Topic of this homework: Line, surface and volume integrals, 
Grad: $\nabla()$, Div: $\nabla \cdot ()$, Curl: $\nabla \times ()$, 
Laplacian, Helmholtz theorem, 
Green’s, Stokes and Divergence Thms

Deliverables: Show your work. Numerical results are not sufficient, expect when specifically requested.

There are two different methods of defining vectors: a) $\mathbf{Z} = [1, 2, 3]^T$, where as before, the script $T$ is transpose of the row-vector to a column, and b) $\mathbf{Z} = \mathbf{x} + 2\mathbf{y} + 3\mathbf{z}$. They represent exactly the same thing, and the rules of computation are identical. An arrow over the top, or the bold font, both indicate a column vector. You must adapt to these two equivalent representations.

Several of problems are adapted from (Greenberg, 1988; Allen, 2017).

1 Scalar fields and $\nabla$:

1. Let $T(x, y)$ be a scalar temperature field in 2 dimensions. Describe or sketch the iso-temperature contours at $T=10$, 20, 30 degrees.
   (a) $T = x^2 + y^2$
   (b) $T = 10(x - y)$

2. Compute the gradient of $T = x^2 + y^2$.

3. Given some scalar field $\phi(x, y, z)$, find the rate of change, in the direction of maximum change. Hint: Take the gradient.

4. Define and then describe the meaning of the directional derivative
   \[
   \frac{\partial T}{\partial \hat{n}} \hat{n} \tag{1}
   \]
   in the direction of unit vector $\hat{n}$.

2 Vector fields

Let $\mathbf{v}(x, y, z) = x\mathbf{x} - y\mathbf{y} \in \mathbb{R}^3$.

1. Describe (or sketch) $\mathbf{v}(x)$.

2. Find the curves of constant scalar speed $s(x)$ of $\mathbf{v}(x)$, defined as the norm of the velocity $s(x) = ||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. 
3 Divergence

1. Find the divergence of the following vector fields:
   
   (a) \( \mathbf{v}(x) = x\hat{x} + xy\hat{y} + \log(z)\hat{z} \)
   
   (b) \( \mathbf{v}(x) = x\hat{x} + xy\hat{y} + \log(z)\hat{z} \) evaluated at \( x = [1, 2, 3]^T \).
   
   (c) \( \mathbf{v}(x, t) = x\hat{x} + xy\hat{y} + \log(z)\hat{z} + t(\hat{x} + \hat{y}) \)

2. Gauss’ Law

Given a closed surface \( S \), the area is denoted as \( |S| \), and the volume enclosed as \( ||S|| \) (see the class notes, Lecture 39, Sect. 1.5.10).

(a) In words, explain:

\[
\int\int\int_{||S||} \nabla \cdot \mathbf{D} \, ||dS|| = \int\int_{S} \mathbf{D} \cdot \hat{n} \, d|S|.
\]  (1)

(b) In words, explain:

\[
\nabla \cdot \mathbf{D} = \lim_{|S| \to 0} \left\{ \frac{\int_{S} \mathbf{D} \cdot \hat{n} \, d|S|}{|S|} \right\} = \rho(x, y, z).
\]  (2)

(c) If \( \phi(x, y) \equiv x + y \), find the unit vector pointing perpendicular (\( \perp \)) to plane \( \phi = 1 \)

(d) Explain the relationship between \( \int_{a}^{b} F' \, dx = F(x)_{a}^{b} = F(b) - F(a) \) and the Divergence Thm.

3. Assuming some vector potential field \( \mathbf{w}(x) \), evaluate \( I = \int_{|S|} \hat{n} \cdot \nabla \times \mathbf{w}(x) \, d|S| \). Explain your reasoning. Hint: DoC/CoG.

4 Curl and Stokes Thm

The curl may be defined as the limit of the integral

\[
\nabla \times \mathbf{H} \equiv \lim_{|S| \to 0} \left\{ \frac{\int_{S} \hat{n} \times \mathbf{H} \, d|S|}{|S|} \right\}.
\]  (1)

Stokes’ theorem (law) says

\[
\int\int_{S} (\nabla \times \mathbf{H}) \cdot \hat{n} \, d|S| = \oint_{C} \mathbf{H} \cdot d\mathbf{l}.
\]  (2)

1. Expand the formula for the curl in rectangular coordinates, about the top row.

2. If \( \mathbf{v}(x, y, z) = \nabla (1/x + 1/y + 1/z) \), find \( \nabla \times \mathbf{v}(x, y, z) \).

3. One of Maxwell’s Equations, under steady state conditions, is \( \nabla \times \mathbf{H} = \mathbf{J} \), where \( \mathbf{H} \) [amps/m] is the magnetic field strength and \( \mathbf{J} \) [amps/m\(^2\)] is the current density. Apply Stokes’ Thm to this equation, and explain the result.
4. Determine when $\nabla \times \mathbf{V}(\mathbf{x})$ is independent of $\mathbf{V}(\mathbf{x})$ Hint: Consider the definition of the triple product $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$.

5. Prove CoG: $\nabla \times \nabla \phi(\mathbf{x}) = 0$

6. Prove DoC: $\nabla \cdot \nabla \times \mathbf{V} = 0$

5  Laplacian

1. If $\vec{v}(x, y, z) = \nabla(1/x + 1/y + 1/z)$, then what is $\nabla \cdot \vec{v}(x, y, z)$? What does this mean?

2. For $u = xe^y$ evaluate
   \[ \nabla^2 u = \nabla^2 xe^y \]
   and
   \[ \nabla \times \nabla u = \]

3. For $\vec{v} = x\hat{x} + y\hat{y} + z\hat{z}$, evaluate
   \[ \nabla \cdot (\nabla \times \vec{v}) = \]
   and
   \[ \nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v} = . \]

6  Compression-Rotation decomposition Thm.

Helmholtz decomposition theorem: Every vector field [e.g., $\vec{v}(x, y, z)$] may be defined as the sum of the gradient of a scalar potential $\Phi(x, y, z)$, and the curl of a vector potential $\vec{A}(x, y, z)$:

\[ \vec{v}(x, y, z) \equiv \nabla \Phi(x, y, z) + \nabla \times \vec{A}(x, y, z). \]

Given the two vector identities CoG $\nabla \times \nabla \Phi = 0$ and DoC $\nabla \cdot \nabla \times \vec{A} = 0$ what can you say about $\Phi$ and $\vec{A}$?

1. First take the (a) divergence and second (b) the curl of $\vec{v}(x, y, z)$. Give an example from Maxwell’s equations (or fluid flow) of the meaning of each result.

2. When is a field irrotational?

3. Classify $\mathbf{v}(x, y, z) = x\hat{x} - y\hat{y} \in \mathbb{R}^3$ as (in)compressible and (ir)rotational by computing $\nabla \cdot \mathbf{v}(\mathbf{x})$ and $\nabla \times \mathbf{v}(\mathbf{x})$ and finding its potential $\Phi(\mathbf{x})$ (i.e., $\mathbf{v}(\mathbf{x}) = \nabla \Phi$).

4. When is a field incompressible?

6.1  Field classification:

Provide your reasoning. Hint: look at Fig. 1.33 of Lec. 40, Sect. 1.5.11.

1. Classify the following fields as (in)compressible or (ir)rotational?
   (a) $\vec{v}(x, y, z) = \nabla(3x^3 + y \cos(xy))$
   (b) $\vec{v}(x, y, z) = xy\hat{x} - z\hat{y} + f(z)\hat{z}$

2. Define the purely rotational (i.e., incompressable) field, and provide your reasoning.
References
