (1 pts) What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?

[Ans:](#)

Chapter 4

Vector differential equations

4.1 Problems VC-1

4.1.1 Topics of this homework:

Vector algebra and fields in \mathbb{R}^3 , gradient and scalar Laplacian operators, definitions of divergence and curl, Gauss's (divergence) and Stokes's (curl) laws, system classification (postulates).

4.1.2 Scalar fields and the ∇ operator

Problem # 1: Let $T(x, y) = x^2 + y$ be an analytic scalar temperature field in two dimensions (single-valued $\in \mathbb{R}^2$).

– 1.1: Find the gradient of $T(\mathbf{x})$ and make a sketch of T and the gradient.

Ans:

– 1.2: Compute $\nabla^2 T(\mathbf{x})$ to determine whether $T(\mathbf{x})$ satisfies Laplace's equation.

Ans:

– 1.3: Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.

Ans:

– 1.4: The heat flux¹ is defined as $\mathbf{J}(x, y) = -\kappa(x, y)\nabla T$, where $\kappa(x, y)$ is a constant that denotes thermal conductivity at the point (x, y) . Given that $\kappa = 1$ everywhere (the medium is homogeneous), plot the vector $\mathbf{J}(x, y) = -\nabla T$ at $x = 2, y = 1$. Be clear about the origin, direction, and length of your result.

Ans:

– 1.5: Find the vector \perp to $\nabla T(x, y)$ —that is, tangent to the iso-temperature contours. Hint: Sketch it for one (x, y) point (e.g., $2, 1$) and then generalize.

Ans:

¹The heat flux is proportional to the change in temperature times the thermal conductivity κ of the medium.

– 1.6: The thermal resistance R_T is defined as the potential drop ΔT over the magnitude of the heat flux $|\mathbf{J}|$. At a single point the thermal resistance is

$$R_T(x, y) = -\nabla T/|\mathbf{J}|.$$

How is $R_T(x, y)$ related to the thermal conductivity $\kappa(x, y)$?

Ans:

Problem # 2: Acoustic wave equation

Note: In this problem, we will work in the frequency domain.

– 2.1: The basic equations of acoustics in one dimension are

$$-\frac{\partial}{\partial x} \mathcal{P} = \rho_o s \mathcal{V} \quad \text{and} \quad -\frac{\partial}{\partial x} \mathcal{V} = \frac{s}{\eta_o P_o} \mathcal{P}.$$

Here $\mathcal{P}(x, \omega)$ is the pressure (in the frequency domain), $\mathcal{V}(x, \omega)$ is the volume velocity (the integral of the velocity over the wavefront with area A), $s = \sigma + \omega j$, $\rho_o = 1.2$ is the specific density of air, $\eta_o = 1.4$, and P_o is the atmospheric pressure (i.e., 10^5 Pa). Note that the pressure field \mathcal{P} is a scalar (pressure does not have direction), while the volume velocity field \mathcal{V} is a vector (velocity has direction).

We can generalize these equations to three dimensions using the ∇ operator

$$-\nabla \mathcal{P} = \rho_o s \mathcal{V} \quad \text{and} \quad -\nabla \cdot \mathcal{V} = \frac{s}{\eta_o P_o} \mathcal{P}.$$

– 2.2: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure \mathcal{P} ,

$$\nabla^2 \mathcal{P} = \frac{s^2}{c_0^2} \mathcal{P},$$

where c_0 is a constant representing the speed of sound.

Ans:

– 2.3: What is c_0 in terms of η_o , ρ_o , and P_o ?

Ans:

– 2.4: Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform [e.g., $dx/dt \leftrightarrow sX(s)$]. For your notation, define the time-domain signal using a lowercase letter, $p(x, y, z, t) \leftrightarrow \mathcal{P}$.

Ans:

4.1.3 Vector fields and the ∇ operator

4.1.4 Vector algebra

Problem # 3: Let $\mathbf{R}(x, y, z) \equiv x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$.

– 3.1: If a , b , and c are constants, what is $\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c)$?

Ans:

– 3.2: If a , b , and c are constants, what is $\frac{d}{dt} (\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c))$?

Ans:

Problem # 4: Find the divergence and curl of the following vector fields:

– 4.1: $\mathbf{v} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}}$

Ans:

– 4.2: $\mathbf{v}(x, y, z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + z^2\hat{\mathbf{z}}$

Ans:

– 4.3: $\mathbf{v}(x, y, z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + \log(z)\hat{\mathbf{z}}$

Ans:

– 4.4: $\mathbf{v}(x, y, z) = \nabla(1/x + 1/y + 1/z)$

Ans:

4.1.5 Vector and scalar field identities

Problem # 5: Find the divergence and curl of the following vector fields:

– 5.1: $\mathbf{v} = \nabla\phi$, where $\phi(x, y) = xe^y$

Ans:

– 5.2: $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Ans:

– 5.3: $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = y\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Ans:

– 5.4: For any differentiable vector field \mathbf{V} , write two vector calculus identities that are equal to zero.

Ans:

– 5.5: What is the most general form a vector field may be expressed in, in terms of scalar Φ and vector \mathbf{A} potentials?

Ans:

Problem # 6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.

– 6.1: Let $\mathbf{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Find $\nabla \cdot (\nabla \times \mathbf{v})$.

Ans:

– 6.2: Let $\mathbf{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Find $\nabla \times (\nabla \sqrt{\mathbf{v} \cdot \mathbf{v}})$

Ans:

– 6.3: Let $\mathbf{v}(x, y, z) = \nabla(x + y^2 + \sin(\log(z)))$. Find $\nabla \times \mathbf{v}(x, y, z)$.

Ans:

4.1.6 Integral theorems

Problem # 7: For each of the following problems, in a few words, identify either Gauss's or Stokes's law, define what it means, and explain the formula that follows the question.

– 7.1: What is the name of this formula?

$$\int_S \hat{\mathbf{n}} \cdot \mathbf{v} \, dA = \int_V \nabla \cdot \mathbf{v} \, dV.$$

Ans:

– 7.2: What is the name of this formula?

$$\int_S (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = \oint_C \mathbf{V} \cdot d\mathbf{R}$$

Give one important application. **Ans:**

– 7.3: Describe a key application of the vector identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}.$$

Ans:

4.2 Problems VC-2

4.2.1 Topics of this homework:

Partial differential equations; fundamental theorem of vector calculus (Helmholtz's theorem); wave equation; Maxwell's equations (ME) and variables (\mathbf{E} , \mathbf{D} ; \mathbf{B} , \mathbf{H}); Second-order vector differentials; Webster horn equation.

Notation: The following notation is used in this homework:

1. $s = \sigma + j\omega$ is the Laplace frequency, as used in the Laplace transform.
2. A Laplace transform pair is indicated by the symbol \leftrightarrow : for example, $f(t) \leftrightarrow F(s)$.
3. π_k is the k th prime; for example, $\pi_k \in \mathbb{P}$, $\pi_k = [2, 3, 5, 7, 11, 13, \dots]$ for $k = 1, \dots, 6$.

4.2.2 Partial differential equations (PDEs): Wave equation

Problem # 1: Solve the wave equation in one dimension by defining $\xi = t \mp x/c$.

– 1.1: Show that d'Alembert's solution, $\varrho(x, t) = f(t - x/c) + g(t + x/c)$, is a solution to the acoustic pressure wave equation in one dimension:

$$\frac{\partial^2 \varrho(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \varrho(x, t)}{\partial t^2},$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions. [Ans:](#)

Problem # 2: Solving the wave equation in spherical coordinates (i.e., three dimensions)

– 2.1: Write the wave equation in spherical coordinates $\varrho(r, \theta, \phi, t)$. Consider only the radial term r (i.e., dependence on angles θ and ϕ is assumed to be zero). Hint: The form of the Laplacian as a function of the number of dimensions is given in Eq. ?? (page ??). Alternatively, look it up on the internet or in a calculus book.

[Ans:](#)

– 2.2: Show that this equation is true:

$$\nabla_r^2 \varrho(r) \equiv \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \varrho(r) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \varrho(r). \quad (\text{VC-2.1})$$

Hint: Expand both sides of the equation. [Ans:](#)

– 2.3: Use the results from Eq. VC-2.1 to show that the solution to the spherical wave equation is

$$\nabla_r^2 \varrho(r, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varrho(r, t) \quad (\text{VC-2.2})$$

$$\varrho(r, t) = \frac{f(t - r/c)}{r} + \frac{g(t + r/c)}{r}. \quad (\text{VC-2.3})$$

[Ans:](#)

– 2.4: Using $f(\xi) = \sin(\xi)u(\xi)$ and $g(\xi) = e^\xi u(\xi)$, write the solutions to the spherical wave equation, where $u(\xi)$ is the Heaviside step function.

Ans:

– 2.5: Sketch this $f(\xi)$ and $g(\xi)$ for several times (e.g., 0, 1, and 2 seconds), and describe the behavior of the pressure $\varrho(r, t)$ as a function of time t and radius r .

Ans:

– 2.6: What happens when the inbound wave reaches the center at $r = 0$?

Ans:

4.2.3 Helmholtz's formula

Every differentiable vector field may be written as the sum of a scalar potential ϕ and a vector potential \mathbf{w} . This relationship is best known as the fundamental theorem of vector calculus (also called Helmholtz's formula):

$$\mathbf{v} = -\nabla\phi + \nabla \times \mathbf{w}. \quad (\text{VC-2.4})$$

This formula seems to be a natural extension of the algebraic products $\mathbf{A} \cdot \mathbf{B} \perp \mathbf{A} \times \mathbf{B}$, since $\mathbf{A} \cdot \mathbf{B} \propto \|\mathbf{A}\| \|\mathbf{B}\| \cos(\theta)$ and $\mathbf{A} \times \mathbf{B} \propto \|\mathbf{A}\| \|\mathbf{B}\| \sin(\theta)$, as developed in Appendix ??, page ??. Thus these orthogonal components have magnitude 1 when we take the norm, due to Euler's identity ($\cos^2(\theta) + \sin^2(\theta) = 1$).

As shown in Table ?? (p. ??), Helmholtz's formula separates a vector field (i.e., $\mathbf{v}(\mathbf{x})$) into compressible and rotational parts:

1. The rotational (e.g., angular) part is defined by the vector potential \mathbf{w} , which requires that $\nabla \times \nabla \times \mathbf{w} \neq 0$. A field is irrotational (conservative) when $\nabla \times \mathbf{v} = 0$, meaning that the field \mathbf{v} can be generated using only a scalar potential, $\mathbf{v} = \nabla\phi$ (note that this is how a conservative field is usually defined, by saying there exists some ϕ such that $\mathbf{v} = \nabla\phi$).²
2. The compressible (e.g., radial) part of a field is defined by the scalar potential ϕ , which requires that $\nabla \cdot \nabla\phi = \nabla^2\phi \neq 0$. A field is incompressible (solenoidal) when $\nabla \cdot \mathbf{v} = 0$, meaning that the field \mathbf{v} can be generated using only a vector potential, $\mathbf{v} = \nabla \times \mathbf{w}$.

The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities: (1) $\nabla \cdot (\nabla \times \mathbf{w}) = 0$ and (2) $\nabla \times (\nabla\phi) = 0$.

Problem # 3: Define the following:

– 3.1: A conservative vector field

Ans:

– 3.2: An irrotational vector field

Ans:

– 3.3: An incompressible vector field

Ans:

– 3.4: A solenoidal vector field

Ans:

– 3.5: When is a conservative field irrotational?

Ans:

²A note about the relationship between the generating function and the test: You might imagine special cases where $\nabla \times \mathbf{w} \neq 0$ but $\nabla \times \nabla \times \mathbf{w} = 0$ (or $\nabla\phi \neq 0$ but $\nabla^2\phi = 0$). In these cases, the vector (or scalar) potential can be recast as a scalar (or vector) potential. For example, consider a field $\mathbf{v} = \nabla\phi_0 + \mathbf{b}$, where $\mathbf{b} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Note that \mathbf{b} can actually be generated by either a scalar potential ($\phi_1 = \frac{1}{2}[x^2 + y^2 + z^2]$, such that $\nabla\phi_1 = \mathbf{b}$) or a vector potential ($\mathbf{w}_0 = \frac{1}{2}[z^2\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}]$, such that $\nabla \times \mathbf{w}_0 = \mathbf{b}$). We find that $\nabla \times \mathbf{v} = 0$; therefore \mathbf{v} must be irrotational. We say this irrotational field is generated by $\nabla\phi = \nabla(\phi_0 + \phi_1)$.

