Chapter 4

Vector differential equations

4.1 Problems VC-1

4.1.1 Topics of this homework:

Vector algebra and fields in $\mathbb{R}^3$, gradient and scalar Laplacian operators, definitions of divergence and curl, Gauss's (divergence) and Stokes's (curl) laws, system classification (postulates).

4.1.2 Scalar fields and the ∇ operator

Problem # *1: Let $T(x, y) = x^2 + y$ be an analytic scalar temperature field in two dimensions (single-valued $\in \mathbb{R}^2$).*

– *1.1: Find the gradient of $T(\boldsymbol{x})$ and make a sketch of T and the gradient.*

Ans:

– *1.2: Compute $\nabla^2 T(\boldsymbol{x})$ to determine whether $T(\boldsymbol{x})$ satisfies Laplace's equation.*

Ans:

– *1.3: Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.*

Ans:

– *1.4: The* heat flux[1] *is defined as $\boldsymbol{J}(x, y) = -\kappa(x, y)\nabla T$, where $\kappa(x, y)$ is a constant that denotes thermal conductivity at the point (x, y). Given that $\kappa = 1$ everywhere (the medium is homogeneous), plot the vector $\boldsymbol{J}(x, y) = -\nabla T$ at $x = 2$, $y = 1$. Be clear about the origin, direction, and length of your result.*

Ans:

– *1.5: Find the vector $\perp$ to $\nabla T(x, y)$—that is, tangent to the iso-temperature contours. Hint: Sketch it for one (x, y) point (e.g., $2, 1$) and then generalize.*

Ans:

[1] The heat flux is proportional to the change in temperature times the thermal conductivity κ of the medium.

– 1.6: The thermal resistance R_T is defined as the potential drop ΔT over the magnitude of the heat flux $|\boldsymbol{J}|$. At a single point the thermal resistance is

$$R_T(x,y) = -\nabla T/|\boldsymbol{J}|.$$

How is $R_T(x,y)$ related to the thermal conductivity $\kappa(x,y)$?

Ans:

Problem # 2: *Acoustic wave equation*

Note: In this problem, we will work in the frequency domain.

– 2.1: The basic equations of acoustics in one dimension are

$$-\frac{\partial}{\partial x}\mathcal{P} = \rho_o s \mathcal{V} \quad \text{and} \quad -\frac{\partial}{\partial x}\mathcal{V} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

Here $\mathcal{P}(x,\omega)$ is the pressure (in the frequency domain), $\mathcal{V}(x,\omega)$ is the volume velocity (the integral of the velocity over the wavefront with area A), $s = \sigma + \omega\jmath$, $\rho_o = 1.2$ is the specific density of air, $\eta_o = 1.4$, and P_o is the atmospheric pressure (i.e., 10^5 Pa). Note that the pressure field $\mathcal{P}$ is a scalar (pressure does not have direction), while the volume velocity field $\mathcal{V}$ is a vector (velocity has direction).

We can generalize these equations to three dimensions using the ∇ operator

$$-\nabla\mathcal{P} = \rho_o s \mathcal{V} \quad \text{and} \quad -\nabla\cdot\mathcal{V} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

– 2.2: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure $\mathcal{P}$,

$$\nabla^2\mathcal{P} = \frac{s^2}{c_0^2}\mathcal{P},$$

where c_0 is a constant representing the speed of sound.

Ans:

– 2.3: What is c_0 in terms of η_0, ρ_0, and P_0?

Ans:

– 2.4: Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform [e.g., $dx/dt \leftrightarrow sX(s)$]. For your notation, define the time–domain signal using a lowercase letter, $p(x,y,z,t) \leftrightarrow \mathcal{P}$.

Ans:

4.1.3 Vector fields and the ∇ operator

4.1.4 Vector algebra

Problem # 3: *Let $\boldsymbol{R}(x,y,z) \equiv x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$.*

– 3.1: If a, b, and c are constants, what is $\boldsymbol{R}(x,y,z)\cdot\boldsymbol{R}(a,b,c)$?

Ans:

– 3.2: If a, b, and c are constants, what is $\frac{d}{dt}\left(\boldsymbol{R}(x,y,z)\cdot\boldsymbol{R}(a,b,c)\right)$?

Ans:

Problem # 4: *Find the divergence and curl of the following vector fields:*

– *4.1:* $\boldsymbol{v} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}}$
Ans:

– *4.2:* $\boldsymbol{v}(x, y, z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + z^2\hat{\mathbf{z}}$
Ans:

– *4.3:* $\boldsymbol{v}(x, y, z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + \log(z)\hat{\mathbf{z}}$
Ans:

– *4.4:* $\boldsymbol{v}(x, y, z) = \nabla(1/x + 1/y + 1/z)$
Ans:

4.1.5 Vector and scalar field identities

Problem # *5: Find the divergence and curl of the following vector fields:*

– *5.1:* $\boldsymbol{v} = \nabla\phi$, *where* $\phi(x, y) = xe^y$
Ans:

– *5.2:* $\boldsymbol{v} = \nabla \times \boldsymbol{A}$, *where* $\boldsymbol{A} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$
Ans:

– *5.3:* $\boldsymbol{v} = \nabla \times \boldsymbol{A}$, *where* $\boldsymbol{A} = y\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + z\hat{\mathbf{z}}$
Ans:

– *5.4: For any differentiable vector field* $\boldsymbol{V}$, *write two vector calculus identities that are equal to zero.*
Ans:

– *5.5: What is the most general form a vector field may be expressed in, in terms of scalar* Φ *and vector* $\boldsymbol{A}$ *potentials?*
Ans:

Problem # *6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.*

– *6.1: Let* $\boldsymbol{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. *Find* $\nabla \cdot (\nabla \times \boldsymbol{v})$.
Ans:

– *6.2: Let* $\boldsymbol{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. *Find* $\nabla \times (\nabla\sqrt{\boldsymbol{v} \cdot \boldsymbol{v}})$
Ans:

– *6.3: Let* $\boldsymbol{v}(x, y, z) = \nabla(x + y^2 + \sin(\log(z))$. *Find* $\nabla \times \boldsymbol{v}(x, y, z)$.
Ans:

4.1.6 Integral theorems

Problem # *7: For each of the following problems, in a few words, identify either Gauss's or Stokes's law, define what it means, and explain the formula that follows the question.*

– *7.1: What is the name of this formula?*

$$\int_{\mathcal{S}} \hat{\mathbf{n}} \cdot \boldsymbol{v} \, dA = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{v} \, dV.$$

Ans:

– *7.2: What is the name of this formula?*

$$\int_S (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{V} \cdot d\boldsymbol{R}$$

Give one important application. **Ans:**

– *7.3: Describe a key application of the vector identity*

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \boldsymbol{\nabla}^{\mathbf{2}}\mathbf{V}.$$

Ans: