1.2 Problems NS-2

Topic of this homework:

Prime numbers, greatest common divisors, the continued fraction algorithm

Continued fractions

Problem # *1: Here we explore the continued fraction algorithm (CFA)*
In its simplest form, the CFA starts with a real number, which we denote as $\alpha \in \mathbb{R}$. Let us work with an irrational real number, $\pi \in \mathbb{I}$, as an example because its CFA representation will be infinitely long. We can represent the CFA coefficients α as a vector of integers n_k, $k = 1, 2, \ldots, \infty$:

$$\begin{aligned}\alpha &= [n_1; n_2, n_3, n_4, \ldots] \\ &= n_1 + \cfrac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \cdots}}}.\end{aligned}$$

The CFA is recursive, with three repeated steps per iteration. For example $\alpha_1 = \pi \approx 3.14159\ldots$, $n_1 = 3$, $r_1 = \pi - 3$, and $\alpha_2 \equiv 1/r_1$.

$$\alpha_2 = 1/0.1416 = 7.0625\ldots$$
$$\alpha_1 = n_1 + \frac{1}{\alpha_2} = n_1 + \frac{1}{n_2 + \frac{1}{\alpha_3}} = \cdots.$$

In terms of a Matlab/Octave script,

```
alpha0 = pi;
K=10;
n=zeros(1,K); alpha=zeros(1,K);
alpha(1)=alpha0;

for k=2:K  %k=1 to K
n(k)=round(alpha(k-1));
%n(k)=fix(alpha(k-1));
alpha(k)= 1/(alpha(k-1)-n(k));
%disp([fix(k), round(n(k)), alpha(k)]); pause(1)
end
disp([n; alpha]);
%Now compare this to matlab's rat() function
rat(alpha0,1e-20)
```

– *1.1: By hand find the first few values of n_k for $\alpha = e^\pi \approx 23.1407$.*
Ans:

– *1.2: For the proceeding question, what is the error (remainder) when you truncate the continued fraction after $n_1, \ldots, n_3$? Give the absolute value of the error and the percentage error relative to the original α.*
Ans:

– *1.3: Use the Matlab/Octave program provided to find the first 10 values of n_k for $\alpha = e^{\pi}$, and verify your result using the Matlab/Octave command `rat()`.*
Ans:

– *1.4: Discuss the similarities and differences between the Euclidean algorithm and the CFA.*
Ans:

Greatest common divisors

Consider using the *Euclidean algorithm* to find the *greatest common divisor* (i.e., GCD; the largest common prime factor) of two numbers (Allen 2020, p. 42). This algorithm may be performed using one of two methods:

Method	Division	Subtraction
On each iteration...	$a_{i+1} = b_i$ $b_{i+1} = a_i - b_i \cdot \text{floor}(a_i/b_i)$	$a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$ $b_{i+1} = \min(a_i, b_i)$
Start with $i = 1$ and terminate when:	$b = 0$ (GCD $= a$)	$b = 0$ (GCD $= a$)

The division method (Matlab's floor function) (Eq. 2.1, Sec. 2.1.2, Ch. 2) is preferred because the subtraction method may require a huge number of itterations steps.

Problem # *2: Understanding the Euclidean algorithm (GCD)*

– *2.1: Find the prime factors of $a = 85$ and $b = 15$.*
Ans:

– *2.2: What is the greatest common prime factor of $a = 85$ and $b = 15$?*
Ans:

– *2.3: By hand, perform the Euclidean algorithm for $a = 85$ and $b = 15$.*
Ans:

– *2.4: By hand, perform the Euclidean algorithm for $a = 75$ and $b = 25$. Is the result a prime number?*
Ans:

– 2.5: Consider the first step of the GCD division algorithm when $a < b$ (e.g., $a = 25$ and $b = 75$). What happens to a and b in the first step? Does it matter if you begin the algorithm with $a < b$ rather than $b < a$?
Ans:

– 2.6: Describe in your own words how the GCD algorithm works. Try the algorithm using numbers that have already been divided into factors (e.g., $a = 5 \cdot 3$ and $b = 7 \cdot 3$).
Ans:

– 2.7: Find the GCD of $2 \cdot \pi_{25}$ and $3 \cdot \pi_{25}$.
Ans: .