– 3.6: When is an incompressible field irrotational?

Ans:

Problem # 4: For each of the following, (i) compute $\nabla \cdot \mathbf{v}$, (ii) compute $\nabla \times \mathbf{v}$, and (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.).

– 4.1: $\mathbf{v}(x, y, z) = -\nabla(3yx^3 + y \log(xy))$

Ans:

– 4.2: $\mathbf{v}(x, y, z) = xy\hat{\mathbf{x}} - z\hat{\mathbf{y}} + f(z)\hat{\mathbf{z}}$

Ans:

– 4.3: $\mathbf{v}(x, y, z) = \nabla \times (x\hat{\mathbf{x}} - z\hat{\mathbf{y}})$

Ans:

4.2.4 Maxwell's Equations

The variables have the following names and defining equations (see Table ??, p. ??):

Symbol	Equation	Name	Units
\mathbf{E}	$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$	Electric field strength	[volts/m]
$\mathbf{D} = \epsilon_o \mathbf{E}$	$\nabla \cdot \mathbf{D} = \rho$	Electric displacement (flux density)	[coul/m ²]
\mathbf{H}	$\nabla \times \mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}$	Magnetic field strength	[amps/m]
$\mathbf{B} = \mu_o \mathbf{H}$	$\nabla \cdot \mathbf{B} = 0$	Magnetic induction (flux density)	[webers/m ²]

Note that $\mathbf{J} = \sigma \mathbf{E}$ is the *current density* (which has units of [amps/m²]). Furthermore, the *speed of light in vacuo* is $c_o = 3 \times 10^8 = 1/\sqrt{\mu_o \epsilon_o}$ [m/s], and the *characteristic resistance of light* $r_o = 377 = \sqrt{\mu_o/\epsilon_o}$ [Ω (i.e., ohms)].

4.2.5 Speed of light

Problem # 5: The speed of light in vacuo is $c_o = 1/\sqrt{\mu_o \epsilon_o} \approx 3 \times 10^8$ [m/s]. The characteristic resistance in vacuo is $r_o = \sqrt{\mu_o/\epsilon_o} \approx 377$ [Ω].

– 5.1: Find a formula for the in-vacuo permittivity ϵ_o and permeability in terms of c_o and r_o . **Ans:** Based on your formula, what are the numeric values of ϵ_o and μ_o ?

Ans:

– 5.2: In a few words, identify the law given by this equation, define what it means, and explain the formula:

$$\int_S \hat{\mathbf{n}} \cdot \mathbf{v} dA = \int_V \nabla \cdot \mathbf{v} dV.$$

Ans:

4.2.6 Application of Maxwell's equations

Problem # 6: The electric Maxwell equation is $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$, where \mathbf{E} is the electric field strength and $\dot{\mathbf{B}}$ is the time rate of change of the magnetic induction field, or simply the magnetic flux density. Consider this equation integrated over a two-dimensional surface \hat{S} , where $\hat{\mathbf{n}}$ is a unit vector normal to the surface (you may also find it useful to define the closed path C around the surface):

$$\iint_S [\nabla \times \mathbf{E}] \cdot \hat{\mathbf{n}} dS = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}} dS.$$

– 6.1: Apply Stokes's theorem to the left-hand side of the equation.

Hint: view this relation in terms of the “integral forms” of the curl. [Ans:](#)

– 6.2: Consider the right-hand side of the equation. How is it related to the magnetic flux Ψ through the surface S ?

[Ans:](#)

– 6.3: Assume the right-hand side of the equation is zero. Can you relate your answer in question 6.1 to one of Kirchhoff's laws?

[Ans:](#)

Problem # 7: The magnetic Maxwell equation is $\nabla \times \mathbf{H} = \mathbf{C} \equiv \mathbf{J} + \dot{\mathbf{D}}$, where \mathbf{H} is the magnetic field strength, $\mathbf{J} = \sigma \mathbf{E}$ is the conductive (resistive) current density, and the displacement current $\dot{\mathbf{D}}$ is the time rate of change of the electric flux density \mathbf{D} . Here we defined a new variable \mathbf{C} as the total current density.

– 7.1: First consider the equation over a two-dimensional surface S :

$$\iint_S [\nabla \times \mathbf{H}] \cdot \hat{\mathbf{n}} dS = \iint_S [\mathbf{J} + \dot{\mathbf{D}}] \cdot \hat{\mathbf{n}} dS = \iint_S \mathbf{C} \cdot \hat{\mathbf{n}} dS.$$

Then apply Stokes's theorem to the left-hand side of this equation. In a sentence or two, explain the meaning of the resulting equation. Hint: What is the right-hand side of the equation? [Ans:](#)

Problem # 8: Consider the next equation in three dimensions. Take the divergence of both sides and integrate over a volume V (closed surface S):

$$\iiint_V \nabla \cdot [\nabla \times \mathbf{H}] dV = \iiint_V \nabla \cdot \mathbf{C} dV.$$

– 8.1: What happens to the left-hand side of this equation? Hint: Can you apply a vector identity? [Ans:](#) Apply the divergence theorem (sometimes known as Gauss's theorem) to the right-hand side of the equation, and interpret your result. Hint: Can you relate your result to one of Kirchhoff's laws?

[Ans:](#)

4.2.7 Second-order differentials

Problem # 9: This problem is about second-order vector differentials.

– 9.1: If $\mathbf{v}(x, y, z) = \nabla \phi(x, y, z)$, then what is $\nabla \cdot \mathbf{v}(x, y, z)$?

[Ans:](#)

– 9.2: Evaluate $\nabla^2 \phi$ and $\nabla \times \nabla \phi$ for $\phi(x, y) = xe^y$.

[Ans:](#)

– 9.3: Evaluate $\nabla \cdot (\nabla \times \mathbf{v})$ and $\nabla \times (\nabla \times \mathbf{v})$ for $\mathbf{v} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$.

[Ans:](#)

– 9.4: When $\mathbf{V}(x, y, z) = \nabla(1/x + 1/y + 1/z)$, what is $\nabla \times \mathbf{V}(x, y, z)$?

[Ans:](#)

– 9.5: When was Maxwell born and when did he die? How long did he live (within ± 10 years)?

[Ans:](#)

4.2.8 Capacitor analysis

Problem # 10: Find the solution to the Laplace equation between two infinite³ parallel plates separated by a distance d . Assume that the left plate at $x = 0$ is at voltage $V(0) = 0$ and the right plate at $x = d$ is at voltage $V_d \equiv V(d)$.

– 10.1: Write Laplace's equation in one dimension for $V(x)$.

Ans:

– 10.2: Write the general solution to your differential equation for $V(x)$.

Ans:

– 10.3: Apply the boundary conditions $V(0) = 0$ and $V(d) = V_d$ to determine the constants in your equation from question 10.2.

Ans:

– 10.4: Find the charge density per unit area ($\sigma = Q/A$, where Q is charge and A is area) on the surface of each plate. Hint: $\mathbf{E} = -\nabla V$, and Gauss's law states that $\iint_S \mathbf{D} \cdot \hat{\mathbf{n}} dS = Q_{\text{enc}}$.

Ans:

– 10.5: Determine the per-unit-area capacitance C of the system.

Ans:

4.2.9 Webster horn equation

Problem # 11: Horns illustrate an important generalization of the solution of the one dimensional wave equation in regions where the properties (i.e., area of the tube) vary along the axis of wave propagation. Classic applications of horns are in vocal tract acoustics, loudspeaker design, cochlear mechanics, and any case that has wave propagation. Write the formula for the Webster horn equation, and explain the variables.

Ans:

³We study plates that are infinite because this means the electric field lines are perpendicular to the plates, running directly from one plate to the other. However, we solve for per-unit-area characteristics of the capacitor.