Chapter 3

Differential equations

3.1 Problems DE-1

3.1.1 Topics of this homework:
Complex numbers and functions (ordering and algebra), complex power series, fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions, multivalued functions (branch cuts and Riemann sheets)

3.1.2 Complex Power Series

Problem # 1: In each case derive (e.g., using Taylor’s formula) the power series of \( w(s) \) about \( s = 0 \) and give the RoC of your series. If the power series doesn’t exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at \( s = 0 \).

- 1.1: \( 1/(1 - s) \)
  [Sol] 1/(1 - s) = \( \sum_{n=0}^{\infty} s^n \), which converges for \(|s| < 1\) (e.g., the RoC is \(|s| < 1\)).

- 1.2: \( 1/(1 - s^2) \)
  [Sol] 1/(1 - s^2) = \( \sum_{n=0}^{\infty} s^{2n} \), which converges for \(|s^2| < 1\). (e.g., the RoC is \(|s| < 1\)). One can also factor the polynomial, thus write it as: \( \frac{1}{1-s^2} \). There are two poles, at \( s = \pm 1 \), and each has an RoC of 1.

- 1.3: \( 1/(1 + s^2) \)
  [Sol] The resulting series is \( 1/(1 + s^2) = 0.5 \sum_{n=0}^{\infty} s^n((-i)^n + (i)^n) \). The RoC is \(|s| < 1\). We can see this by considering the poles of the function at \( s = \pm i \); both poles are 1 from \( s = 0 \), the point of expansion. An alternative is to write the function as \( 1/(1 - (is)^2) = \sum (is)^n \).

- 1.4: \( 1/s \)
  [Sol] If you try to do a Taylor expansion at \( s = 0 \), the first term, \( w(0) \to \infty \). Thus, the Taylor series expansion in \( s \) does not exist.

- 1.5: \( 1/(1 - |s|^2) \)
  [Sol] The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed.

Problem # 2: Consider the function \( w(s) = 1/s \)

- 2.1: Expand this function as a power series about \( s = 1 \). Hint: Let \( 1/s = 1/(1 - 1 + s) = 1/(1 - (1 - s)) \).
**The power series is**

\[ w(s) = \sum_{n=0}^{\infty} (-1)^n (s - 1)^n, \]

which converges for \(|s - 1| < 1\).

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s, s, 'ExpansionPoint', 1)`, which is equivalent to the shorthand `syms s; taylor(1/s, s, 1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to \(z = -1\) via the Möbius “translation” \(s = z + 1\), and expand using the Taylor series

\[ \frac{1}{s} = \frac{1}{1 + z} = \sum_{n=0}^{\infty} (-z)^n. \]

Next back-substitute \(z = s - 1\) giving

\[ \frac{1}{s} = \sum_{n=0}^{\infty} (-1)^n (s - 1)^n. \]

It follows that the RoC is \(|z| = |s - 1| < 1\), as provided by Matlab/Octave.

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**Problem # 3: Consider the function \(w(s) = 1/(2 - s)\)**

- **3.1:** Expand \(w(s)\) as a power series in \(s^{-1} = 1/s\) about \(s^{-1} = 1\).
  
  **Hint:** Multiply top and bottom by \(s^{-1}\).

  **Sol:** \(1/(2 - s) = -s^{-1}/(1 - 2s^{-1}) = -s^{-1} \cdot 2^n s^{-n}\). The RoC is \(|2/s| < 1\), or \(|s| > 2\).

- **3.2:** Find the inverse function \(s(w)\). Where are the poles and zeros of \(s(w)\), and where is it analytic?
  
  **Sol:** Solving for \(s(w)\) we find \(2 - s = 1/w \) and \(s = 2 - 1/w = (2w - 1)/w\). This has a pole at 0 and a zero at \(w = 1/2\). The RoC is therefore from the expansion point out to, but not including \(w = 0\).

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**Problem # 4: Summing the series**

The Taylor series of functions have more than one region of convergence.

- **4.1:** Given some function \(f(x)\), if \(a = 0.1\), what is the value of \(f(a) = 1 + a + a^2 + a^3 + \cdots\)?
  
  **Show your work. Sol:** To sum this series, we may use the fact that

  \[ f(a) - af(a) = (1 + a + a^2 + a^3 + \cdots) - a(1 + a + a^2) = 1 + a(1 - 1) + a^2(1 - 1) + \cdots \]

  This gives \((1 - a)f(a) = 1\), or \(f(a) = 1/(1 - a)\). Now since \(a = .1\), the sum is \(1/(1 - 0.1) = 1.11\).

- **4.2:** Let \(a = 10\). What is the value of \(f(a) = 1 + a + a^2 + a^3 + \cdots\)?
  
  **Sol:** In this case the series clearly does not converge. To make it converge we need to write a formula for \(y = 1/x\) rather than for \(x\).

  \[ f(1/y) - f(1/y)/a = (1 + 1/a + 1/a^2 + 1/a^3 + \cdots) - 1/a(1 + 1/a + 1/a^2) = 1 + (1 - 1)/a + (1 - 1)/a^2 + \cdots \]

  This gives \(f(1/a) = -a^{-1}/(1 - a^{-1})\). Now since \(a = 10\), the series sums to \(f(10) = -0.1/(1 - 0.1) = -1/9\).
3.1.3 Cauchy-Riemann Equations

**Problem # 5:** For this problem \( j = \sqrt{-1} \), \( s = \sigma + \omega j \), and \( F(s) = u(\sigma, \omega) + jv(\sigma, \omega) \). According to the fundamental theorem of complex calculus (FTCC), the integration of a complex analytic function is independent of the path. It follows that the derivative of \( F(s) \) is defined as

\[
\frac{dF}{ds} = \frac{d}{ds} [u(\sigma, \omega) + jv(\sigma, \omega)]. \tag{DE-1.1}
\]

If the integral is independent of the path, then the derivative must also be independent of the direction:

\[
\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \omega}. \tag{DE-1.2}
\]

The Cauchy-Riemann (CR) conditions

\[
\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}
\]

may be used to show where Equation DE-1.2 holds.

- **5.1:** Assuming Equation DE-1.2 is true, use it to derive the CR equations.

**Sol:** First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations. ■

- **5.2:** Merge the CR equations to show that \( u \) and \( v \) obey Laplace’s equations

\[ \nabla^2 u(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma, \omega) = 0. \]

**Sol:** Take partial derivatives with respect to \( \sigma \) and \( \omega \) and solve for one equation in each of \( u \) and \( v \). ■

What can you conclude?

**Sol:** We can conclude that the real and imaginary parts of complex analytic functions must obey these conditions. ■

**Problem # 6:** Apply the CR equations to the following functions. State for which values of \( s = \sigma + \omega j \) the CR conditions do or do not hold (e.g., where the function \( F(s) \) is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

- **6.1:** \( F(s) = e^s \)

**Sol:** CR conditions hold everywhere. ■

- **6.2:** \( F(s) = 1/s \)

**Sol:** CR conditions are violated at \( s = 0 \). The function is analytic everywhere except \( s = 0 \). ■

3.1.4 Branch cuts and Riemann sheets

**Problem # 7:** Consider the function \( w^2(z) = z \). This function can also be written as \( w_{\pm}(z) = \sqrt{z} \pm \). Assume \( z = re^{i\theta} \) and \( w(z) = \rho e^{i\theta/2} \).

- **7.1:** How many Riemann sheets do you need in the domain \( z \) and the range \( w \) to fully represent this function as single-valued?

**Sol:** There is one sheet for \( z \) and two sheet for \( w = \pm \sqrt{z} \). When any point in the domain \( z \) (being mapped to \( w(z) \)) crosses the \( z \) branch cut, the codomain (range) \( w_{\pm}(z) \) switches from the \( w_+ \) sheet to the \( w_- \) sheet. \( w(z) \) remains analytic on the cut. Look at Fig. 4.4 in Chap. 4 (p. 148) to see how this works. ■

- **7.2:** Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

**Sol:** Above we show the mapping for the square root function \( w(z) = \sqrt{z} = \sqrt{re^{i\theta/2}} \). ■
7.3: Use *zviz.m* to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.

**Sol:** The sheet for the positive root is shown in Fig 3.2 (page 106 of the Oct 24 version of the class notes.) Two view the two sheets use Matlab command `zviz sqrt(W) -sqrt(W)`. ■

7.4: Where does *zviz.m* place the branch cut for this function?

**Sol:** Typically the cut is placed along the negative real $z$ axis $\phi = \pm \pi$. This is Matlab’s/Octave’s default location. In the figure above, it has been placed along the positive real axis, $\phi = 0 = 2\pi$. ■

7.5: Must the branch cut necessarily be in this location?

**Sol:** No, it can be moved, at will. It must start from $z = 0$ and end at $|z| \to \infty$. The cut may be moved when using *zviz.m* by multiplying $z$ by $e^{j\phi}$. For example, $zviz W = \sqrt{j*Z}$ rotates the cut by $\pi/2$. The colors of $w(z)$ (angle maps to color) always ‘jump’ at the branch cut, as you make the transition across the cut. ■

Problem # 8: Consider the function $w(z) = \log(z)$.

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8.1: Describe with a sketch and then discuss the branch cut for $f(z)$.

**Sol:** From the plot of $zviz w(z) = \log(z)$ of Lecture 18, we see a branch cut going from $w = 0$ to $w = -\infty$. If we express $z$ in polar coordinates ($z = re^{i\phi}$), then

$$w(z) = \log(r) + i\phi = u(x,y) + i(v(x,y)),$$

where $r(x,y) = |z| = \sqrt{x^2 + y^2}$ and $\phi = \angle z = \phi(x,y)$. Thus a zero in $w(z)$ appears at $z = 1 + 0i$, and only appears on the principle sheet of $z$ (between $-\pi < \angle z = \phi < \pi$), because this is the only place where $\phi = 0$. As the angle $\phi$ increases, the imaginary part of $w = \angle z$, which increases without bound. Thus $w$ is like a spiral stair case, or cork-screw. If $\rho = 1$ and $\phi \neq 0$, $w(r) = \log(1) + i\phi$ is not zero, since the angle is not zero. ■

8.2: What is the inverse of the function $z(f)$? Does this function have a branch cut? If so, where is it?

**Sol:** $z(w) = e^w$ is a single valued function, so a branch cut is not appropriate. Only multi-valued functions require a branch cut. ■

8.3: Using *zviz.m*, show that

$$\tan^{-1}(z) = -\frac{j}{2} \log \frac{j-z}{j+z}.$$  \hspace{1cm} (DE-1.3)

In Fig. 4.1 (p. 134) these two functions are shown to be identical.

**Sol:** Use the Matlab commands `zviz atan(Z)` and `zviz -(j/2)*log((j+Z)./(j-Z))`. ■

8.4: Algebraically justify Eq. DE-1.3. Hint: Let $w(z) = \tan^{-1}(z)$ and $z(w) = \tan w = \sin w/\cos w$; then solve for $e^{wj}$.

**Sol:** Following the hint gives

$$z(w) = -j e^{wj} - e^{-wj} = -j e^{2wj}/e^{2wj} + 1.$$

Solving this bilinear equation for $e^{2wj}$ gives

$$e^{2wj} = \frac{1 + jz}{1 - jz} = \frac{j - z}{j + z}.$$

Taking the log and using our definition of $w(z)$ we find

$$w(z) = \tan^{-1}(z) = -\frac{j}{2} \log \frac{j-z}{j+z}.$$ ■
3.1.5 A Cauer synthesis of any Brune impedance

**Problem # 9:** One may synthesize a transmission line (ladder network) from a positive real impedance $Z(s)$ by using the continued fraction method. To obtain the series and shunt impedance values, we can use a residue expansion. Here we shall explore this method.

- **9.1:** Starting from the Brune impedance $Z(s) = \frac{1}{s+1}$, find the impedance network as a ladder network.
  **Sol:** Taking the reciprocal we find the sum of two shunt admittances, and capacitor and resistor $Y(s) = s + 1$. The impedance is $Z(s) = \frac{s}{s+1}$.

- **9.2:** Use a residue expansion in place of the CFA floor function (Sec. 2.4.4, p. 31) for polynomial expansions. Find the residue expansion of $H(s) = \frac{s^2}{s + 1}$ and express it as a ladder network.
  **Sol:** Verify that $Z(s) = \frac{s}{s+1} = s - 1 + \frac{1}{s+1}$. Thus the Cauer synthesis is a series combination $s - 1$ (an inductor $L = 1$ and a resistor $R = -1$ ohms) and a shunt $1 || s$ (i.e., $Y(s) = 1 + s$, a resistor of $R = 1$ in parallel with a capacitor $C = 1$.) It would appear that $Z(s)$ is not a positive real impedance.

- **9.3:** Discuss how the series impedance $Z(s, x)$ and shunt admittance $Y(s, x)$ determine the wave velocity $\kappa(s, x)$ and the characteristic impedance $z_0(s, x)$ when (1) $Z(s)$ and $Y(s)$ are both independent of $x$; (2) $Y(s)$ is independent of $x$, $Z(s, x)$ depends on $x$; (3) $Z(s)$ is independent of $x$, $Y(s, x)$ depends on $x$; and (4) both $Y(s, x), Z(s, x)$ depend on $x$.
  **Sol:** In the most general case $z_0(s, x) = \sqrt{Z(s, x)/Y(s, x)}$ and $\kappa(s, x) = \sqrt{Z(s, x)Y(s, x)}$. The general equations for $z_0(s, x)$ and $\kappa(s, x)$ are given in Mason (1927), and discussed in Appendix D (p. 241). When $z_0$ and $\kappa$ depend on $x$, the area function $A(x)$ of the WHEN will depend on $x$. Thus the eigenfunction will critically depend on the characteristic impedance and the propagation function.

  For example, $\kappa(s)$ can be independent of the area because it cancels out in the product. This is called the case of constant $k$ because the speed of sound is independent of the area function. It follows that the area function only depends on $z_0(s, x)$.

  This shows that a Cauer synthesis may be implemented with the residue expansion replacing the floor function in the CFA. This seems to solve Brune’s network synthesis problem.