3.3 Problems DE-3

3.3.1 Topics of this homework: Brune impedance

lattice transmission line analysis

3.3.2 Brune Impedance

**Problem #1: Residue form**

A Brune impedance is defined as the ratio of the force $F(s)$ to the flow $V(s)$ and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)}$$

(De-3.1)

with

$$D(s) = \prod_{k=1}^{K} (s - s_k)$$

and

$$c_k = \lim_{s \to s_k} (s - s_k)D(s) = \prod_{n' = 1}^{K-1} (s - s_n).$$

The prime on the index $n'$ means that $n = k$ is not included in the product.

1.1: Find the Laplace transform ($\mathcal{L}$) of a (1) spring, (2) dashpot, and (3) mass.

Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: (1) Hooke’s law $f(t) = Kx(t)$, (2) dashpot resistance $f(t) = Rv(t)$, and (3) Newton’s law for mass $f(t) = M\frac{dv(t)}{dt}$.

**Ans:**

1.2: Take the Laplace transform ($\mathcal{L}$) of Eq. DE-3.2 and find the total impedance $Z(s)$ of the mechanical circuit.

$$M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + K x(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s).$$

(De-3.2)

**Ans:**

1.3: What are $N(s)$ and $D(s)$ (see Eq. DE-3.1)?

**Ans:**
1.4: Assume that \( M = R = K = 1 \) and find the residue form of the admittance \( Y(s) = 1/Z(s) \) (see Eq. DE-3.1) in terms of the roots \( s_{\pm} \). Hint: Check your answer with Octave’s/Matlab’s residue command.

\[ \text{Ans:} \]

1.5: By applying Eq. 4.5.3 (page 149), find the inverse Laplace transform \( \mathcal{L}^{-1} \). Use the residue form of the expression that you derived in question 1.4.

\[ \text{Ans:} \]

### 3.3.3 Transmission-line analysis

**Problem # 2: Train-mission-line**

We wish to model the dynamics of a freight train that has \( N \) such cars and study the velocity transfer function under various load conditions.

As shown in Fig. 4.11, the train model consists of masses connected by springs.

\[ \text{Figure 3.2: Depiction of a train consisting of cars treated as masses } M \text{ and linkages treated as springs of stiffness } K \text{ or compliance } C = 1/K. \text{ Below it is the electrical equivalent circuit for comparison. The masses are modeled as inductors and the springs as capacitors to ground. The velocity is analogous to a current and the force } f_n(t) \text{ to the voltage } \phi_n(t). \text{ The length of each cell is } \Delta \text{ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL. This method, called a Cauer synthesis, is based on the ABCD transmission line method of Sec. 3.8 (p. 105).} \]

Use the ABCD method (see the discussion in Appendix B.3, p. 228) to find the matrix representation of the system of Fig. 4.11. Define the force on the \( n \)th train car \( f_n(t) \leftrightarrow F_n(\omega) \) and the velocity \( v_n(t) \leftrightarrow V_n(\omega) \).

Break the model into cells consisting of three elements: a series inductor representing half the mass \( (M/2) \), a shunt capacitor representing the spring \( (C = 1/K) \), and another series inductor representing half the mass \( (L = M/2) \), transforming the model into a cascade of symmetric \( (A = D) \) identical cell matrices \( T(s) \).

2.1: Find the elements of the ABCD matrix \( T \) for the single cell that relate the input node 1 to output node 2

\[
\begin{bmatrix} F \\ V \end{bmatrix}_1 = T \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2. \]  

(DE-3.3)
3.3. PROBLEMS DE-3

Ans:

– 2.2: Express each element of \( T(s) \) in terms of the complex Nyquist ratio \( s/s_c < 1 \) \((s = 2\pijf, \ s_c = 2\pijf_c)\). The Nyquist wavelength sampling condition is \( \lambda_c > 2\Delta \). It says the critical wavelength \( \lambda_c > 2\Delta \). Namely it is defined in terms the minimum number of cells \( 2\Delta \), per minimum wavelength \( \lambda_c \).

The Nyquist wavelength sampling theorem says that there are at least two cars per wavelength.

Proof: From the figure, the distance between cars \( \Delta = c_oT_o \) [m], where

\[
c_o = \frac{1}{\sqrt{MC}} \text{ [m/s].}
\]

The cutoff frequency obeys \( f_c\lambda_c = c_o \). The Nyquist critical wavelength is \( \lambda_c = c_o/f_c > 2\Delta \). Therefore the Nyquist sampling condition is

\[
f < f_c \equiv \frac{c_o}{\lambda_c} = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}} \text{ [rad/sec].} \quad (DE-3.4)
\]

Finally, \( s_c = j2\pi f_c \).

Ans:

– 2.3: Use the property of the Nyquist sampling frequency \( \omega < \omega_c \) (Eq. DE-3.4) to remove higher order powers of frequency

\[
1 + \left( \frac{s}{s_c} \right)^0 \approx 1
\]

(DE-3.5)

to determine a band-limited approximation of \( T(s) \).

Ans:

Problem # 3: Now consider the cascade of \( N \) such \( T(s) \) matrices and perform an eigenanalysis.

– 3.1: Find the eigenvalues and eigenvectors of \( T(s) \) as functions of \( s/s_c \).

Ans:
Problem # 4: Find the velocity transfer function $H_{12}(s) = \frac{V_2}{V_1}|_{F_2=0}$.

- 4.1: Assuming that $N = 2$ and $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv \frac{V_2}{V_1}$. From the results of the $T$ matrix, find

$$H_{21}(s) = \frac{V_2}{V_1}|_{F_2=0}$$

Express $H_{12}$ in terms of a residue expansion. \textbf{Ans:}

- 4.2: Find $h_{21}(t) \leftrightarrow H_{21}(s)$.
\textbf{Ans:}

- 4.3: What is the input impedance $Z_2 = \frac{F_2}{V_2}$, assuming $F_3 = -r_0V_3$?
\textbf{Ans:}

- 4.4: Simplify the expression for $Z_2$ as follows:
1. Assuming the characteristic impedance $r_0 = \sqrt{M/C}$,
2. terminate the system in $r_0$: $F_2 = -r_0V_2$ (i.e., $-V_2$ cancels).
3. Assume higher-order frequency terms are less than 1 ($|s/s_c| < 1$).
4. Let the number of cells $N \to \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteristic impedance $r_0$, the input impedance $Z_1(s) = r_0$. Thus, when we simplify the expression for $T(s)$, it should be equal to $r_0$. Show that this is true for this setup.

\textbf{Ans:}
4.5: State the ABCD matrix relationship between the first and \( N \)th nodes in terms of the cell matrix. Write out the transfer function for one cell, \( H_{21} \).

\textbf{Ans:}

4.6: What is the velocity transfer function \( H_{N1} = \frac{V_N}{V_1} \)?

\textbf{Ans:}