Problem # *3: Coprimes*

– 3.1: Define the term coprime.
Ans:

– 3.2: How can the Euclidean algorithm be used to identify coprimes?
Ans:

– 3.3: Give an important application of the Euclidean algorithm.
Ans:

– 3.4: Write a Matlab function, `function x = my_gcd(a,b)`*, that uses the Euclidean algorithm to find the GCD of any two inputs* `a` *and* `b`*. Test your function on the (a, b) combinations from the previous problem. Include a printout (or hand-write) your algorithm to turn in.*
Hints and advice:

- Don't give your variables the same names as Matlab functions! Since `gcd` is an existing Matlab/Octave function, if you use it as a variable or function name, you won't be able to use `gcd` to check your `gcd()` function. Try `clear all` to recover from this problem.
- Try using a "while" loop for this exercise (see Matlab documentation for help).
- You may need to use some temporary variables for `a` and `b` in order to perform the algorithm.

Ans:

Algebraic generalization of the GCD (Euclidean) algorithm

Problem # 4: *In this problem we are looking for integer solutions $(m, n) \in \mathbb{Z}$ to the equations $ma + nb = gcd(a, b)$ and $ma + nb = 0$ given positive integers $(a, b) \in \mathbb{Z}^+$.*
Note that this requires that either m or n be negative. These solutions may be found using the Euclidean algorithm only if (a, b) are coprime ($a \perp b$). Note that integer (whole number) polynomial relations such as these are known as *Diophantine equations*. Such equations (e.g., $ma + nb = 0$) are linear Diophantine equations, possibly the simplest form of such relations.

Example: `gcd(2,3)= 1`: For $(a, b) = (2, 3)$, the result is

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & 1\\3 & -2\end{bmatrix}}_{m \quad n}\begin{bmatrix}2\\3\end{bmatrix}.$$

Thus from the above equation we find the solution (m, n) to the integer equation

$$2m + 3n = \texttt{gcd}(2, 3) = 1;$$

namely, $(m, n) = (-1, 1)$ (i.e., $-2+3 = 1$). There is also a second solution $(3, -2)$ (i.e., $3\cdot 2-2\cdot 3 = 0$) that represents the terminating condition. Thus these two solutions are a pair and the solution exists only if (a, b) are coprime ($a \perp b$).
Subtraction method: This method is more complicated than the division algorithm because at each stage we must check whether $a < b$. Define

$$\begin{bmatrix}a_0\\b_0\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}, \qquad Q = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}, \qquad S = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix},$$

where Q sets $a_{i+1} = a_i - b_i$ and $b_{i+1} = b_i$ assuming $a_i > b_i$, and S is a swap matrix that swaps a_i and b_i if $a_i < b_i$. Using these matrices, we implement the algorithm by assigning

$$\begin{bmatrix}a_{i+1}\\b_{i+1}\end{bmatrix} = Q\begin{bmatrix}a_i\\b_i\end{bmatrix} \text{ for } a_i > b_i, \qquad \begin{bmatrix}a_{i+1}\\b_{i+1}\end{bmatrix} = QS\begin{bmatrix}a_i\\b_i\end{bmatrix} \text{ for } a_i < b_i.$$

The result of this method is a cascade of Q and S matrices. For $(a, b) = (2, 3)$, the result is

$$\begin{bmatrix}1\\1\end{bmatrix} = \underbrace{\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}}_{S}\begin{bmatrix}2\\3\end{bmatrix} = \underbrace{\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}}_{m \quad n}\begin{bmatrix}2\\3\end{bmatrix}.$$

Thus we find two solutions (m, n) to the integer equation $2m + 3n = \text{gcd}(2, 3) = 1$.

– *4.1: By inspection, find at least one integer pair (m, n) that satisfies $12m+15n = 3$.*
Ans:

– *4.2: Using matrix methods for the Euclidean algorithm, find integer pairs (m, n) that satisfy $12m + 15n = 3$ and $12m + 15n = 0$. Show your work!!!*
Ans:

– *4.3: Does the equation $12m + 15n = 1$ have integer solutions for n and m? Why or why not?*
Ans:

Problem # *5: Matrix approach:*
It can be difficult to keep track of the a's and b's when the algorithm has many steps. We need an alternative way to run the Euclidean algorithm using matrix algebra. Matrix methods provide a more transparent approach to the operations on (a, b). Thus the Euclidean algorithm can be classified in terms of standard matrix operations. Write out the indirect matrix approach discussed at the end of Sec. **??** (Eq. **??**).
Ans:

Prime numbers

Problem # *6: Every integer may be written as a product of primes.*

– *6.1: Write the numbers* $1,000,000$*,* $1,000,004$*, and* $999,999$ *in the form* $N = \prod_k \pi_k^{\beta_k}$. Hint: Use Matlab/Octave to find the prime factors.
Ans:

– *6.2: Give a generalized formula for the natural logarithm of a number* $\ln(N)$ *in terms of its primes* π_k *and their multiplicities* β_k*. Express your answer as a* sum *of terms.*
Ans:

Problem # *7: Using the computer*

– *7.1: Explain why the following brief Matlab/Octave program returns the prime numbers* π_k *between 1 and 100.*

```
n=2:100;
k = isprime(n);
n(k)
```
Ans:

– *7.2: How many primes are there between* 2 *and* $N = 100$*?*
Ans:

Problem # *8: Prime numbers may be identified using a sieve.*

– *8.1: By hand, complete the sieve of Eratosthenes for* $n = 1, \ldots, 49$*. Start by writing out a table of the integers 1-50, as 5 rows of 10 numbers. Starting with the first prime,* $p_k = 2$*,* $k = 1$, circle it *and* cross out *all multiples (e.g.,* $2\pi_k = r, 3\pi_k = 6, \cdots, 24 * \pi_2 = 48.$*). Then repeat for the second, third, and higher primes* π_2*. When done, only the circled primes should remain. Be sure you look up the definition of a prime.*
Ans:

– *8.2: What is the largest number you need to consider before only primes remain? Look up the definition of the Matlab/Octave* floor *function (e.g,* $\lfloor \pi \rfloor = 3$*).*
Ans:

– *8.3: Generalize: For* $n = 1, \ldots, N$*, what is the largest number you need to consider before only the primes remain?*
Ans:

– *8.4: Write each of these numbers as a product of primes: 22, 30, 34, 43, 44, 48, 49.* **Ans:**

– *8.5: Find the largest prime $\pi_k \le 100$. Do not use Matlab/Octave other than to check your answer.* Hint: Write the numbers starting with 100 and count backward: 100, 99, 98, 97, ... Cross off the even numbers, leaving $99, 97, 95, \dots$. Pull out a factor (only one is necessary to show that it is not prime).
Ans:

– *8.6: Find the largest prime $\pi_k \le 1000$. Do not use Matlab/Octave other than to check your answer.*
Ans:

– *8.7: Explain why $\pi_k^{-s} = e^{-s \ln \pi_k}$.*
Ans:

Problem # *9:CFA of ratios of large primes*

– *9.1:* ***(4pts)*** *Expand 23/7 as a continued fraction. Express your answer in bracket notation* (e.g., $\pi = [3., 7, 16, \cdots]$). Show your work. **Ans:**

– 9.2:Starting from the primes below 10^6, form the CFA of π_j/π_k with $j = 78498$ and $k < j$.
Ans:

– 9.3: Look at other ratios of prime numbers and look for a pattern in the CFA of the ratios of large primes. What is the most obvious conclusion? **Ans:**

*– 9.4: **(1pts)** Try the Matlab/Octave functions `rats(23/7)`, `rats(3.2857)`, and `rats(3.2856)`. What an you conclude?*
Ans:

*– 9.5: **(2pts)** Can $\sqrt{2}$ be represented as a finite continued fraction? Why or why not?*
Ans:

*– 9.6: **(2pts)** What is the CFA for $\sqrt{2} - 1$?*

$$\text{Hint:} \qquad \sqrt{2} + 1 = \frac{1}{\sqrt{2} - 1} = [2; 2, 2, 2, \cdots].$$

Ans:

– 9.7: Show that

$$\frac{1}{1 - \sqrt{a}} = a^{\frac{11}{2}} + a^{\frac{9}{2}} + a^{\frac{7}{2}} + a^{\frac{5}{2}} + a^{\frac{3}{2}} + \sqrt{a} + a^5 + a^4 + a^3 + a^2 + a + 1 = 1 - a^6$$

```
syms a,b
b= taylor(1/( 1-sqrt(a) ))
simplify((1-sqrt(a))*b) = 1-a^6
```

Use symbolic analysis to show this, then explain. **Ans:**