3.3 Problems DE-3

3.3.1 Topics of this homework: Brune impedance

lattice transmission line analysis

3.3.2 Brune Impedance

Problem # *1: Residue form*

A Brune impedance is defined as the ratio of the force $F(s)$ to the flow $V(s)$ and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)} \tag{DE-3.1}$$

with

$$D(s) = \prod_{k=1}^{K}(s - s_k) \qquad \text{and} \qquad c_k = \lim_{s \to s_k}(s - s_k)D(s) = \prod_{n'=1}^{K-1}(s - s_n).$$

The prime on the index n' means that $n = k$ is not included in the product.

– *1.1: Find the Laplace transform ($\mathcal{LT}$) of a (1) spring, (2) dashpot, and (3) mass.* Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: (1) Hooke's law $f(t) = Kx(t)$, (2) dashpot resistance $f(t) = Rv(t)$, and (3) Newton's law for mass $f(t) = Mdv(t)/dt$. **Ans:**

– *1.2: Take the Laplace transform ($\mathcal{LT}$) of Eq. DE-3.2 and find the total impedance $Z(s)$ of the mechanical circuit.*

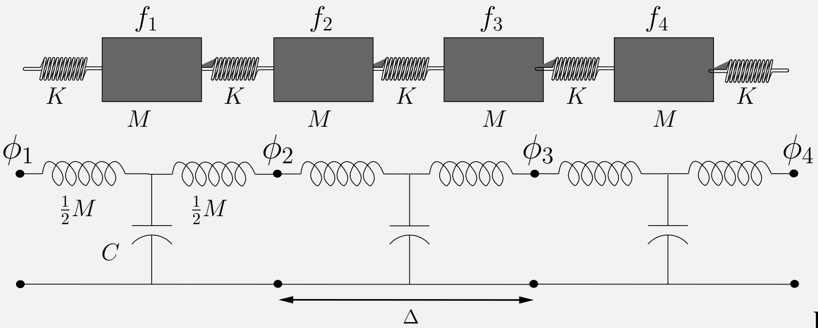
$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s). \tag{DE-3.2}$$

Ans:

– *1.3: What are $N(s)$ and $D(s)$ (see Eq. DE-3.1)?*
Ans:

– *1.4: Assume that $M = R = K = 1$ and find the residue form of the admittance $Y(s) = 1/Z(s)$ (see Eq. DE-3.1) in terms of the roots $s_\pm$. Hint: Check your answer with Octave's/Matlab's* `residue` *command.*
Ans:

– *1.5: By applying Eq.* **??** *(page* **??***), find the inverse Laplace transform ($\mathcal{LT}^{-1}$). Use the residue form of the expression that you derived in question 1.4.*
Ans:



Depiction of a train consisting of cars treated as masses M and linkages treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit for comparison. The masses are modeled as inductors and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is Δ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL. This method, called a *Cauer synthesis*, is based on the ABCD transmission line method of Sec. **??** (p. **??**).

3.3.3 Transmission-line analysis

Problem # *2:(14 pts)* **Train-mission-line** *We wish to model the dynamics of a freight train that has N such cars and study the* velocity transfer function *under various load conditions.*

As shown in Fig. 3.3.2, the train model consists of masses connected by springs.

Problem # *3: Transfer functions*

Use the ABCD method (see the discussion in Appendix **??**, p. **??**) to find the matrix representation of the system of Fig. 3.3.2. Define the force on the nth train car $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$. Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($\mathcal{A} = \mathcal{D}$) identical cell matrices $\mathcal{T}(s)$.

– 3.1: Find the elements of the ABCD matrix $\mathcal{T}$ for the single cell that relate the input node 1 to output node 2

$$\begin{bmatrix} F \\ V \end{bmatrix}_1 = \mathcal{T} \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2 . \tag{DE-3.3}$$

Ans:

– 3.2: Express each element of $\mathcal{T}(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi \jmath f$, $s_c = 2\pi \jmath f_c$). The Nyquist wavelength sampling condition is $\lambda_c > 2\Delta$. It says the critical wavelength $\lambda_c > 2\Delta$. condition is $\lambda_c > 2\Delta$.[a] It says the critical wavelength $\lambda_c > 2\Delta$. Namely it is defined in terms the minimum number of cells 2Δ, per minimum wavelength λ_c.

The Nyquist wavelength sampling theorem says that there are at least two cars per wavelength.
Proof: From the figure, the distance between cars $\Delta = c_o T_o$ [m], where

$$c_o = \frac{1}{\sqrt{MC}} \quad \text{[m/s]}.$$

The cutoff frequency obeys $f_c \lambda_c = c_o$. The Nyquist critical wavelength is $\lambda_c = c_o/f_c > 2\Delta$. Therefore the Nyquist sampling condition is

$$f < f_c \equiv \frac{c_o}{\lambda_c} = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}} \quad \text{[rad/sec]}. \tag{DE-3.4}$$

Finally, $s_c = \jmath 2\pi f_c$.
Ans:

[a] The history of this relation has been traced back to 1841, as discussed by (**?**, Chap. I,II, Eq. 4.7).

– 3.3: Use the property of the Nyquist sampling frequency *$\omega < \omega_c$ (Eq. DE-3.4) to remove higher order powers of frequency*

$$1 + \cancelto{0}{\left(\frac{s}{s_c}\right)^2} \approx 1 \tag{DE-3.5}$$

to determine a band-limited approximation of $\mathcal{T}(s)$.
Ans:

Problem # 4: *(4 pts) Now consider the cascade of N such $\mathcal{T}(s)$ matrices and perform an eigenanalysis.*

– 4.1: (4 pts) Find the eigenvalues and eigenvectors of $\mathcal{T}(s)$ as functions of s/s_c.
Ans:

Problem # 5: *(14 pts) Find the velocity transferfunction $H_{12}(s) = V_2/V_1|_{F_2=0}$.*

– 5.1: (3 pts) Assuming that $N = 2$ and $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the $\mathcal{T}$ matrix, find

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

Express H_{12} in terms of a residue expansion.
Ans:

– 5.2: (2 pts) Find $h_{21}(t) \leftrightarrow H_{21}(s)$.
Ans:

– 5.3: (2 pts) What is the input impedance $Z_2 = F_2/V_2$, assuming $F_3 = -r_0 V_3$?
Ans:

– 5.4: (5 pts) Simplify the expression for Z_2 as follows:

1. Assuming the *characteristic impedance* $r_0 = \sqrt{M/C}$,
2. terminate the system in r_0: $F_2 = -r_0 V_2$ (i.e., $-V_2$ cancels).
3. Assume higher-order frequency terms are less than 1 ($|s/s_c| < 1$).
4. Let the number of cells $N \to \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteristic impedance r_0, the input impedance $Z_1(s) = r_0$. Thus, when we simplify the expression for $\mathcal{I}(s)$, it should be equal to r_0. Show that this is true for this setup.
Ans:

– 5.5: (1 pts) State the ABCD matrix relationship between the first and Nth nodes in terms of the cell matrix. Write out the transfer function for one cell, H_{21}.
Ans:

– 5.6: (1 pts) What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?
Ans: