Chapter 17

Fourier Series, Fourier Integral, Fourier Transform

17.1 Introduction

This chapter is about a number of methods associated with the French applied mathematician Joseph Fourier (1768–1830), methods which will be central when we turn next to the subject of partial differential equations in Chapters 18–20. In fact, series of trigonometric functions, which we now call Fourier series, were already being actively studied—by Euler, Lagrange, d'Alembert, Daniel Bernoulli, and others—even when Fourier was born. For instance, d'Alembert had already given integral formulas for computing the coefficients in those series, and various specific “Fourier series” had been put forward. Yet, there was considerable debate regarding the class of functions that could be successfully expanded in such series, and various sets of sufficient conditions were slow to appear, the first being given by Peter Gustav Lejeune–Dirichlet around 1829. Part of the difficulty was that even the meaning of the term “function” was not yet clear or agreed upon. But it is also true that mathematical issues were deep and elusive. For instance, in 1873, Paul Du Bois–Reymond put forward an example of a function that is continuous on \((-\pi, \pi)\), yet has a Fourier series that fails to converge at any point in that interval! Thus, the subject of Fourier series has been one of the most fertile in the development of modern pure and applied mathematics.

Although Fourier neither invented “Fourier” series nor settled the outstanding fundamental questions, he did use them fruitfully, especially for problems regarding the conduction of heat in solids, governed by the PDE

\[ \alpha^2 \nabla^2 T = \frac{\partial T}{\partial t} \]

that is derived in Section 16.8 and to which we return in Chapter 18. In claiming that an “arbitrary” function could be represented by a Fourier series Fourier overstated the case, and his work was faulted for lack of rigor. Yet he had the insight to
see the power of these new methods. His work advanced the use of Fourier methods and solidified techniques that would be needed to solve problems in field theory and that continue to be developed even today.

To explain what this chapter is about we ask you to recall, first, the importance of being able to expand a given vector in terms of a set of orthogonal base vectors. Likewise, we will find in this chapter that a given (sufficiently well-behaved) function can be expanded in terms of a set of "orthogonal functions." Fourier and his contemporaries did not have vector space concepts available to them but, essentially, they were seeking sets of orthogonal functions to be used as base vectors in an infinite-dimensional function space. That task is much more difficult than for finite-dimensional spaces. For instance, if we have \( n \) orthogonal vectors in an \( n \)-dimensional space (where \( n \) is finite), then they provide a basis. However, in an infinite-dimensional space an infinite set of orthogonal functions may, but need not, be a basis. For suppose we have an infinite set of orthogonal functions that is a basis. If we remove one of them, then we have an infinite number of them left, but that diminished set will not be a basis. Further, expansions in an infinite-dimensional space are infinite series, so subtle matters arise regarding their convergence and manipulation.

Fourier series are introduced and developed in Section 17.3, though not from a vector space point of view. The alternative, and more modern, vector space approach is given subsequently in Section 17.6. By then, the nagging question arises: Where does the set of orthogonal functions, which comprise the individual terms in a Fourier series, "come from"? Are there other such sets? The answer is provided, in Sections 17.7 and 17.8, by the Sturm–Liouville theory. There, it is revealed that such sets of orthogonal base vectors are generated as the eigenfunctions of second-order differential equation eigenvalue problems known as Sturm–Liouville problems. Thus, Sections 17.7 and 17.8 are the function space analog of Section 11.3 on symmetric matrices, wherein we found that the eigenvectors of a real symmetric \( n \times n \) matrix provide an orthogonal basis for \( n \)-space.

In Section 17.9 we will let the period of the periodic functions under consideration tend to infinity, and will find that the Fourier series representation gives way to a Fourier integral representation. The Fourier integral gives us, by a mere rearrangement, the Fourier transform, in Section 17.10, which is very much analogous to the Laplace transform that we studied in Chapter 5. In fact, in the final section, 17.11, we show how to derive the Laplace transform from the Fourier transform.

For interesting historical (and mathematical) accounts, we suggest the little book by R. L. Jeffery [Trigonometric Series (Toronto: University of Toronto Press, (1956)] as well as the historical treatise by Morris Kline [Mathematical Thought from Ancient to Modern Times (New York: Oxford, 1972)].

17.2 Even, Odd, and Periodic Functions

Before taking up our study of Fourier series, in the next section, we need to define even, odd, and periodic functions.

Let \( f \) be defined on an \( x \) interval, finite or infinite, that is centered at \( x = 0 \). We say that \( f \) is an even function if

\[
\begin{align*}
    f(-x) &= f(x),
  \end{align*}
\]

and an odd function if

\[
\begin{align*}
    f(-x) &= -f(x),
  \end{align*}
\]

for all \( x \) in that interval. That is, the graph of \( f \) is symmetric about \( x = 0 \) if \( f \) is even, and antisymmetric about \( x = 0 \) if \( f \) is odd. Examples are shown in Fig. 1. For example, \( 5, x^2, 3x^4, \cos x, \sin |x|, \) and \( e^{-x^2} \) are even, and \( x, 3x^3, 2x^5, \sin x, \) and \( x \cos x \) are odd.

There are several useful algebraic properties of even and odd functions, such as the following:

\[ \begin{align*}
    \text{even} + \text{even} &= \text{even}, \\
    \text{even} \times \text{even} &= \text{even}, \\
    \text{odd} + \text{odd} &= \text{odd}, \\
    \text{odd} \times \text{odd} &= \text{even}, \\
    \text{even} \times \text{odd} &= \text{odd}. 
\end{align*} \]

To prove (3e), for example, let \( F(x) \) be even and let \( G(x) \) be odd. Then \( F(-x)G(-x) = F(x)[-G(x)] = -F(x)G(x) \), in accord with (2).

In addition, two useful integral properties are as follows. If \( f \) is even, then

\[
\int_{-A}^{A} f(x) \, dx = 2 \int_{0}^{A} f(x) \, dx. \quad (f \text{ even})
\]

and if \( f \) is odd, then

\[
\int_{-A}^{A} f(x) \, dx = 0. \quad (f \text{ odd})
\]

for if we interpret the integrals in (4a) as areas (positive above the \( x \) axis, negative below it), then the area \( \int_{-A}^{0} f(x) \, dx \) is equal to the \( \int_{0}^{A} f(x) \, dx \) due to the symmetry of the graph of \( f \). And in the case of (4b) the areas \( \int_{-A}^{0} f(x) \, dx \) and \( \int_{0}^{A} f(x) \, dx \) are negatives of each other, due to the antisymmetry of the graph of \( f \), and hence cancel.
17.2. Even, Odd, and Periodic Functions

Alternatively, (4a) and (4b) follow directly from (1) and (2), respectively. For example, if \( f \) is odd, then

\[
\int_{-A}^{A} f(x) \, dx = \int_{0}^{A} f(x) \, dx + \int_{0}^{A} f(x) \, dx
\]

\[
= \int_{A}^{0} f(-t) \, (-dt) + \int_{0}^{A} f(x) \, dx \quad (x = -t)
\]

\[
= \int_{0}^{A} f(-t) \, dt + \int_{0}^{A} f(x) \, dx
\]

\[
= \int_{0}^{A} -f(t) \, dt + \int_{0}^{A} f(x) \, dx \quad \text{(oddness of } f \text{)}
\]

\[
= -\int_{0}^{A} f(x) \, dx + \int_{0}^{A} f(x) \, dx \quad (t = x)
\]

\[
= 0,
\]

as stated in (4b).

Note carefully that a given function is not necessarily even or odd; it may be both even and odd, or it may be neither. Every function can be uniquely decomposed into the sum of an even function, say \( f_e \), and an odd function, say \( f_o \), as demonstrated by the simple identity

\[
f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}
\]

\[
\equiv f_e(x) + f_o(x),
\]

(6)

for observe that

\[
f_o(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x)
\]

and, similarly, that \( f_e(-x) = f_e(x) \).

**EXAMPLE 1.** Surely \( f(x) = e^x \) is neither even nor odd, since (Fig. 2) it is neither symmetric nor antisymmetric about \( x = 0 \). Putting \( f(x) = e^x \) and \( f(-x) = e^{-x} \) into (6) gives

\[
f_e(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad f_o(x) = \frac{e^x - e^{-x}}{2}
\]

as the even and odd parts of \( e^x \), respectively. In fact, we recognize these functions as \( \cosh x \) and \( \sinh x \), so it is interesting that we can think of \( \cosh x \) and \( \sinh x \) as the even and odd parts of \( e^x \), respectively.

\[\text{Figure 2. Even and odd parts of } e^x.\]

Notice that (6) is reminiscent of other decompositions that occur in mathematics, whereby a mathematical object (such as a function, matrix, or vector) is broken...
into the sum of two complementary parts. For instance, every square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix (Exercise 6, Section 10.3), every vector in 3-space can be expressed as the sum of a vector along a given line and a vector perpendicular to that line, and every \( C^1 \) vector field can be expressed as the sum of an irrotational field and a solenoidal field (Exercise 10, Section 16.10).

Next, suppose that for a given function \( f \) there exists a positive constant \( T \) such that

\[
    f(x + T) = f(x)
\]

for every \( x \) in the domain of \( f \). Then we say that \( f \) is a periodic function of \( x \), with period \( T \). Sometimes we say that \( f \) is \( T \)-periodic.

**EXAMPLE 2.** For example, \( \sin x \) is periodic with period \( 2\pi \) because \( \sin (x + 2\pi) = \sin x \cos 2\pi + \sin 2\pi \cos x = \sin x \) for all \( x \).

In graphical terms, one can think of the graph of a periodic function \( f \) as generated by stamping it out one period at a time, as with an inked woodblock.

**EXAMPLE 3.** The function \( f \) shown in Fig. 3 is seen to be periodic with period \( T = 4 \), for if the segment \( BCD \), for instance, is "stamped out" indefinitely to the right and left we generate the graph of \( f \). There is nothing special about choosing the segment \( BCD \) for this purpose: \( ABC \), or any other segment of length 4, would do as well.

Notice that if \( f \) is periodic with period \( T \), it is necessarily periodic with period \( 2T, 3T, 4T, \ldots \) as well. For example, \( f(x + 2T) = f((x + T) + T) = f(x + T) = f(x) \), so that \( f \) is periodic with period \( 2T \). Of all these possible periods, if there exists a smallest one, that period is called the fundamental period. Thus, \( \sin x \) (in Example 2) is periodic with period \( 2\pi, 4\pi, 6\pi, \ldots \), and its fundamental period is \( 2\pi \): \( f \) in Example 3 is periodic with period \( 4, 8, 12, \ldots \), and its fundamental period is 4.

In contrast, observe that if \( f(x) = \) constant, then \( f \) is periodic and every \( T > 0 \) is a period. Thus, there exists no smallest period, so \( f \) does not have a fundamental period.

**Closure.** Even, odd, and periodic functions will be basic to our study of Fourier series, to follow. The defining properties are (1), (2), and (7), respectively.
EXERCISES 17.2

1. (a) Prove (3a). (b) Prove (3b).
   (c) Prove (3c). (d) Prove (3d).
2. Provide a proof of (4a) that is analogous to the proof of (4b) given in (5).
3. Prove that
   (a) \( f \) is both even and odd if and only if it is identically zero.
   (b) If \( f \) is even (and integrable), then \( F(x) = \int_0^x f(t) \, dt \) is odd.
   (c) If \( f \) is odd (and integrable), then \( F(x) = \int_0^x f(t) \, dt \) is even.
   (d) If \( f \) is even (and differentiable), then \( F(x) = df/dx \) is odd.
   (e) If \( f \) is odd (and differentiable), then \( F(x) = df/dx \) is even.
4. Prove the decomposition formula (6). HINT: Start with \( f(x) = f_e(x) + f_o(x) \) and change \( x \) to \(-x\).
5. Determine \( f_e(x) \) and \( f_o(x) \). Is \( f \) even? Odd? Neither?
   (a) \( 2 - 5x \)
   (b) \( \sin (x + 2) \)
   (c) \( x/(x^2 + x + 3) \)
   (d) \( xe^{-x} \)
   (e) \( x/(x + 2) \)
   (f) \( x^2 \cos (x^3) - 8 \)
   (g) \( x^4 + x^3 + x^2 + x + 1 \)
   (h) \( \ln (1 + x^2) \)
   (i) \( e^{-2 \sin x} \)
   (j) \( \sin (\sin x) \)
   (k) \( \cos (\sin x) \)
   (l) \( e^{(-x)/(x^2 + 1)} \)
6. If \( F \) is even and \( G \) is odd, show that
   (a) \( 1/F(x) \) is even
   (b) \( 1/G(x) \) is odd
   (c) \( F(G(x)) \) is even
   (d) \( G(F(x)) \) is even.
7. Show that if \( f \) is odd, then it is necessarily true that \( f(0) = 0 \).
8. Let \( f \) be even (and not identically zero), and let \( g \) be odd (and not identically zero). If \( f \) and \( g \) are defined on a common interval, show that \( f \) and \( g \) are necessarily linearly independent on that interval.
9. Show that if \( f(x) = g(x) \) (over a common \( x \) interval), we can equate even and odd parts: \( f_e(x) = g_e(x) \) and \( f_o(x) = g_o(x) \).
10. Show that if \( f \) is even and \( g \) is odd, and \( f + g = 0 \) (over a common \( x \) interval), then \( f(x) = 0 \) and \( g(x) = 0 \).
11. Determine the fundamental period in each case. Also, draw the graphs of \( f_e \) and \( f_o \).

12. Determine whether or not the given function (defined on \(-\infty < x < \infty\)) is periodic. If it is, find its fundamental period (if it has one).

\[
\begin{align*}
\text{(g)} & \quad x^4 \\
\text{(d)} & \quad \sin (\omega x + \phi) \\
\text{(e)} & \quad \cos 6x \\
\text{(f)} & \quad \sin 2x \\
\text{(g)} & \quad \tan x \\
\text{(h)} & \quad \sinh x \\
\text{(i)} & \quad \cosh x \\
\text{(j)} & \quad \cos^2 x \\
\text{(k)} & \quad \sin^2 x \\
\text{(l)} & \quad \sin x \cos 2x \\
\text{(m)} & \quad e^{\sin 3x} \\
\text{(n)} & \quad \sinh (2x) \\
\text{(o)} & \quad \sin (\sin x) \\
\text{(p)} & \quad \cos (\sin x) \\
\text{(q)} & \quad \sin (8\pi \cos x) \\
\text{(r)} & \quad \sin (4x) \\
\text{(s)} & \quad \sin |x| \\
\text{(t)} & \quad \cos |x| \\
\text{(u)} & \quad x \sin x
\end{align*}
\]

13. The following functions are periodic. Determine the fundamental period in each case. (The \( a_n \)'s and \( b_n \)'s are nonzero constants.)

\[
\begin{align*}
\text{(g)} & \quad a_0 + a_1 \cos x \\
\text{(b)} & \quad a_0 + a_1 \cos x + a_2 \cos 2x \\
\text{(c)} & \quad 6 \cos x - 3 \sin x \\
\text{(d)} & \quad \cos 5x + 5 \sin x \\
\text{(e)} & \quad a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x \\
\text{(f)} & \quad a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\
\text{(g)} & \quad a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)
\end{align*}
\]
14. Show that if \( f \) is periodic with period \( T \), then
\[
\int_0^T f(x) \, dx = \int_{A}^{A+T} f(x) \, dx
\]
for any finite value of \( A \).

15. Show that if \( f(x) \) is periodic with period \( T \), then
(a) so is its derivative \( f'(x) \)
(b) so is its integral \( \int_0^T f(t) \, dt \), if and only if \( \int_0^T f(t) \, dt = 0 \)

16. Let \( f(x) = 1 \) whenever \( x \) is rational, and let \( f(x) = 0 \) whenever \( x \) is irrational. Show that \( f \) is periodic with period \( T \) where \( T \) is any (positive) rational number. (Thus there exists no smallest period, so \( f \) does not have a fundamental period.) Is it also true that \( f \) is periodic with period \( T \), where \( T \) is any (positive) irrational number? Explain.

17. Show that if \( f \) is periodic with period \( T \), then \( g(f(x)) \) is, too. Give two examples.

### 17.3 Fourier Series of a Periodic Function

17.3.1. Fourier series. Recall that if \( f(x) \) is infinitely differentiable at a point \( x = a \), then it has a Taylor series about that point,

\[
TS_f \bigg|_{x=a} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n. \tag{1}
\]

For that series to be useful, we need two things. First, we need it to converge on some interval \( I \). Second, if it does converge on \( I \), then we need its sum function to be the same as the original function \( f(x) \). If that is the case, then we say that the Taylor series \( TS_f \bigg|_{x=a} \) represents \( f \) on \( I \), and we can write

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \tag{2}
\]
on \( I \). We know from experience that Taylor series representations are useful in many ways.

Similarly, there are other types of representations that are useful as well. In this chapter we are concerned with the representation of periodic functions, not by Taylor series but by trigonometric series, that is, by series of the form

\[
a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right). \tag{3}
\]

First, observe that (3) is a periodic function with fundamental period \( 2\ell \). To see this, note that (3) is a linear combination of the functions \( 1, \cos \left( \frac{\pi x}{\ell} \right), \sin \left( \frac{\pi x}{\ell} \right), \)

*For instance, we saw in Example 2 of Section 13.5 that it is possible for \( TS_f \bigg|_{x=a} \) to converge, but not to \( f(x) \).
\[ \cos \left( \frac{2\pi x}{\ell} \right), \sin \left( \frac{2\pi x}{\ell} \right), \text{and so on, and that these functions are periodic with the following periods:} \]

\[ \begin{align*}
\cos \frac{\pi x}{\ell} & \quad \text{and} \quad \sin \frac{\pi x}{\ell} : \quad 2\ell, 4\ell, 6\ell, 8\ell, \ldots, \\
\cos \frac{2\pi x}{\ell} & \quad \text{and} \quad \sin \frac{2\pi x}{\ell} : \quad \ell, 2\ell, 3\ell, 4\ell, \ldots, \\
\cos \frac{3\pi x}{\ell} & \quad \text{and} \quad \sin \frac{3\pi x}{\ell} : \quad \frac{2\ell}{3}, \frac{4\ell}{3}, \frac{8\ell}{3}, \ldots, 
\end{align*} \]

and so on. The smallest period shared by all the terms is \( 2\ell \) [underlined in (4)], so the fundamental period of (3) is \( 2\ell \). Thus, perhaps the trigonometric series (3) can be used to represent periodic functions of period \( 2\ell \). (When we say of period \( 2\ell \), we shall mean of fundamental period \( 2\ell \).)

Specifically, if \( f(x) \) is periodic, of period \( 2\ell \), then we define the Fourier series of \( f \), say \( \text{FS} f \), as

\[ \text{FS} f = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right), \]

where the coefficients are given by the Euler formulas

\[ \begin{align*}
a_0 & = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \, dx, \\
a_n & = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx, \quad n = 1, 2, \ldots, \\
b_n & = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx, \quad n = 1, 2, \ldots
\end{align*} \]

and are known as the Fourier coefficients of \( f \).

For \( \text{FS} f \) to represent \( f \) we need the series to converge, and we need its sum function to be the same as the original function \( f(x) \). Various theorems are available, that give sufficient conditions on \( f \) for \( \text{FS} f \) to represent \( f \). One such theorem, that is easily applied and which covers the vast majority of periodic functions that arise in applications, is as follows.\(^1\)

**THEOREM 17.3.1 Fourier Convergence Theorem**

Let \( f \) be \( 2\ell \)-periodic, and let \( f \) and \( f' \) be piecewise continuous on \([-\ell, \ell]\). Then

\(^*\) Actually, the fundamental period of the function (3) may be less than \( 2\ell \). For example, if all the coefficients except for \( a_3 \) are zero, then the fundamental period is \( 2\ell/3 \). Thus, we should say that (3) is always periodic with period \( 2\ell \); its fundamental period is at most \( 2\ell \), but may be less.

the Fourier series given by (5) converges to \( f(x) \) at every point \( x \) at which \( f \) is continuous, and to the mean value \( [f(x^+) + f(x^-)]/2 \) at every point \( x \) at which \( f \) is discontinuous.

Piecewise continuity is defined in Section 5.2. By \( f(x^+) \) and \( f(x^-) \) we mean the right- and left-hand limits of \( f \),

\[
f(x^+) \equiv \lim_{h \to 0^+} f(x + h) \quad \text{and} \quad f(x^-) \equiv \lim_{h \to 0^-} f(x - h),
\]

where \( h \to 0 \) through positive values. [If \( f(x^+) = f(x^-) = f(x) \), then \( f \) is continuous at \( x \), otherwise it is discontinuous there.]

**EXAMPLE 1. Square Wave.** Consider the "square wave" \( f \) shown in Fig. 1, where \( f(x) \) is defined as 2 at \( x = 0, \pm \pi, \pm 2\pi, \ldots \), as indicated by the heavy dots. Since the period referred to in Theorem 17.3.1 and in (5) is 2\( \ell \), and the period is seen from Fig. 1 to be 2\( \pi \), it follows that \( \ell = \pi \). Both \( f \) and \( f' \) are piecewise continuous on \( [-\pi, \pi] \), so the theorem applies. Let us use (5) to work out the Fourier series of \( f \) and examine its convergence to the square wave \( f \) using computer plots of the partial sums of the series.

First, (5b) gives

\[
a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx
\]

\[
= \frac{1}{2\pi} \left[ \int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} 4 \, dx \right] = 2.
\]

Actually, \( a_0 = 2 \) could have been seen by inspection because the right-hand side of (5b) is, by definition, the average value of \( f \) over one period and, from Fig. 1, we can see that for our square wave that average value is 2.

Next,

\[
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n\pi x \, dx
\]

\[
= \frac{1}{\pi} \left[ \int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} 4 \cos n\pi x \, dx \right] = 4 \sin n\pi \left| \frac{\pi}{n \pi} \right| = 0
\]

since \( \sin n\pi = 0 \) and \( \sin 0 = 0 \). Finally,

\[
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n\pi x \, dx
\]

\[
= \frac{1}{\pi} \left[ \int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} 4 \sin n\pi x \, dx \right] = \frac{4}{n\pi} (1 - \cos n\pi).
\]

We can leave (9) as it stands, but it is best to note that \( \cos n\pi = (-1)^n \), so that we can reduce (9) to

\[
b_n = \frac{4}{n\pi} [1 - (-1)^n] = \left\{ \begin{array}{ll} \frac{8}{n\pi}, & n = 1, 3, \ldots \\ 0, & n = 2, 4, \ldots \end{array} \right.
\]
Thus the Fourier series representation of \( f \) is

\[
f(x) = 2 + \frac{8}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin nx,
\]

where "1, 3, ..." tells us to omit the terms corresponding to \( n = 2, 4, \ldots \). If preferred, (11) can be expressed, equivalently, as

\[
f(x) = 2 + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin (2n - 1)x}{2n - 1},
\]

with \( \sum_{1}^{\infty} \) denoting the usual summation over \( n = 1, 2, 3, \ldots \).

Observe from Fig. 1 that the defined values of \( f(x) \) at the jump discontinuities \( x = n\pi \) coincide with the mean values \([f(x^+) + f(x^-)]/2 = 2\), so Theorem 17.3.1 assures us that (12) is indeed an equality for every value of \( x \).

Is it not remarkable that a linear combination of sines and cosines, each of which is beautifully smooth (infinitely differentiable for all \( x \), i.e., of class \( C^\infty \)), can sum to a function with jump discontinuities? To obtain insight as to how this convergence to \( f \) is accomplished, it will be illuminating to plot some of the partial sums of the series. Writing out (12) as

\[
f(x) = 2 + \frac{8}{\pi} \sin x + \frac{8}{3\pi} \sin 3x + \frac{8}{5\pi} \sin 5x + \cdots,
\]

we define the partial sums of the series as

\[
s_1(x) = 2, \\
s_2(x) = 2 + \frac{8}{\pi} \sin x, \\
s_3(x) = 2 + \frac{8}{\pi} \sin x + \frac{8}{3\pi} \sin 3x,
\]

and so on: that is, \( s_N(x) \) is the sum of the first \( N \) terms.

We see from Fig. 2 that although the first partial sum \( s_1(x) \) is merely a constant it "does the best it can" by equalling the average value of \( f \). If we are to provide the needed correction to \( s_1(x) \) we need to pull it up on the right and push it down on the left, as

![Figure 2](image-url)
CHAPTER 17. Fourier Series, Fourier Integral, Fourier Transform

Figure 3. Gibbs phenomenon.

Figure 4. Modified f.

indicated by the broad arrows in the figure. A sin x term can accomplish that (a cos x term is not appropriate since we need an antisymmetric correction, not a symmetric one) and an amplitude a bit greater than 2 seems optimal. That correction is precisely what is provided by the second term in the series, so \( s_2(x) \) begins to take on the desired shape. To correct \( s_2(x) \), in turn, we need to push it up and down according to the six thin arrows. Such correction dictates the need for a third sine harmonic, and such a term is indeed forthcoming as the third term in the series. By adding more and more terms of the series, the graph comes closer and closer to the square wave \( f \). Observe also that the graph of every partial sum passes through the mean value \( 2 \) at each jump discontinuity, which result is also clear from (12) since \( \sin n\pi = 0 \) for each \( n \).

Next, consider the graph of \( s_N(x) \) for larger \( N \), for instance for \( N = 20 \) (Fig. 3). As \( x \) increases from zero, \( s_{20}(x) \) rises sharply from the mean value \( 2 \), overshoots the value 4, and settles down close to 4—until \( x \) approaches \( \pi \), where the same sort of overshoot is followed by a steep descent to the mean value \( 2 \) at \( x = \pi \). By periodicity, these results repeat over each period.

Strangely, the overshoot does not diminish as \( N \) increases. For instance, the peak values of \( s_2(x) \), \( s_4(x) \), and \( s_{20}(x) \) are 4.548, 4.376, and 4.360, respectively, and it can be shown that the overshoot approaches a limiting value of around 9% of the jump (the jump is 4, so 9% of 4 is 0.36) as \( N \to \infty \). This persistent 9% overshoot occurs not only in this example but in the Fourier series representation of any function with a jump discontinuity, and is known as the Gibbs phenomenon.\(^*\)

In view of this overshoot, one may well wonder how convergence to the square wave is attained, for are there not always \( x \) locations at which the error is around 9% no matter how many terms are summed? The key point is that when we say that \( \lim_{N \to \infty} S_N(x) = f(x) \) it is \( N \) that is varying; \( x \) is fixed. Picking any \( x \) point (Fig. 3), as close to the origin as we like, as \( N \) is increased the overshoot “spike” eventually moves to the left of \( x \), and subsequent values of \( S_N(x) \) do settle down and converge to \( f(x) \). Nonetheless, the Gibbs phenomenon is of great practical importance because it implies that the convergence of Fourier series may be painfully slow (and expensive) in the vicinity of a jump discontinuity.\(^1\)

COMMENT 1. We have already noted that the Fourier series of the square wave \( f(x) \) shown in Fig. 1 converges to \( f(x) \) at every point \( x \). Suppose, instead, we define \( f \) as shown in Fig. 4. That is, the modified \( f \) is 0 at \( x = 0, \pm 2\pi, \pm 4\pi \), ... , it is 4 at \( x = \pm \pi, \pm 3\pi \), ..., and it is \( 10^9 \) (not shown to scale) at \( x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \), .... The Fourier series of the modified \( f \), say \( f_{\text{mod}} \), will be identical to the Fourier series of \( f \), because the Fourier coefficients are computed, according to (5b,c,d), by integrals, and changing the value of the integrands at a number of isolated points does not change the value of those integrals.

\(^*\)Josiah Willard Gibbs (1839–1903) was one of the first important American mathematician physicists, and is especially well known for his work on vector analysis and thermodynamics. The Gibbs phenomenon was noted by Gibbs in 1899 (Nature, Vol. 59, p. 606) and explained in 1906 by Maxime Böcher (Annals of Mathematics, Vol. 2, No. 7, p. 81).

\(^1\)When studying infinite series in the calculus one learns to classify series as convergent or divergent, with the implication that convergent series are “good” and divergent series are “bad.” Practically speaking, however, a series may converge— but so slowly as to be almost worthless. Thus, interest exists in developing acceleration techniques that permit us to transform a given slowly convergent series into a more rapidly convergent one. For a numerical technique to suppress the Gibbs phenomenon, see Forman Acton, Numerical Methods That Work (Washington, D.C.: Mathematical Assn. of America, 1990).
Thus, the Fourier series of \( f_{\text{mod}} \) will converge to \( f_{\text{mod}}(x) \) for every point \( x \) except points such as \( x = 0, \frac{\pi}{2}, \) and \( \pi \), where it will converge to 2, 4, and 2, respectively. Such pointwise discrepancies will cause no problem in applications. Thus, from this point forward (except in the exercises to this section) we will not bother to show the heavy dots, as we did in Figs. 1 and 4, and will simply show these graphs as in Fig. 5.

Thus far, the concepts of even and odd functions, introduced in Section 17.2, have not been used here. Suppose now that \( f \) is an even 2\(\ell\)-periodic function. Then the integrands \( f(x) \) and \( f(x) \cos \left(\frac{n\pi x}{\ell}\right) \) in (5b) and (5c) are even functions, and the integrand \( f(x) \sin \left(\frac{n\pi x}{\ell}\right) \) in (5c) is odd, so these Euler formulas simplify to

\[
a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \, dx = \frac{1}{\ell} \int_0^{\ell} f(x) \, dx,
\]

\[
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx = 2 \frac{1}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx,
\]

\[
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx = 0.
\]

Observe that only the constant and cosine terms survive, there being no need for sine terms in representing an even function. Similarly, if \( f \) is odd, then

\[
a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \, dx = 0,
\]

\[
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx = 0,
\]

\[
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx = 2 \frac{1}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx,
\]

and the constant and cosine terms drop out.

More generally, \( f \) is neither even nor odd. The constant and cosine terms in the Fourier series of \( f(x) \) represent the even part \( f_e(x) \), and the sine terms represent the odd part \( f_o(x) \). That is,

\[
f(x) = \left[ a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} \right] + \left[ \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell} \right]
\]

and

\[
f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}, \quad (17a)
\]

\[
f_o(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}. \quad (17b)
\]
EXAMPLE 2. Let us work out the Fourier series of the periodic function \( f \), the graph of which is given in Fig. 6. Its period is 16, so \( \ell = 8 \). We can compute \( a_0 \) by (5b), but we can see from the figure (dividing the net area over one period by 16) that the average height is \(-\frac{1}{4}\) so \( a_0 = -\frac{1}{4} \). Further, we see that \( f \) is even, in which case (14) gives \( b_n = 0 \) and

\[
\begin{align*}
\alpha_n &= \frac{2}{\ell} \int_0^\ell f(x) \cos \frac{n\pi x}{\ell} \, dx = \frac{2}{8} \int_0^8 f(x) \cos \frac{n\pi x}{8} \, dx \\
&= \frac{1}{4} \left[ \int_0^4 (2 - x) \cos \frac{n\pi x}{8} \, dx + \int_4^8 (x - 6) \cos \frac{n\pi x}{8} \, dx + \int_6^8 0 \cos \frac{n\pi x}{8} \, dx \right] \\
&= \frac{1}{4} \left[ \frac{1}{n^2/8} \sin \frac{n\pi x}{8} - \frac{1}{(n\pi/8)^2} \left( \cos \frac{n\pi x}{8} + \frac{n\pi}{8} \sin \frac{n\pi x}{8} \right) \right]_0^4 \\
&\quad + \frac{1}{4} \left[ \cos \frac{n\pi x}{8} + \frac{n\pi x}{8} \sin \frac{n\pi x}{8} \sin \frac{n\pi x}{8} - \frac{1}{(n\pi/8)^2} \left( \cos \frac{n\pi x}{8} + \frac{n\pi}{8} \sin \frac{n\pi x}{8} \right) \right]_4^6 \\
&= \frac{16}{n^2\pi^2} \left( 1 - 2 \cos \frac{n\pi}{2} + \cos \frac{3n\pi}{4} \right). \quad (18)
\end{align*}
\]

Thus,

\[
f(x) = -\frac{1}{4} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - 2 \cos \frac{n\pi}{2} + \cos \frac{3n\pi}{4}}{n^2} \cos \frac{n\pi x}{8}, \quad (19)
\]

with the equality holding for all \( x \), without exception.

COMMENT. In Example 1 we simplified \( \sin n\pi \) and \( \cos n\pi \) as 0 and \((-1)^n\), respectively, but the \( \cos (n\pi/2) \) and \( \cos (3n\pi/4) \) terms in (19) are trickier (depending on \( n \), \( \cos (n\pi/2) \) takes on the values 0, \( \pm 1 \), and \( \cos (3n\pi/4) \) takes on the values 0, \( \pm 1 \), \( \pm \sqrt{2}/2 \), so we will leave them intact. \( \blacksquare \)

Recall from the calculus that the \textbf{p-series},

\[
\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad (20)
\]

converges if \( p > 1 \) and diverges if \( p \leq 1 \), the borderline divergent case of \( p = 1 \) corresponding to the \textbf{harmonic series}

\[
\sum_{n=1}^{\infty} \frac{1}{n}. \quad (21)
\]

Further, you may recall that the \textit{alternating} harmonic series

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n}. \quad (22)
\]
is convergent—barely convergent in that it converges extremely slowly,* and is only conditionally convergent rather than absolutely convergent. The Fourier series (11) is similar to the alternating series (22) and, like (22), converges "by the skin of its teeth,"† which result should not be surprising since (11) amounts to the representation of a function having jump discontinuities in terms of a collection of infinitely smooth sines. The function $f$ in Example 2 is better behaved in that it is continuous and, sure enough, the coefficients in (19) die out faster, proportional to $1/n^2$. That pattern persists: as we consider periodic functions that are better and better behaved ($C^0, C^1, C^2, \ldots$) we find that their Fourier coefficients die out (as $n \to \infty$) more and more rapidly. In general, if a periodic function is $C^N$, we can expect its Fourier coefficients to tend to zero at least as fast as $1/n^{N+2}$. As an extreme case, observe that the periodic function $f(x) = 5 \sin 3x$, say, is infinitely smooth (i.e., it is of class $C^\infty$), and its Fourier coefficients do tend to zero infinitely fast. That is, its Fourier series is simply one term, $FS\{5 \sin 3x\} = 5 \sin 3x$.

17.3.2. Euler’s formulas. We chose to state the Fourier convergence theorem and to move quickly into examples to solidify the Fourier series concept. Now let us back up and show where Euler’s formulas (5b,c,d) come from. Accepting that a (sufficiently well-behaved) $2\ell$-periodic function $f$ can be represented in the Fourier series form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right),$$

we focus attention on how to compute the coefficients $a_0$, $a_n$, and $b_n$. We will need the following elementary integral formulas:

$$\int_{-\ell}^{\ell} \cos \frac{m\pi x}{\ell} \cos \frac{n\pi x}{\ell} \, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \neq 0 \\ 2\ell & m = n = 0 \end{cases} \quad (24a)$$

$$\int_{-\ell}^{\ell} \sin \frac{m\pi x}{\ell} \sin \frac{n\pi x}{\ell} \, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \neq 0 \end{cases} \quad (24b)$$

$$\int_{-\ell}^{\ell} \cos \frac{m\pi x}{\ell} \sin \frac{n\pi x}{\ell} \, dx = 0 \quad \text{for all } m, n. \quad (24c)$$

where $m$ and $n$ are integers. The three zero results in (24) are due to cancellation of positive and negative areas; the two nonzero results, in (24a) and (24b), occur when $m = n$, in which case the integrands are squared quantities and no such cancellation can occur.

*For three-significant-figure accuracy we need to sum around $10^3$ terms; for six-significant-figure accuracy we need around $10^6$ terms, and so on.

†The $1/n$ decay, in (22) and (11), is not enough to induce convergence. Rather, these series converge (according to the Dirichlet test) because $(-1)^n$ and $\sin nx$ have bounded partial sums; that is, there exist finite numbers $A$ and $B$ such that $|\sum_{n=1}^{N} (-1)^n| < A$ and $|\sum_{n=1}^{N} \sin nx| < B$ for all $N$'s, no matter how large.

For definiteness, let us solve (23) for $a_2$. To do so, multiply both sides of (23) by $\cos \left(\frac{2\pi x}{\ell}\right)$ and integrate over one period. That step gives

$$
\int_{-\ell}^{\ell} f(x) \cos \left(\frac{2\pi x}{\ell}\right) dx = \int_{-\ell}^{\ell} \left[ a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \right] \cos \left(\frac{2\pi x}{\ell}\right) dx
$$

$$
= a_0 \int_{-\ell}^{\ell} \cos \left(\frac{2\pi x}{\ell}\right) dx + a_1 \int_{-\ell}^{\ell} \cos \left(\frac{n\pi x}{\ell}\right) \cos \left(\frac{2\pi x}{\ell}\right) dx
$$

$$
+ b_1 \int_{-\ell}^{\ell} \sin \left(\frac{n\pi x}{\ell}\right) \cos \left(\frac{2\pi x}{\ell}\right) dx + a_2 \int_{-\ell}^{\ell} \cos \left(\frac{3\pi x}{\ell}\right) \cos \left(\frac{2\pi x}{\ell}\right) dx
$$

$$
+ b_2 \int_{-\ell}^{\ell} \sin \left(\frac{3\pi x}{\ell}\right) \cos \left(\frac{2\pi x}{\ell}\right) dx + a_3 \int_{-\ell}^{\ell} \cos \left(\frac{5\pi x}{\ell}\right) \cos \left(\frac{2\pi x}{\ell}\right) dx + \cdots
$$

$$
= 0 + 0 + a_2 \ell + 0 + 0 + \cdots
$$

$$
= a_2 \ell,
$$

so

$$
a_2 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \left(\frac{2\pi x}{\ell}\right) dx,
$$

where the zeros following the third equal sign follow from (24a,b,c), and the $a_2 \ell$ term follows from (24a) with $m = n = 2$. [The first zero follows from (24a) with $m = 0$ and $n = 2$.] More generally, to solve for $a_n$, multiply (23) by $\cos \left(\frac{n\pi x}{\ell}\right)$ and integrate over one period; to solve for $b_n$, multiply by $\sin \left(\frac{n\pi x}{\ell}\right)$ instead, and to solve for $a_0$ multiply by 1 instead. These steps, with the help of (24), give the Euler formulas (5b,c,d).

The interchange in the order of summation and integration, in the second step in (25) needs justification, but we will postpone discussion of such technical points until Section 17.5.

There is a strong analogy between the calculation of the coefficients $c_n$ in the expansion

$$
\mathbf{v} = \sum_{n=1}^{N} c_n \mathbf{e}_n
$$

of a vector $\mathbf{v}$ in an $N$-dimensional vector space, in terms of an orthogonal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$. To solve for $c_2$, for instance, we take advantage of the orthogonality of the basis vectors, and dot both sides with $\mathbf{e}_2$. That step gives

$$
\mathbf{v} \cdot \mathbf{e}_2 = c_1 \mathbf{e}_1 \cdot \mathbf{e}_2 + c_2 \mathbf{e}_2 \cdot \mathbf{e}_2 + c_3 \mathbf{e}_3 \cdot \mathbf{e}_2 + \cdots + c_N \mathbf{e}_N \cdot \mathbf{e}_2
$$

$$
= 0 + c_2 \mathbf{e}_2 \cdot \mathbf{e}_2 + 0 + \cdots + 0
$$

$$
= c_2 \mathbf{e}_2 \cdot \mathbf{e}_2,
$$

so

$$
c_2 = \frac{\mathbf{v} \cdot \mathbf{e}_2}{\mathbf{e}_2 \cdot \mathbf{e}_2}
$$
and similarly for \( c_1, c_2, c_4, \ldots, c_N \). In fact, in Sections 17.6 and 17.7 we will take another look at Fourier series from exactly that point of view. There, we will understand (23) as the expansion of a vector \( f \) in an infinite-dimensional function space in terms of the base vectors \( 1, \cos (\pi x/\ell), \sin (\pi x/\ell), \cos (2\pi x/\ell), \sin (2\pi x/\ell), \ldots \), which are orthogonal by virtue of equations (24); in fact, (24) are often called orthogonality relations.

### 17.3. Applications.

Fourier series are indispensable in our study of PDE’s (partial differential equations) in the next three chapters. Here, we give two applications to physical systems that are governed by ODE’s and subjected to periodic forcing functions. Our approach will be formal, by which we mean that although we believe the results to be correct, we will not rigorously justify all of the steps involved in their derivation.

#### Example 3. Periodically Driven Oscillator.

Consider the driven mechanical oscillator shown in Fig. 7 and governed by the differential equation of motion

\[
mx'' + cx' + kx = F(t),
\]

where \( m, c, k \) are the mass, damping coefficient (associated with some combination of viscous damping due to a film of lubricating oil between the mass and the table, and air resistance), and spring stiffness, respectively. Let \( m = 1 \text{ kg}, c = 0.04 \text{ kg/sec}, \) and \( k = 15 \text{ kg/sec}^2 \), and let \( F(t) \) (in newtons) be as shown in Fig. 8. \( F \) consists of an endless sequence of pulses, each having unit area (except the first, which is only a “half pulse”). In mechanics, \( \int_{t_1}^{t_2} F(t) \, dt \) is the impulse delivered by the force \( F \) between times \( t_1 \) and \( t_2 \), so \( F \) consists of a periodic sequence of unit impulses, of period 2\( \pi \). Thus, \( \ell = \pi \) in (5). Even though the starting time is \( t = 0 \), so \( t \geq 0 \), we can think of \( F \) as the extended function shown in Fig. 9, which is even. Thus, if we expand \( F \) in a Fourier series we have, for its coefficients, \( a_0 = \text{average value} = 1/(2\pi), b_n = 0 \) because \( F \) is even, and, from (14),

\[
a_n = \frac{2}{\pi} \int_0^\pi F(t) \cos nt \, dt = \frac{2}{\pi} \int_0^\infty \frac{1}{2n} \cos nt \, dt = \frac{\sin n\alpha}{n\pi a}.
\]

Thus, (30) becomes

\[
x'' + 0.04x' + 15x = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{\sin n\alpha}{n\pi a} \cos nt.
\]

Recall, from the general theory in Chapter 3, that if

\[
L[x] = f_1 + \cdots + f_N,
\]

where \( L \) is shorthand for a linear second-order differential operator, then the general solution of (33) is of the form

\[
x(t) = C_1x_{n1}(t) + C_2x_{n2}(t) + \cdots + x_{N}(t),
\]

where \( C_1, C_2 \) are arbitrary constants, \( x_{n1} \) and \( x_{n2} \) are linearly independent homogeneous solutions, and \( x_{p1}, \ldots, x_{pN} \) are particular solutions corresponding to the forcing functions
Chapter 17. Fourier Series, Fourier Integral, Fourier Transform

\[ f_1, \ldots, f_N, \text{ respectively. If two initial conditions are prescribed, they enable us to determine } C_1 \text{ and } C_2. \]

In the present example \( x_{h1} \) and \( x_{h2} \) are oscillatory, but with a slow exponential decay due to the 0.04s damping term in (32). As \( t \to \infty \) those terms decay to zero, leaving us with the particular solution as the steady-state response. Let us limit our attention to finding, and discussing, the steady-state response.

Our plan is to find a particular solution corresponding to each forcing term on the right side of (32), using the method of undetermined coefficients, and then to superimpose those solutions, as in (34). Consider

\[ x'' + 0.04x' + 15x = \cos nt, \quad (35) \]

and seek \( x_p \) in the form \( A \cos nt + B \sin nt \). Putting that form into the left-hand side of (35) enables us to solve for \( A \) and \( B \), and we obtain

\[ x_p(t) = \frac{15 - n^2}{(15 - n^2)^2 + 0.0016n^2} \cos nt + \frac{0.04n}{(15 - n^2)^2 + 0.0016n^2} \sin nt. \quad (36) \]

Further, we find that a particular solution corresponding to the \( 1/2\pi \) forcing term in (32) is \( x_p(t) = 1/30\pi \), so the desired steady-state response is, by linearity and superposition,

\[ x(t) = \frac{1}{30\pi} + \sum_{n=1}^{\infty} \frac{\sin n\pi}{n\pi} \left( \frac{15 - n^2}{(15 - n^2)^2 + 0.0016n^2} \cos nt + \frac{0.04n}{(15 - n^2)^2 + 0.0016n^2} \sin nt \right). \quad (37) \]

Choosing a specific value for \( a \), we can use (37) to compute \( x(t) \). However, for purposes of understanding it will be useful to express the square-bracketed term in (37), equivalently, in the form \( A_n \cos (nt + \phi_n) \), where the amplitude \( A_n \) and the phase \( \phi_n \) are

\[ A_n = \frac{1}{\sqrt{(15 - n^2)^2 + 0.0016n^2}} \quad \text{and} \quad \phi_n = \tan^{-1} \left( \frac{0.04n}{n^2 - 15} \right). \quad (38) \]

Let \( a = \pi/8 \), say. Then, using (38) we can write out (37) as

\[ x(t) = 0.0106 + 0.0222 \cos (t - 0.0029) + 0.0261 \cos (2t - 0.0073) + 0.0416 \cos (3t - 0.0200) + 0.2001 \cos (4t + 3.3002) + 0.0150 \cos (5t + 3.1616) + \cdots. \quad (39) \]

Consider the terms on the right-hand side of (39). The first, 0.0106, is the response to the \( 1/2\pi \) forcing term in (32), the second is the response to the \( n = 1 \) term in (32), the third is the response to the \( n = 2 \) term in (32), and so on. Observe that if we ignore the 0.04x damping term in (35) then the natural frequency of the oscillator is \( \sqrt{15} \) and that the \( \cos nt \) forcing function comes close to that natural frequency when \( n = 4 \). Sure enough, the \( n = 4 \) term in (39) has by far the largest amplitude, 0.2001. Observe further that the phase \( \phi_n \) is rather small for \( n < 4 \). That is, the response term is almost in phase with the forcing term

---

*If this sounds unfamiliar we urge you to review Section 3.5.
because the damping is light. However, for \( n \geq 4 \) the \( \phi_n \)'s are approximately \( \pi \). That is, when the frequency of the forcing terms exceeds the approximate natural frequency \( \sqrt{15} \) the phase increases to around \( \pi \), so that the response is around 180° out of phase with the forcing function.*

Plotting the steady-state response \( x(t) \), given by (39), in Fig. 10, we can see how the response is dominated by the 0.2001 \( \cos(4t + 3.3002) \) term, as discussed above. Observe that it suffices to plot \( x(t) \) over any \( 2\pi \) interval since it is 2\( \pi \)-periodic.

**COMMENT 1.** Why did we expand \( F(t) \) in a Fourier series? That step gave us \( F(t) \) as a linear combination of elementary functions, the response to each of which was readily found, with the total response then built up by superposition. Alternatively, the Laplace transform method would also have been convenient. (See Exercise 17 for other ideas.)

**COMMENT 2.** We stated that our solution would be formal rather than rigorous. The point that we did not justify is as follows. The expression (34) satisfies (33) because

\[
L[x] = L \left[ C_1 x_{h1} + C_1 x_{h2} + \sum_{n=1}^{N} x_{pn} \right]
= C_1 L[x_{h1}] + C_2 L[x_{h2}] + \sum_{n=1}^{N} L[x_{pn}]
= 0 + 0 + \sum_{n=1}^{N} L[x_{pn}] = \sum_{n=1}^{N} f_n.
\]

However, in the present case \( N = \infty \), so the step

\[
L \left[ \sum_{n=1}^{\infty} x_{pn} \right] = \sum_{n=1}^{\infty} L[x_{pn}]
\]

amounts to an interchange in the order of two limit operations, the derivatives in \( L \) and the infinite series. The validity of such interchange will not be covered until Section 17.5.

**COMMENT 3.** Two limiting cases are of interest (and are available to us because we left \( a \) as a parameter). As \( a \to \pi \), \( F(t) \) tends to the constant \( 1/2\pi \). In that case the series in (32) vanishes and the steady-state response is simply \( x(t) = 1/30\pi \). More subtle is the case where \( a \to 0 \) since then \( F(t) \) becomes a sequence of hammer blows, each imparting unit impulse. Discussion of this case is left for the exercises.

**COMMENT 4.** The electrical analog of the forced mechanical oscillator, governed by (30) and shown in Fig. 7, is the equation

\[
LQ'' + RQ' + \frac{1}{C} Q = E(t)
\]

governing the circuit shown in Fig. 11, where \( Q(t) \) is the charge on the capacitor. *

---

*See Fig. 2 in Section 3.8. To understand the present example you may need to review both Sections 3.5 and 3.8.
EXAMPLE 4. Infinite Beam on Elastic Foundation. Consider an infinitely long beam on an elastic foundation, sketched in Fig. 12. The constant $k$ is called the foundation modulus (i.e., the spring stiffness per unit $x$-length) and the square wave $w(x)$ is a prescribed periodic loading (force per unit length).* Physically, the beam might be a train track, with the elastic foundation used to model the track bed. We wish to determine the vertical deflection $u(x)$.

![Figure 12. Infinite beam on elastic foundation.](image)

According to the classical Euler beam theory\footnote{That is, the downward load between any points $x_1$ and $x_2$ ($> x_1$) is $\int_{x_1}^{x_2} w(x) \, dx$.} the deflection $u(x)$ resulting from a load distribution $p(x)$ newtons per meter satisfies the fourth-order differential equation

$$EIu''' = p(x),$$

where $E$ and $I$ are physical constants of the beam; $EI$ is called the flexural rigidity of the beam (and is considered here as a known constant) since $u'''$, and hence the deflection $u$, is inversely proportional to $EI$. Now $p(x)$ is the net loading and consists of the applied periodic loading $w(x)$ downward and the spring force $ku(x)$ upward. (We neglect the weight of the beam, for simplicity.) Thus, $p(x) = w(x) - ku(x)$, and (43) becomes

$$EIu''' + ku = w(x).$$

This problem is similar to the preceding one in that they both involve differential equations with periodic forcing functions, but in this case the independent variable is $x$ and the interval is $-\infty < x < \infty$. As in Example 3, we begin by expanding the forcing function in a Fourier series,

$$w(x) = \frac{w_0}{2} + \frac{2w_0}{\pi} \sum_{n=1}^{\infty} \frac{\sin \left(\frac{n\pi}{2}\right)}{n} \cos \left(\frac{n\pi x}{2a}\right).$$

Rather than find the response to each forcing term and adding them up, let us use a slightly different procedure (which could have been used in Example 3 as well). Namely, anticipating that $u(x)$ will be an even periodic function, of the same period as $w(x)$ (i.e., $4a$), let us seek $u(x)$ in the form

$$u(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi x}{2a}\right).$$

\[^{\dagger}\text{See, for example, S. Timoshenko, Strength of Materials, Part I (Princeton, NJ: D. Van Nostrand, 1955).}\]
Formally differentiating (46) termwise four times, putting this result and (45) into (44), and formally equating coefficients of each cosine harmonic gives

\[
k a_0 = \frac{w_0}{2}\]

and

\[
\left( EI \frac{n^4 \pi^4}{16a^4} + k \right) a_n = \frac{2w_0}{\pi} \frac{\sin (n\pi/2)}{n},
\]

so

\[
a_0 = \frac{w_0}{2k}, \quad a_n = \frac{2w_0}{\pi} \frac{\sin (n\pi/2)}{n\left[EI(n^4 \pi^4/16a^4) + k\right]} \quad (n \geq 1)
\]

Thus we have the formal solution

\[
u(x) = \frac{w_0}{2k} + \frac{32w_0a^4}{\pi \left(\pi^4 EI + 16a^4 k\right)} \sum_{n=1}^{\infty} \frac{\sin (n\pi/2)}{n\left[EI(n^4 \pi^4/16a^4) + k\right]} \cos \frac{n\pi x}{2a}.
\]

COMMENT 1. It is striking that the terms in the series die out so rapidly, proportional to \(1/n^2\) as \(n \to \infty\), so that we can expect merely the first couple of terms of (50) to give a good approximation to \(u(x)\),

\[
u(x) \approx \frac{w_0}{2k} + \frac{32w_0a^4}{\pi \left(\pi^4 EI + 16a^4 k\right)} \frac{\pi x}{2a}.
\]

How are we to understand the input \(w(x)\) being discontinuous and the output \(u(x)\) being quite smooth? Physically, we don’t expect \(u(x)\) to be discontinuous or “kinky” just because the loading \(w(x)\) is, because a train track is too “stiff” for that. Mathematically, observe that in essence (though not procedurally, unless \(k = 0\)) we solve (44) for \(u\) by four integrations of \(w\). Now, what happens when we repeatedly integrate a discontinuous function? Consider a Heaviside step function, for simplicity, in place of \(w\). Integrating from \(-\infty\), say, to a variable point \(x\), gives

\[
\int_{-\infty}^{x} H(\xi) \, d\xi \begin{cases} 
0, & x < 0 \\
x, & x > 0 
\end{cases} = xH(x),
\]

which is a “ramp” function. Integrating \(xH(x)\), in turn, gives \((x^2/2)H(x)\); integrating \((x^2/2)H(x)\) gives \((x^3/6)H(x)\), and so on, as displayed in Fig. 13. That is, integration is a smoothing operation: \(H(x)\) is discontinuous, its integral \(xH(x)\) is \(C^0\) (continuous), the integral of the latter is \(C^1\), the integral of the latter is \(C^2\), and so on.* From this rough argument, we expect the response \(u(x)\) to the discontinuous load \(w(x)\) to be \(C^3\), so its Fourier coefficients should tend to zero like \(1/n^3\), and that is precisely what is revealed by (50).

COMMENT 2. Actually, (50) is a particular solution of (44). To obtain the general solution we need to add the homogeneous solution

\[
e^{\beta x}(A \sin \beta x + B \sin \beta x) + e^{-\beta x}(C \sin \beta x + D \sin \beta x),
\]

*Conversely, differentiation has the opposite effect. For instance, \((x^3/6)H(x)\) is smooth but has a singular behavior at the origin, that is brought to light by repeated differentiations.

Figure 13. The smoothing effect of integration.
Chapter 17. Fourier Series, Fourier Integral, Fourier Transform

where $\beta \equiv \sqrt{k/E}/\ell$. To determine $A, B, C, D$ we need some sort of boundary conditions at the ends of the interval, that is, as $x \to \pm \infty$. By way of such conditions, let us require $u$ to be bounded as $x \to \pm \infty$, as would be a reasonable requirement for a loaded train track. Then the behavior $e^{\beta x} \to \infty$ as $x \to +\infty$ implies that we need $A = B = 0$, and the behavior $e^{-\beta x} \to \infty$ as $x \to -\infty$ implies that we need $C = D = 0$. Thus, (53) is eliminated entirely, and we are left with (50).

COMMENT 3. Thus, this example is a boundary-value problem in which the homogeneous solution drops out by virtue of the boundary conditions. Example 3 is an initial-value problem in which the homogeneous solution is not zero, but tended to zero as $t \to \infty$, leaving the particular solution as the steady-state solution.

17.3.4. Complex exponential form for Fourier series. Using the definitions

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

it is possible to re-express the Fourier series formula (5) in terms of complex exponentials, as follows:

$$\text{FS } f = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell},$$

where

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell} \, dx.$$  \hspace{1cm} (55b)

Although the $c_n$'s and exponentials in (55a) are complex, the series does have a real-valued sum.

Notice that the usual definition

$$\sum_{n=1}^{\infty} A_n \equiv \lim_{N \to \infty} \sum_{n=1}^{N} A_n$$

does not apply to (55a) because the lower limit is infinite as well. From our derivation of (55), below, we will see that the appropriate meaning of the series in (55a) is

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell} \equiv \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{in\pi x/\ell}.$$  \hspace{1cm} (56)

Let us proceed:

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

$$= \lim_{N \to \infty} \sum_{n=0}^{N} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$
\[\begin{align*}
&= \lim_{N \to \infty} \sum_{n=0}^{N} \left( a_n e^{in\pi x/\ell} + e^{-i\pi x/\ell} \right) + \frac{b_n}{2} \left( e^{i\pi x/\ell} - e^{-i\pi x/\ell} \right) \\
&= \lim_{N \to \infty} \left[ \sum_{n=0}^{N} \left( a_n - \frac{ib_n}{2} \right) e^{in\pi x/\ell} + \sum_{n=0}^{N} \left( a_n + \frac{ib_n}{2} \right) e^{-i\pi x/\ell} \right]. \quad (57)
\end{align*}\]

Changing \(n\) to \(-n\) in the second sum, which is permissible because \(n\) is merely a dummy summation index,

\[\begin{align*}
\text{FS } f &= \lim_{N \to \infty} \sum_{n=0}^{N} \left( a_n - \frac{i b_n}{2} \right) e^{in\pi x/\ell} + \sum_{n=0}^{N} \left( a_{-n} + \frac{i b_{-n}}{2} \right) e^{i\pi x/\ell} \\
&= \lim_{N \to \infty} \sum_{n=-N}^{N} c_n e^{i\pi x/\ell} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{i\pi x/\ell}. \quad (58)
\end{align*}\]

For \(n = 0\),

\[c_0 = a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \, dx, \quad (59)\]

for \(n > 0\),

\[c_n = \frac{a_n - \frac{i b_n}{2}}{2} = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \left( \cos \frac{n\pi x}{\ell} - i \sin \frac{n\pi x}{\ell} \right) \, dx = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-i\pi x/\ell} \, dx, \quad (60)\]

and for \(n < 0\),

\[c_n = \frac{a_{-n} + \frac{i b_{-n}}{2}}{2} = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \left[ \cos \left( -\frac{n\pi x}{\ell} \right) + i \sin \left( -\frac{n\pi x}{\ell} \right) \right] \, dx = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-i\pi x/\ell} \, dx. \quad (61)\]

All three cases, (59)–(61), agree with (55b), as was to be shown.

The complex form (55) is sometimes favored, especially for electrical engineering applications. An advantage of (55) over (5) is that (55) contains only the exponentials and one set of coefficients (the \(c_n\)'s), whereas (5) contains both cosines and sines and two sets of coefficients (the \(a_n\)'s and \(b_n\)'s).

**EXAMPLE 5.** Let us return to the square wave of Example 1 and work out its complex Fourier series. With \(\ell = \pi\), and \(f\) displayed in Fig. 1,

\[c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-i\pi x/\ell} \, dx = \frac{1}{2\pi} \left[ \int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} 4 e^{-i\pi x} \, dx \right] \]
Chapter 17. Fourier Series, Fourier Integral, Fourier Transform

\[ F(\pi) = \frac{2}{\pi} \left( e^{-in\pi} \right) = \begin{cases} 0, & n = \pm 2, \pm 4, \ldots \\ -\frac{4i}{n\pi}, & n = \pm 1, \pm 3, \ldots \end{cases} \tag{62} \]

For \( n = 0 \), the quantity following the third equal sign is indeterminate; it is \((1 - 1)/0 = 0/0\). In that case, in place of \( \int 4e^{-in\pi} \, dx = 4e^{-in\pi}/(-in) \) we should use \( \int 4e^0 \, dx = 4x \), which gives \( c_0 = 2 \). Thus,

\[ f(x) = 2 + \sum_{n=-\infty}^{\infty} \left( -\frac{4i}{n\pi} \right) e^{inx}, \tag{63} \]

and it is not difficult to verify that (63) is equivalent to (11).

**Closure.** The Fourier series of a \( 2\ell \)-periodic function \( f \), given by (5), converges to \( f(x) \) at every point \( x \) at which \( f \) is continuous, if \( f \) and \( f' \) are piecewise continuous on \([-\ell, \ell]\). These conditions are sufficient, not necessary, and are met by virtually all functions of applied interest. If \( f \) is discontinuous at \( x \), then the series converges to the mean value \([f(x^+) + f(x^-)]/2\). If the latter does not equal the defined value of \( f(x) \), then a discrepancy exists at that point. However, such pointwise discrepancies will be inconsequential in applications.

Beyond the question of convergence, the speed of convergence depends on how well-behaved the function is and is of great practical importance. Surely it matters to us whether we need to add 100,000 terms to achieve a desired accuracy, or whether three or four terms will suffice.

Specifically, if a periodic function is \( C^N \), then we can expect its Fourier coefficients to tend to zero at least as fast as \( 1/n^{N+2} \). In the loaded beam example (Example 4), for instance, the solution \( u(x) \) is \( C^3 \), so its Fourier coefficients decay like \( 1/n^3 \). Because of that rapid decay, it suffices to retain only the first two or three terms of the series.

The Fourier coefficients of a given periodic function often contain such quantities as \( \cos n\pi \), \( \sin n\pi \), \( \cos (n\pi/2) \), \( \sin (n\pi/2) \), \( \cos (3n\pi/4) \), and so on. Of these, \( \cos n\pi = (-1)^n \) and \( \sin n\pi = 0 \), but the others are not so readily expressed in algebraic form, so we will leave them intact, and we suggest that you do the same – in working out the exercises.

**EXERCISES 17.3**

1. Are these functions piecewise continuous on \([0, \pi]\)? Explain.
   - (a) \( \sin^2 x \)
   - (b) \( \tan x \)
   - (c) \( \sin \frac{1}{x} \)
   - (d) \( \cos \frac{1}{x} \)
   - (e) \( e^{-x} \)
   - (f) \( 1/(x - 1) \)
   - (g) \( 1/x \)
   - (h) \( \sqrt{x} \)

2. (a) Derive (24a). HINT: \( \cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)] \)
   - (b) Derive (24b). HINT: \( \sin A \sin B = \frac{1}{2} [\cos (A - B) - \cos (A + B)] \)
   - (c) Derive (24c).
3. If both left- and right-hand derivatives of \( f \) exist at \( x_0 \) and are equal, does that imply that \( f \) is differentiable at \( x_0 \) ? Explain.

4. Work out the Fourier series of \( f \), given over one period as follows. At which values of \( x \), if any, does the series fail to converge to \( f(x) \)? To what values does it converge at those points?

(a) \( x \) on \([-\pi, \pi]\) 

(b) \( |x| \) on \([-\pi, \pi]\) 

(c) \( |x| \) on \([-\pi, \pi]\) 

(d) \( 50 \) on \((0, 2]\) 

(e) \( 50 \) on \([-8, -2) \cup (2, 8]\) 

(f) \( \sin x \) on \([0, \pi]\) 

(g) \( \cos x \) on \([0, \pi]\) 

(h) \( e^{-x} \) on \([0, \pi]\) 

(i) \( \sin^2 x \) on \([0, \pi]\) 

(j) \( 20 + 3 \sin 4x \) for all \( x \) 

(k) \( x \) on \([0, \pi]\), \( 0 \leq x \leq 2 \) 

(l) \( \cos^2 x \) on \([0, \pi]\) 

(m) \( \sin^2 x \) on \([0, \pi]\) 

(n) \( e^{-x} \) on \([0, \pi]\) 

(o) \( 100 \) on \([0, \pi]\) 

5. Work out the Fourier series of the \( 2\pi \)-periodic function \( f \) defined on \(-\pi < x \leq \pi\) as follows, using computer software (such as Maple int command) to evaluate the integrals.

HINT: For parts (c) and (d) you will need to use l'Hôpital's rule to simplify the result.

6. Obtain a computer plot of the partial sums of the Fourier series (19) of the periodic function shown in Fig. 6, for

(a) \( n = 2 \) 

(b) \( n = 5 \) 

(c) \( n = 10 \) 

(d) \( n = 20 \) 

(e) \( n = 30 \) 

(f) \( n = 50 \) 

7. (a) Use (11) to show that

\[
\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n \pi}{2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.
\]  

(b) If you were to use (7.1) to compute \( \pi \), how many terms would be needed to obtain accuracy to six significant figures? Explain.

8. Let \( f \) be the periodic function shown in the figure, each segment of which is a semicircle of radius \( \pi \). Show that its Fourier series is

\[
f(x) = \frac{\pi^2}{4} + \frac{\pi^2}{2} \sum_{n=1}^{\infty} \left[ J_0(n\pi) + J_2(n\pi) \right] \cos nx,
\]  

where the \( J_n \)'s denote Bessel functions of the first kind.

HINT: You may use any of these formulas:

\[
\begin{align*}
\cos (a \sin \theta) &= J_0(2a \cos \theta) + 2 J_2(2a \cos \theta) \cos 2\theta + \ldots, \\
n \sin (a \sin \theta) &= 2 J_1(2a \cos \theta) \sin \theta + 2 J_3(2a \cos \theta) \sin 3\theta + \ldots, \\
n \cos (a \cos \theta) &= J_0(2a \cos \theta) + 2 J_2(2a \cos \theta) \cos 2\theta - \ldots, \\
n \sin (a \cos \theta) &= 2 J_1(2a \cos \theta) \cos \theta - 2 J_3(2a \cos \theta) \cos 3\theta - \ldots,
\end{align*}
\]

where the \( J_n \)'s are shorthand for \( J_{n}(a) \). NOTE: Observe that Theorem 17.3.1 does not apply in this case since \( f' \) is not piecewise continuous on \([-\pi, \pi]\), but it follows from a stronger version of that theorem that the series in (8.1) does converge to \( f(x) \) for all \( x \).

9. If the Fourier coefficients of a periodic function \( f(x) \) are \( a_n \) \((n = 0, 1, 2, \ldots)\) and \( b_n \) \((n = 1, 2, \ldots)\), what are the Fourier coefficients \( A_n, B_n \), say, of the shifted periodic function \( f(x - c) \)?

10. We have been interpreting the period \( 2\ell \), in equations (5), to be the fundamental period. However, surely (5) should yield the same result if we use twice the fundamental period, and so on. The purpose of this exercise is not to prove that claim but merely to illustrate its truth through a concrete example. Specifically, use (5) to determine the Fourier series of the square wave shown in Fig. 1, using \( 2\ell = 4\pi \) (rather than \( 2\pi \)), and show that you obtain exactly the same final result as was given in (11).

11. (Polynomials) It is a useful fact that if \( p(x) \) is an even polynomial (i.e., containing only even-integer powers of \( x \)) on \((-\ell, \ell)\), then the \( b_n \)'s are zero and

\[
a_n = \frac{2\ell}{n^2 \pi^2} (-1)^n \left[ p'(\ell) - \frac{\ell^2}{n^2 \pi^2} p''(\ell) + \frac{\ell^4}{n^4 \pi^4} p^{(4)}(\ell) - \ldots \right]
\]  

for \( n \geq 1 \); and if \( p(x) \) is an odd polynomial on \((-\ell, \ell)\), then the \( a_n \)'s are zero and

\[
b_n = \frac{2}{n \pi} (-1)^n \left[ p(\ell) - \frac{\ell^2}{n^2 \pi^2} p''(\ell) + \frac{\ell^4}{n^4 \pi^4} p^{(4)}(\ell) - \ldots \right]
\]  

for \( n \geq 1 \).
(a) Derive (11.1) through the $p^n$ term.
(b) Derive (11.2) through the $p^n$ term.

12. Use (11.1) and/or (11.2) in Exercise 11 to obtain the Fourier series of the given periodic function. NOTE: (11.1) does not hold for $n = 0$, so you need to compute $a_0$ in the usual way.

(g) $f(x) = x$ on $(-3, 3)$
(b) $f(x) = x^2$ on $(-3, 3)$
(c) $f(x) = 6 + 2x - x^3$ on $(-2, 2)$
(d) $f(x) = x^2(x^2 - 8)$ on $(-2, 2)$
(e) $f(x) = x(x^2 - 3)$ on $(-1, 1)$
(f) $f(x) = 2 + x - x^2$ on $(-3, 3)$
(g) $f(x) = x + x^2$ on $(-1, 1)$

13. (Relaxation oscillator) There exist a great many periodic occurrences, called relaxation oscillations, which are characterized by a slow buildup and a rapid discharge. Examples include epidemics, economic crises, the sound generated by the bowing of a violin string, shivering from the cold, menstruation, the beating of the heart, and the discharge of a capacitor through a neon tube. Here we consider the latter (see the following figure). The resistance $R$, capacitance $C$, and voltage $E_0$ are constants. If $Q(t)$ is the charge on the capacitor, then $i(t) = dQ/dt$ is the current in the $E_0, R, C$ loop, and Kirchhoff's law gives

$$R \frac{dQ}{dt} + \frac{1}{C} Q = E_0.$$  

(a) Solve (13.1) subject to the initial condition $Q(0) = 0$.
(b) When the voltage $V/C$ on the capacitor, and hence on the neon tube, reaches a certain level, say $E_0/2$, the neon tube fires, and the charge $Q$ on the capacitor drops to zero. The firing is so rapid compared to the "buildup time" that we may assume it to be instantaneous. From your solution to part (a), show that the neon tube fires at $t = RC \ln 2$.
(c) The cycle then repeats indefinitely. Sketch and label the graph of the periodic function $Q(t)$, and work out its Fourier series.

14. The voltage $E(\theta)$ is maintained at 100 volts on the top edge of the disk and at 20 volts on the bottom edge. Expand $E(\theta)$ in a Fourier series. HINT: Sketch the graph of $E(\theta)$ on $-\infty < \theta < \infty$.

15. (RMS current) If a steady electric current $i$ flows through a resistor of resistance $R$, the power delivered (i.e., the rate of doing work) is equal to $i^2R$. In many applications $i$ is not a constant, but a periodic function of the time $t$. In such cases one defines the average power as

$$\text{average power} = \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)R \, dt,$$  

where $T$ is the fundamental period of $i(t)$. Expressing $i(t)$ as a Fourier series,

$$i(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T} \right),$$  

show that the

$$\text{average power} = \left[ a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right] R,$$  

and that the steady current that is equivalent to $i(t)$, in that it will deliver the same power, is

$$I_{\text{rms}} = \sqrt{a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)}.$$  

$I_{\text{rms}}$ is known as the root-mean square (rms) current.

16. (Complex form) Work out the complex exponential form of the Fourier series (55), for the given periodic function, defined over one period as follows.

(a) $f(x) = 50$ on $|x| \leq 1$, and 0 on $0 < |x| \leq 2$
(b) $f(x) = e^x$ on $0 \leq x < 2$
(c) $f(x) = e^{-x}$ on $-3 \leq x < 3$
(d) $f(x) = 6 \sin x$ (for all $x$)
(e) $f(x) = 4 - 5 \cos 2x$ (for all $x$)
(f) $f(x) = -100$ on $-5 \leq x < 0$, and 100 on $0 \leq x < 5$
(g) $f(x) = 20 + 5 \pi x$ on $-2 \leq x < 2$

17. (Example 3) (a) In Example 3 we expand $F(t)$ in a Fourier series. Could we have used a Taylor series, profitably, instead? (b) It is proposed that we solve (30) in a piecewise manner. That is, since $F(t)$ is piecewise constant we might do well to consider the $t$ intervals $0 < t < a$, $a < t < 2\pi - a$, $2\pi - a < t < 2\pi + a$, and so on. Solving $mx'' + cx' + kx$
= \frac{1}{2a}$ on $0 < t < a$, subject to initial conditions $x(0) = x_0$ and $x'(0) = x'_0$, say, we can then use the final values $x(a)$ and $x'(a)$ from the first interval as the initial conditions for the next interval, $a < t < 2\pi - a$, on which $m a'' + c x' + k x = 0$, and so on. You don’t need to carry out this method; rather, we merely ask you to assess whether it is a good idea for determining the steady-state response.

(c) Would the method used in Example 3 work if (30) were modified to

$$m a'' + c x' + \alpha x + \beta x^3 = F(t)$$

Explain.

(d) If we let $a \to 0$ in Example 3, then (see Fig. 8) $F(t)$ becomes a series of delta functions (unit impulses), at $t = 0, 2\pi, 4\pi, \ldots$. (Actually, the one at $t = 0$ will be half a delta function.) Find the Fourier series representation of the steady-state response.

(c) Same as part (d), but find a closed form expression of the steady-state response $x(t)$ over one period, and sketch its graph.

18. Consider an undamped, driven mechanical oscillator, governed by the equation

$$m x'' + k x = F(t).$$

Solve for the steady-state response $x(t)$, if $F(t)$ is periodic and defined (over one period) as follows. Let $m = k = 1$.

(a) $F(t) = 100$ on $0 \leq t < 2$, and 0 on $2 \leq t < 4$
(b) $F(t) = 30$ on $0 \leq t < 2$, and $-30$ on $2 \leq t < 4$
(c) $F(t) = 5 t$ on $0 \leq t < 1$, and $10 - 5 t$ on $1 \leq t < 2$
(d) $F(t) = 5 t$ on $0 \leq t < 1$, and $5 t - 10$ on $1 \leq t < 2$
(e) $F(t) = 20$ on $0 \leq t < 2$, and $10$ on $2 \leq t < 4$, and 0 on $4 \leq t < 6$
(f) $F(t) = 10 t$ on $0 \leq t < 3$, and 0 on $3 \leq t < 4$

---

17.4 Half- and Quarter-Range Expansions

It often happens in applications, especially when we solve partial differential equations by the method of separation of variables (Chapters 18–20), that we need to expand a given function $f$ in a Fourier series, where $f$ is defined only on a finite interval such as the function $f$ whose graph is shown in Fig. 1. and which is defined only on $0 < x < L$. But in that case $f$ is not periodic, so how can we expand it in a Fourier series?

The idea is to extend the domain of definition of $f$ to $-\infty < x < \infty$ and to define an “extended function,” say $f_{ext}$, so that $f_{ext}$ is periodic, with $f_{ext}(x) = f(x)$ on the original interval $0 < x < L$. There are an infinite number of such extensions, two of which are shown in Fig. 2. Each of these is periodic, the first with period $2L$ and the second with period $L$. Their Fourier series are different, but each of them converges to the original function $f$ on the original interval $0 < x < L$.

How, then, do we know which extension to use? We shall see that the choice will be dictated by the context, so let us not worry about that right now. We will always need to choose from among four extensions which are known as half- and quarter-range cosine and sine extensions and which are based on symmetry or antisymmetry about the endpoints $x = 0$ and $x = L$. These are shown, for the representative function $f$ of Fig. 1, in Fig. 3. For instance, in Fig. 3a $f_{ext}$ is symmetric about $x = 0$ and also about $x = L$, hence the two $S$'s below the $x$ axis.

---

*We use the open interval notation throughout this section since the values of $f(x)$ at the endpoints will not affect the Fourier coefficients, as we learned in the foregoing section.
Because of its symmetry about \( x = 0 \), \( f_{\text{ext}} \) is an even function, and its Fourier series will contain only cosines, no sines. Further, its period is \( 2L \), so \( L \) is half the period. Thus, it is customary to designate this case as the “half-range cosine extension,” which we denote in this text by the letters HRC. In Fig. 3b, \( f_{\text{ext}} \) is antisymmetric about \( x = 0 \) and \( x = L \) the period is \( 2L \), and we have the half-range sine extension, denoted by HRS. In Fig. 3c \( f_{\text{ext}} \) is symmetric about \( x = 0 \) and antisymmetric about \( x = L \), the period is \( 4L \) (so \( L \) is only a quarter of the period), and we have the quarter-range cosine extension, denoted by QRS. Similarly for the quarter-range sine extension shown in Fig. 3d and denoted by QRS.

Let us derive the Fourier series for these cases. For the half-range cosine case the period is \( 2L \), so \( \ell = L \) (where we carry over the \( \ell \) notation from Section 17.3) and

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f_{\text{ext}}(x) \, dx = \frac{1}{L} \int_{0}^{L} f(x) \, dx, \tag{1a}
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f_{\text{ext}}(x) \cos \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \tag{1b}
\]

\[
b_n = \frac{1}{L} \int_{-L}^{L} f_{\text{ext}}(x) \sin \frac{n\pi x}{L} \, dx = 0. \tag{1c}
\]

The last step in (1a) follows because \( f_{\text{ext}}(x) \) is symmetric about \( x = 0 \). Similarly, the symmetry of \( f_{\text{ext}}(x) \cos \left( \frac{n\pi x}{L} \right) \) about \( x = 0 \) gives (1b), and the antisymmetry of \( f_{\text{ext}}(x) \sin \left( \frac{n\pi x}{L} \right) \) about \( x = 0 \) gives (1c). Thus, we can write

\[
f_{\text{ext}}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad (-\infty < x < \infty) \tag{2}
\]

with \( a_0 \) and \( a_n \) given by (1a) and (1b).

Understand that the extension was only an artifice, to make possible a Fourier representation of the original function \( f \) on the original interval \( 0 < x < L \). With the expansion in hand, we can now limit our attention to the \( 0 < x < L \) interval and write

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad (0 < x < L) \tag{3}
\]

\[
a_0 = \frac{1}{L} \int_{0}^{L} f(x) \, dx, \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx.
\]

We call (3) the \textbf{half-range cosine expansion} of \( f \). Observe that the final formulas in (3) contain no artifacts of the extension, only \( f \) defined on \( 0 < x < L \).

By a similar process we obtain the \textbf{half-range sine expansion}

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad (0 < x < L) \tag{4}
\]

\[
b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx,
\]
the quarter-range cosine expansion

\[
 f(x) = \sum_{n=1,3,\ldots}^{\infty} a_n \cos \frac{n\pi x}{2L}, \quad (0 < x < L)
\]

\[
 a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{2L} \, dx.
\]

and the quarter-range sine expansion

\[
 f(x) = \sum_{n=1,3,\ldots}^{\infty} b_n \sin \frac{n\pi x}{2L}, \quad (0 < x < L)
\]

\[
 b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{2L} \, dx.
\]

We have already derived (3). Let us now derive (6) and leave (4) and (5) for the exercises. We see from Fig. 3d that the period is \(4L\), so \(\ell = 2L\). Thus

\[
 f_{\text{ext}}(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{2L} + b_n \sin \frac{n\pi x}{2L} \right).
\]

The antisymmetry of \(f_{\text{ext}}\) about \(x = 0\) implies that \(a_0 = 0\) and \(a_n = 0\) for each \(n = 1, 2, \ldots\). Next,

\[
 b_n = \frac{1}{2L} \int_{-2L}^{2L} f_{\text{ext}}(x) \sin \frac{n\pi x}{2L} \, dx = \frac{2}{2L} \int_0^{2L} f_{\text{ext}}(x) \sin \frac{n\pi x}{2L} \, dx
\]

because both \(f_{\text{ext}}(x)\) and \(\sin \frac{n\pi x}{2L}\) are symmetric about \(x = 0\), so their product is symmetric about \(x = 0\). Now consider the symmetry or antisymmetry about \(x = L\). We see from Fig. 3d that \(f_{\text{ext}}(x)\) is symmetric about that point. Further, it is easily verified that \(\sin \frac{n\pi x}{2L}\) is symmetric about that point if \(n\) is odd, and antisymmetric about that point if \(n\) is even. Thus, the integrand in the second integral in (8) is symmetric about \(x = L\) (which is the midpoint of the integration interval) if \(n\) is odd, and antisymmetric about that point if \(n\) is even. Hence (8) gives

\[
 b_n = \frac{1}{L} \int_0^{2L} f_{\text{ext}}(x) \sin \frac{n\pi x}{2L} \, dx
\]

\[
 = \begin{cases} 
 0 & n \text{ even} \\
 \frac{2}{L} \int_0^{L} f_{\text{ext}}(x) \sin \frac{n\pi x}{2L} \, dx & n \text{ odd}
\end{cases}
\]

*Understand the difference between symmetric/antisymmetric and even/odd. A graph can be symmetric about any \(x\) point. If, in particular, the graph of \(f\) is symmetric/antisymmetric about the origin, then \(f\) is even/odd, respectively.
Finally, we can drop the subscripted “ext” in the final integral in (9) because the interval of integration is $0 < x < L$, over which interval $f_{\text{ext}}(x) = f(x)$. Putting these expressions for $a_0, a_n$, and $b_n$ into (7) gives (6).

**EXAMPLE 1.** To illustrate, let us expand the function $f$, displayed in Fig. 4, in half- and quarter-range cosine and sine series.

HRC: From (3),

$$a_0 = \frac{1}{L} \int_0^L F dx = \frac{FL}{L} = L,
\quad a_n = \frac{2}{L} \int_0^L F \cos \frac{n\pi x}{L} dx = \frac{2F}{n\pi} \sin \frac{n\pi x}{L} \bigg|_{x=0}^{x=L} = 0,$$

so the HRC expansion of $f$ is simply

$$f(x) = F,$$

which results from the extension shown in Fig. 5a.

HRS: From (4),

$$b_n = \frac{2}{L} \int_0^L F \sin \frac{n\pi x}{L} dx = -\frac{2F}{n\pi} (\cos n\pi - 1),$$

so the HRS expansion of $f$ is

$$f(x) = \frac{2F}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n} \sin \frac{n\pi x}{L},$$

which results from the extension shown in Fig. 5b.

QRC: From (5),

$$a_n = \frac{2}{L} \int_0^L F \cos \frac{n\pi x}{2L} dx = \frac{4F}{n\pi} \sin \frac{n\pi}{2},$$

so the QRC expansion of $f$ is

$$f(x) = \frac{4F}{\pi} \sum_{n=1,3,5,...}^{\infty} \sin \frac{(n\pi/2)}{n} \cos \frac{n\pi x}{2L},$$

which results from the extension shown in Fig. 5c.

QRS: From (6),

$$b_n = \frac{2}{L} \int_0^L F \sin \frac{n\pi x}{2L} dx = \frac{4F}{n\pi} (1 - \cos \frac{n\pi}{2}) = \frac{4F}{n\pi},$$

because $\cos \frac{n\pi}{2} = 0$ if $n$ is odd, and $n$ is indeed odd in (6). Thus, the QRS expansion of $f$ is

$$f(x) = \frac{4F}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{2L},$$
which results from the extension shown in Fig. 5d.

The series (11), (13), (15), and (17) converge to the functions shown in Fig. 5a,b,c,d, respectively, but on the interval \(0 < x < L\) they all converge to the function \(f\) defined in Fig. 4.

COMMENT. Observe that \(n\) runs from 1 to infinity continuously in the half-range formulas (3) and (4), but only through the odd values 1, 3, \ldots in the quarter-range formulas (5) and (6). In the present example, the sum in the HRS expansion (13) runs through \(n = 1, 3, \ldots\) because \(1 - \cos n\pi = 1 - (-1)^n = 0\) for even \(n\)'s, but in general the sums in the half-range expansions run through \(n = 1, 2, 3, \ldots\).

Closure. Functions defined only on a finite interval, say \(0 < x < L\), can be periodically extended in an infinite number of ways. Of these, only four possible extensions will prove to be of interest, the half- and quarter-range extensions, which lead to the half- and quarter-range cosine and sine series (3)–(6). Application of these results occur repeatedly in Chapters 18–20 when we solve partial differential equations by the method of separation of variables.

**EXERCISES 17.4**

1. We derived the HRC and QRS formulas, (3) and (6), respectively. In similar fashion, derive the
   (a) HRS formulas (4)          (b) QRC formulas (5)

2. For the given function, prepare labeled sketches of the HRC, HRS, QRC, and QRS extensions, and derive the corresponding expansions.
   (g) \(f(x) = 25\) on \(0 < x < 1\), and \(0\) on \(1 < x < 2\)

3. \(f(x) = 5x\) on \(0 < x < 4\)
4. \(f(x) = \sin x\) on \(0 < x < \pi\)
5. \(f(x) = \sin x\) on \(0 < x < \pi/2\)
6. \(f(x) = 1 - x\) on \(0 < x < 1\)
7. \(f(x) = 0\) on \(0 < x < 4\), and \(50\) on \(4 < x < 5\)
8. \(f(x) = \sin x\) on \(0 < x < \pi\), and \(0\) on \(\pi < x < 2\pi\)
9. \(f(x) = 2 + x\) on \(0 < x < 3\)

**17.5 Manipulation of Fourier Series (Optional)**

In Examples 3 and 4 of Section 17.4 we differentiated Fourier series termwise and equated coefficients of trigonometric series on the left- and right-hand sides of an equation. Our approach is formal (i.e., we do not rigorously justify the steps) since we did not yet have theorems in place with which to justify those steps. This section is intended to provide the necessary theorems.
Fundamental is the concept of uniform convergence. Consider a series
\[
\sum_{n=1}^{\infty} a_n(x),
\]
and let \( s_n(x) \) be its \( n \)th partial sum
\[
s_n(x) = a_1(x) + \cdots + a_n(x).
\]
We say that (1) converges uniformly to \( s(x) \) on an interval \( a \leq x \leq b \) if to each \( \epsilon > 0 \) (i.e., no matter how small) there corresponds an \( N(\epsilon) \), independent of \( x \), such that \( |s_n(x) - s(x)| < \epsilon \) for all \( n > N \) and for all \( x \) in the interval \( a \leq x \leq b \).

The situation is illustrated in Fig. 1. Supposing that \( s(x) \) is as shown, choose an arbitrarily small \( \epsilon > 0 \) and draw an "\( \epsilon \) band" about \( s(x) \), the band between \( s(x) - \epsilon \) and \( s(x) + \epsilon \). If (1) converges uniformly, then (no matter how small we choose \( \epsilon \) to be) there must be some integer \( N \) such that the graph of \( s_n(x) \) lies entirely within the \( \epsilon \) band for all \( n \)'s greater than \( N \).

\[\text{EXAMPLE 1.} \quad \text{Let (1) be the Fourier series} \]
\[f(x) = 2 + \frac{8}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin nx\]
of the square wave, one period of which is shown in Fig. 2. We know that that series converges to the square wave shown in the figure: 0 for \(-\pi < x < 0\), 2 at \( x = 0 \) (denoted by the heavy dot), and 4 for \( 0 < x < \pi \). Is that convergence uniform? The answer depends upon the interval under consideration. Over \( 1 \leq x \leq 2 \), for instance, the answer is yes, which we state without proof; for any \( \epsilon > 0 \), no matter how small, we can force the graph of \( s_n(x) \) to fall within the \( \epsilon \) band by taking \( n \) sufficiently large.

However, over any interval containing a jump discontinuity of \( f \), such as \(-1 \leq x \leq 1 \), the answer is no because the partial sums are continuous, so the graph of \( s_n(x) \) must inevitably break out of the \( \epsilon \) bands (at \( P \) and \( Q \)) in order to pass through the heavy dot. In addition, there are breakouts due to the Gibbs phenomenon, the spike-like overshoots and undershoots near the jump discontinuities.

Of course we cannot rely on pictures, we need an analytical test for uniformity.

\[\text{The concept of uniform convergence is due to Karl Weierstrass (1841) and G. G. Stokes (1847). Generally speaking, it is probably true that major theorems attract the most acclaim in mathematics but, in truth, the fundamental definitions – such as the definitions of the limit of a function, the derivative of a function, the uniform convergence of a series, and so on, are of comparable importance. It is easy to invent definitions, but not all definitions are fruitful. Consider, for instance, the way all of the differential and integral calculus, as well as other branches of analysis, rest upon the shoulders of the limit concept.}
\[\text{Contrast this definition with the definition of convergence of (1) at } x; \ (1) \text{ converges to } s(x) \text{ at } x \text{ if to each } \epsilon > 0 \text{ (i.e., no matter how small) there corresponds an } N(\epsilon) \text{ such that } |s_n(x) - s(x)| < \epsilon \text{ for all } n > N.\]
of convergence. A useful sufficient condition for uniform convergence is as follows.

**Theorem 17.5.1 Weierstrass M-Test**

If \( \sum_{n=1}^{\infty} M_n \) is a convergent series of positive constants and \( |a_n(x)| \leq M_n \) on an x interval I, then \( \sum_{n=1}^{\infty} a_n(x) \) is uniformly (and absolutely) convergent on I.

**Example 2.** The series

\[
\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}
\]

converges uniformly on \(-\infty < x < \infty\), according to the Weierstrass M-test, because \( |(\cos nx)/n^2| \leq 1/n^2 \) on \(-\infty < x < \infty\), and \( \sum_{n=1}^{\infty} 1/n^2 \) is convergent; specifically, it is a convergent p-series. That is, \( M_n = 1/n^2 \) in this case.

We can now state a useful theorem on the termwise differentiation of an infinite series.

**Theorem 17.5.2 Termwise Differentiation of Series**

Let \( \sum_{n=1}^{\infty} a_n(x) \) converge on an x interval I. Then

\[
\frac{d}{dx} \sum_{n=1}^{\infty} a_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} a_n(x)
\]

*Recall from the calculus that the p-series \( \sum_{n=1}^{\infty} 1/n^p \) converges if \( p > 1 \) and diverges if \( p \leq 1 \). If \( p = 1 \) it becomes the well known (and divergent) harmonic series.*
if the series on the right converges uniformly on $I$.\footnote{For proof of this theorem and additional discussion of this issue, see T. M. Apostol, *Mathematical Analysis* (Reading, MA: Addison–Wesley, 1957), Chap. 13.}

That is, we can interchange the order of the two limit operations (the infinite series is the limit of the sequence of partial sums, and the derivative is the limit of a difference quotient) if the $a_n(x)$'s are well enough behaved, specifically, if $\sum a'_n(x)$ converges uniformly on $I$. That condition is stated as sufficient, not necessary. To put (5) into perspective, it might be helpful to recall the analogous result for interchanging the order of differentiation and integration: if $a$ and $b$ are constants, then

$$\frac{d}{dx} \int_a^b f(t, x) \, dt = \int_a^b \frac{\partial f}{\partial x}(t, x) \, dt$$

(6)

if $f$ is sufficiently well behaved, namely, if $f$ and $\partial f/\partial x$ are continuous on the relevant rectangle in the $x, t$ plane (see the Leibniz rule, Theorem 13.8.1).

**EXAMPLE 3.** The series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$$

(7)

can be differentiated termwise on $-\infty < x < \infty$. That is,\footnote{This example is due to A. C. Thompson.}

$$\frac{d}{dx} \sum_{n=1}^{\infty} \frac{\sin nx}{n^3} = \sum_{n=1}^{\infty} \frac{d}{dx} \left( \frac{\sin nx}{n^3} \right) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2},$$

(8)

because the series on the right converges uniformly for all $x$, as shown in Example 2. \[\]

Besides the termwise differentiations employed in the representative Examples 3 and 4 of Section 17.3, we also equated the corresponding coefficients of trigonometric series on the left- and right-hand sides of an equation. Justification of that step is provided by the following theorem.

**THEOREM 17.5.3 Uniqueness of Trigonometric Series**

If

$$a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{\ell} + B_n \sin \frac{n\pi x}{\ell} \right),$$

(9)

where the trigonometric series on the left- and right-hand sides converge to the
17.5. Manipulation of Fourier Series

same sum for all \( x \), then \( a_0 = A_0, a_n = A_n, \) and \( b_n = B_n \) for each \( n \).*

Notice the wording "trigonometric series" for the theorem. Recall that a trigonometric series is a series of the form given by the left- and right-hand sides of (9). Why don't we say Fourier series instead? Because not every convergent trigonometric series is a Fourier series. For example,

\[
\sum_{n=1}^{\infty} \frac{1}{\ln (n + 1)} \sin nx
\]

is a trigonometric series, and it can be shown (by a Dirichlet test) to converge for all \( x \), yet it is not a Fourier series.* That is, there does not exist a \( 2\pi \)-periodic function \( f \) such that

\[
\int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0, \quad (n = 0, 1, 2, \ldots) \tag{11a}
\]

\[
\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\ln (n + 1)}, \quad (n = 1, 2, \ldots) \tag{11b}
\]

Thus, every Fourier series is a trigonometric series, but not every convergent trigonometric series is a Fourier series. This result is in interesting contrast with the result, encountered later in this text (Theorem 24.2.8), that every convergent power series is the Taylor series of its sum function. However, if a trigonometric series, with period \( 2\ell \), converges uniformly on \([-\ell, \ell]\) (or, equivalently, on \(-\infty < x < \infty\)), then it is the Fourier series of its sum function.†

Let us illustrate these results with a practical application that is similar to Examples 3 and 4 in Section 17.3.

**EXAMPLE 4.** Find a particular solution to the differential equation

\[
x'' + 0.5x = f(t), \quad (0 < t < \infty) \tag{12}
\]

where the forcing function \( f(t) \) is the \( 2\pi \)-periodic function shown in Fig. 3. The Fourier of \( f \) is

\[
f(t) = \frac{8F}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin (2n-1)t, \tag{13}
\]

and if we seek \( x(t) \) in the form

\[
x(t) = \sum_{n=1}^{\infty} b_n \sin (2n-1)t \tag{14}
\]

---


Chapter 17. Fourier Series, Fourier Integral, Fourier Transform

we find (Exercise 1), formally, that

\[ x(t) = \frac{8F}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n - 1)^2[(2n - 1)^2 - 0.5]} \sin(2n - 1)t. \tag{15} \]

We can make the process rigorous either by justifying each of the steps in the derivation of (15a), using Theorems 17.5.1–17.5.3, or we can proceed formally to (15) and then rigorously verify that the latter does satisfy the given differential equation (12). Let us do the latter.

According to Theorem 17.5.2 we can differentiate (15) termwise and obtain

\[ x'(t) = \frac{8F}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n - 1)^2[(2n - 1)^2 - 0.5]} \cos(2n - 1)t \tag{16} \]

because the latter series is uniformly convergent on \(0 < t < \infty\). That is,

\[ \left| \frac{(-1)^n}{(2n - 1)^2[(2n - 1)^2 - 0.5]} \cos(2n - 1)t \right| \leq \frac{1}{(2n - 1)^2[(2n - 1)^2 - 0.5]} = M_n, \tag{17} \]

and \( \sum_{n=1}^{\infty} M_n \) is convergent because \( M_n \sim 1/8n^3 \) as \( n \to \infty \), where \( \sum_{n=1}^{\infty} 1/n^3 = \frac{1}{4} \sum_{n=1}^{\infty} 1/n^3 \) is \( \frac{1}{4} \) times a convergent \( p \)-series, with \( p = 3 \). Likewise, we can differentiate (16) termwise and obtain

\[ x''(t) = \frac{8F}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n - 1)^2 - 0.5} \sin(2n - 1)t \tag{18} \]

because the latter series is uniformly convergent on \(0 < t < \infty\). That is,

\[ \left| \frac{(-1)^n}{(2n - 1)^2 - 0.5} \sin(2n - 1)t \right| \leq \frac{(-1)^n}{(2n - 1)^2 - 0.5} = M_n, \tag{19} \]

where \( \sum_{n=1}^{\infty} M_n \) is convergent because \( M_n \sim 1/4n^2 \) as \( n \to \infty \).

Finally, putting (18), (15), and (13) into (12) and adding the two series on the left-hand side termwise\(^1\) is found to produce an identity, so the verification of (15) is complete.

COMMENT. Observe that each time we differentiate (15) we weaken the convergence because we pull out a \( 2n - 1 \) from the sine or cosine. Since \( f \) was \( C^0 \) (i.e., continuous), the terms in (13) were of order \( O(1/n^2) \) and, consequently, the terms in (15) were \( O(1/n^2) \), as \( n \to \infty \). Thus, the terms in \( x'(t) \) were \( O(1/n^2) \) and those in \( x''(t) \) were \( O(1/n^2) \), the \( n^2 \) being sufficient to render the series uniformly convergent so as to justify the termwise differentiation of (16). If \( f(t) \) had been even better behaved such as \( C^1 \) or \( C^2 \), then so much the better, but if it had been discontinuous such as a square wave, then our efforts at justification would have failed because then the terms in the \( x''(t) \) series would die out like

\(^1\)Recall from the calculus that if \( \sum a_n \) is a series of positive constants and \( a_n \sim Kb_n \) as \( n \to \infty \), for some positive constant \( K \), then the series \( \sum a_n \) and \( \sum b_n \) both converge or both diverge.

\(^1\)Recall from the calculus that if \( \sum a_n \) and \( \sum b_n \) are convergent, then \( \sum a_n + \sum b_n = \sum (a_n + b_n) \), and the latter is convergent as well.
O(1/n), which would not be fast enough to ensure uniform convergence by the Weierstrass M-test (since the series \( \sum \frac{1}{n} \) is divergent).  

Actually, the idea of finding solutions to differential equations that are in the form of infinite series is not new to us since we have already studied power series solutions in Chapter 4. Indeed, the technical issues that arose there are the same: the termwise differentiation of a power series, the termwise addition of two series, and the fact that if two power series are equal, then their corresponding coefficients must be equal. These matters are handled by parts (a), (b), (c), respectively, of Theorem 4.2.3. There, we saw that power series are very easy to work with, especially in that a power series may be differentiated (or integrated) any number of times (within its interval of convergence). Of course, that fact is not surprising since a convergent power series is the Taylor series of its sum function, and for a function to have a Taylor series it must be extremely well-behaved, for instance, infinitely differentiable \((C^\infty)\). However, in applications like Example 4, above, power series are of no help because \(f(t)\) is merely continuous; it is not even once differentiable because of the kinks in its graph at \(t = \pi/2, 3\pi/2, 5\pi/2, \ldots\). But since \(f\) is periodic we are able to use Fourier series. 

The fact that the Fourier series (15) of \(x(t)\) can be differentiated termwise only twice is not surprising since \(x(t)\) is not infinitely differentiable. It is only \(C^2\) (since the input \(f\) is \(C^0\) and \(x\) is obtained from \(f\), in effect, by two integrations).

Though we have not used termwise integration of Fourier series we would be remiss if we did not include the following theorem.

**THEOREM 17.5.4** Termwise Integration of Fourier Series

If a Fourier series is integrated termwise between any finite limits, the resulting series converges to the integral of the periodic function corresponding to the original series.

**EXAMPLE 5.** Let us integrate the square wave \(f\) shown in Fig. 4, say, from 0 to any point \(x\). From Example 1 of Section 17.3 we have, for the Fourier series of \(f\).

\[
f(x) = 2 + \frac{8}{\pi} \sum_{n=1,3, \ldots}^{\infty} \frac{1}{n} \sin nx. \tag{20}
\]

Thus,

\[
\int_{0}^{x} f(\xi) \, d\xi = \int_{0}^{x} \left[ 2 + \frac{8}{\pi} \sum_{n=1,3, \ldots}^{\infty} \frac{1}{n} \sin n\xi \right] \, d\xi
\]

\[
= \int_{0}^{x} 2 \, d\xi + \frac{8}{\pi} \sum_{n=1,3, \ldots}^{\infty} \frac{1}{n} \int_{0}^{x} \sin n\xi \, d\xi
\]

Figure 4. Square wave \(f\).
where the second equality follows from Theorem 17.5.4. Further, the final right-hand side, in (21) must, according to the theorem, converge to \( \int_0^\infty f(\xi) \, d\xi \), the graph of which can be inferred by inspection of Fig. 4 and is drawn in Fig. 5.

**COMMENT.** The series (21) is not quite a Fourier series because of the \( 2x \) term, which arises because the average value of \( f \) [corresponding to the 2 in (20)] is nonzero.

**Closure.** This section is about technical matters related to the manipulation of Fourier series, especially regarding their termwise differentiation. The latter is important because functions represented by Fourier series are often not very well behaved; for instance, they may be \( C^1 \), \( C^0 \), or even discontinuous. And since differentiation makes bad behavior even worse, one is well advised to approach the differentiation of Fourier series with caution. Theorem 17.5.2 tells us that termwise differentiation is permissible if the resulting series is uniformly convergent, although that condition is stated as sufficient, not necessary. In turn, Theorem 17.5.1 gives us a simple and well known test for uniformity of convergence, the Weierstrass M-test.

Whereas each termwise differentiation of a Fourier series pulls a factor of \( n \) out of the sine and cosine terms, thus retarding the speed of convergence, termwise integration introduces factors of \( 1/n \) and therefore enhances the convergence. In fact, Theorem 17.5.4 tells us that every Fourier series can be safely integrated termwise.

---

**EXERCISES 17.5**

1. Assuming the form (14), derive the solution (15).

2. Show that the given series is uniformly convergent on the given interval.
   - (a) \( \sum_{1}^{\infty} e^{-nx} \sin nx \) on \( 2 < x < 5 \)
   - (b) \( \sum_{1}^{\infty} \frac{\sin 2nx}{n^2 - 2n + 2} \) on \( -\infty < x < \infty \)
   - (c) \( \sum_{0}^{\infty} \frac{e^{-nx}}{n^2 + 5} \) on \( 0 < x < \infty \)
   - (d) \( \sum_{3}^{\infty} \frac{1}{n^2 + x^2} \) on \( -\infty < x < \infty \)
   - (e) \( \sum_{1}^{\infty} \ln \left( 1 + \frac{x^2}{n^2} \right) \) on every finite interval

3. Show that \( \sum_{1}^{\infty} n^{-x} \) converges uniformly on \( x_0 \leq x < \infty \), for every \( x_0 > b \).

4. Differentiate the given series termwise, and verify the validity of that step if \( x \) is in the given interval (which is not necessarily the broadest one possible).
   - (a) \( \sum_{1}^{\infty} \frac{1}{1 + x^n} \)
   - (b) \( \sum_{1}^{\infty} n(\cos x)^n \)
   - (c) \( \sum_{1}^{\infty} n^5(\sin x)^n \)
   - (d) \( \sum_{6}^{\infty} (5x)^n \)
   - (e) \( \sum_{1}^{\infty} \frac{x^n}{n^3} \)
   - (f) \( \sum_{1}^{\infty} (x^2 + 2x - 1)^n \)
   - (g) \( \sum_{1}^{\infty} (x^2 - 2x)^{3n} \)

*See Comment 1 in Example 4, Section 17.3.*
17.6 Vector Space Approach

An elegant and more modern approach to Fourier series is available within the vector space context. Vector space is the subject of Section 9.6, which section you may wish to review before continuing.

Specifically, consider the function space \( C_p[a, b] \) of all real-valued piecewise-continuous functions defined on \([a, b]\), that is, on \(a \leq x \leq b\). First, let us verify that \( C_p[a, b] \) is indeed a vector space. Let \( f = f(x) \) and \( g = g(x) \) be any two functions* in \( C_p[a, b] \), and let \( \alpha \) be any (real) scalar. We define the sum \( f + g \) and the scalar multiple \( \alpha f \) by

\[
 f + g \equiv f(x) + g(x), \quad \alpha f \equiv \alpha f(x),
\]

respectively. Observe that if \( f \) and \( g \) are piecewise continuous on \([a, b]\) then so is \( f + g \), so that \( C_p[a, b] \) is closed under vector addition. Similarly, if \( f \) is piecewise continuous on \([a, b]\) then so is \( \alpha f \), so that \( C_p[a, b] \) is closed under scalar multiplication. Furthermore, we define the zero vector \( 0 \) as the function which is identically zero, so that \( f + 0 = f(x) + 0 = f(x) = f \). And we define the negative inverse of \( f \) as \( -f(x) \) as \(-f \equiv -f(x)\), in which case we have \( f + (-f) = f(x) + [-f(x)] = 0 = 0 \).

With these definitions of vector addition, scalar multiplication, the zero vector, and the negative inverse, we can see that all of the requirements listed in Definition 9.6.1 are satisfied, so that \( C_p[a, b] \) is indeed a vector space.

Next, we wish to introduce an inner product for \( C_p[a, b] \) and we choose the

\[
 (f, g) = \int_a^b f(x)g(x)\,dx.
\]

6. Evaluate \( \int_0^1 f(x)\,dx \) to three significant figures, and verify the validity of the termwise integration.

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( \int_0^1 f(x),dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) = \sin t ) on ( 0 \leq t \leq \pi )</td>
<td>0.500</td>
</tr>
<tr>
<td>( f(t) = \frac{1}{2} \cos t ) on ( 0 \leq t \leq \pi )</td>
<td>0.383</td>
</tr>
<tr>
<td>( f(t) = \sin t ) on ( 0 \leq t \leq \pi ) and ( 0 \leq t \leq 2\pi )</td>
<td>0.500</td>
</tr>
<tr>
<td>( f(t) =</td>
<td>5 - 5t</td>
</tr>
<tr>
<td>( f(t) = 0 ) on ( 0 \leq t \leq \pi/2 ) and ( \cos t ) on ( \pi/2 \leq t \leq 3\pi/2 )</td>
<td>0.500</td>
</tr>
</tbody>
</table>

\( * \) By \( "f = f(x)" \) we mean that the function \( f \) is the function whose values are \( f(x) \). Notation becomes tricky here since there are now three quantities to be distinguished: the function \( f \) considered as a mapping, the values \( f(x) \) of that function, and the vector \( f \).
inner product of \( f \) and \( g \) as

\[
\langle f, g \rangle \equiv \int_a^b f(x)g(x) \, dx,
\]

which is introduced in Example 4 of Section 9.6. Recall that \( f \) and \( g \) are orthogonal if \( \langle f, g \rangle = 0 \). That the norm \( \|f\| \) of \( f \) is defined as \( \|f\| = \sqrt{\langle f, f \rangle} \), and that \( f \) is said to be normalized (i.e., scaled so as to have unit norm) if \( \|f\| = 1 \).

Finally, recall from our study of best approximation, in Section 9.10, the following important result: If \( f \) is any vector in a normed inner product vector space \( \mathcal{S} \) with natural norm \( \|f\| = \sqrt{\langle f, f \rangle} \), and \( \{\mathbf{e}_1, \ldots, \mathbf{e}_N\} \) is an ON (orthonormal) set in \( \mathcal{S} \), then the best approximation of \( f \) within span \( \{\mathbf{e}_1, \ldots, \mathbf{e}_N\} \) is given by the orthogonal projection of \( f \) onto span \( \{\mathbf{e}_1, \ldots, \mathbf{e}_N\} \), namely, by

\[
f \approx (\langle f, \mathbf{e}_1 \rangle \mathbf{e}_1 + \cdots + (\langle f, \mathbf{e}_N \rangle \mathbf{e}_N = \sum_{n=1}^N (\langle f, \mathbf{e}_n \rangle \mathbf{e}_n.
\]

To apply these results to Fourier series, let \( \mathcal{S} \) be \( C[a, b] \), with the inner product and norm defined above, let \( a = -\ell \) and \( b = \ell \), and consider the vectors

\[
\mathbf{e}_1 = , \quad \mathbf{e}_2 = \cos \frac{\pi x}{\ell}, \quad \mathbf{e}_3 = \sin \frac{\pi x}{\ell}, \ldots, \quad \mathbf{e}_{2k} = \cos \frac{k\pi x}{\ell}, \quad \mathbf{e}_{2k+1} = \sin \frac{k\pi x}{\ell}
\]

in \( C[-\ell, \ell] \). The set \( \{\mathbf{e}_1, \ldots, \mathbf{e}_N\} \) is orthogonal by virtue of the inner product definition (2) and the integrals (24a,b,c) in Section 17.3. For instance,

\[
\langle \mathbf{e}_2, \mathbf{e}_3 \rangle = \int_{-\ell}^{\ell} \cos \frac{\pi x}{\ell} \sin \frac{\pi x}{\ell} \, dx = 0.
\]

Furthermore,

\[
\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \int_{-\ell}^{\ell} 1 \, dx = 2\ell,
\]

\[
\langle \mathbf{e}_2, \mathbf{e}_2 \rangle = \int_{-\ell}^{\ell} \cos^2 \frac{\pi x}{\ell} \, dx = \ell,
\]

\[
\vdots
\]

\[
\langle \mathbf{e}_{2k+1}, \mathbf{e}_{2k+1} \rangle = \int_{-\ell}^{\ell} \sin^2 \frac{k\pi x}{\ell} \, dx = \ell.
\]

\[1\] If we were considering a complex function space (i.e., where the functions are complex-valued and the scalars are complex), then we would use the inner product

\[
\langle f, g \rangle \equiv \int_a^b f(x)\overline{g}(x) \, dx
\]

instead of (2), in accordance with the discussion in Section 12.2, where \( \overline{g}(x) \) is the complex conjugate of \( g(x) \). For example, if \( g(x) = 6 + 5i \), then \( \overline{g}(x) = 6 - 5i \).
so that the normalized $e_n$'s are

\[
\hat{e}_1 = \frac{1}{\|e_1\|} e_1 = \frac{1}{\sqrt{\langle e_1, e_1 \rangle}} e_1 = \frac{1}{\sqrt{2\ell}} e_1, \\
\hat{e}_2 = \frac{1}{\|e_2\|} e_2 = \frac{1}{\sqrt{\langle e_2, e_2 \rangle}} e_2 = \frac{1}{\sqrt{\ell}} \cos \frac{\pi x}{\ell}, \\
\vdots \\
\hat{e}_{2k+1} = \frac{1}{\sqrt{\ell}} \sin \frac{k\pi x}{\ell}. 
\]

(7)

Since the set $\{\hat{e}_1, \ldots, \hat{e}_{2k+1}\}$ is ON, it is LI (linearly independent), and since $k$ (and hence $2k+1$) is arbitrarily large, it follows that we have an arbitrarily large number of LI vectors in $C_p[-\ell, \ell]$, so the latter is infinite dimensional.

Returning to (3), we are approximating a given $f = f(x)$, in $C_p[-\ell, \ell]$, in the form

\[
f(x) \approx c_1 \frac{1}{\sqrt{2\ell}} + c_2 \frac{1}{\sqrt{\ell}} \cos \frac{\pi x}{\ell} + c_3 \frac{1}{\sqrt{\ell}} \sin \frac{\pi x}{\ell} + \cdots \\
+ c_{2k} \frac{1}{\sqrt{\ell}} \cos \frac{k\pi x}{\ell} + c_{2k+1} \frac{1}{\sqrt{\ell}} \sin \frac{k\pi x}{\ell},
\]

(8)

where

\[
c_1 = \langle f, \hat{e}_1 \rangle = \int_{-\ell}^{\ell} f(x) \frac{1}{\sqrt{2\ell}} dx = \frac{1}{\sqrt{2\ell}} \int_{-\ell}^{\ell} f(x) dx \\
c_2 = \langle f, \hat{e}_2 \rangle = \int_{-\ell}^{\ell} f(x) \frac{1}{\sqrt{\ell}} \cos \frac{\pi x}{\ell} dx = \frac{1}{\sqrt{\ell}} \int_{-\ell}^{\ell} f(x) \cos \frac{\pi x}{\ell} dx \\
\vdots \\
c_{2k+1} = \langle f, \hat{e}_{2k+1} \rangle = \frac{1}{\sqrt{\ell}} \int_{-\ell}^{\ell} f(x) \sin \frac{k\pi x}{\ell} dx.
\]

(9)

Equivalently, we can write

\[
f(x) \approx a_0 + a_1 \frac{\cos \frac{\pi x}{\ell}}{\ell} + b_1 \frac{\sin \frac{\pi x}{\ell}}{\ell} + \cdots + a_k \frac{\cos \frac{k\pi x}{\ell}}{\ell} + b_k \frac{\sin \frac{k\pi x}{\ell}}{\ell} \\
= a_0 + \sum_{n=1}^{k} \left( a_n \frac{\cos \frac{n\pi x}{\ell}}{\ell} + b_n \frac{\sin \frac{n\pi x}{\ell}}{\ell} \right),
\]

(10)

where

\[
a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) dx, \\
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx, \\
b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx.
\]

(11)
Let us review the situation. The right-hand side of (10) is the orthogonal projection of \( f \) on the \((2k + 1)\)-dimensional subspace of \( C_\ell[-\ell, \ell] \), which is the span of the ON vectors
\[
\frac{1}{\sqrt{2\ell}} \cos \frac{\pi x}{\ell}, \quad \frac{1}{\sqrt{\ell}} \sin \frac{\pi x}{\ell}, \ldots, \quad \frac{1}{\sqrt{\ell}} \cos \frac{k\pi x}{\ell}, \quad \frac{1}{\sqrt{\ell}} \sin \frac{k\pi x}{\ell}.
\]

With the coefficients in (10) equal to the by-now-familiar Fourier coefficients, as given in (11), (10) is the best possible approximation to \( f(x) \) of the form (8). That is, it is the best approximation in the sense of minimizing the norm of the error vector
\[
\left\| f - \left( c_1 \frac{1}{\sqrt{2\ell}} + c_2 \frac{1}{\sqrt{\ell}} \cos \frac{\pi x}{\ell} + \cdots + c_{2k+1} \frac{1}{\sqrt{\ell}} \sin \frac{k\pi x}{\ell} \right) \right\|^2
\]
\[
= \int_{-\ell}^{\ell} \left[ f(x) - \left( c_1 \frac{1}{\sqrt{2\ell}} + c_2 \frac{1}{\sqrt{\ell}} \cos \frac{\pi x}{\ell} + \cdots + c_{2k+1} \frac{1}{\sqrt{\ell}} \sin \frac{k\pi x}{\ell} \right) \right]^2 dx.
\]

The integral in (13) is known as the squared error. Thus (10) is the best approximation in the sense of minimizing the squared error and is called the least-square approximation of \( f(x) \) with respect to the ON set (4).

We can well expect the squared error to diminish as we increase \( k \), but the key question is: Does it tend to zero as \( k \to \infty \)? It does.

**THEOREM 17.6.1 Vector Convergence**

If \( f(x) \) is piecewise continuous on \([-\ell, \ell]\) and \( a_0, a_1, b_1, a_2, b_2, \ldots \) are the Fourier coefficients defined by (11), then
\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)
\]
holds in the sense of vector (least-square) convergence, namely,
\[
\lim_{k \to \infty} \int_{-\ell}^{\ell} \left[ f(x) - a_0 - \sum_{n=1}^{k} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \right]^2 dx = 0.
\]

Proof of this sophisticated theorem is well beyond our present scope.

Note that the meaning of (14) here is different from its meaning in preceding sections. Specifically, in preceding sections equation (14) held in the pointwise sense, namely, that at a specific single fixed value of \( x \) we have
\[
\lim_{k \to \infty} \left[ a_0 + \sum_{n=1}^{k} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \right] = f(x)
\]
or, equivalently,

$$\lim_{k \to \infty} \left[ f(x) - a_0 - \sum_{n=1}^{k} \left( a_n \cos \frac{n \pi x}{\ell} + b_n \sin \frac{n \pi x}{\ell} \right) \right] = 0. \quad (17)$$

In contrast, in the present section (14) is understood as a vector equation, which holds in the sense that

$$\lim_{k \to \infty} \int_{-\ell}^{\ell} \left[ f(x) - a_0 - \sum_{n=1}^{k} \left( a_n \cos \frac{n \pi x}{\ell} + b_n \sin \frac{n \pi x}{\ell} \right) \right]^2 dx = 0. \quad (18)$$

The truth of (17) does not imply the truth of (18), nor does the truth of (18) imply the truth of (17); they are logically independent statements (Exercise 4).

To better appreciate the distinction, observe that the pointwise convergence addressed in Theorem 17.3.1 is convergence in a local sense, at a specific point \( x \), whereas the vector convergence addressed in Theorem 17.6.1 is convergence in a global sense. That is, the squared error integrated over the entire interval tends to zero as \( k \to \infty \).

We do not wish to imply that one form of convergence is inherently better or more correct than the other; they are simply different.

**Closure.** In effect, Theorem 17.6.1 tells us that the infinite set of orthogonal vectors \( \{1, \cos \frac{\pi x}{\ell}, \sin \frac{\pi x}{\ell}, \cos \frac{2 \pi x}{\ell}, \sin \frac{2 \pi x}{\ell}, \ldots\} \) comprises an orthogonal basis for \( C_p[-\ell, \ell] \). Couldn’t we say that that set is a basis because \( C_p[-\ell, \ell] \) is infinite-dimensional and there is an infinite number of vectors in the set? No. For suppose we remove one (or more) of the vectors from the set. Then we still have an infinite set of orthogonal vectors, but they are not a basis. For instance, if we remove the \( \cos (\pi x/\ell) \) vector and take \( f(x) \) to be \( 6 \cos (\pi x/\ell) \), say, then (14) would be

$$6 \cos \frac{\pi x}{\ell} = 0 + 0 + 0 + \ldots,$$

which is surely incorrect.

**EXERCISES 17.6**

1. Corresponding to the approximation (10), let the error vector be

$$\mathbf{E} = f(x) - a_0 - \sum_{n=1}^{k} \left( a_n \cos \frac{n \pi x}{\ell} + b_n \sin \frac{n \pi x}{\ell} \right), \quad (1.1)$$

(a) Show that

(b) Deduce, from (1.2), the Bessel inequality
2a_n^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^{L} |f(x)|^2 \, dx,

(1.3)

which holds for each \( k = 1, 2, \ldots \). NOTE: Since \( \| E \| \to 0 \) as \( k \to \infty \), according to Theorem 17.6.1, we obtain

\[
2a_n^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{L} \int_{-L}^{L} |f(x)|^2 \, dx,
\]

which is known as the Parseval equality, and which is an infinite-dimensional function space version of the familiar Pythagorean theorem. We can draw an interesting and useful conclusion from (1.4). The integral on the right converges because \( f \) has been assumed piecewise continuous on \([-\ell, \ell]\). Thus, the series on the left converges and has nonnegative terms. For such a series we know that its \( n \)th term must tend to zero as \( n \to \infty \). Hence, it follows that \( a_n \to 0 \) and \( b_n \to 0 \) as \( n \to \infty \).

2. Use equation (1.2), above, to compute \( \| E \| \), for \( k = 1, 2, \ldots, 8 \), where the \( 2\pi \)-periodic function \( f \) is defined on \(-\pi < x < \pi\) as follows.

(a) \( f(x) = |x| \)
(b) \( f(x) = x \)
(c) \( f(x) = \cos^2 x \)
(d) \( f(x) = x^2 \)
(e) \( f(x) = | \sin x | \)
(f) \( f(x) = | \cos x | \)

(g) \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \)

(h) \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases} \)

(i) \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 100, & \pi < x < \pi/2 \\ 50, & \pi/2 < x < \pi \\ \sin x, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases} \)

3. Let \( f \) and \( g \) both be members of \( C_p[-\ell, \ell] \), and \( f(x) = g(x) \) on \([-\ell, \ell] \) except at a finite number of points, at which \( f(x) \neq g(x) \). Show that \( f = g \) in spite of these pointwise differences; i.e., show that \( \| f - g \| = 0 \), so that within the vector space framework \( f \) and \( g \) are indistinguishable.

4. (Vector convergence and pointwise convergence) Below Theorem 17.6.1, we emphasized that vector and pointwise convergence are independent, neither one implies the other. The purpose of this exercise is to illustrate that claim through a simple example. Consider, in place of the messy \( \| \cdot \|^2 \) integrand in (18), the sequence \( F_k(x) \) defined in parts (a), (b), and (c).

(a) First, consider the sequence displayed below.

\[ \lim_{k \to \infty} F_k(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases} \neq 0. \]

(b) With \( F_k(x) \) as shown below, instead, show that \( \lim_{k \to \infty} \int_{-\ell}^{\ell} F_k(x) \, dx = 0 \), whereas

\[ \lim_{k \to \infty} F_k(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \neq 0. \]

(c) Let \( F_k(x) \) be the same as in part (b), except that the height of the “mountain” is \( 1/k \) instead of 1. Show that in this case both limits are zero. CONCLUSION: The statements \( \lim_{k \to \infty} \int_{-\ell}^{\ell} F_k(x) \, dx = 0 \) and \( \lim_{k \to \infty} F_k(x) = 0 \) on \([-\ell, \ell] \) are independent: the truth of one does not imply the truth of the other.

5. Beginning with the expression

\[
\| E \|^2 = \int_{-\ell}^{\ell} \left[ f(x) - a_0 \sum_{n=1}^{k} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \right]^2 \, dx
\]
for the square of the norm of the error vector $E$ associated with the approximation (10), seek the optimal choice of $a_0, a_n, b_n$ by setting $\partial \|E\|^2 / \partial a_0 = 0, \partial \|E\|^2 / \partial a_n = 0$, and $\partial \|E\|^2 / \partial b_n = 0$. Show that that step produces the expressions for $a_0, a_n, b_n$ given in (11).

---

17.7 The Sturm–Liouville Theory

17.7.1. Sturm–Liouville problem. In Section 17.6 we found that the sines and cosines present in Fourier series constitute an orthogonal basis for the relevant infinite-dimensional function space. Where do such bases come from? Are there others as well? In this section we discover that such bases arise as the eigenfunctions of Sturm–Liouville eigenvalue problems, just as real symmetric $n \times n$ matrices provide us with sets of eigenvectors that are orthogonal bases for $n$-space. Thus, Fourier series will be found to be just one (extremely important) example within a broader Sturm–Liouville theory.

By a **Sturm–Liouville problem** we mean a linear homogeneous second-order differential equation

$$[p(x) y']' + q(x) y + \lambda w(x) y = 0, \quad (a < x < b) \quad (1a)$$

with homogeneous boundary conditions of the form

$$\begin{align*}
\alpha y(a) + \beta y'(a) &= 0, \\
\gamma y(b) + \delta y'(b) &= 0, \\
\end{align*} \quad (1b)$$

where $a, b$ are finite, where $p, p', q, w$ are continuous on $[a, b]$, and where $p(x) > 0$ and $w(x) > 0$ on $[a, b]$. These conditions, as well as the precise form of (1a) and (1b), are important, and should be carefully noted. Further, $\alpha$ and $\beta$ are not both zero, $\gamma$ and $\delta$ are not both zero, and $a, b, p(x), q(x), w(x), \alpha, \beta, \gamma, \delta$ are all real.

Also critical is that (1) is a boundary-value problem, not an initial-value problem, because the conditions (1b) are imposed at both ends of the interval. If, in place of (1b), we imposed homogeneous initial conditions $y(a) = 0$ and $y(b) = 0$, then (1) would admit the trivial solution $y(x) = 0$, and that solution would be unique (Theorem 3.3.1). However, we saw in Section 3.3.2 that boundary-value problems can admit no solution, a unique solution, or a nonunique solution. Thus, even though the Sturm–Liouville boundary-value problem (1) surely

---

*The theory presented in this section was discovered by Charles Sturm (1803–1855) and Joseph Liouville (1809–1882), and published in 1836–1837. Sturm, a professor of mechanics at the Sorbonne, became involved in this work as an outgrowth of his studies of the partial differential equations governing the flow of heat in a bar of nonuniform density. He was joined in this work by his friend Liouville, who was a professor of mathematics at the Collège de France and who is also well known for his work on complex variable theory and on transcendental numbers. Note the spelling Liouville (not Louisville), and the pronunciation "lu-ee-vill" (not "lu-ee-vil").*
admits the trivial solution, our interest is in finding nontrivial solutions. Though
\( a, b, p(x), q(x), w(x) \), \( \alpha, \beta, \gamma, \delta \) are all specified, \( \lambda \) is a free parameter. Any value
of \( \lambda \) that permits the existence of nontrivial solutions of (1) is called an eigenvalue
of (1), and the corresponding nontrivial solution is called an eigenfunction of (1).
Thus, (1) is an eigenvalue problem, analogous to the matrix eigenvalue problem
studied in Chapter 11.

**EXAMPLE 1.** Consider the case

\[
y'' + \lambda y = 0, \quad (0 < x < L) \quad (2a)
y(0) = 0, \quad y(L) = 0. \quad (2b)
\]

Comparing (2) with (1) we see that \( p(x) = w(x) = 1, q(x) = 0, a = 0, b = L, \alpha = \gamma = 1, \)
and \( \beta = \delta = 0 \), all of which satisfy the conditions listed below (1).

To see if (2) admits nontrivial solutions, let us not be intimidated by the fact it is an
"eigenvalue problem." After all, (2a) is a simple differential equation. Its general solution
is evidently

\[
y(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x. \quad (3)
\]

However, if \( \lambda = 0 \), then (3) reduces to \( y(x) = A \), which is not a general solution of (2a).
Thus, for the special case \( \lambda = 0 \) we return to (2a), which becomes \( y'' = 0 \), and determine
the general solution to be \( C + Dx \). Thus, in place of (3) we write

\[
y(x) = \begin{cases} 
A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x, & \lambda \neq 0 \\
C + Dx, & \lambda = 0
\end{cases} \quad (4a,b)
\]

Treating the cases \( \lambda = 0 \) and \( \lambda \neq 0 \) separately, consider \( \lambda = 0 \) first. Applying the
boundary conditions (2b) to \( y(x) = C + Dx \) gives

\[
y(0) = 0 = C, \quad (5a)
y(L) = 0 = C + DL, \quad (5b)
\]

so \( C = D = 0 \). Thus, the only solution corresponding to \( \lambda = 0 \) is the trivial solution
\( y(x) = 0 \), so \( \lambda = 0 \) is not an eigenvalue (in this example).

*Note that we don’t yet know \( \lambda \). If \( \lambda > 0 \), then (3) is the general solution of (2), but does (3) hold
if \( \lambda < 0 \)? Shouldn’t we have a \( \cosh \) and \( \sinh \) in that case? If \( \lambda < 0 \), (3) becomes,

\[
y(x) = A \cos (i \sqrt{\lambda} x) + B \sin (i \sqrt{\lambda} x) 
= A \cosh (\sqrt{\lambda} x) + iB \sinh \sqrt{\lambda} x,
\]

so we do have an arbitrary linear combination of \( \cosh \) and \( \sinh \), as we should. (We could rename \( iB \)
as \( C \) if we don’t like the \( i \).) Thus, there is no need to treat the cases \( \lambda > 0 \) and \( \lambda < 0 \) separately,
although some authors prefer to do that. In fact, (3) holds even if \( \lambda \) is complex, but we will find that
for Sturm–Liouville problems \( \lambda \) is always real.
Turning to the case $\lambda \neq 0$, the boundary conditions give
\[
\begin{align*}
y(0) &= 0 = A, \\
y(L) &= 0 = A \cos \sqrt{\lambda} L + B \sin \sqrt{\lambda} L.
\end{align*}
\]
Since (6a) gives $A = 0$, (6b) gives $B \sin \sqrt{\lambda} L = 0$, so either $B = 0$ or $\sin \sqrt{\lambda} L = 0$ and $B$ is arbitrary. We rule out the choice $B = 0$ since then we would have $A = B = 0$ and hence the trivial solution $y(x) = 0$. Rather, $B$ is arbitrary and
\[
\sin \sqrt{\lambda} L = 0.
\]
Solving (7), we have $\sqrt{\lambda} L = n\pi$ for $n = 0, 1, 2, \ldots$. Of these values, discard $n = 0$ because it gives $\lambda = 0$, which case has already been considered. Thus we have the eigenfunctions
\[
y(x) = B \sin \frac{n\pi x}{L},
\]
where $B$ is arbitrary and $n = 1, 2, \ldots$. The negative values of $n$ can be discarded as well since the positive and negative choices do not lead to distinct (i.e., linearly independent) eigenfunctions. For example, $n = +2$ gives $\sin (2\pi x/L)$, and $n = -2$ gives $\sin (-2\pi x/L) = -\sin (2\pi x/L)$, and since the scale factor $B$ is arbitrary it can absorb the minus sign. Let us set $B = 1$, say, for definiteness. The upshot is that we have the infinite set of eigenvalues and eigenfunctions
\[
\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad \phi_n(x) = \sin \frac{n\pi x}{L}
\]
for $n = 1, 2, \ldots$, where we use the symbol $\phi_n$ to denote the $n$th eigenfunction (analogous to the special symbol $e_n$ that we used in the matrix case).

**COMMENT 1.** It may be useful to recast the solution of (6) in matrix form because the matrix approach is more convenient in algebraically-more-difficult cases. Re-expressing (6) as
\[
\begin{bmatrix}
1 & 0 \\
\cos \sqrt{\lambda} L & \sin \sqrt{\lambda} L
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]
we see that we have a nontrivial solution of (10) [and hence of (2)] if and only if the determinant of the coefficient matrix is zero:
\[
\begin{bmatrix}
1 & 0 \\
\cos \sqrt{\lambda} L & \sin \sqrt{\lambda} L
\end{bmatrix}
= \sin \sqrt{\lambda} L = 0,
\]
which is the same as (7). Next, solve (11) for the $\lambda$’s, put them back in (10), and solve (10), say by Gauss elimination, for the nontrivial solutions for $A$ and $B$. That step gives $A = 0$ and $B = 0$, arbitrary, as above.

**COMMENT 2.** We call (11) the characteristic equation corresponding to the eigenvalue problem (2). In the $n \times n$ matrix case the characteristic equation is always an $n$th degree algebraic equation; in the Sturm–Liouville case it is always transcendental (an infinite-degree algebraic equation), with an infinite number of distinct roots.*

Sturm–Liouville problems such as (2) arise throughout Chapters 18–20, when

*To see that (11) is an infinite-degree algebraic equation, replace $\sin \sqrt{\lambda} L$ by its Taylor series.
we solve partial differential equations by the method of separation of variables — for instance in connection with unsteady heat conduction in a rod and the lateral motion of a vibrating string. Sometimes, however, they arise directly. For instance, a classical problem in structural mechanics is the determination of the buckling load of a structure. Consider, as a simple structure, the column shown in Fig. 1, of length \(L\), pinned at both ends and subjected to a downward load (i.e., force) \(P\). As we increase \(P\) nothing happens — the column remains straight, until a certain value of \(P\) is reached, which we call the critical load or buckling load and which we denote as \(P_{cr}\). Under that load the column bends (and probably collapses). Clearly, it is important to be able to predict \(P_{cr}\). To do so, we use Euler beam theory,\(^*\) which tells us that the lateral deflection \(y(x)\) is governed by the boundary-value problem

\[
EIy'' + Py = 0, \quad (0 < x < L)
\]

\[
y(0) = 0, \quad y(L) = 0,
\]

where \(E\) and \(I\) are physical constants of the column: \(E\) is Young's modulus of the material and \(I\) is the cross-sectional inertia. We see that (12) is the same as (2), with \(\lambda = P/EI\). Surely, \(y(x) = 0\) satisfies (12), but that solution is of no interest because it does not correspond to buckling. From Example 1 we recall that nontrivial solutions occur for \(\lambda = P/EI = \pi^2/L^2, 4\pi^2/L^2, 9\pi^2/L^2, \ldots\). The smallest of these, \(P/EI = \pi^2/L^2\), gives the buckling load

\[
P_{cr} = \frac{\pi^2 EI}{L^2},
\]

and the corresponding eigenfunction \(\sin (\pi x/L)\) gives (to within an arbitrary scale factor) the shape of the corresponding buckling mode, which is somewhat as sketched in Fig. 1. The analysis gives the inception of buckling and does not give insight into the dynamical process of collapse. The formula (13) was published first by Euler in 1757.

Since we will be concerned with orthogonal bases in function space, we will need an inner product \((f, g)\) between two functions (i.e., vectors) \(f\) and \(g\). It will be convenient to use the inner product\(^1\)

\[
(f, g) \equiv \int_a^b f(x) g(x) w(x) \, dx,
\]

where the weight function \(w(x)\) is the \(w(x)\) in the Sturm–Liouville equation (1a). If \((f, g) = 0\), then \(f\) and \(g\) are orthogonal.


\(^1\)If \(f\) and \(g\) are complex-valued functions such as \(e^{ix}\), then (14) fails to meet the conditions required of any dot or inner product [116] in Section 9.6 and should be modified as \(\int_a^b f(x) \bar{g}(x) w(x) \, dx\). We will face up to that detail in the optional Section 17.7.2 but, otherwise, will continue to use (14) because it will turn out that complex-valued functions need not arise. Also, note that whether we write \((f, g)\) or \(\langle f, g \rangle\) doesn’t matter, as long as we understand that \(f\) and \(g\) are here being considered as vectors.
We have the following major theorem regarding the eigenvalues and eigenfunctions of the Sturm–Liouville eigenvalue problem (1), with the restrictions on \( p(x), q(x), w(x), a, b, \alpha, \beta, \gamma, \delta \) stated earlier.

**THEOREM 17.7.1 Sturm–Liouville Theorem**

Let \( \lambda_n \) and \( \phi_n(x) \) denote any eigenvalue and corresponding eigenfunction of the Sturm–Liouville eigenvalue problem (1), respectively.

(a) The eigenvalues are real.

(b) The eigenvalues are simple. That is, to each eigenvalue there corresponds only one linearly independent eigenfunction. Further, there are an infinite number of eigenvalues, and they can be ordered so that \( \lambda_1 < \lambda_2 < \lambda_3 < \ldots \), where \( \lambda_n \to \infty \) as \( n \to \infty \).

(c) Eigenfunctions corresponding to distinct eigenvalues are orthogonal. That is, if \( \lambda_j \neq \lambda_k \), then \( \langle \phi_j, \phi_k \rangle = 0 \).

(d) Let \( f \) and \( f' \) be piecewise continuous on \( a \leq x \leq b \). If \( a_n = \langle f, \phi_n(x) \rangle / \langle \phi_n, \phi_n \rangle \), then the series \( \sum_{n=1}^{\infty} a_n \phi_n(x) \) converges to \( f(x) \) if \( f \) is continuous at \( x \), and to the mean value \( [f(x+) + f(x-)]/2 \) if \( f \) is discontinuous at \( x \), for each point \( x \) in the open interval \( a < x < b \).

This theorem is analogous to the several individual theorems given in Section 11.3 for \( n \times n \) real symmetric matrix eigenvalue problems. Parts of it are proved in the optional Section 17.7.2. Part (d) says that

\[
f(x) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n(x)
\]

(15)

holds at each point \( x \) in \( a < x < b \) at which \( f \) is continuous. If, at a discontinuity, \( f(x) \) does not happen to equal the mean value \( [f(x+) + f(x-)]/2 \), then the equality in (15) does not hold at that point. To remind us of the possibility of such pointwise discrepancies, some authors write an "ae" above the equal sign, to mean equal "almost everywhere," but we will simply write the eigenfunction expansion of \( f \) as we have in (15), without such reminders. Observe that (d) is a pointwise convergence statement. Various other statements are available regarding both pointwise and vector convergence, but (d) will suffice for most practical purposes and for our purposes in this text.

**EXAMPLE 2.** Consider the results of Example 1 in the light of Theorem 17.7. Since \( \lambda_n = n^2 \pi^2 / L^2 \), the eigenvalues are real, there are an infinite number of them, and \( \lambda_n \to \infty \) as \( n \to \infty \). Further, each eigenvalue is simple because the single eigenfunction
\[ \phi_n(x) = \sin \left( \frac{n\pi x}{L} \right) \]
corresponds to each eigenvalue \( \lambda_n \). Finally, with the weight function \( w(x) = 1 \) we have

\[ \langle \phi_m, \phi_n \rangle = \int_0^L \sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) \, dx = 0 \quad (16) \]

for \( m \neq n \), by virtue of the Euler formula (24b) in Section 17.3. Each of these results is in accord with Theorem 17.7.1.

To illustrate the eigenfunction expansion (15), let \( f(x) = x \). Then

\[ \langle f, \phi_n \rangle = \int_0^L x \sin \left( \frac{n\pi x}{L} \right) \, dx = \frac{(-1)^n L^2}{n\pi}, \quad (17a) \]

and

\[ \langle \phi_n, \phi_n \rangle = \int_0^L \sin^2 \left( \frac{n\pi x}{L} \right) \, dx = \frac{L}{2}, \quad (17b) \]

so we have

\[ f(x) = x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left( \frac{n\pi x}{L} \right) \quad (18) \]

**COMMENT 1.** Carefully observe that part (d) of the theorem does NOT require \( f \) to satisfy the homogeneous Sturm–Liouville boundary conditions, which are \( y(0) = 0 \) and \( y(L) = 0 \) in the present example. In fact, \( f(L) \) is \( L \), not 0. Nonetheless, we do obtain the convergence that is guaranteed by the theorem, over \( 0 < x < L \) (actually, over \( 0 \leq x < L \) in this example), as hinted at in Fig. 2, where we compare \( f(x) = x \) with the fifth and tenth partial sums, \( s_5(x) \) and \( s_{10}(x) \).

**COMMENT 2.** Observe also that, corresponding to the present Sturm–Liouville problem, the eigenfunction expansion (15), namely,

\[ f(x) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{n\pi x}{L} \right), \quad (19a) \]

where

\[ a_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx, \quad (19b) \]

is actually the half-range sine expansion of \( f \), studied in Section 17.4. Other choices of the boundary conditions, in place of \( y(0) = 0 \) and \( y(L) = 0 \), will result in eigenfunctions that produce the half-range cosine and quarter-range sine and cosine expansions (Exercise 2). □

There is a small flaw in our procedure. To solve the characteristic equation (7), we recalled that \( \sin x = 0 \) has roots at \( x = n\pi \) on the real axis. Might there be complex roots as well? It's true that we know in advance that \( \lambda \) is real, but if it is real and negative then the argument of the sine is purely imaginary. Thus, we need to search for roots of \( \sin z = 0 \) (where \( z = x + iy \)) not only along the real axis but along the imaginary axis as well. Doing so, observe that \( \sin iy = i \sinh y = 0 \) is equivalent to \( \sinh y = 0 \), which admits only the root \( y = 0 \). Thus, we did not miss
any roots, and all is well. The following theorem could have saved us this extra trouble.

**Theorem 17.7.2 Nonnegative Eigenvalues**

If \( q(x) \leq 0 \) on \([a, b]\) and \( [p(x)\phi_n(x)\phi'_n(x)]_a^b \leq 0 \) for the eigenfunction \( \phi_n(x) \), then not only is \( \lambda_n \) real, \( \lambda_n \geq 0 \).

Applying Theorem 17.7.2 to the problem in Example 1, observe that \( q(x) \leq 0 \) on \([0, L]\) because \( q(x) = 0 \). Further, \( p(x) = 1 \) and the \( \phi_n(x) \)'s satisfy the boundary conditions (2b), so

\[
[p(x)\phi_n(x)\phi'_n(x)]_a^b = \phi_n(L)\phi'_n(L) - \phi_n(0)\phi'_n(0) = 0. \tag{20}
\]

Hence, not only are the \( \lambda_n \)'s real, they are also nonnegative.

**Example 3.** Find the eigenvalues and eigenfunctions for the Sturm–Liouville problem

\[
\begin{align*}
y'' + \lambda y &= 0, & (0 < x < 1) \tag{21a} \\
y(0) - 2y'(0) &= 0, & y(1) = 0. \tag{21b}
\end{align*}
\]

We speak of the boundary condition at \( x = 0 \) as being of **mixed** type because both \( y \) and \( y' \) are present: that is, both \( \alpha \) and \( \beta \) are nonzero in (1b). Such boundary conditions do arise in applications such as unsteady heat conduction, as we shall see in Chapter 18.

As in Example 1, solution of (21a) gives

\[
y(x) = \begin{cases} 
A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x, & \lambda \neq 0 \\
C + Dx, & \lambda = 0.
\end{cases} \tag{22a,b}
\]

Imposing the boundary conditions (21b) on the \( C + Dx \) solution gives \( C - 2D = 0 \) and \( C + D = 0 \), so \( C = D = 0 \). Hence, \( \lambda = 0 \) is not an eigenvalue. For the \( \lambda \neq 0 \) case, the boundary conditions give

\[
\begin{bmatrix} 1 \\ \cos \sqrt{\lambda} \end{bmatrix} = \begin{bmatrix} 2\sqrt{\lambda} \\ \sin \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{23}
\]

To obtain nontrivial solutions (i.e., where \( A \) and \( B \) are not both zero), set the determinant equal to zero, which step gives the characteristic equation \( \sin \sqrt{\lambda} + 2\sqrt{\lambda} \cos \sqrt{\lambda} = 0 \) or, more conveniently,

\[
\tan \sqrt{\lambda} = -2\sqrt{\lambda}. \tag{24}
\]

Applying Theorem 17.7.2, with \( q(x) = 0 \) and \( p(x) = 1 \),

\[
(p\phi_n\phi'_n)^1_0 = \phi_n(1)\phi'_n(1) - \phi_n(0)\phi'_n(0) = (0) - 2[\phi'_n(0)]^2 \quad [\text{from (21b)}]
\leq 0, \tag{25}
\]
so the $\lambda_n$'s are nonnegative. Hence, it suffices to look for solutions of (24) along a real $\sqrt{\lambda}$ axis. Plotting the left- and right-hand sides of (24) in Fig. 3, we can see the roots $\sqrt{\lambda_n}$ fall as indicated. (The intersection at the origin can be discarded because we already determined that $\lambda = 0$ is not an eigenvalue.) We can find any number of roots using computer software such as Maple and obtain, for the first few,

\[ \sqrt{\lambda_1} = 1.83660, \quad \sqrt{\lambda_2} = 4.81584, \quad \sqrt{\lambda_3} = 7.91705, \ldots, \quad (26) \]

or

\[ \lambda_1 = 3.3731, \quad \lambda_2 = 23.1923, \quad \lambda_3 = 62.6797, \ldots, \quad (27) \]

and so on. The graphs in Fig. 3, or even a freehand sketch of them, reveal that $\sqrt{\lambda_n} \sim (2n - 1)\pi/2$ as $n \to \infty$.

With the eigenvalues determined, we return to (23) to find the nontrivial solutions for $A$ and $B$ and hence for $y(x)$. With $\lambda$ satisfying (24), the second row of the coefficient matrix is a multiple of the first, so of the two scalar equations implied by (23) the second can be discarded, leaving the one equation

\[ A - 2\sqrt{\lambda_n} B = 0 \quad (28) \]

on the two unknowns $A$ and $B$. Thus, $A = 2\sqrt{\lambda_n} B$, where $B$ remains arbitrary, so

\[
y(x) = A \cos \sqrt{\lambda_n} x + B \sin \sqrt{\lambda_n} x \]

\[ = B(2\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + \sin \sqrt{\lambda_n} x), \quad (29)\]

and hence the eigenfunctions are

\[ \phi_n(x) = 2\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x + \sin \sqrt{\lambda_n} x, \quad (30) \]

where the $\lambda_n$'s are given by (27).

COMMENT. Actually, if we extended the graphs in Fig. 3 over $-\infty < \sqrt{\lambda} < 0$ we would find the additional roots $-\sqrt{\lambda_1}$ and $-\sqrt{\lambda_2}$ and so on. These can be discarded just as we discarded the $n = -1, -2, \ldots$ cases in Example 1 because they contribute nothing new.
For instance, the right-hand side of (30) is an odd function of $\sqrt{\lambda_n}$, so changing $\sqrt{\lambda_n}$ to $-\sqrt{\lambda_n}$ merely scales $\phi_n(x)$ by a factor of $-1$. 

**EXAMPLE 4.** As a final example, consider the problem

$$y'' - 2y' + \lambda y = 0, \quad (0 < x < \pi)$$

$$y(0) = 0, \quad y(\pi) = 0.$$  

(31a)  

(31b)

It appears that (31) may not be a Sturm–Liouville problem at all since the written-out version of (1a) is

$$py'' + p'y' + qy + \lambda wy = 0.$$  

(32)

That is, the coefficient of $y'$ needs to be the derivative of the coefficient of $y''$. Yet, $-2$ in (31a) is not the derivative of 1. However, let us multiply (31a) by a yet-to-be-determined function $\sigma(x)$, giving

$$\sigma y'' - 2\sigma y' + \lambda \sigma y = 0$$

(33)

such that the coefficient $-2\sigma$ of $y'$ is the derivative of the coefficient $\sigma$ of $y''$: 

$$\sigma' = -2\sigma.$$  

(34)

Solving, $\sigma(x) = Ce^{-2x}$ and we can take $C = 1$ without loss. Putting that $\sigma$ into (33) does give the standard Sturm–Liouville form,

$$(e^{-2x}y')' + \lambda e^{-2x}y = 0.$$  

(35)

Since the factor $\sigma = e^{-2x}$ in (33) is everywhere nonzero, the solution of (31a) and (35) are identical, so the two equations are equivalent.

That step was important for two reasons. First, it establishes the problem as being of Sturm–Liouville type so that we can make use of Theorems 17.7.1 and 17.7.2. Second, it enables us to identify $p(x)$ and $w(x)$: $p(x) = w(x) = e^{-2x}$, so we see that $p(x) > 0$ and $w(x) > 0$ on the closed interval $0 \leq x \leq \pi$, as required by the theory. And of course we will need to know the weight function $w(x)$ in the inner product if we are to carry out any eigenfunction expansions.

To find the eigenvalues and eigenfunctions, seek an exponential solution form $y(x) = e^{rx}$. Putting that form into (31a), we find that $r = 1 \pm \sqrt{1 - \lambda}$, so the general solution of (31a) is

$$y(x) = e^r \left( Ae^{\sqrt{1-\lambda}x} + Be^{-\sqrt{1-\lambda}x} \right),$$

(36)

unless $\lambda = 1$, in which case the two solutions in (36) coalesce into one. Thus, we need to distinguish the two cases $\lambda \neq 1$ and $\lambda = 1$, and write the general solution as

$$y(x) = \begin{cases} 
  e^r [C \sinh(\sqrt{1-\lambda}x) + D \cosh(\sqrt{1-\lambda}x)], & \lambda \neq 1 \\
  e^r (E + Fx), & \lambda = 1 
\end{cases}$$

(37a,b)

where the sinh, cosh combination will be a bit more convenient than the positive and negative exponentials in (36) because the $y(0) = 0$ boundary condition will give $D = 0$ and will thereby knock out one of the two terms.
Applying the boundary conditions to (37b) gives \( E = F = 0 \), so \( \lambda = 1 \) is not an eigenvalue of (31). Applying them to (37a) gives \( D = 0 \) and the characteristic equation
\[
\sinh (\sqrt{1 - \lambda \pi}) = 0,
\]
with \( C \) remaining arbitrary. For \( \lambda < 1 \), (38) has no roots. For \( \lambda > 1 \), write
\[
\sinh (\sqrt{1 - \lambda \pi}) = \sinh (i \sqrt{1 - \lambda \pi}) = i \sin (\sqrt{1 - \lambda \pi}) = 0,
\]
so \( \sqrt{1 - \lambda \pi} = n\pi \) for \( n = 1, 2, \ldots \). Thus, the eigenvalues are
\[
\lambda_n = 1 + n^2. \quad (n = 1, 2, \ldots)
\]
Further,
\[
y(x) = Ce^x \sinh (\sqrt{1 - \lambda \pi} x) = Ce^x \sinh (i \sqrt{1 - \lambda \pi} x) = iCe^x \sin (\sqrt{1 - \lambda \pi} x) = iCe^x \sin n x,
\]
so the eigenfunctions are
\[
\phi_n(x) = e^x \sin nx.
\]
Finally, the eigenfunction expansion of a given function \( f(x) \) on \( 0 < x < \pi \) is
\[
f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \quad (0 < x < \pi)
\]
\[
a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_0^\pi f(x) e^x \sin nx e^{-2x} dx}{\int_0^\pi (e^x \sin nx)^2 e^{-2x} dx}
\]
\[
= \frac{\int_0^\pi f(x) e^{-x} \sin nx dx}{\int_0^\pi e^{-2x} \sin^2 nx dx} = \frac{2}{\pi} \int_0^\pi f(x) e^{-x} \sin nx dx.
\]
\[
\text{COMMENT. We did not use Theorem 17.7.2 in this example because it addresses the distinction } \lambda > 0, \lambda < 0, \text{ whereas here we were concerned with the cases } \lambda > 1, \lambda < 1. \]

In Examples 1—4 it turned out that the separately-considered \( \lambda \)'s \( [\lambda = 0 \text{ in } (4), \lambda = 0 \text{ in } (22), \text{ and } \lambda = 1 \text{ in } (37)] \) turned out not to be eigenvalues. Do not discard such values out of hand because they might, in other examples, turn out to be eigenvalues. For instance, you will find that if the boundary conditions in Example

*Here we use the definitions of \( \sinh(\cdot) \) and \( \sin(\cdot) \):

\[
\sinh it = \frac{e^{it} - e^{-it}}{2} = i \frac{e^{it} - e^{-it}}{2i} = i \sin it.
\]
are changed to \( y'(0) = 0 \) and \( y'(L) = 0 \), then \( \lambda = 0 \) is indeed an eigenvalue, with the eigenfunction \( \phi(x) = 1 \).

17.7.2. Lagrange identity and proofs. (Optional) We will derive a Lagrange identity, and use it to prove parts of Theorem 17.7.1. Proof of Theorem 17.7.2 is left for the exercises. We assume elementary knowledge of the Cartesian representation of complex numbers \( z = x + iy \) and the complex conjugate \( \bar{z} = x - iy \), as covered in Section 21.2.

When we introduced the inner product (14), we noted that if we are to admit complex-valued functions then we should modify the inner product as

\[
\langle f, g \rangle = \int_a^b f(x) \overline{g}(x) \, w(x) \, dx.
\]  

That is, we continue to ask \( p(x), q(x), w(x), \alpha, \beta, \gamma, \delta \) to be real, but it is not at all obvious that assumption implies that the eigenvalues and eigenfunctions must be real. For instance, the real matrix

\[
A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\]

has the complex eigenvalues \( \lambda = 1 \pm i \) and complex eigenvectors as well.

To begin, observe that the properties

\[
\begin{align*}
\langle f, g \rangle &= \overline{\langle g, f \rangle}, \\
\langle \mu f, g \rangle &= \mu \langle f, g \rangle, \\
\langle f, \mu g \rangle &= \overline{\mu} \langle f, g \rangle,
\end{align*}
\]  

follow immediately from (44), where \( \mu \) is any scalar. Since \( \langle f, g \rangle \neq \langle g, f \rangle \) in general, according to (43a), it is sometimes useful to specify that \( f \) comes first and \( g \) comes second in \( \langle f, g \rangle \). In what follows we say that \( \langle f, g \rangle \) is \textit{g pre-dotted} with \( f \) or \( f \) \textit{post-dotted} with \( g \), which terminology is not standard.

Let us express (1a) in operator form as

\[
L[y] = \lambda y,
\]  

where \( L \) is the differential operator

\[
L = -\frac{1}{w} \left[ \frac{d}{dx} \left( p \frac{d}{dx} \right) + q \right],
\]  

and let \( u \) and \( v \) be any functions having continuous second derivatives on \( [a, b] \) and satisfying the homogeneous boundary conditions (1b). Then

\[
\langle L[u], v \rangle = \int_a^b -\frac{1}{w} [(pu')' + qu]\overline{v} \, dx
= -\int_a^b [(pu')' + qu]\overline{v} \, dx,
\]  

and

\[
\langle L[u], v \rangle = \overline{\langle L[v], u \rangle}
\]
and integrating the \( (pu')' \) term by parts twice, so as to undo the derivatives on \( u \), gives

\[
\langle L[u], v \rangle = [p(u'v - u'v)]_a^b - \int_a^b [(pu')' + qv]u \, dx. \tag{49}
\]

Remembering that \( p(x), q(x), \) and \( w(x) \) are real, we can re-express the last term in (49) as

\[
\int_a^b [(pu')' + qv]u \, dx = \int_a^b \frac{1}{w} [(pu')' + qv]u \, w \, dx \\
= \int_a^b \frac{1}{w} [(pu')' + qv] \, u \, w \, dx \\
= - \int_a^b u \, L[v] \, w \, dx \\
= -\langle u, L[v] \rangle. \tag{50}
\]

Further, the boundary term in (49) is zero because \( u \) and \( v \) satisfy the homogeneous boundary conditions (1b). For instance, suppose \( \alpha = \gamma = 1 \) and \( \beta = \delta = 0 \) in (1b), so \( u(a) = 0, u(b) = 0, v(a) = 0, \) and \( v(b) = 0. \) It follows from the latter two that \( \bar{v}(a) = 0 \) and \( \bar{v}(b) = 0, \) so

\[
[p(u'v - u'v)]_a^b = (0 - 0) - (0 - 0) = 0; \tag{51}
\]

similarly for any \( \alpha \) and \( \beta \) (not both zero) and any \( \gamma \) and \( \delta \) (not both zero), verification of which claim is left for the exercises. Thus, (49) becomes

\[
\langle L[u], v \rangle = \langle u, L[v] \rangle, \tag{52}
\]

which formula is known as the **Lagrange identity**.

**Proof of Theorem 17.7.1, part (a):** Let \( \lambda \) be an eigenvalue of (1) and \( \phi \) a corresponding eigenfunction, so

\[
L[\phi] = \lambda \phi. \tag{53}
\]

Post-dotting (53) with \( \phi \) and pre-dotting (53) with \( \phi \) gives

\[
\langle L[\phi], \phi \rangle = \langle \lambda \phi, \phi \rangle = \lambda \langle \phi, \phi \rangle \tag{54a}
\]

from (45b), and

\[
\langle \phi, L[\phi] \rangle = \langle \phi, \lambda \phi \rangle = \lambda \langle \phi, \phi \rangle \tag{54b}
\]

from (45c), respectively. Subtracting (54) from (54a) gives, by virtue of the Lagrange identity,

\[
0 = \langle \lambda - \lambda \rangle \langle \phi, \phi \rangle = (\lambda - \lambda) \| \phi \|^2. \tag{55}
\]
The factor $||\phi||^2$ is nonzero because $\phi$ is an eigenfunction, so it follows from (55) that $\lambda - \lambda_j = 0$. Thus, $\lambda = \lambda_j$ so $\lambda$ is real, as was to be proved. 

**Proof of Theorem 17.7.1, part (b):** Let $\phi_j$ and $\phi_k$ be eigenfunctions corresponding to distinct eigenvalues $\lambda_j$ and $\lambda_k$, respectively. Thus,

$$L[\phi_j] = \lambda_j \phi_j \quad \text{and} \quad L[\phi_k] = \lambda_k \phi_k. \quad (56a, b)$$

If we dot $\phi_k$ into each side of (56a) and dot each side of (56b) into $\phi_j$ [i.e., we pre-dot (56a) with $\phi_k$ and post-dot (56b) with $\phi_j$], we obtain

$$\langle \phi_k, L[\phi_j] \rangle = \langle \phi_k, \lambda_j \phi_j \rangle = \lambda_j \langle \phi_k, \phi_j \rangle, \quad (57)$$

and

$$\langle L[\phi_k], \phi_j \rangle = \langle \lambda_k \phi_k, \phi_j \rangle = \lambda_k \langle \phi_k, \phi_j \rangle, \quad (58)$$

respectively. The left-hand sides are equal by virtue of the Lagrange identity, and $\lambda_j = \lambda_k$ because the $\lambda$'s are real, so subtraction of (57) from (58) gives

$$(\lambda_j - \lambda_k) \langle \phi_k, \phi_j \rangle = 0. \quad (59)$$

Finally, $\lambda_j - \lambda_k \neq 0$ by assumption, so it follows that $\langle \phi_k, \phi_j \rangle = 0$, as was to be proved. 

Before closing this section let us explain the significance of the Lagrange identity (52). The operator $L$ is the differential operator

$$L = \frac{1}{w} \left[ \frac{d}{dx} \left( p \frac{d}{dx} \right) + q \right] \quad (60a)$$

on the domain $\mathcal{D}$ of functions, $u$ say, that are defined and have continuous second derivatives on $[a, b]$ and that satisfy the homogeneous boundary conditions

$$\alpha u(a) + \beta u'(a) = 0, \quad \gamma u(b) + \delta u'(b) = 0. \quad (60b)$$

*Note that the definition of the domain $\mathcal{D}$ is part of the definition of $L$, just as the domain of definition is part of the definition of a function. For instance, the function whose values are $\sin x$ on $0 \leq x \leq \pi$ is different from the function whose values are $\sin x$ on $-\pi \leq x \leq 2\pi$. Thus, the operator is the differential operator (60a) plus the domain of definition. In this discussion, although not elsewhere in this text, we distinguish between the differential operator [namely, the "action" (60a)] and operator [namely, the action (60a) plus the domain of definition $\mathcal{D}$]. By the way, why do we ask functions $u(x)$ in $\mathcal{D}$ to have continuous second derivatives? They need to be twice differentiable so that $L[u]$ exists because $L$ is a second-order differential operator. Further, we wish to ensure that the inner product integral $\langle L[u], v \rangle$ exists, and we can do that by asking the integrand to be continuous on $[a, b]$. In fact, $L[u]$ will be continuous if $u$ has a continuous second derivative.
More generally, the equation
\[ (L[u], v) = \langle u, L^*[v] \rangle \]  
(61)

is used to define the Hermitian conjugate (or adjoint) \( L^* \) of the operator \( L \), relative to whatever inner product is chosen. Let us illustrate.

**EXAMPLE 5.** Find the Hermitian conjugate of the operator consisting of the differential operator
\[
L = \frac{d^2}{dx^2} + \frac{d}{dx} + 1
\]
(62a)
on the domain \( D \) of real-valued functions defined and having continuous second derivatives on \([0, \pi]\) and satisfying the homogeneous initial conditions
\[
u(0) = 0, \quad \nu'(0) = 0,
\]
(62b)
say. Begin with the left-hand side of (61),
\[
(L[u], v) = \int_0^\pi \left( u'' + u' + u \right) v \, dx.
\]
(63)
Integrating the \( u''v \) term by parts twice, the \( u'v \) term once, leaving the \( uv \) term intact, and using (62b), gives
\[
(L[u], v) = \left. (u'v - uv' + uv) \right|_0^\pi + \int_0^\pi (v'' - v' + v) \, dx
\]
\[
= \left[ u'(\pi) + u(\pi) v(\pi) - [u(\pi)] v'(\pi) + \langle u, L^*[v] \rangle, \right.
\]
(64)
where, from the \( v'' - v' + v \) in the integral, we can infer that
\[
L^* = \frac{d^2}{dx^2} - \frac{d}{dx} + 1.
\]
(65a)

To obtain the boundary conditions associated with \( L^* \) we see, by comparing (64) with (61), that we need the boundary terms in (64) to drop out. Whereas \( u(0) = 0 \) and \( u'(0) = 0 \), the bracketed quantities \( u'(\pi) + u(\pi) \) and \( u(\pi) \) are not prescribed, so we must have both
\[
v(\pi) = 0, \quad v'(\pi) = 0.
\]
(65b)
Thus, the Hermitian conjugate operator is the differential operator (65a) on the domain \( D^* \) of real-valued functions defined and having continuous second derivatives on \([0, \pi]\) and satisfying the conditions (65b).

If the operator and its Hermitian conjugate (or adjoint) are identical, then we say that it is Hermitian (or self-adjoint). Thus, the operator in Example 5 is not
Hermitian because the $L^*$ in (65a) differs from the $L$ in (62a) (by the minus sign in front of the $d/dx$) and also because the boundary conditions (65b) on functions in $D^*$ are different from the boundary conditions (62b) on functions in $D$. Either of these differences would be sufficient to conclude that the operator is not Hermitian.

**EXAMPLE 6. Matrix Case.** Find the Hermitian conjugate of a real $n \times n$ matrix operator $A$, defined on the vector space $\mathbb{R}^n$, with the dot product $x \cdot y = x^T y$, where $x$ and $y$ are $n$-dimensional column vectors. $A^*$ is defined by requiring that

$$\langle Ax \rangle \cdot y = x \cdot \langle A^* y \rangle$$

for all $x$'s in the domain $D$ of $A$ and for all $y$'s in the domain $D^*$ of $A^*$. Since $A$ is $n \times n$ and $x$ is $n \times 1$, the $Ax$ in (66) is $n \times 1$. For the dot product on the left to be defined we need $y$ to be $n \times 1$ as well. On the right, $x$ is $n \times 1$, so we need $A^* y$ to be $n \times 1$. Since $y$ is $n \times 1$, $A^*$ needs to be $n \times n$. Thus, like $A$, $A^*$ is an $n \times n$ matrix operator defined on $\mathbb{R}^n$. To determine $A^*$, begin with the left-hand side of (66),

$$\langle Ax \rangle \cdot y = (Ax)^T y = x^T A^T y = x \cdot (A^T y),$$

so the Hermitian conjugate $A^*$ of $A$ is

$$A^* = A^T.$$  

Thus, $A$ is Hermitian if and only if $A^T = A$, that is, if $A$ is symmetric. Just as the eigenvalue problem for the Hermitian Sturm-Liouville problem is of great importance, so is the eigenvalue problem for real symmetric (hence Hermitian) matrices and, indeed, that case is singled out for study in Section 11.3.

**Closure.** The Sturm-Liouville eigenvalue problem is the differential equation analog of the matrix eigenvalue problem $Ax = \lambda x$, where $A$ is real and symmetric (hence Hermitian). In both cases the eigenvalues are real and the eigenvectors provide an orthogonal basis for the relevant vector space. The expansion formula (15) is the analog of formula (23) in Section 9.9.

Although the Sturm-Liouville and matrix eigenvalue problems are closely related (if $A$ is Hermitian), the Sturm-Liouville case is much more subtle because the vector space is infinite-dimensional and expansions are, in general, infinite series. For instance, in a five-dimensional space five orthogonal vectors necessarily constitute a basis, but in an infinite-dimensional space an infinite number of orthogonal vectors need not constitute a basis. Thus, part (d) of Theorem 17.7.1 is a deep result, and we put it forward without proof.

If we were to generalize the real symmetric matrix and Sturm-Liouville eigenvalue problems further, we would study Hermitian operators, with matrix, ordinary differential, partial differential, and integral operators merely occurring as special cases. Such study is beyond our present scope and falls within the domain of mathematics known as functional analysis. Rather than being exceptional, the operators encountered in applications are often Hermitian.
EXERCISES 17.7

1. Identify \( p(x), q(x), w(x), \alpha, \beta, \gamma, \delta \), solve for the eigenvalues and eigenfunctions, and work out the eigenfunction expansion of the given function \( f \). If the characteristic equation is too difficult to solve analytically, state that and proceed with the rest of the problem as though the \( \lambda_n \)’s were known.

(a) \( y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0, \quad f(x) = 100 \)
(b) \( y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0, \quad f(x) = 1 \)
(c) \( y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0, \quad f(x) = \begin{cases} 1, & 0 \leq x < L/2 \\ 0, & L/2 \leq x \leq L \end{cases} \)
(d) \( y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) + y'(L) = 0, \quad f(x) = 50 \)
(e) \( y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(\pi) = 0, \quad f(x) = 10 \)
(f) \( y'' + \lambda y = 0, \quad y'(-1) = 0, \quad y'(1) = 0, \quad f(x) = \begin{cases} 0, & -1 \leq x < 0 \\ 50, & 0 < x \leq 1 \end{cases} \)
(g) \( y'' + \lambda y = 0, \quad y(0) - 2y'(0) = 0, \quad y'(2) = 0, \quad f(x) = 100 \)

2. We pointed out that the Sturm-Liouville problem in Exercise 1 generated the half-range sine expansion studied in Section 17.4. Modify the boundary conditions in that example so as to generate, instead, the

(a) half-range cosine expansion
(b) quarter-range sine expansion
(c) quarter-range cosine expansion

3. (Obtaining Sturm–Liouville form) We observed, in Exercise 4, that the equation

\[
A(x)y'' + B(x)y' + C(x)y + \lambda D(x)y = 0 \tag{3.1}
\]

is in the standard Sturm-Liouville form (1a) only if \( B(x) = A'(x) \). Show that if \( A(x) \neq 0 \) on \([a,b]\) and \((B - A')/A\) is continuous on \([a,b]\), then we can recast \((3.1)\) in the form \((1a)\) by multiplying \((3.1)\) by

\[
\sigma(x) = e^{\int \frac{(B - A')}{A} \, dx}. \tag{3.2}
\]

4. Use the results of Exercise 3 to recast each of the following differential equations in the Sturm–Liouville form \((1a)\). Identify \( p(x), q(x), \) and \( w(x) \).

(a) \( xy'' + 5y' + \lambda xy = 0 \)
(b) \( y'' + 2y' + xy + \lambda x^2 y = 0 \)
(c) \( y'' + y' + \lambda y = 0 \)
(d) \( y'' - y' + \lambda xy = 0 \)
(e) \( x^2 y'' + xy' + \lambda x^2 y = 0 \)
(f) \( y'' + (\cot x)y' + \lambda y = 0 \)

5. Use computer software to find \( \lambda_1, \ldots, \lambda_8 \) from \((24)\), to seven significant figures.

6. Consider the eigenvalue problem

\[
y'' + \lambda y = 0, \quad (0 < x < 1) \quad 2y(0) - y(1) + 4y'(1) = 0, \quad y(0) + 2y'(1) = 0. \]

Explain why the latter problem is not of Sturm-Liouville type. Using computer software, determine any two eigenvalues.
HINT: You should obtain the characteristic equation

\[ \sin \sqrt{\lambda} = 2 \sqrt{\lambda}. \]

Although the latter equation has no roots on a real \( \sqrt{\lambda} \) axis, we need to search in the complex plane. With \( z = x + iy \) write \( \sin z = 2z \), use the identity \( \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y \) and obtain the equations \( \sin x \cosh y = 2x, \cos x \sinh y = 2y \) on \( x \) and \( y \). Then, use computer software to find any two solution pairs for \( x \) and \( y \), and hence for \( \lambda \).

7. Show that for

\[
\begin{align*}
y'' + \lambda y &= 0, & (0 < x < 1) \\
y(0) - y(1) &= 0, & y'(0) + y'(1) = 0
\end{align*}
\]
every \( \lambda \) (real or complex) is an eigenvalue! Is the latter a Sturm–Liouville system? Explain.

8. Consider the eigenvalue problem

\[
x^2 y'' + xy' + \lambda y = 0, & (1 < x < \alpha) \\
y(1) = 0, & y(\alpha) = 0.
\]

(a) Show that the eigenvalues and eigenfunctions are

\[
\lambda_n = \frac{n^2 \pi^2}{(\ln a)^2}, \quad \phi_n(x) = \sin \left( n \pi \frac{\ln x}{\ln a} \right)
\]

for \( n = 1, 2, \ldots \).

(b) Show that the eigenfunction expansion of a given function \( f \) is of the form

\[
f(x) = \sum_{n=1}^{\infty} c_n \sin \left( n \pi \frac{\ln x}{\ln a} \right),
\]

where

\[
c_n = \frac{\int_1^{\alpha} f(x) \sin \left( n \pi \frac{\ln x}{\ln a} \right) dx}{\int_1^{\alpha} \sin^2 \left( n \pi \frac{\ln x}{\ln a} \right) dx}.
\]

HINT: You will need to get the differential equation into Sturm–Liouville form, as discussed in Exercise 3, before you can identify the weight function \( w(x) \) for the inner product.

9. Expand the function

\[
f(x) = \begin{cases} x^1, & 0 \leq x < 2 \\ 0, & 2 \leq x \leq \pi \end{cases}
\]
in terms of the eigenfunctions of the given eigenvalue problem. Use computer software, such as the Maple int command, to evaluate the expansion coefficient \( a_n \) as a function of \( n \).

(a) \( y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0 \)

(b) \( y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0 \)

(c) \( y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0 \)

10. Prove that the Sturm–Liouville eigenvalues are simple, as stated in part (b) of Theorem 17.7.1. HINT: Suppose that \( \phi_1 \) and \( \phi_2 \) are two eigenfunctions corresponding to an eigenvalue \( \lambda \) of (1), and suppose \( \alpha \neq 0 \) in (1b). Then the Wronskian \( W(x) \) of \( \phi_1 \) and \( \phi_2 \), evaluated at \( x = \alpha \), is

\[
W(\alpha) = \begin{vmatrix} \phi_1(\alpha) & \phi_2(\alpha) \\ \phi'_1(\alpha) & \phi'_2(\alpha) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ -\alpha \phi'_1(\alpha) & -\alpha \phi'_2(\alpha) \end{vmatrix} = 0.
\]

On the other hand, if \( \alpha \) does equal zero in (1b) then (1b) becomes \( y''(0) = 0 \), so

\[
W(\alpha) = \begin{vmatrix} \phi_1(\alpha) & \phi_2(\alpha) \\ \phi'_1(\alpha) & \phi'_2(\alpha) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0
\]

once again. Show that it follows from these results and Liouville’s formula that \( W(x) = 0 \) on \( [a, b] \), and cite an appropriate theorem which then implies that \( \phi_1 \) and \( \phi_2 \) must be linearly dependent on \( [a, b] \).

11. (Real eigenfunctions) Show that if \( \phi(x) \) is an eigenfunction of a Sturm–Liouville problem, then \( \phi(x) \) is either a real-valued function or else it is a complex constant times a real-valued function. HINT: Show that if \( \phi(x) \) is an eigenfunction corresponding to an eigenvalue \( \lambda \), then so is \( \bar{\phi}(x) \). Then use part (b) of Theorem 17.7.1 (namely, that the eigenvalues are simple) to show that \( \phi(x) = c \bar{\phi}(x) \), where \( c \) is a constant. Expressing the latter equation in the (polar) form \( A(x)e^{iB(x)} = Ce^{iD}A(x)e^{-iB(x)} \), show that \( B(x) \) is, at most, a constant.

12. Prove Theorem 17.7.2. HINT: You may assume that the eigenfunctions are real (proof of which is outlined in Exercise 10). Multiply each term in \( (py')' + qy + \lambda wy = 0 \) by \( \overline{y} \), and integrate over the \( [a, b] \) interval. Thus, show that

\[
\lambda \|y\|^2 = -(py') - \int_a^b p|y'|^2 dx = -\int_a^b q|y|^2 dx,
\]

and examine the signs of the individual terms.
13. (Column buckling with lateral restraint) Consider the buckling of the column of length \( L \) and stiffness \( EI \) shown in the figure. It is fixed into the floor such that \( y(0) = y'(0) = 0 \), and it is restrained laterally, at the free end, by a spring of stiffness \( k \). Then it turns out that the lateral deflection \( y(x) \) is governed by the eigenvalue problem

\[
\begin{align*}
y'''' + \lambda y'' &= 0, \quad (0 < x < L) \\
y(0) &= y'(0) = y''(L) = 0, \\
y''(L) &= -\lambda y'(L) + \kappa y(L),
\end{align*}
\]

where \( \lambda = P/EI \) and \( \kappa = k/EI \).

(a) Show that the characteristic equation is

\[
(\Lambda^2 - \kappa L)\Lambda \cos \Lambda L + \kappa \sin \Lambda L = 0, \tag{13.2}
\]

where \( \Lambda = \sqrt{\lambda} \), and that the corresponding eigenfunctions (buckling modes) are (to within an arbitrary scale factor)

\[
y = \sin \Lambda x - \tan \Lambda L \cos \Lambda x - \Lambda x + \tan \Lambda L. \tag{13.3}
\]

(b) Solve (13.2) for the critical buckling load, \( P_{cr} \), for the case where \( \kappa = 0 \).

(c) Is (13.1) a Sturm–Liouville problem? Explain.

14. (Buckling of linearly tapered column) Consider a column of circular cross section, the radius of which varies linearly with \( x \). It extends over \( a < x < b \), as shown in the figure.

\[
x^2 y'' + \lambda y = 0, \quad (a < x < b) \\
y(a) = 0, \quad y(b) = 0.
\]

is pinned at both ends, and is loaded axially by a force \( P \). Then the cross-sectional moment of inertia \( I \) is not a constant; it is given by \( I(x) = I_0(x/b)^4 \), where the constant \( I_0 \) is the value of \( I(x) \) at \( x = b \), so the eigenvalue problem governing buckling is found to be

\[
x^2 y'' + \lambda y = 0, \quad (a < x < b) \\
y(a) = 0, \quad y(b) = 0, \tag{14.1}
\]

where \( \lambda \equiv b^4 P/EI_0 \).

(a) Verify that (for the case \( \lambda \neq 0 \)) the general solution of the differential equation can be expressed as

\[
y(x) = x \left[ A \sin \left( \frac{\sqrt{\lambda}}{x} \right) + B \cos \left( \frac{\sqrt{\lambda}}{x} \right) \right]. \tag{14.2}
\]

(b) Applying the boundary conditions, show that the eigenvalues and eigenfunctions are

\[
\lambda_n = \left( \frac{n\pi ab}{L} \right)^2, \tag{14.3a}
\]

\[
\phi_n(x) = x \sin \left[ \frac{n\pi b}{L} \left( 1 - \frac{a}{x} \right) \right], \tag{14.3b}
\]

for \( n = 1, 2, \ldots \), and that the buckling load is \( P_{cr} = \pi^2 EI_0 a^2/b^2 L^2 \), where \( L = b - a \).

15. (Buckling of nonlinearly tapered column) Although not wishing to give undue prominence to the subject of the buckling of columns, we include this final exercise on buckling, which we believe is interesting and challenging. If, in the problem of Exercise 14, the column radius is proportional to \( \sqrt{x} \) rather than to \( x \), then \( I(x) = I_0(x/b)^2 \), and we have

\[
x^2 y'' + \lambda y = 0, \quad (a < x < b) \\
y(a) = 0, \quad y(b) = 0. \tag{15.1}
\]
where $\lambda \equiv b^2 P/E I_0$. Show that the buckling load is

$$P_{cr} = \frac{E I_0}{b^2} \left\{ \frac{1}{4} + \frac{\pi^2}{\left[ \ln (b/a) \right]^2} \right\}. \quad (15.2)$$

16. We show in (51) that the boundary term in (49) is zero for the special simple case where $\alpha = \gamma = 1$ and $\beta = \delta = 0$. Prove that the boundary term is zero for any $\alpha$ and $\beta$ (not both zero) and for any $\gamma$ and $\delta$ (not both zero).

17. Find the Hermitian conjugate (i.e., the adjoint) of the given operator and state whether the given operator is Hermitian (self-adjoint). If it is not, state why it is not. In each case, the interval is $0 \leq x \leq 1$, and the inner product is

$$\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx.$$ 

(a) $L = \frac{d}{dx}, \quad u(0) = 0$
(b) $L = \frac{d}{dx}, \quad u(1) = 0$

(c) $L = \frac{d^2}{dx^2}, \quad u(0) = u'(0) = 0$
(d) $L = \frac{d^2}{dx^2}, \quad u'(0) = u'(1) = 0$
(e) $L = \frac{d^2}{dx^2} + 3, \quad u'(0) = u'(1) = 0$
(f) $L = \frac{d^2}{dx^2} + \frac{d}{dx}, \quad u(0) = u(1) = 0$
(g) $L = \frac{d^2}{dx^2} + \frac{d^2}{dx^2} + 2 \frac{d}{dx}, \quad u(0) = u(1) = u'(0) = u'(1) = 0$
(h) $L = \frac{d^2}{dx^2} - 1, \quad u(0) + u'(0) = u'(1) = 0$
(i) $L = \frac{d^2}{dx^2}, \quad u'(0) = u(1) + 5u'(1) = 0$
(j) $L = \frac{d^2}{dx^2}, \quad u(0) - u(1) = u'(0) - u'(1) = 0$

---

### 17.8 Periodic and Singular Sturm–Liouville Problems

The Sturm–Liouville problem studied in Section 17.7 consists of the linear homogeneous second-order differential equation

$$[p(x)y']' + q(x)y + \lambda w(x)y = 0, \quad (a < x < b) \quad (1a)$$

with homogeneous boundary conditions of the form

$$\alpha y(a) + \beta y'(a) = 0,$$
$$\gamma y(b) + \delta y'(b) = 0. \quad (1b)$$

where $a, b$ are finite, where $p, p', q, w$ are continuous on $[a, b]$, and where $p(x) > 0$ and $w(x) > 0$ on $[a, b]$. The latter is generally known as a regular Sturm–Liouville problem, and many powerful results followed, as given in Theorems 17.7.1 and 17.7.2. We say that the boundary conditions (1b) are separated since one condition applies at $x = a$ and the other at $x = b$.

If any of the conditions cited above are not met, then the results obtained in Theorems 17.7.1 and 17.7.2 may not hold. Among the nonregular versions of the Sturm–Liouville problem, two are especially prominent and are the subject of this section: the Sturm–Liouville problem with periodic boundary conditions, and the singular Sturm–Liouville problem. In these cases the conditions cited above are met, except as noted below.
**Periodic boundary conditions.** In this case we have, in place of the separated boundary conditions \((1b)\), the nonseparated conditions

\[
\begin{align*}
y(a) &= y(b), \\
y'(a) &= y'(b),
\end{align*}
\tag{2}
\]

which are known as *periodic boundary conditions*, for reasons that will become clear when we work an example.

**Singular case.** In this case \(p(x)\) [and possibly \(w(x)\)] vanishes at one or both endpoints, so that \(p(x) > 0\) and \(w(x) > 0\) holds on the open interval \((a, b)\) rather than on the closed interval \([a, b]\). Further, the boundary conditions are modified as follows.

- **\(p(a) = 0\) [and \(p(b) \neq 0\)]**: Then the boundary conditions are
  \[
  \begin{align*}
y \text{ bounded at } a, \\
y'(b) &= 0.
\end{align*}
  \tag{3}
  \]

- **\(p(b) = 0\) [and \(p(a) \neq 0\)]**: Then the boundary conditions are
  \[
  \begin{align*}
  \alpha y(a) + \beta y'(a) &= 0, \\
y \text{ bounded at } b.
\end{align*}
  \tag{4}
  \]

- **\(p(a) = p(b) = 0\)**: Then the boundary conditions are
  \[
  \begin{align*}
y \text{ bounded at } a, \\
y \text{ bounded at } b.
\end{align*}
  \tag{5}
  \]

By \(y\) being bounded at \(a\), for example, we mean that \(\lim_{x \to a^-} y(x)\) exists (and is therefore finite).

For these cases we have the following results.

---

**THEOREM 17.8.1 Periodic and Singular Cases**

Let \(\lambda_n\) and \(\phi_n(x)\) denote any eigenvalue and corresponding eigenfunction of a Sturm–Liouville problem with periodic boundary conditions, given by \((2)\), or a singular Sturm–Liouville problem (as defined above).

\(\lambda_n\) are real.
Periodic and Singular Sturm–Liouville Problems

(b) If \( q(x) \leq 0 \) on \([a, b]\) and \([p(x)\phi_n(x)\phi_n'(x)]_a^b \leq 0\) for the eigenfunction \( \phi_n(x) \), then not only is \( \lambda_n \) real, it is also nonnegative: \( \lambda_n \geq 0 \).

(c) Eigenfunctions corresponding to distinct eigenvalues are orthogonal. That is, if \( \lambda_j \neq \lambda_k \), then \( \langle \phi_j, \phi_k \rangle = 0 \).

As for the regular case, positive statements can be made about the completeness of the sets of orthogonal eigenfunctions generated by these problems, in the sense of their being bases for the eigenfunction expansion representation of sufficiently well behaved functions on the interval \( a < x < b \).

**EXAMPLE 1. Periodic Boundary Conditions.** Consider the Sturm–Liouville problem

\[
\begin{align*}
y'' + \lambda y &= 0, \quad (-L < x < L) \\
y(-L) &= y(L), \quad y'(-L) = y'(L).
\end{align*}
\]

We begin with the general solution

\[
y(x) = \begin{cases} 
A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x, & \lambda \neq 0 \\
C + Dx, & \lambda = 0.
\end{cases}
\]

For \( \lambda = 0 \) the boundary conditions (6b) give \( C - DL = C + DL \) and \( D = D \), so \( D = 0 \) and \( C \) is arbitrary. Thus, \( y(x) = C \). So \( \lambda = 0 \) is an eigenvalue and has the eigenfunction \( \phi(x) = 1 \). For \( \lambda \neq 0 \) the boundary conditions (6b) give

\[
\begin{align*}
(\sin \sqrt{\lambda} L)B &= 0, \\
(\sin \sqrt{\lambda} L)A &= 0.
\end{align*}
\]

Thus, either \( A = B = 0 \), which result we reject because it gives only the trivial solution \( y(x) = 0 \), or else

\[
\sin \sqrt{\lambda} L = 0
\]

and \( A, B \) are arbitrary. Since \( q(x) = 0 \), and

\[
[p(x)\phi_n(x)\phi_n'(x)]_L^L = \phi_n(L)\phi_n'(L) - \phi_n(-L)\phi_n'(-L)
\]

\[
= \phi_n(L)\phi_n'(L) - \phi_n(L)\phi_n'(L) = 0,
\]

it follows from part (b) of Theorem 17.8.1 that \( \lambda_n \geq 0 \). Thus, \( \sqrt{\lambda} \) in (9) is real and (9) has the roots \( \sqrt{\lambda} L = n\pi \), so the eigenvalues and eigenfunctions are

\[
\begin{align*}
\lambda_0 &= 0, \quad \phi_0(x) = 1 \\
\lambda_n &= \left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = \cos \frac{n\pi x}{L} \quad \text{and} \quad \sin \frac{n\pi x}{L}
\end{align*}
\]

for \( n = 1, 2, \ldots \).

Remember that there is nothing inappropriate about an eigenvalue being zero. It is the eigenfunction that is to be nontrivial, and \( \lambda_0 = 0 \) does give the nontrivial solution \( \phi_0(x) = 1 \).
Observe that the eigenvalues $\lambda_1, \lambda_2, \ldots$ are non-simple since each one has two linearly independent eigenfunctions. This result could not have occurred in a regular Sturm–Liouville problem, which must have simple eigenvalues [part (b) of Theorem 17.7.1], and is due to the periodic boundary conditions.

Noting that the weight function is $w(x) = 1$, the eigenfunction expansion of a given function $f$ on $(-L, L)$ takes the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

$$a_0 = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx,$$

$$a_n = \frac{\langle f, \cos \frac{n\pi x}{L} \rangle}{\langle \cos \frac{n\pi x}{L}, \cos \frac{n\pi x}{L} \rangle} = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx,$$

$$b_n = \frac{\langle f, \sin \frac{n\pi x}{L} \rangle}{\langle \sin \frac{n\pi x}{L}, \sin \frac{n\pi x}{L} \rangle} = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.$$

The result (11) is seen to be the same as the classical Fourier series of a $2L$-periodic function, which we studied in Section 17.3. Thus the expansion (11a) of the function $f$ shown in Fig. 1, for example, will be the same as the classical Fourier series of the $2L$-periodic function shown in Fig. 2, which result explains why we call (2) "periodic boundary conditions."

COMMENT. As usual, the function $f$ being expanded (e.g., $f$ in Fig. 1) does not itself need to satisfy the boundary conditions imposed on the eigenfunctions [(6b) in this case].

EXAMPLE 2. A Bessel Equation. Consider the singular Sturm–Liouville problem

$$(xy')' + \lambda xy = 0, \quad (0 < x < L)$$

$$g(0) \text{ bounded, } g(L) = 0.$$  

Since (12a) is already in the standard Sturm–Liouville form $(py')' + qy + \lambda wy = 0$, we can see that $p(x) = x$, $q(x) = 0$, and $w(x) = x$. And since $p(x)$ and $w(x)$ vanish at the left endpoint $x = 0$, the problem (12) is singular; hence the boundary condition adopted at $x = 0$ is simply a boundedness condition.
To solve (12a), we notice that (12a) would be a Bessel equation of order zero if the \( \lambda \) were not present. Let us try to convert (12a) to a Bessel equation by a simple scaling, \( x = \alpha t \), where \( \alpha \) is to be determined. Under that change of variables (12a) becomes

\[(tT')' + \alpha^2 \lambda T = 0,\]  

where \( T(t) \equiv y(x(t)) = y(\alpha t) \), and where the primes denote \( d/dt \). Thus, we can remove the \( \lambda \) by choosing \( \alpha = 1/\sqrt{\lambda} \) (tentatively assuming that \( \lambda \neq 0 \)), so \( t = \sqrt{\lambda} x \). Then (13) becomes \((tT')' + T = 0\), with the general solution \( T(t) = AJ_0(t) + BY_0(t) \). Or, reverting to \( x \) and \( y \),

\[y(x) = AJ_0(\sqrt{\lambda} x) + BY_0(\sqrt{\lambda} x).\]  

However, for \( \lambda = 0 \) the latter fails to provide the general solution of (12a) because \( Y_0(0) = -\infty \) is undefined. But if \( \lambda = 0 \) then (12a) becomes \((xy')' = 0\) which can be integrated to give \( y(x) = C + D \ln x \). Thus, let us write the general solution of (12a) as

\[y(x) = \begin{cases} 
AJ_0(\sqrt{\lambda} x) + BY_0(\sqrt{\lambda} x), & \lambda \neq 0 \\
C + D \ln x, & \lambda = 0.
\end{cases}\]  

(15a,b)

For \( \lambda = 0 \), the boundedness condition requires that \( D = 0 \), and then \( y(L) = 0 \) gives \( C = 0 \), so \( y(x) = 0 \). Therefore, \( \lambda = 0 \) is not an eigenvalue. For \( \lambda \neq 0 \), the boundedness condition requires that \( B = 0 \) (because \( Y_0 \to -\infty \) as its argument tends to zero), so

\[y(x) = AJ_0(\sqrt{\lambda} x).\]  

(16)

Then, the other boundary condition gives

\[y(L) = 0 = AJ_0(\sqrt{\lambda} L).\]  

(17)

If \( A = 0 \), then (16) becomes the trivial solution, so let us satisfy (17) by asking that

\[J_0(\sqrt{\lambda} L) = 0\]  

(18)

instead. Now, \( q(x) = 0 \), and

\[\left[p(x)\phi_n(x)\phi'_n(x)\right]_0^L = \left[x\phi_n(x)\phi'_n(x)\right]_0^L\]  

(19)

is zero because \( \phi_n(L) = 0 \) and the \( x \) factor is zero at \( x = 0 \), so Theorem 17.8.1 tells us that not only is \( \lambda \) real, it is also nonnegative. Thus, the argument of \( J_0 \) in (18) is real, and it suffices to look for roots of (18) on the real axis. If we denote the zeros of \( J_0(x) \) as \( x = z_1, z_2, \ldots \) (Fig. 3), then (18) gives \( \sqrt{\lambda} L = z_n \), so the eigenvalues and eigenfunctions of (12) are

\[\lambda_n = \left(\frac{z_n}{L}\right)^2 \text{ and } \phi_n(x) = J_0 \left(\frac{z_n x}{L}\right)\]

(20)

*We can conclude that \( x\phi_n\phi'_n \) vanishes at \( x = 0 \), provided that \( \phi_n \) and \( \phi'_n \) are finite there. We know that \( \phi_n \) is finite there because that is our boundary condition, at \( x = 0 \), in (12b). Thus, we need to also ask that \( \phi'_n \) be finite there. But by the time we reach (19) we already have the form (16) for the eigenfunctions, even if we don’t yet know the \( \lambda \)’s, and the derivative of the right-hand side of (16) is bounded at \( x = 0 \); indeed, it is even zero.
Chapter 17. Fourier Series, Fourier Integral, Fourier Transform

for \( n = 1, 2, \ldots \). The zeros \( z_n \) are tabulated, and the first several are as follows:

\[
\begin{align*}
z_1 &= 2.405, & z_2 &= 5.520, & z_3 &= 8.654, & z_4 &= 11.792, & z_5 &= 14.931, \\
& & & & & & (21)
\end{align*}
\]

Further, the eigenfunction expansion of a given function \( f \), on \( 0 < x < L \), is given by

\[
f(x) = \sum_{n=1}^{\infty} a_n J_0 \left( \frac{z_n}{L} x \right),
\]

where, recalling that the weight function in the inner product is \( w(x) = x \),

\[
a_n = \frac{\langle f(x), J_0 \left( \frac{z_n}{L} x \right) \rangle}{\langle J_0 \left( \frac{z_n}{L} x \right), J_0 \left( \frac{z_n}{L} x \right) \rangle} = \frac{\int_0^L f(x) J_0 \left( \frac{z_n}{L} x \right) x \, dx}{\int_0^L \left[ J_0 \left( \frac{z_n}{L} x \right) \right]^2 x \, dx}. \tag{23}
\]

The integral in the denominator of (23) is evaluated in Exercise 7 of Section 4.6, so the final expression for \( a_n \) is

\[
a_n = \frac{2}{L^2 J_1(z_n)^2} \int_0^L f(x) J_0 \left( \frac{z_n}{L} x \right) x \, dx, \tag{24}
\]

where \( J_0 \) and \( J_1 \) are Bessel functions of the first kind, of orders 0 and 1, respectively.

**EXAMPLE 3. A Legendre Equation.** The Sturm–Liouville problem

\[
(1 - x^2)y'' - 2xy' + \lambda y = 0 \quad (-1 < x < 1) \tag{25a}
\]

\[
y(-1) \text{ bounded.} \quad y(1) \text{ bounded} \tag{25b}
\]

is also singular because \( p(x) = 1 - x^2 \) vanishes at both endpoints. Hence, we apply the boundedness boundary conditions. In fact, (25a) is the Legendre equation, which is the subject of Section 4.4. There, we found that solutions of (25a) that are bounded on \(-1 \leq x \leq 1\) are possible only if \( \lambda = n(n + 1) \), for \( n = 0, 1, 2, \ldots \), and those nontrivial solutions are the Legendre polynomials \( P_n(x) \). Thus, the eigenvalues and eigenfunctions of (25) are

\[
\lambda_n = n(n + 1), \quad \phi_n(x) = P_n(x). \tag{26}
\]

for \( n = 0, 1, 2, \ldots \).

The eigenfunction expansion of a given function \( f \), on \(-1 \leq x \leq 1\), is given by

\[
f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \tag{27}
\]

where, since the weight function is \( w(x) = 1 \),

\[
a_n = \frac{\langle f(x), P_n(x) \rangle}{\langle P_n(x), P_n(x) \rangle} = \frac{\int_{-1}^1 f(x) P_n(x) \, dx}{\int_{-1}^1 P_n^2(x) \, dx}. \tag{28}
\]
The integral in the denominator of (28) was found [(18) in Section 4.4] to be \(2/(2n+1)\), so

\[
a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)\,dx.
\]

For instance, let \(f\) be the “ramp” \(f(x) = xH(x)\), where \(H\) is the Heaviside function. Then (29) becomes

\[
a_n = \frac{2n+1}{2} \int_{0}^{1} xP_n(x)\,dx,
\]

so

\[
a_0 = \frac{1}{2} \int_{0}^{1} x\,dx = \frac{1}{4},
\]

\[
a_1 = \frac{3}{2} \int_{0}^{1} x^2\,dx = \frac{1}{2},
\]

and so on. Thus,

\[
f(x) = \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) - \frac{3}{32} P_3(x) + \frac{13}{256} P_5(x) - \cdots.
\]

For comparison, we have plotted both \(f(x)\) and the partial sum \(s_n(x)\) of the first \(n\) nonvanishing terms on the right-hand side of (31), for \(n = 2\) and \(n = 5\), in Fig. 4.

Because of their close relationship with Fourier series, we call (22) and (31) **Fourier-Bessel** and **Fourier-Legendre** series, respectively. Such expansions will be needed in Chapters 18–20.

**Closure.** We have studied the Sturm–Liouville problem with periodic boundary conditions [specifically, the nonseparated conditions (2)], and the singular Sturm–Liouville problem [where \(p(x) > 0\) and \(w(x) > 0\) hold on \(\alpha < x < b\) rather than on \(\alpha \leq x \leq b\), and where a boundedness boundary condition is imposed at an endpoint at which \(p(x)\) vanishes] because those cases are not covered by the theorems given in Section 17.7, and because they are important cases. Essentially, the upshot is that “all is well”: we still obtain real eigenvalues and sets of orthogonal eigenfunctions that can be used to expand functions over the \((\alpha, b)\) interval.

Observe that for the interval \(0 < x < 1\), say, each of the Sturm–Liouville problems

\[
y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0
\]

and

\[
(1 - x^2)y'' - 2xy' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = \text{bounded}
\]

generates an orthogonal basis of eigenfunctions, and that one could write down an unlimited number of other Sturm–Liouville problems that generate orthogonal bases over the same interval. If we wish to expand a given function on that interval.
then how do we know which basis to use? As we shall see in Chapters 18–20, that decision will be based on the mathematical context.

**EXERCISES 17.8**

1. We derived the solution (14) of (12a) by introducing a change of variables $x = t/\sqrt{\lambda}$. Derive it using the method explained in Section 4.6.6. Instead,

$$y'' + \lambda y = 0, \quad y(0) = y(4), \quad y'(0) = y'(4), \quad f(x) = H(x - 2).$$

2. Find the eigenvalues and eigenfunctions, and work out the eigenfunction expansion of the given $f$. NOTE: As usual, $H(x)$ denotes the Heaviside step function. Use computer software, if you wish, to evaluate any needed integrals.

(a) $y'' + \lambda y = 0, \quad y(0) = y(4), \quad y'(0) = y'(4), \quad f(x) = H(x - 2)$

(b) $y'' + \lambda y = 0, \quad y(-1) = y(5), \quad y'(-1) = y'(5), \quad f(x) = x + 2$

(c) $x^2y'' + xy' + \lambda y = 0, \quad y(1) = y(2), \quad y'(1) = y'(2), \quad f(x) = 6$

(d) $(1 - x^2)y'' - 2xy' + \lambda y = 0, \quad y(1) = 0, \quad y(0) = 0, \quad y(1) \text{ bounded,} \quad f(x) = 4; \quad \text{evaluate only the first three nonvanishing terms in the expansion of}$

(e) $(1 - x^2)y'' - 2xy' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) \text{ bounded,} \quad f(x) = x; \quad \text{evaluate only the first three nonvanishing terms in the expansion of}$

(f) $(1 - x^2)y'' - 2xy' + \lambda y = 0, \quad y(-1) \text{ bounded,} \quad y'(0) = 0, \quad f(x) = x - 2$

3. Expand $f(x) = H(x)$, on $-1 < x < 1$, in terms of the eigenfunctions of the Sturm–Liouville problem

$$(1 - x^2)y'' - 2xy' + \lambda y = 0,$$

where $y(-1)$ and $y(1)$ are bounded. Plot both $f(x)$ and the sum of the first four nonvanishing terms of that expansion.

4. Expand $f(x) = 1 - x$ on $0 < x < 1$, in terms of the eigenfunctions of the Sturm–Liouville problem

$$(1 - x^2)y'' - 2xy' + \lambda y = 0,$$

where $y'(0) = 0$ and $y(1)$ is bounded. Plot both $f(x)$ and the sum of the first three nonvanishing terms of that expansion.

5. Determine the eigenvalues (or at least the characteristic equation for them), eigenfunctions, and weight function of the Sturm–Liouville problem

$$x^2y'' + xy' + (\lambda x^2 - 9)y = 0,$$

where $y(0)$ is bounded and $y(L) = 0$.

6. (Chebyshev polynomials) Consider the eigenvalue problem

$$(1 - x^2)y'' - xy' + \lambda y = 0, \quad (-1 < x < 1)$$

where $y(-1)$, $y'(-1)$, $y(1)$, and $y'(1)$ are to be bounded; (6.1) is the Chebyshev equation, after the Russian mathematician Pafnui Chebyshev (1821–1894), often transliterated as Tchebichef.

(a) Show that under the change of variables $x = \cos \theta$ the equation (6.1) becomes

$$\theta'' + \lambda \theta = 0, \quad (0 < \theta < \pi)$$

where $\Theta(\theta) \equiv y(x(\theta)) = y(\cos \theta)$. Thus, the general solution, in terms of $\theta$, is

$$\Theta(\theta) = \begin{cases} A \cos \sqrt{\lambda} \theta + B \sin \sqrt{\lambda} \theta, & \lambda \neq 0 \\ C + D\theta, & \lambda = 0. \end{cases}$$

(b) Similarly, the solutions $\cos \sqrt{\lambda} \theta$, $\sin \sqrt{\lambda} \theta$, 1, and $\theta$ in (6.3) are bounded at $\theta = 0$ ($x = 1$) and $\theta = \pi$ ($x = -1$). However, show (by chain differentiation) that $y'(x)$ is bounded at $x = \pm 1$ only if $B = D = 0$ and $\sqrt{\lambda} = n = 1, 2, \ldots$. Thus, the eigenfunctions of (6.1) are, in terms of $\theta$, $\cos \theta$, $\sin \theta$, and 1 or, equivalently, $\cos n\theta$ for $n = 0, 1, 2, \ldots$. In terms of the original $x$ variable, the eigenvalues and eigenfunctions of (6.1) are

$$\lambda_n = n^2, \quad T_n(x) = \cos (n \cos^{-1} x), \quad (n = 0, 1, 2, \ldots)$$

the $T$ in honor of Chebyshev.

(c) Though not obvious, it turns out that $T_n(x)$ is an $n$th-degree polynomial in $x$. Show that the first several are as follows:

$$\lambda_n = n^2, \quad T_n(x) = \cos (n \cos^{-1} x), \quad (n = 0, 1, 2, \ldots)$$
\[ T_0(x) = 1, \]
\[ T_1(x) = x, \]
\[ T_2(x) = 2x^2 - 1, \]
\[ T_3(x) = 4x^3 - 3x, \]
\[ T_4(x) = 8x^4 - 8x^2 + 1, \]
\[ T_5(x) = 16x^5 - 20x^3 + 5x. \]

HINT: Use the trigonometric identities
\[
\begin{align*}
\cos 2\theta &= 2\cos^2 \theta - 1, \\
\cos 3\theta &= 4\cos^3 \theta - 3\cos \theta, \\
\cos 4\theta &= 8\cos^4 \theta - 8\cos^2 \theta + 1, \\
\cos 5\theta &= 16\cos^5 \theta - 20\cos^3 \theta + 5\cos \theta,
\end{align*}
\]
and so on.

(d) Get (6.1) into the standard Sturm–Liouville form by multiplying through by a suitably chosen function \(\sigma(x)\) (that is, nonzero on \(-1 < x < 1\)), and thus show that the weight function is
\[
w(x) = \frac{1}{\sqrt{1 - x^2}},
\]
(6.6)
(c) Theorem 17.8.1 guarantees that, with respect to the weight function (6.6), \(\langle T_m(x), T_n(x) \rangle = 0\) for \(m \neq n\). Nonetheless, prove that result directly, by evaluating the integral
\[
\langle T_m, T_n \rangle = \int_{-1}^{1} T_m(x)T_n(x) \frac{1}{\sqrt{1 - x^2}} dx.
\]
Further, show by direct integration that for \(m = n\) we have
\[
\langle T_n, T_n \rangle = \int_{-1}^{1} T_n^2(x) \frac{1}{\sqrt{1 - x^2}} dx = \begin{cases} 
\pi, & m = n = 0 \\
\pi/2, & m = n \neq 0.
\end{cases}
\]  
(6.7)
(f) Thus, the eigenfunction expansion of a given function \(f\), defined on \(-1 < x < 1\), is
\[
f(x) = \sum_{n=0}^{\infty} a_n T_n(x), \quad (-1 < x < 1)
\]
(6.8)
where
\[
a_n = \frac{\langle f, T_n \rangle}{\langle T_n, T_n \rangle} = \begin{cases} 
\frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} dx, & n = 0 \\
\frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1 - x^2}} dx, & n = 1, 2, \ldots.
\end{cases}
\]  
(6.9)
Use (6.9) to evaluate the \(a_n\)'s for the case where \(f(x) = H(x)\).
(g) Plot \(H(x)\) and (by computer) the partial sum of the series obtained in part (f), through \(n = 5\).
(h) Derive the values
\[
T_0(1) = 1, \quad T_0(-1) = (-1)^n,
\]
(6.10)
and the recursion formula
\[
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).
\]  
(6.11)
(i) Use (6.11), and the \(T_n\)'s given in (6.5), to derive \(T_0(x)\) and \(T_1(x)\).
(j) It can be shown that
\[
\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \quad (-1 < t < 1)
\]
(6.12)
so the left-hand member of (6.12) is called a generating function of the \(T_n\)'s. By working out the Taylor series of the left-hand side, verify (6.12) through \(n = 2\).

17.9 Fourier Integral

If a function \(f\) defined on \(-\infty < x < \infty\) is periodic (and sufficiently well-behaved), then it can be represented by a Fourier series. We have begun to see, and will continue to see in Chapters 18–20, that Fourier series representation is of great importance. Sometimes we work with functions, defined on \(-\infty < x < \infty\), that are not periodic, such as \(f(x) = e^{-x^2}\), the graph of which is given in Fig. 1.
Evidently, we cannot expand such functions in Fourier series if they are not periodic. Yet, we can think of $f$ as periodic but with an infinite period. Thus, to extend the Fourier series concept to nonperiodic functions we will now consider the limiting case of the classical Fourier series

$$f(x) = \sum_{n=0}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

(1a)

where

$$a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) \, dx, \quad a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx,$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx,$$

(1b)

as $\ell \to \infty$. We cannot simply set $\ell = \infty$ in (1), as that would yield nonsense. Rather one needs to carry out a careful limit process, as $\ell$ tends to $\infty$.

First, note that the $n\pi/\ell$'s in (1a) are the frequencies – spatial or temporal, depending on whether $x$ is a space variable or time. The set of all of the frequencies,

$$0, \frac{\pi}{\ell}, \frac{2\pi}{\ell}, \frac{3\pi}{\ell}, \ldots$$

is called the **frequency spectrum**. To see what happens to the frequency spectrum as $\ell$ increases, consider the cases where $\ell = \pi$, $2\pi$, and $10\pi$. The corresponding frequency spectra are as follows:

$$\ell = \pi : \quad n\pi/\ell = 0, 1, 2, 3, 4, \ldots$$

$$\ell = 2\pi : \quad n\pi/\ell = 0, 0.5, 1.0, 1.5, 2.0, \ldots$$

$$\ell = 10\pi : \quad n\pi/\ell = 0, 0.1, 0.2, 0.3, 0.4, \ldots$$

Observe that as $\ell$ increases the discrete spectrum becomes more and more dense, and approaches a continuous spectrum (from 0 to $\infty$) as $\ell \to \infty$. Therefore, we can expect that as $\ell \to \infty$ the summation in (1a), on the discrete variable $n$, will give way to an integration on a continuous variable, say $\omega$. In fact, if $f$ is sufficiently well-behaved (e.g., see Theorem 17.9.1 below) one can show that

$$f(x) = \int_{0}^{\infty} \left[ a(\omega) \cos \omega x + b(\omega) \sin \omega x \right] \, d\omega,$$

(2a)

*Although we usually split out the $a_0$ term, here we include it in the sum, merely to increase the resemblance between (1a) and (2a), below.*
where

\begin{align}
  a(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x \, dx, \\
  b(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x \, dx.
\end{align}

(2b)

The right-hand side of (2a) is called the Fourier integral of \( f \), which we denote as \( \mathcal{F} f \), and (2a) is called the Fourier integral representation of \( f \).

**THEOREM 17.9.1 Fourier Integral Theorem**

Let \( f \) be defined on \(-\infty < x < \infty\), let \( f \) and \( f' \) be piecewise continuous on every finite interval \([-\ell, \ell]\) (i.e., for \( \ell \) arbitrarily large), and let \( \int_{-\infty}^{\infty} |f(x)| \, dx \) be convergent. Then the Fourier integral of \( f \) converges to \( f(x) \) at every point \( x \) at which \( f \) is continuous, and to the mean value \([f(x^+) + f(x^-)]/2\) at every point \( x \) at which \( f \) is discontinuous.

**Proof**: Rigorous proof of this theorem is well beyond our present scope, and the following is put forward only as a heuristic derivation. If we change the dummy integration variable to \( \xi \) in (1b), to distinguish it from the fixed point \( x \) in (1a), insert (1b) into (1a), and use the identity \( \cos A \cos B + \sin A \sin B = \cos (A-B) \) for greater compactness, then the right-hand side of (1a), namely, \( \mathcal{F} f \), becomes

\[
\mathcal{F} f = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(\xi) \, d\xi + \sum_{n=1}^{\infty} \frac{1}{\ell} \int_{-\ell}^{\ell} f(\xi) \cos \frac{n\pi}{\ell} (\xi - x) \, d\xi.
\]

(3)

Denote the two terms on the right-hand side as \( I \) and \( J \), respectively. For \( I \) we have

\[
|I| = \left| \frac{1}{2\ell} \int_{-\ell}^{\ell} f(\xi) \, d\xi \right| \leq \frac{1}{2\ell} \int_{-\infty}^{\infty} |f(\xi)| \, d\xi \to 0
\]

(4)

*Normally, a singular integral of the form \( \int_{-\infty}^{\infty} F(x) \, dx \) is understood to mean

\[
\int_{-\infty}^{\infty} F(x) \, dx = \lim_{A \to \infty, B \to \infty} \int_{-A}^{B} F(x) \, dx,
\]

where \( A \) and \( B \) tend to infinity independently. However, the integrals in \( (2b) \) are to be understood in the more forgiving sense,

\[
\int_{-\infty}^{\infty} F(x) \, dx = \lim_{A \to \infty} \int_{-A}^{A} F(x) \, dx.
\]

To see that the latter is "more forgiving" than the former, observe that if \( F(x) = x \), for instance, then with the former interpretation the integral is divergent but with the latter interpretation it converges to zero.
as $\ell \to \infty$ because $\int_{-\infty}^{\infty} |f(\xi)| \, d\xi$ is finite, by assumption, and $1/(2\ell) \to 0$ as $\ell \to \infty$. Thus, $I \to 0$ as $\ell \to \infty$. In $J$, let $\pi/\ell \equiv \Delta \omega$. Then

$$J = \frac{1}{\pi} \sum_{n=1}^{\infty} \left[ \int_{-\ell}^{\ell} f(\xi) \cos n \Delta \omega (\xi - x) \, d\xi \right] \Delta \omega$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \left[ \int_{-\infty}^{\infty} f(\xi) \cos \omega (\xi - x) \, d\xi \right] \omega \, d\omega$$  \hspace{1cm} (5)

as $\ell \to \infty$. To understand the last step, observe that the sum in (5) is a Riemann sum, and as $\Delta \omega \to 0$ (i.e., as $\ell \to \infty$, because $\Delta \omega = \pi/\ell$) it yields a Riemann integral. That is, we partition the interval $0 < \omega < \infty$ into equal parts of dimension $\Delta \omega = \pi/\ell$, and call $\omega_1 = \pi/\ell, \omega_2 = 2\pi/\ell, \omega_3 = 3\pi/\ell$, and so on, and use the general Riemann integral formula

$$\lim_{\Delta \omega \to 0} \sum_{n=1}^{\infty} F(n\Delta \omega) \Delta \omega = \lim_{\Delta \omega \to 0} \sum_{n=1}^{\infty} F(\omega_n) \Delta \omega = \int_{0}^{\infty} F(\omega) \, d\omega.$$  \hspace{1cm} (6)

(See Fig. 2.) Finally, expressing $\cos \omega (\xi - x) = \cos \omega_1 \cos \omega x + \sin \omega_1 \sin \omega x$, we have

$$FS f = I + J$$

$$= 0 + \frac{1}{\pi} \int_{0}^{\infty} \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \cos \omega \xi \, d\xi \right] \cos \omega x \, d\omega$$

$$+ \int_{0}^{\infty} \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \sin \omega \xi \, d\xi \right] \sin \omega x \, d\omega$$

$$= \int_{0}^{\infty} [a(\omega) \cos \omega x + b(\omega) \sin \omega x] \, d\omega$$  \hspace{1cm} (7)

as $\ell \to \infty$. That is, as $\ell \to \infty$ the Fourier series tends to the Fourier integral. We reiterate that our approach has been heuristic, not rigorous.  

**EXAMPLE 1.** *Rectangular Pulse.* Let $f$ be the rectangular pulse shown in Fig. 3. This $f$ does satisfy the conditions of the theorem. According to (2b),

$$a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x \, dx = \frac{1}{\pi} \int_{-1}^{1} \cos \omega x \, dx = \frac{2}{\pi} \frac{\sin \omega}{\omega}$$  \hspace{1cm} (8a)

and

$$b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x \, dx = 0$$  \hspace{1cm} (8b)

because the integrand $f(x) \sin \omega x$ is an odd function (recall that even $\times$ odd = odd). Thus, the Fourier integral representation of $f$ is, from (2) and (8),

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \omega}{\omega} \cos \omega x \, d\omega.$$  \hspace{1cm} (9)
Just as it was illuminating to plot the partial sums $s_n(x)$ of Fourier series, it should be illuminating to see how the partial integral

$$f_\Omega(x) = \frac{2}{\pi} \int_0^\Omega \frac{\sin \omega}{\omega} \cos \omega x \, d\omega$$

(10)

converges to $f(x)$ as $\Omega \to \infty$. Though the latter integral is not elementary, it can be evaluated in terms of the sine integral function

$$Si(x) = \int_0^x \frac{\sin t}{t} \, dt,$$

(11)

either analytically (Exercise 1) or using computer software. Using Maple, for instance, the command

$$\text{int}((2/\pi) \times (1/y) \times \sin(y) \times \cos(y \times x), \ y = 0..z);$$

(where $z$ is used in place of $\Omega$) gives

$$f_\Omega(x) = \frac{1}{\pi} \{Si[\Omega(x + 1)] - Si[\Omega(x - 1)]\},$$

(12)

which we’ve plotted in Fig. 4 for $\Omega = 4, 16, \text{and } 128$. Evidently $f_\Omega(x)$ does converge to the pulse $f(x)$, but the convergence is slow near the jump discontinuities; the limiting case as $\Omega \to \infty$ is left for Exercise 3. As in the case of Fourier series, the Fourier integral exhibits the Gibbs phenomenon at such discontinuities.

![Figure 4. Convergence of $f_\Omega(x)$ to $f(x)$ as $\Omega \to \infty$.](image)

It is also interesting to plot the Fourier coefficient $a(\omega)$, given by (8a), since $a(\omega)$ tells us the harmonic content of $f$, that is, the amplitude or "amount" of each $\cos \omega x$ harmonic present (Fig. 5).

**EXAMPLE 2.** Infinite Beam on Elastic Foundation. As a physical application of the Fourier integral let us consider the same problem as contained in Example 4 of Section 17.3 (which we urge you to review), but instead of the periodic loading suppose we have the nonperiodic rectangular pulse loading shown here in Fig. 6. Recall that the beam’s
deflection \( u(x) \) is governed by the differential equation
\[
EIu''' + ku = w(x),
\]  
(13)
where \( E, I, k \) are physical constants, and
\[
w(x) = \begin{cases} 
  w_0, & |x| < 1 \\
  0, & |x| > 1
\end{cases}
\]  
(14)
is the rectangular pulse applied load distribution. Proceeding essentially as in Example 4 of Section 17.3, we first express \( w(x) \) in Fourier integral form. Since \( w(x) \) is merely \( f(x) \) in Example 1, scaled by \( w_0 \), we conclude from (9) that
\[
w(x) = \frac{2w_0}{\pi} \int_{0}^{\infty} \frac{\sin \omega}{\omega} \cos \omega x \, d\omega.
\]  
(15)
Next, we seek \( u(x) \) in the Fourier integral form
\[
u(x) = \int_{0}^{\infty} a(\omega) \cos \omega x \, d\omega,
\]  
(16)
where \( a(\omega) \) remains to be determined. We have omitted the \( b(\omega) \sin \omega x \) since \( u(x) \) will evidently be a symmetric (even) function of \( x \); that is, if we did include that term we would find that \( b(\omega) = 0 \).

Formally differentiating (16) under the integral sign four times, and putting that result and (15) and (16) into (13) gives
\[
\int_{0}^{\infty} (EI\omega^3 + k)a(\omega) \cos \omega x \, d\omega = \frac{2w_0}{\pi} \int_{0}^{\infty} \frac{\sin \omega}{\omega} \cos \omega x \, d\omega.
\]  
(17)
Then, formally equating the coefficients of each cosine harmonic gives
\[
(EI\omega^3 + k)a(\omega) = \frac{2w_0}{\pi} \sin \frac{\omega}{\omega}
\]  
(18)
or
\[
a(\omega) = \frac{2w_0}{\pi} \frac{\sin \omega}{\omega} \frac{1}{(EI\omega^3 + k)}
\]  
(19)
so that (16) becomes
\[
u(x) = \frac{2w_0}{\pi} \int_{0}^{\infty} \frac{\sin \omega}{\omega} \frac{\cos \omega x}{(EI\omega^3 + k)} \, d\omega.
\]  
(20)
17.10. Fourier Transform

Our purpose in this section is to recast the Fourier integral representation of a function \( f \) as a pair of formulas, the first giving the so-called Fourier transform of \( f \), and the second giving the inverse of that transform. Once the transform and its inverse

This integral can be evaluated analytically using complex variable techniques (the residue theorem), but for our present purposes it will suffice to let (20) stand as it is.  

Closure. We obtain the Fourier integral representation of nonperiodic functions defined on \(-\infty < x < \infty\) by taking the limit of the Fourier series formula as the period tends to infinity. We limit discussion to two examples because we plan to use the Fourier integral only as a stepping stone to the Fourier transform. The latter is more highly developed as a methodology, like the Laplace transform, and is the subject of the next section.

EXERCISES 17.9

1. Use (11) to derive (12) from (10).
2. Derive the Fourier integral representations of the following functions. At which points, if any, does the Fourier integral fail to converge to \( f(x) \)? To what value does the integral converge at those points?
   (g) \( f(x) = \begin{cases} 100, & 0 \leq x \leq 2 \\ 0, & x < 0, x > 2 \end{cases} \)
   (b) \( f(x) = \begin{cases} x, & 0 \leq x < L \\ 0, & x < 0, x \geq L \end{cases} \)
   (c) \( f(x) = \begin{cases} x, & |x| \leq L \\ 0, & |x| > L \end{cases} \)
   (d) \( f(x) = \begin{cases} -x, & -5 < x \leq 0 \\ 0, & x \leq -5, x > 0 \end{cases} \)
   (e) \( f(x) = \begin{cases} |x|, & -1 < x < 2 \\ 0, & x \leq -1, x \geq 2 \end{cases} \)
   (f) \( f(x) = \begin{cases} 5, & 0 \leq x \leq 9 \\ 0, & x < 0, 3 < x < 6, x > 9 \end{cases} \)
   (g) \( f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \)
   (h) \( f(x) = \begin{cases} e^{x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \)

   (i) \( f(x) = e^{-|x|} \)

3. (a) Show that \( Si(x) \) is an odd function of \( x \).
   (b) Using the known integral
   \[
   Si(\infty) = \int_{0}^{\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2}
   \]
   and recalling (11), show that
   \[
   \lim_{\delta \to \infty} f_\delta(x) = \begin{cases} 1, & |x| < 1 \\ 1/2, & |x| = 1 \\ 0, & |x| > 1 \end{cases}
   \]
   so that the Fourier integral (9) does converge to the rectangular pulse and (in accordance with Theorem 17.9.1) to the average values at the two jump discontinuities.

4. Comparing (2) with the classical Fourier series (1), it might appear that \( a(0) \) is analogous to the \( a_0 \) term in the Fourier series and represents the average value of \( f \). Is it true that \( a(0) \) is the average value of \( f \) where, by the average value of \( f \) we mean \( \lim_{A \to \infty} \frac{1}{2A} \int_{-A}^{A} f(x) \, dx \)? Explain.
are derived, in Section 17.10.1, discussion will closely parallel our discussion, in Chapter 5, of the Laplace transform.

17.10.1. Transition from Fourier integral to Fourier transform. Our starting point is the Fourier integral formula

\[ f(x) = \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^{\infty} [a(\omega) \cos \omega x + b(\omega) \sin \omega x] \, d\omega \right\} \, d\omega, \quad (1a) \]

where

\[ a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x \, dx, \]
\[ b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x \, dx. \quad (1b) \]

Just as one can express a Fourier series in complex exponential form (Section 17.3.4), one can express the Fourier integral (1) in complex exponential form. To obtain that form put (1b) into (1a). [First we change the dummy integration variable \( x \) in (1b) to \( \xi \), say, to avoid confusing that variable with the \( \omega \)'s occurring in (1a), which denote the fixed point at which \( f(x) \) is being computed.] Thus,

\[ f(x) = \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^{\infty} f(\xi) \cos \omega \xi \cos \omega x + \sin \omega \xi \sin \omega x \right\} \, d\omega \]
\[ = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(\xi) \cos \omega (\xi - x) \, d\xi \, d\omega, \quad (2) \]

since \( \cos (A - B) = \cos A \cos B + \sin A \sin B \). To introduce complex exponentials, re-express (2) as

\[ f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(\xi) \frac{e^{i\omega(\xi-x)} + e^{-i\omega(\xi-x)}}{2} \, d\xi \, d\omega \]
\[ = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} f(\xi) e^{i\omega(\xi-x)} \, d\xi \, d\omega + \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi-x)} \, d\xi \, d\omega. \]

To combine the two terms on the right-hand side, let us change the dummy integration variable from \( \omega \) to \(-\omega\) in the first. Thus,

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi-x)} \, d\xi \, (-d\omega) \]
\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi-x)} \, d\xi \, d\omega \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi-x)} \, d\xi \, d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi-x)} \, d\xi \, d\omega, \]

so

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{-i\omega(\xi-x)} \, d\xi \, d\omega \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\xi) e^{-i\omega \xi} \, d\xi \right] e^{i\omega x} \, d\omega. \quad (3) \]
The latter can be split apart as

\[ f(x) = a \int_{-\infty}^{\infty} c(\omega) e^{i\omega x} \, d\omega \quad (4a) \]

and

\[ c(\omega) = b \int_{-\infty}^{\infty} f(\xi) e^{-i\omega \xi} \, d\xi, \quad (4b) \]

if the constants \( a, b \) are such that \( ab = 1/2\pi \). We can make (4) resemble (1) more closely by choosing \( a = 1 \) and \( b = 1/2\pi \), but we will choose \( a = 1/2\pi \) and \( b = 1 \). There is no longer a need to distinguish \( x \) and \( \xi \), because the \( x \)'s are confined to (4a) and the \( \xi \)'s to (4b). Thus, to minimize nomenclature and to mimic the form of (1), we write

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\omega) e^{i\omega x} \, d\omega, \quad (5a) \]

\[ c(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx. \quad (5b) \]

Rather than thinking of (5a) as the Fourier integral of \( f \) and (5b) as giving (give or take the factor of \( 1/2\pi \)) the Fourier coefficients \( c(\omega) \), we can think of (5a,b) as a transform pair; (5b) defines the **Fourier transform** \( c(\omega) \) of the given function \( f(x) \), and (5a) is called the **inversion formula** because putting \( c(\omega) \) in and integrating gives us back \( f(x) \). It is standard to use the notation \( f(\omega) \), in place of \( c(\omega) \), for the transform, so we rewrite (5) in final form as

\[
F\{ f(x) \} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx,
\]

(6a)

and

\[
F^{-1}\{ \hat{f}(\omega) \} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega.
\]

(6b)

Thus, the Fourier transform and inversion formulas are not mysterious; together, they simply amount to the Fourier integral representation, expressed in complex exponential form, and conditions imposed on \( f \) are the same as in Theorem 17.9.1.

Let us illustrate the calculation of the transform \( \hat{f}(\omega) \) of \( f(x) \).

**EXAMPLE 1.** *Rectangular Pulse.* Consider the rectangular pulse \( f(x) = H(x + 1) - H(x - 1) \), where \( H \) denotes the Heaviside function. The graph of \( f \) is given in Fig. 1.

Using (6a), the Fourier transform of \( f \) is

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} [H(x + 1) - H(x - 1)] e^{-i\omega x} \, dx = \int_{-1}^{1} e^{-i\omega x} \, dx = \left. \frac{e^{-i\omega x}}{-i\omega} \right|_{-1}^{1} = \frac{2\sin \omega}{\omega}.
\]

(7)

*Some authors, perhaps out of a greater sense of fair play, choose \( a = b = 1/\sqrt{2\pi} \).*
COMMENT. We could illustrate the inversion formula as well, by putting \( f(\omega) = (2\sin \omega) / \omega \) into the integrand of (6b), integrating, and showing that the result is the rectangular pulse \( f \) that we started with. In fact, that integral can be evaluated by using the residue theorem of the complex integral calculus, but we won’t study that theorem until Chapter 24.

EXAMPLE 2. Evaluate the Fourier transform of
\[
f(x) = H(x)e^{-ax}, \quad (a > 0)
\]
the graph of which is given in Fig. 2. From (6a),
\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} H(x)e^{-ax}e^{-i\omega x} \, dx = \int_{0}^{\infty} e^{-(a+i\omega)x} \, dx
\]
\[
= \left. \frac{e^{-(a+i\omega)x}}{a + i\omega} \right|_{0}^{\infty} = \frac{1}{a + i\omega} - \frac{1}{a + i\omega} \lim_{x \to \infty} e^{-(a+i\omega)x}
\]
\[
= \frac{1}{a + i\omega}.
\]
To explain the last step in (9), recall that the magnitude (or modulus) \( |z| \) of a complex number \( z = x + iy \) (Fig. 3) is \( |z| = \sqrt{x^2 + y^2} \), and that the modulus of a product is the product of the moduli \( |z_1z_2| = |z_1||z_2| \). Then
\[
|e^{-(a+i\omega)x}| = |e^{-ax}e^{-i\omega x}| = |e^{-ax}||e^{-i\omega x}|
\]
\[
= e^{-ax}|\cos \omega x - i\sin \omega x|
\]
\[
= e^{-ax}\sqrt{\cos^2 \omega x + \sin^2 \omega x}
\]
\[
= e^{-ax} \to 0
\]
as \( x \to \infty \).

As with the Laplace transform, it is convenient to use tables for both the transform and its inverse, as far as possible. The table provided here, in Appendix D, is brief, but will suffice for present purposes. Much more extensive tables are available, as well as powerful computer software; the relevant commands, using Maple, are given at the end of this section.

17.10.2. Properties and applications. The Fourier transform admits a number of useful properties, our discussion of which will closely parallel our analogous discussion for the Laplace transform, given in Chapter 5. We will assume, without reiteration, that the functions being transformed satisfy the conditions given in

---

Linearity of the transform and its inverse. For any scalars $\alpha$ and $\beta$, and any functions $f$ and $g$,

$$F\{\alpha f + \beta g\} = \alpha F\{f\} + \beta F\{g\}$$  \hspace{1cm} (11)

and

$$F^{-1}\{\alpha \hat{f} + \beta \hat{g}\} = \alpha F^{-1}\{\hat{f}\} + \beta F^{-1}\{\hat{g}\}.$$  \hspace{1cm} (12)

Of course, $F^{-1}\{\hat{f}\}$ is $f$ and $F^{-1}\{\hat{g}\}$ is $g$. Proofs of (11) and (12) follow the same lines as the proofs of Theorems 5.3.1 and 5.3.2, respectively.

Transform of $n$th derivative. If $f(x), f'(x), \ldots, f^{(n-1)}(x)$ all tend to zero as $x \to \pm \infty$, and $\int_{-\infty}^{\infty} |f^{(j)}(x)| \, dx$ converges for each $j = 0, 1, \ldots, n$, then

$$F\{f^{(n)}(x)\} = (i\omega)^n \hat{f}(\omega).$$  \hspace{1cm} (n = 0, 1, 2, \ldots)  (13)

Proof: Let us prove (13) by induction. First, observe that (13) holds for $n = 0$, by definition. To complete the proof by induction we need to establish that if it holds for $n = k$ then it also holds for $n = k + 1$. To do so, integrate by parts:

$$F\{f^{(k+1)}(x)\} = \int_{-\infty}^{\infty} f^{(k+1)}(x)e^{-i\omega x} \, dx$$

$$= \left[ f^{(k)}(x)e^{-i\omega x} \right]_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f^{(k)}(x)e^{-i\omega x} \, dx$$

$$= \left[ f^{(k)}(x)e^{-i\omega x} \right]_{-\infty}^{\infty} + i\omega [(i\omega)^k \hat{f}(\omega)]$$

$$= \left[ f^{(k)}(x)e^{-i\omega x} \right]_{-\infty}^{\infty} + (i\omega)^{k+1} \hat{f}(\omega),$$  \hspace{1cm} (14)

the third equality following from the assumption that (13) holds for $n = k$. Now, $|f^{(k)}(x)e^{-i\omega x}| = |f^{(k)}(x)| e^{-i\omega x} = |f^{(k)}(x)|$, and since $f^{(k)}(x) \to 0$ as $x \to \pm \infty$, by assumption, the boundary term in (14) drops out. Thus $F\{f^{(k+1)}(x)\} = (i\omega)^{k+1} \hat{f}(\omega)$ so (13) does hold for $n = k + 1$, which result completes our proof by induction.

Fourier convolution. We denote the Fourier convolution of functions $f$ and $g$ as $f \ast g$. It too is a function of $x$, defined as

$$(f \ast g)(x) \equiv \int_{-\infty}^{\infty} f(x - \xi)g(\xi) \, d\xi.$$  \hspace{1cm} (15)

Then the Fourier convolution theorem states that

$$F\{f \ast g\} = \hat{f}(\omega)\hat{g}(\omega),$$  \hspace{1cm} (16)
or, equivalently, that

$$F^{-1}\{\hat{f} \hat{g}\} = f * g.$$  \hspace{1cm} (17)

**Proof:** It suffices to prove (16) or (17), because of their equivalence. Let us prove (16). We have

$$F\{f * g\} = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x - \xi)g(\xi) \, d\xi \right\} e^{-i\omega x} \, dx$$

$$= \int_{-\infty}^{\infty} g(\xi) \, d\xi \int_{-\infty}^{\infty} f(x - \xi) e^{-i\omega x} \, dx$$

$$= \int_{-\infty}^{\infty} g(\xi) \, d\xi \int_{-\infty}^{\infty} f(\eta) e^{-i\omega(\xi+\eta)} \, d\eta$$

$$= \left( \int_{-\infty}^{\infty} g(\xi) e^{-i\omega \xi} \, d\xi \right) \left( \int_{-\infty}^{\infty} f(\eta) e^{-i\omega \eta} \, d\eta \right)$$

$$= \hat{g}(\omega) \hat{f}(\omega),$$  \hspace{1cm} (18)

as was to be shown. In the second equality we reversed the order of integration,* and the third equality follows from the change of variables $x - \xi = \eta$. \[ \square \]

It is easy to show that the Fourier convolution is commutative,

$$f * g = g * f,$$  \hspace{1cm} (19)

proof of which is left for the exercises. That is, it doesn’t matter whether we take the argument of $f$ to be $x - \xi$ and the argument of $g$ to be $\xi$, in (15), or visa versa.

**Translation formulas, $x$-shift and $\omega$-shift.** It is also easy to derive the shift (or translation) formulas

$$F\{f(x - a)\} = e^{-i\omega a} \hat{f}(\omega)$$  \hspace{1cm} (20)

and

$$F^{-1}\{f(\omega - a)\} = e^{iax} f(x),$$  \hspace{1cm} (21)

proofs of which are left for the exercises.

Each of the properties (11), (12), (13), (16), (17), (20), and (21) corresponds to an analogous property of the Laplace transform. Properties (11) and (12) are identical to the corresponding properties of the Laplace transform. The derivative property (13) is similar to the property

$$L\{f^{(n)}(t)\} = s^n \hat{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0),$$  \hspace{1cm} (22)

*Sufficient conditions for the validity of this reversal are that $f$ and $g$ both be absolutely integrable, i.e., that $\int_{-\infty}^{\infty} |f(x)| \, dx$ and $\int_{-\infty}^{\infty} |g(x)| \, dx$ both converge. But these conditions have already been assumed. For detailed discussion, see T. M. Apostol, *Mathematical Analysis* (Reading, MA: Addison-Wesley, 1957), p.491.
but does not include any boundary values analogous to the initial conditions  
\[ f(0), \ldots, f^{(n-1)}(0) \]  
in (22), because the boundaries are at  \( x = \pm \infty \), and it is assumed that  \( f(x), f'(x), \ldots, f^{(n-1)}(x) \) all tend to zero as  \( x \to \pm \infty \). The convolution properties (16) and (17) are identical to the corresponding ones for the Laplace transform, but keep in mind that the Fourier and Laplace convolutions are not quite the same; in contrast with the Fourier convolution (15), the Laplace convolution was defined as  
\[ (f * g)(t) \equiv \int_0^t f(t - \tau)g(\tau) \, d\tau. \]  
That is, the integration limits are different.

In the examples to follow, we illustrate the use of these various properties and the table in Appendix D.

**EXAMPLE 3.** Given  
\[ f(x) = 4e^{-|x|} - 5e^{-3|x+2|}, \]  
evaluate its transform \( \hat{f}(\omega) \). First, the linearity property (11) gives  
\[ F\{4e^{-|x|} - 5e^{-3|x+2|}\} = 4F\{e^{-|x|}\} - 5F\{e^{-3|x+2|}\}. \]  
Next, entry 4 in Appendix D gives  
\[ F\{e^{-|x|}\} = \frac{2}{\omega^2 + 1}, \]  
and entry 11 (with  \( a = 3 \) and  \( b = 6 \)) gives  
\[ F\{e^{-3|x+2|}\} = F\{e^{-3|x+6|}\} = F\{e^{-3x+6}\} = \frac{1}{3} e^{\frac{2\omega}{\omega+3}}, \]  
\[ = \frac{1}{3} \left( \frac{2}{\omega^2 + 1} \right), \]  
\[ = \frac{1}{3} \left( \frac{2}{\left(\frac{\omega}{3}\right)^2 + 1} \right). \]  
From (25)–(27), it follows that  
\[ \hat{f}(\omega) = 4 \frac{2}{\omega^2 + 1} - 5 \left( \frac{2}{3} \right) \frac{e^{\frac{2\omega}{\omega^2 + 1}}} \]  
\[ = \frac{8}{\omega^2 + 1} - \frac{30e^{\frac{2\omega}{\omega^2 + 1}}}{\omega^2 + 9}. \]
EXAMPLE 4. Given

\[ f(x) = xe^{-4x^2}, \]  

(28)
evaluate \( \hat{f}(\omega) \). First,

\[ F\{e^{-4x^2}\} = F\{e^{-(2x)^2}\} \]
\[ = \frac{1}{2} \left( F\{e^{-x^2}\} \right) \Bigg|_{x \to \omega/2} \]  
(entry 11, \( a = 2, b = 0 \))
\[ = \frac{1}{2} \sqrt{\pi} e^{-\omega^2/4} \Bigg|_{\omega \to \omega/2} \]  
(entry 5)
\[ = \frac{\sqrt{\pi}}{2} e^{-\omega^2/16}. \]

(29)

Next, entry 16 (with \( n = 1 \)) gives

\[ \hat{f}(\omega) = i \frac{d}{d\omega} \left( \frac{\sqrt{\pi}}{2} e^{-\omega^2/16} \right) = -i \frac{\sqrt{\pi}}{16} \omega e^{-\omega^2/16}. \]

EXAMPLE 5. Given \( \hat{f}(\omega) = e^{-2|\omega|} \), evaluate \( f(x) \). With \( a = 2 \), entry 1 gives

\[ F^{-1} \left\{ \frac{\pi}{2} e^{-2|\omega|} \right\} = \frac{1}{x^2 + 4}. \]

(30)

Then it follows from the linearity property (12), with \( \alpha = \pi/2 \) and \( \beta = 0 \), that (30) can be re-expressed as

\[ \frac{\pi}{2} F^{-1} \left\{ e^{-2|\omega|} \right\} = \frac{1}{x^2 + 4}. \]

(31)

so

\[ F^{-1} \left\{ e^{-2|\omega|} \right\} = \frac{2}{\pi} \frac{1}{x^2 + 4}, \]

(32)
is the desired inverse function \( f(x) \).  

EXAMPLE 6. Evaluate

\[ F^{-1} \left\{ \frac{1}{\omega^2 + 4\omega + 13} \right\}. \]

First, completing the square, write \( \omega^2 + 4\omega = (\omega^2 + 4\omega + 4) - 4 = (\omega + 2)^2 - 4 \), so (33) becomes

\[ F^{-1} \left\{ \frac{1}{(\omega + 2)^2 + 9} \right\}. \]

(34)

Now, entry 4 (with \( a = 3 \)) and linearity give

\[ F^{-1} \left\{ \frac{1}{\omega^2 + 9} \right\} = \frac{1}{6} e^{-3|x|}. \]

(35)

Finally, from entry 12, with \( a = 1 \) and \( b = 2 \), and (35), we have

\[ F^{-1} \left\{ \frac{1}{(\omega + 2)^2 + 9} \right\} = e^{-2x} \left( \frac{1}{6} e^{-3|x|} \right). \]

(36)
as the desired result.

COMMENT. Alternatively, we could have used partial fractions, as follows. Solving \( \omega^2 + 4\omega + 13 = 0 \) gives \( \omega = -2 \pm 3i \). Thus,

\[
\frac{1}{\omega^2 + 4\omega + 13} = \frac{1}{(\omega + 2 - 3i)(\omega + 2 + 3i)} = \frac{1}{6} \frac{1}{\omega + 2 - 3i} - \frac{1}{6} \frac{1}{\omega + 2 + 3i} = \frac{1}{6} \frac{1}{i\omega + (3 + 2i)} + \frac{1}{6} \frac{1}{-i\omega + (3 - 2i)}.
\]

Observe that the first term after the third equal sign can be inverted by entry 2, because \( \text{Re}(3 + 2i) = 3 > 0 \), but the second cannot, because \( \text{Re}(-3 + 2i) = -3 < 0 \). Thus, in the final step we rearranged that term so that it can be inverted by entry 3, because \( \text{Re}(3 - 2i) = 3 > 0 \). Finally, using the linearity property (12), together with entries 2 and 3 gives

\[
F^{-1} \left\{ \frac{1}{\omega^2 + 4\omega + 13} \right\} = \frac{1}{6} H(x)e^{-(3+2i)x} + \frac{1}{6} H(-x)e^{(3-2i)x} = \frac{1}{6} e^{-3x} e^{-2ix} + \frac{1}{6} e^{-3x} e^{2ix} = \frac{1}{6} e^{-3x} (1 + e^{2ix}).
\]

as obtained in (36). \( \square \)

**EXAMPLE 7.** Evaluate

\[
F^{-1} \left\{ \frac{1}{\omega^2 + 1} \right\}.
\]

The latter can be evaluated directly from entry 4 (together with the linearity property), as \( e^{-|x|}/2 \). However, for pedagogical purposes let us use this example to illustrate the convolution property. First, factor \( 1/(\omega^2 + 1) \) as

\[
\frac{1}{\omega^2 + 1} = \frac{1}{(\omega + i)(\omega - i)} = \frac{1}{(1 - i\omega)(1 + i\omega)}.
\]

the final form in (40) being more convenient than the intermediate form because each factor can be inverted, using entries 3 and 2, respectively:

\[
F^{-1} \left\{ \frac{1}{1 - i\omega} \right\} = H(-x)e^{x},
\]

\[
F^{-1} \left\{ \frac{1}{1 + i\omega} \right\} = H(x)e^{-x}.
\]

*If \( z = x + iy \) is a complex number, then \( x \) and \( y \) are called the real and imaginary parts of \( z \), and are denoted as \( \text{Re} z \) and \( \text{Im} z \), respectively.*
Then the Fourier convolution formula (17) gives

\[ F^{-1}\left\{ \frac{1}{w^2 + 1} \right\} = [H(-x)e^x] * [H(x)e^{-x}] \]

\[ = \int_{-\infty}^{\infty} H(-x + \xi)e^{\xi - \xi}H(\xi)e^{-\xi} \, d\xi \]

\[ = \begin{cases} 
  e^x \int_{-\infty}^{\infty} e^{-2\xi} \, d\xi, & x > 0 \\
  e^x \int_{0}^{\infty} e^{-2\xi} \, d\xi, & x < 0 
\end{cases} \]

\[ = \begin{cases} 
  \frac{1}{2}e^{-x}, & x > 0 \\
  \frac{1}{2}e^x, & x < 0 
\end{cases} \]

\[ = \frac{1}{2}e^{-|x|}. \quad (42) \]

To understand the third equality in (42), it is useful to plot \(H(\xi - x)\) and \(H(\xi)\), both for \(x > 0\) and for \(x < 0\), as we have in Fig. 4. Namely, we see from Fig. 4 that their product \(H(x)e^x\) is \(H(\xi)\) if \(x > 0\), and \(H(\xi)\) if \(x < 0\).

**COMMENT.** Alternatively, we could have applied partial fractions to the right-hand side of (40) and proceeded as in the comment in Example 6.

**EXAMPLE 8.** Infinite Beam on Elastic Foundation, Revisited. In Example 2 in Section 17.9 we investigated the deflection \(u(x)\) of an infinitely long beam resting on an elastic foundation and subjected to a load \(w(x)\) newtons per meter (Fig. 5), governed by the differential equation

\[ EIu''' + ku = w(x), \quad (43) \]

where \(E, I, k\) are physical constants. There we used the Fourier integral method, whereby \(w(x)\) was expanded in a Fourier integral, and \(u(x)\) was sought in the form of a Fourier integral. Here, we use the Fourier transform instead.

Taking the Fourier transform of (43) (i.e., multiplying each term by \(e^{-i\omega x} \, dx\) and integrating from \(x = -\infty\) to \(x = +\infty\)), we have

\[ F\{EIu''' + ku\} = F\{w\}. \quad (44) \]

It follows that

\[ EI F\{u'''\} + k F\{u\} = F\{w\} \quad (45) \]

by the linearity property (11), and

\[ EI(i\omega)^3 \dot{u} + ku = \dot{w} \quad (46) \]

by (13), assuming that \(u, u', u'', u'''\) all tend to zero as \(x \to \pm\infty\) and that \(\int_{-\infty}^{\infty} |u^{(j)}(x)| \, dx\) converges for \(j = 0, 1, 2, 3, 4\). Now solving (46) for \(\ddot{u}\),

\[ \ddot{u} = \frac{\dot{w}}{EI\omega^4 + k}. \quad (47) \]
To invert (47), let us write the right-hand side in the more suggestive product form

\[ \hat{u} = \left( \frac{1}{EI\omega^4 + k} \right) \hat{w}, \]

(48)

The inverse of the product is not the product of the two inverses, but is the Fourier convolution of the two inverses,

\[ u(x) = F^{-1} \left\{ \frac{1}{EI\omega^4 + k} \right\} \ast F^{-1} \{ \hat{w} \}. \]

(49)

From entry 8 in Appendix D, and the linearity of \( F^{-1} \), we obtain

\[ F^{-1} \left\{ \frac{1}{EI\omega^4 + k} \right\} = \frac{\alpha}{\sqrt{2k}} e^{-\alpha|x|} \sin \left( \alpha|x| + \frac{\pi}{4} \right), \]

(50)

where \( \alpha = \sqrt{\frac{k}{4 EI}} \), and of course \( F^{-1} \{ \hat{w} \} = w(x) \), so (49) gives

\[ u(x) = \frac{\alpha}{\sqrt{2k}} \int_{-\infty}^{\infty} e^{-\alpha|x-\xi|} \sin \left( \alpha|x| + \frac{\pi}{4} \right) w(\xi) \, d\xi \]

(51)

as the desired solution.

COMMENT 1. It is always important to check our results. An excellent partial check of a result is provided by any special case for which the exact solution is known. In the present example, such a special case is provided by the case where \( w(x) = \text{constant} \equiv W \), for then surely the solution \( u(x) \) will be a constant too. Specifically, with \( w(x) = W \), and \( u \) a constant, (43) gives \( u(x) = W/k \). If we set \( w(\xi) = W \) in (51), and integrate, do we obtain the same result? Yes, but we leave that calculation for the exercises.

In fact, observe that the correctness of (51) for the case where \( w(x) \) is a constant is a surprise since some of the assumptions that were built into our solution are violated in that case. For instance, the transform \( \hat{w} \) of \( w(x) = W \) does not even exist, because the transform integral does not converge. That is,

\[ \hat{w} = \int_{-\infty}^{\infty} We^{-i\omega x} \, dx = W \lim_{A \to \infty} \int_{-A}^{A} e^{-i\omega x} \, dx \]

\[ = 2W \lim_{A \to \infty} \frac{\sin \omega A}{\omega}, \]

(52)

which limit does not exist. Without elaborating, we state that (51) ends up being correct, for this case, even though (44) is not (since \( F\{ w \} = \hat{w} \) does not exist), thanks to the interchange in the order of integration that underlies the convolution property. In any case, the moral is that it is often best to proceed formally to a solution. If we can verify that the solution thus obtained does satisfy all of the specified requirements (such as a differential equation and boundary conditions), then there is no need to worry about any lack of rigor in the intermediate steps.

COMMENT 2. If \( w \) is a delta function, \( w(x) = \delta(x) \), then \( u(x) \) is the response to a unit load at \( x = 0 \) (Fig. 6).\(^1\) Remember that

\[ * \text{See our proof of the convolution formula, following (17).} \]

\[ ^1 \text{A unit load because } \int_{-\infty}^{\infty} w(x) \, dx = \int_{-\infty}^{\infty} \delta(x) \, dx = 1. \]

---

Figure 6. Response to a point unit load at \( x = 0 \).
Chapter 17. Fourier Series, Fourier Integral, Fourier Transform

\[ \int_{-\infty}^{\infty} f(x) \delta(x) \, dx = f(0) \]  

(53)

if \( f \) is continuous, by the definition of the delta function. Thus, if \( w(x) = \delta(x) \) in (51) then we obtain the response

\[ u(x) = \frac{\alpha}{\sqrt{2k}} e^{-\alpha|x|} \sin \left( \alpha |x| + \frac{\pi}{4} \right), \]  

(54)

which is plotted in Fig. 6 for the representative case where \( \alpha = 1 \).

COMMENT 3. Let us re-express (51) in the compact form

\[ u(x) = \int_{-\infty}^{\infty} K(x - \xi) w(\xi) \, d\xi. \]  

(55)

That is, the output function \( u \) is given by the action of an integral operator on the input function \( w \), the operator being multiplication by \( K \) followed by integration from \(-\infty\) to \( \infty \). We have written \( K(x - \xi) \) rather than \( K(\xi, x) \) to show that \( K \) depends only on the difference between \( x \) and \( \xi \). We call \( K \) the kernel of the integral operator. Since \( K \) is a function of the difference \( x - \xi \), we call it a difference kernel. The physical significance of \( K \) is revealed by the fact that if the loading \( w(\xi) \) in (51) is a point unit load \( \delta(\xi) \) at the origin, then the resulting deflection \( u(x) \) is, according to (53),

\[ u(x) = \int_{-\infty}^{\infty} K(x - \xi) \delta(\xi) \, d\xi = K(x). \]  

(56)

That is, \( K(x) \) is the response function (54), the graph of which is shown in Fig. 6 (for \( \alpha = 1 \)), so \( K(x - \xi) \) in (55) is the deflection due to a point unit load at \( \xi \). If the load between \( \xi \) and \( \xi + d\xi \) is \( w(\xi) \, d\xi \) rather than unity (Fig. 7), then the contribution \( du \) to the deflection \( u \), due to that bit of loading, is \( K(x - \xi) \) scaled by \( w(\xi) \, d\xi \),

\[ du = K(x - \xi) w(\xi) \, d\xi. \]  

(57)

Adding all of these \( du \)'s gives the integral in (55). Thus, we can now understand (55) as a superposition principle, whereby \( u(x) \) is the sum, or superposition, of the individual contributions \( du \) due to each point load \( w(\xi) \, d\xi \). That superposition result is a consequence of the linearity of (43).

Additional physical ODE (ordinary differential equation) applications, to the deflection of a loaded string and to the steady-state concentration distribution in a stream subjected to an input of pollutant are given in the exercises. We urge you to at least read through those exercises even if you do not work them.

Closure. We have seen that the Fourier transform and the corresponding inversion formula are actually just a restatement of the Fourier integral representation of a nonperiodic function defined on \(-\infty < x < \infty \). We have also seen that the

---

*To further illustrate the idea of the kernel of an integral operator, we note that the kernel of the Laplace transform operator is \( e^{-at} \).
Fourier transform methodology is analogous, and in some aspects identical, to the Laplace transform methodology studied in Chapter 5. Given that similarity, the important question arises, as to which of the two transform methods to use in a given ODE application. The general guideline is as follows:

- The Laplace transform is tailored to initial-value problems on a semi-infinite interval \(0 < t < \infty\).
- The Fourier transform is tailored to boundary-value problems on an infinite interval \(-\infty < x < \infty\).

The independent variables are usually \(t\) (time) and \(x\) (linear dimension), but not always. Further, there does exist a "bilateral" Laplace transform defined on an infinite interval, the Laplace transform can sometimes be used for boundary-value problems on finite intervals, and the Fourier transform can (and will, in Section 17.11) be adapted to boundary-value problems on a semi-infinite interval \(0 < x < \infty\). But the guideline given above provides the general rule of thumb for selecting one transform or the other.

In closing, we note that to compute the Fourier transform \(\hat{f}(\omega)\) of \(f(x)\) requires an integration of \(f\) over \((-\infty, \infty)\). One can evaluate the integral numerically by sampling the integrand at discrete \(x\) points and obtain what is known as the discrete Fourier transform. Further, there exist algorithms known collectively as the fast Fourier transform (i.e., the FFT) that provide a more efficient method of calculating the sum in the discrete Fourier transform. For an introduction to these methods we refer the interested reader to Peter V. O’Neil’s Advanced Engineering Mathematics, 3rd ed. (Belmont, CA: Wadsworth, 1991).

**Computer software.** To obtain the Fourier transform of \(e^{-5|x|}\), say, using Maple, enter

```
readlib(fourier);
```

to access the Fourier transform and inverse transform commands. Then enter

```
fourier(exp(-5*abs(x)), x, w);
```

and return. The result is

\[
\frac{10}{25 + w^2}
\]

which agrees with entry 4 in Appendix D. To invert the latter, enter

```
invfourier(10/(25 + w^2), w, x);
```

and return. The result is

\[
e^{-5x} \text{Heaviside}(x) + e^{5x} \text{Heaviside}(-x)
\]

which does reduce to \(e^{-5|x|}\).
1. Using (6a), derive the result

\[ F\{H(x)e^{-a x}\} = \frac{1}{a + i \omega} \]

if \( \Re a > 0 \). (This case was worked in Example 2, but there \( a \) was considered to be real. Here, allow for \( a \) to be complex, with \( \Re a > 0 \).)

2. Using (6a), derive the result

\[ F\{H(-x)e^{a x}\} = \frac{1}{a - i \omega} \]

if \( \Re a > 0 \).

3. Using (6a), derive the result

\[ F\{e^{-a|x|}\} = \frac{2a}{\omega^2 + a^2} \]

if \( a > 0 \).

4. Given \( \hat{f}(\omega) \), use (6b) to evaluate the inverse, \( f(x) \).
   (a) \( e^{-|\omega|} \quad (a > 0) \)
   (b) \( H(\omega + a) - H(\omega - a) \quad (a > 0) \)
   (c) \( [H(\omega) - H(\omega - 1)]\omega \)
   (d) \( H(\omega)e^{-a \omega} \quad (\Re a > 0) \)
   (e) \( [H(\omega + 1) - H(\omega - 1)]\omega \)
   (f) \( \delta(\omega - a) \)

5. Derive the following entry in Appendix D, by showing that the transform of the function given in the \( f \) column is the result given in the \( \hat{f} \) column.
   (a) Entry 10
   (b) Entry 11
   (c) Entry 12
   (d) Entry 13
   (e) Entry 14
   (f) Entry 15
   (g) Entry 17, by formally differentiating both sides of

\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, dx \]

a sufficient number of times, with respect to \( \omega \)
   (h) Entry 19, using integration by parts

6. Evaluate the following using Appendix D. You may need to use more than one entry. Cite, by number, any entries that you use.

(a) \( F\{4x^2e^{-3|x|}\} \)
(b) \( F\{x^2e^{-x^2}\} \)
(c) \( F\{\cos 3x\} \)
(d) \( F\{\sin 2x\} \)
(e) \( F\{\frac{3+2e^{-3|x|}}{x^2+2}\} \)
(f) \( F\{e^{-|x|} + e^{-|x+2|}\} \)
(g) \( F^{-1}\{\frac{4\sin \omega}{\omega} - \frac{1}{\sqrt{\omega}}\} \)
(h) \( F^{-1}\{e^{-2\omega-3}\} \)
(i) \( F^{-1}\{\frac{9}{2\omega + i}\} \)
(j) \( F^{-1}\{e^{-a^2+4\omega}\} \)
(k) \( F^{-1}\{e^{-|\omega|}\cos \omega\} \)
(l) \( F^{-1}\{e^{-\omega^3}\} \)
(m) \( F^{-1}\{\frac{1}{\omega^2 + i\omega + 2}\} \)
(n) \( F^{-1}\{\frac{\omega}{\omega^2 + 1}\} \)

7. (a)–(m) Use computer software to obtain (if possible) the transform or inverse transform called for in the corresponding part of Exercise 6.

8. We claimed, in Comment 1 in Example 8, that if \( \omega(x) = W \) then the integral on the right-hand side of (51) is \( W/k \). Verify that claim by evaluating the integral. HINT: Break the integral into two parts, one from \(-\infty\) to \( x \) and the other from \( x \) to \( \infty \), then make a suitable change of variables in each.

9. (Preservation of evenness and oddness)
   (a) Show that \( f(x) \) is an even function of \( x \) if and only if \( \hat{f}(\omega) \) is an even function of \( \omega \).
   (b) Show that \( f(x) \) is an odd function of \( x \) if and only if \( \hat{f}(\omega) \) is an odd function of \( \omega \).

10. (Extension of transform tables) It follows from the results in Exercise 9 that the transform of the even and odd parts of \( f(x) \) are the even and odd parts of \( \hat{f}(\omega) \), respectively. This result can be used to obtain more information from a given transform table. For example, consider entry 2 in Appendix D. Breaking \( f \) and \( \hat{f} \) into even and odd parts, show that

\[ F\{e^{-a|x|}\} = \frac{2a}{\omega^2 + a^2} \quad (10.1) \]

and

\[ F\{(\text{sgn } x)e^{-a|x|}\} = \frac{-2i\omega}{\omega^2 + a^2} \quad (10.2) \]

where
\[ \text{sgn} \, x \equiv \begin{cases} +1, & x > 0 \\ -1, & x < 0, \end{cases} \]  
\[ (10.3) \]

which is read as "sign of \( x \)." Of these two results, observe that (10.1) is identical to entry 4, and that (10.2) is not contained in Appendix D.

11. (Extension of transform tables) Another idea that enables us to extend a given Fourier transform table is that of reciprocity, namely, the reciprocity relations

\[ F\{f(x)\} = 2\pi f(-\omega) \]  
\[ (11.1) \]

and

\[ F^{-1}\{f(-\omega)\} = \frac{f(x)}{2\pi}, \]  
\[ (11.2) \]

(a) Derive the relations (11.1) and (11.2).

(b) To illustrate, use (11.1) and entry 4, in Appendix D, to show that

\[ F\left\{ \frac{2a}{x^2 + a^2} \right\} = 2\pi e^{-a|\omega|}, \quad (a > 0) \]

or, equivalently,

\[ F\left\{ \frac{1}{x^2 + a^2} \right\} = \frac{\pi}{a} e^{-a|\omega|}, \quad (a > 0) \]

(In this case the result does not extend our table since it already appears as Entry 1.)

(c) Use (11.1) and entry 9 to show that

\[ F\left\{ \frac{\sin \alpha x}{x} \right\} = \pi [H(\omega + a) - H(\omega - a)], \quad (a > 0) \]

(d) Use (11.1) and entry 3 to show that

\[ F\left\{ \frac{1}{a - ix} \right\} = 2\pi H(\omega) e^{-a\omega}. \quad (\text{Re} \, a > 0) \]

12. (Deflection of loaded string) Related to the problem of a beam on an elastic foundation is the analogous problem for a flexible string. Imagine a string (of negligible mass per unit length) stretched along the \( x \) axis, over \(-\infty < x < \infty\), by a tension \( T \) newtons, and let \( w(x) \) be an applied load distribution (newtons/meter), as sketched in the figure. If the displacement \( u(x) \) of the string is resisted by a distributed spring of stiffness \( k \) (newtons per meter per meter), and the slope \( u'(x) \) is sufficiently small over \(-\infty < x < \infty\), then \( u(x) \) is accurately governed by the differential equation \( Tu'' - ku = -w \), or

\[ u'' - \alpha^2 u = -f(x), \quad (-\infty < x < \infty) \]  
\[ (12.1) \]

where \( \alpha^2 = k/T \) and \( f(x) = w(x)/T \). (Here we consider the static deflection. In Chapter 20 we will derive the governing differential equation for the not-necessarily-static case.) Assume that \( w(x) \) is sufficiently localized for \( u(x) \) to satisfy the boundary conditions

\[ u \to 0 \quad \text{and} \quad u' \to 0 \]  
\[ (12.2) \]

as \( x \to \pm \infty \), as well.

13. (Pollution in river) Suppose that a manufacturing plant discharges a certain pollutant into an initially clear river at the rate \( Q \) grams/second. We wish to determine the resulting steady-state distribution of pollutant in the river, i.e., its concentration \( c \) (grams/meter\(^3\)). Measure \( x \) along the river, positive downstream, with origin at the plant site, as shown in the figure. The river flows with velocity \( U \) (meters/second), and has a cross-sectional area \( A \), both of which, for simplicity, we assume to be constant. Also for simplicity, suppose that \( c \) is a function of \( x \) only; i.e., it is a constant over each cross section of the stream. This is evidently a poor approximation near the plant, where we expect appreciable across-stream and vertical variations in \( c \), but it should suffice if we are concerned mostly with the concentration variation far upstream and downstream (for \( |x| \) greater than several river widths, say). Then it can be shown that \( c(x) \) is governed by the differential equation

\[ u'' - \alpha^2 u = -f(x), \quad (-\infty < x < \infty) \]  
\[ (12.1) \]
\[ ke'' - U e' - \beta c = - \frac{Q}{A} \delta(x), \quad (13.1) \]

where \( k \) (meters\(^2\)/second) is a diffusion constant, \( \beta \) (grams per second per gram) is a chemical decay constant, and \( \delta(x) \) is a delta function. [Physically, (13.1) expresses a mass balance between the input \( Q \delta(x)/A \), the transport of pollutant by diffusion, \( ke'' \), the transport of pollutant by convection with the moving stream, \( U e' \), and by disappearance through chemical decay, \( \beta c \).

\[ c(x) = F^{-1} \left\{ \frac{Q}{A} \frac{1}{k \omega^2 + i U \omega + \beta} \right\}. \quad (13.2) \]

(b) Expanding \( 1/(k \omega^2 + i U \omega + \beta) \) in partial fractions, and then using Appendix D, show that (13.2) gives

\[ c(x) = \begin{cases} c_0 e^{-\Omega_+ x}, & x < 0 \\ c_0 e^{-\Omega_- x}, & x > 0 \end{cases} \quad (13.3) \]

where

\[ c_0 = \frac{Q}{A \sqrt{U^2 + 4k \beta}} \]

\[ \Omega_{\pm} = \frac{1}{2k} \left( \pm \sqrt{U^2 + 4k \beta} - U \right). \]

(c) Sketch the graph of \( c(x) \) and state the qualitative effect of increasing \( \beta \).

17.11 Fourier Cosine and Sine Transforms, and Passage from Fourier Integral to Laplace Transform (Optional)

17.11.1. Cosine and sine transforms. Recall from Section 17.4 that if a problem is defined on a finite interval, say \( 0 < x < L \), then the concepts of periodicity and Fourier series are not directly applicable. However, by fictitiously extending both the domain and the functions involved to the infinite interval \(-\infty < x < \infty\), so that the extended functions are periodic, we are able to use Fourier series representations of those functions. Depending upon the symmetries and/or antisymmetries about \( x = 0 \) and \( x = L \), we obtain half- or quarter-range cosine or sine expansions.

Similarly, we sometimes encounter problems defined on a semi-infinite interval, say \( 0 < x < \infty \). Fictitiously extending both the domain and the functions involved, to the infinite interval \(-\infty < x < \infty\), we will be able to use Fourier integral representations of those functions. Specifically, given a function \( f \) defined on \( 0 < x < \infty \) (such as the one shown in Fig. 1a), we shall be interested in two particular extensions, one that is even and one that is odd, as indicated in Figs.1b and 1c, respectively.

Denoting the extended functions as \( f_{\text{ext}} \), consider first the even extension (Fig. 1b). Then the Fourier transform of \( f_{\text{ext}} \) is

\[ \overline{f_{\text{ext}}}(\omega) = \int_{-\infty}^{\infty} f_{\text{ext}}(x)e^{-i\omega x} \, dx \]
17.11. Fourier Cosine and Sine Transforms, and Passage from Fourier Integral to Laplace Transform

\[ \begin{align*}
&= \int_{-\infty}^{\infty} f_{\text{ext}}(x)(\cos \omega x - i \sin \omega x) \, dx = 2 \int_{0}^{\infty} f_{\text{ext}}(x) \cos \omega x \, dx, \quad (1a)
\end{align*} \]

where the last equality holds because \( f_{\text{ext}}(x) \cos \omega x \) is an even function of \( x \) and \( f_{\text{ext}}(x) \sin \omega x \) is odd, and the inversion formula gives

\[ \begin{align*}
f_{\text{ext}}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_{\text{ext}}(\omega) e^{i \omega x} \, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}_{\text{ext}}(\omega) (\cos \omega x + i \sin \omega x) \, d\omega \\
&= \frac{1}{2\pi} \int_{0}^{\infty} \hat{f}_{\text{ext}}(\omega) \cos \omega x \, d\omega, \quad (1b)
\end{align*} \]

where the last equality holds because \( \hat{f}_{\text{ext}}(\omega) \) is an even function of \( \omega \) [as can be seen from (1a)] and \( \sin \omega x \) is an odd function of \( \omega \), so that the term \( \hat{f}_{\text{ext}}(\omega) \sin \omega x \) in the integrand is odd. Putting (1a) into (1b) gives the single statement

\[ \begin{align*}
f_{\text{ext}}(x) &= \int_{0}^{\infty} \left\{ \frac{2}{\pi} \int_{0}^{\infty} f_{\text{ext}}(\xi) \cos \omega \xi \, d\xi \right\} \cos \omega x \, d\omega, \quad (-\infty < x < \infty) \quad (2)
\end{align*} \]

Since (2) holds on \(-\infty < x < \infty\), it also holds on the original interval \( 0 < x < \infty \). For \( 0 < x < \infty \) we can drop the subscripted “ext” on the left-hand side, because on that interval \( f_{\text{ext}}(x) = f(x) \). Further, we can drop the “ext” on the right-hand side because the \( \xi \) integral is on \( 0 < x < \infty \), not \(-\infty < x < \infty\). Then (2) becomes

\[ \begin{align*}
f(x) &= \frac{2}{\pi} \int_{0}^{\infty} \left\{ \int_{0}^{\infty} f(\xi) \cos \omega \xi \, d\xi \right\} \cos \omega x \, d\omega, \quad (0 < x < \infty) \quad (3)
\end{align*} \]

which can be re-expressed, equivalently, as the Fourier cosine transform

\[\boxed{\begin{align*}
F_C \{ f(x) \} &= \hat{f}_C(\omega) = \int_{0}^{\infty} f(x) \cos \omega x \, dx \\
\end{align*}}\]  

and its inverse

\[\boxed{\begin{align*}
F^{-1}_C \{ \hat{f}_C(\omega) \} &= f(x) = \frac{2}{\pi} \int_{0}^{\infty} \hat{f}_C(\omega) \cos \omega x \, d\omega, \\
\end{align*}}\]

on \( 0 < x < \infty \).

Similarly, an odd extension of \( f \) gives the Fourier sine transform

\[\boxed{\begin{align*}
F_S \{ f(x) \} &= \hat{f}_S(\omega) = \int_{0}^{\infty} f(x) \sin \omega x \, dx \\
\end{align*}}\]

and its inverse

\[\boxed{\begin{align*}
F^{-1}_S \{ \hat{f}_S(\omega) \} &= f(x) = \frac{2}{\pi} \int_{0}^{\infty} \hat{f}_S(\omega) \sin \omega x \, d\omega, \\
\end{align*}}\]

on \( 0 < x < \infty \).
on $0 < x < \infty$ (Exercise 1). Sufficient conditions on $f$, for (4) and (5) to hold, are that $f$ and $f'$ be piecewise continuous on $0 \leq x < \infty$, and that $\int_0^\infty |f(x)| \, dx$ converge (i.e., that $f$ be absolutely integrable on $0 < x < \infty$).

Next, we could derive the various properties of the cosine and sine transforms, that are analogous to those derived for the Fourier transform. However, for brevity let it suffice to state that both the cosine and sine transforms, and their inverses, are linear, and that the transforms of the derivative $f'$ are

$$
F_C\{f'(x)\} = \omega \hat{f}_S(\omega) - f(0),
$$

$$
F_S\{f'(x)\} = -\omega \hat{f}_C(\omega),
$$

(6a,b)

if we assume additionally that $f(x) \to 0$ as $x \to \infty$. Let us derive (6a) and leave (6b) for the exercises. Integrating by parts,

$$
F_C\{f'(x)\} = \int_0^\infty f'(x) \cos \omega x \, dx
$$

$$
= \left[ f(x) \cos \omega x \right]_0^\infty + \omega \int_0^\infty f(x) \sin \omega x \, dx
$$

$$
= 0 - f(0) + \omega \hat{f}_S(\omega).
$$

(7)

The transforms of higher-order derivatives can be obtained by repeated use of (6). For instance, replacing the function $f$ by the function $f'$ in (6) gives

$$
F_C\{f''(x)\} = \omega F_S\{f'(x)\} - f'(0) = \omega [\omega \hat{f}_C(\omega)] - f'(0)
$$

(8a)

and

$$
F_S\{f''(x)\} = -\omega F_C\{f'(x)\} = -\omega [\omega \hat{f}_S(\omega) - f(0)],
$$

(8b)

so

$$
F_C\{f''(x)\} = -\omega^2 \hat{f}_C(\omega) - f'(0),
$$

$$
F_S\{f''(x)\} = -\omega^2 \hat{f}_S(\omega) + \omega f(0),
$$

(9a,b)

if $f$ and $f'$ tend to zero as $x \to \infty$. Similarly for higher-order derivatives. For convolution properties, see Exercise 10 and Appendix E.

Short Fourier cosine and sine transform tables are given in Appendix E.

**EXAMPLE 1.** Consider the boundary-value problem

$$
u'' - 9u = 50e^{-2x}, \quad (0 < x < \infty) \tag{10a}$$

$$u(0) = u_0, \quad u(\infty) \text{ bounded.} \tag{10b}$$

To solve (10) using an integral transform we need to choose among the Laplace, Fourier cosine, and Fourier sine transforms, all of these being candidates because they are semi-infinite transforms; that is, they apply when the domain is semi-infinite ($0 < x < \infty$ in
this case). The Laplace transform will be inconvenient at best, because it is tailored to initial value problems whereas \((10)\) is of boundary-value type. In choosing between the Fourier cosine and sine transforms, the key is in \((9)\). Taking a cosine transform of \((10a)\) we will, according to \((9a)\), need to know \(u'(0)\), yet the latter is not prescribed; taking a sine transform of \((10a)\) we will, according to \((9a)\), need to know \(u(0)\), and the latter is prescribed.

Thus, take the Fourier sine transform of \((10a)\), using the linearity of the transform, property \((9b)\), and entry 1.5 in Appendix E:

\[
F_S \{ u'' - 9u \} = F_S \{ 50e^{-2x} \},
\]

\[
F_S \{ u'' \} - 9F_S \{ u \} = 50F_S \{ e^{-2x} \},
\]

\[
-\omega^2 \hat{u}_S + \omega u_0 - 9\hat{u}_S = 50 \frac{\omega}{\omega^2 + 4}.
\]

Solving this linear algebraic equation for \(\hat{u}_S\) gives

\[
\hat{u}_S(\omega) = u_0 \frac{\omega}{\omega^2 + 9} - 50 \frac{\omega}{(\omega^2 + 4)(\omega^2 + 9)}.
\]

By partial fractions,

\[
\frac{1}{(\omega^2 + 4)(\omega^2 + 9)} = \frac{1}{5} \frac{1}{\omega^2 + 4} - \frac{1}{5} \frac{1}{\omega^2 + 9},
\]

so

\[
\hat{u}_S(\omega) = (u_0 + 10) \frac{\omega}{\omega^2 + 9} - 10 \frac{\omega}{\omega^2 + 4}.
\]

Then, using the linearity of the inverse transform and entry 1.5, gives

\[
u(x) = F_S^{-1} \left\{ (u_0 + 10) \frac{\omega}{\omega^2 + 9} - 10 \frac{\omega}{\omega^2 + 4} \right\}
= (u_0 + 10)F_S^{-1} \left\{ \frac{\omega}{\omega^2 + 9} \right\} - 10F_S^{-1} \left\{ \frac{\omega}{\omega^2 + 4} \right\}
= (u_0 + 10)e^{-3x} - 10e^{-2x}.
\]

COMMENT 1. Note that we used property \((9b)\) tentatively since \((9b)\) presupposes that both \(u(x) \to 0\) and \(u'(x) \to 0\) as \(x \to \infty\). Now that \(u(x)\) is in hand, in \((15)\), we can check, and we see that these conditions are met. Or, more directly, we can simply verify that \((15)\) does satisfy \((10a)\) and \((10b)\).

COMMENT 2. If \(u'(0)\) were prescribed, in \((10b)\), in place of \(u(0)\), then we would use the Fourier cosine transform instead.

17.11.2. Passage from Fourier integral to Laplace transform. We have seen that the Fourier transform is merely a restatement of the Fourier integral, the Fourier integral is a limiting case of the Fourier series of a periodic function (as the period tends to infinity), and the Fourier series is as an eigenfunction expansion corresponding to a periodic Sturm–Liouville problem. On the other hand, the Laplace transform and its inversion formula were given in Chapter 5 without derivation and
may seem unrelated to these Fourier methods. In this final subsection we will show that the Laplace transform and its inverse can be derived from the Fourier integral!

We begin with the Fourier integral in the complex exponential form given by equation (3) in Section 17.10,

\[
F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F(\tau)e^{-i\omega \tau} d\tau \right\} e^{i\omega t} d\omega, \quad (-\infty < t < \infty) \quad (16)
\]

where we use \( t \) in place of \( x \) because we will end up with the Laplace transform, in which the independent variable is traditionally taken to be \( t \) (because in applications it usually corresponds to the time). Further, it will be convenient to use \( \hat{F} \) in place of \( F \). In (16), let

\[
F(t) = \begin{cases} 
    e^{-\gamma t} f(t), & t > 0 \\
    0, & t < 0,
\end{cases} \quad (17)
\]

where \( \gamma \) is a real constant which is sufficiently positive so that \( e^{-\gamma \omega} \) "clobbers" \( f(t) \) as \( t \to \infty \). Specifically, suppose that \( f \) is of exponential order (defined in section 5.2) as \( t \to \infty \). Then a sufficiently positive \( \gamma \) can indeed be found such that \( e^{-\gamma t} f(t) \) dies out exponentially fast as \( t \to \infty \). Of course, whereas \( e^{-\gamma t} \) helps as \( t \to +\infty \), it hurts as \( t \to -\infty \). Thus, we simply "shut off \( F \)" for \( t < 0 \) by defining it to be zero for \( t < 0 \), in (17). The resulting \( F \) easily satisfies the condition \( \int_{-\infty}^{\infty} |F(t)| dt < \infty \) contained in the Fourier integral theorem.

Putting (17) into (16) gives

\[
H(t) e^{-\gamma t} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{0}^{\infty} e^{-\gamma \tau} f(\tau)e^{-i\omega \tau} d\tau \right\} e^{i\omega t} d\omega \quad (18)
\]

where \( H \) is the Heaviside function. Thus,

\[
H(t) f(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left\{ \int_{0}^{\infty} e^{-(\gamma+i\omega) \tau} f(\tau) d\tau \right\} e^{(\gamma+i\omega)t} d\omega \quad (19)
\]

which form suggests changing variables from \( \omega \) to \( s \) according to

\[
s = \gamma + i\omega. \quad (20)
\]

Thus,

\[
H(t) f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left\{ \int_{0}^{\infty} e^{-s \tau} f(\tau) d\tau \right\} e^{st} ds, \quad (21)
\]

where \( \int_{\gamma-i\infty}^{\gamma+i\infty} \) denotes an integration along a vertical line in a complex \( s \) plane (Fig. 2). If we define the Laplace transform of \( f \) as

\[
L\{f(t)\} = \bar{f}(s) = \int_{0}^{\infty} f(t)e^{-st} dt, \quad (22)
\]
where we have changed the dummy integration variable from $\tau$ to $t$, then (21) gives the inversion formula as

$$L^{-1}\{f(s)\} = H(t)f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f(s) e^{st} \, ds.$$  (23)

Normally we are interested only in $0 < t < \infty$, so we can replace $H(t)$ by 1 in (23). However, for completeness we observe that while the integral in (23) gives $f(t)$ for $t > 0$, it gives 0 for $t < 0$, which result is, after all, in accordance with our definition $F(t) = 0$ for $t < 0$, in (17).

**Closure.** In this section we have combined two topics, each being too brief to justify a full section. First we introduced the Fourier cosine and sine transforms, for problems defined on the semi-infinite interval $0 < x < \infty$, which results are analogous to the half-range cosine and sine Fourier series. The Laplace transform is also a semi-infinite transform, but it is tailored to *initial-value* problems whereas the Fourier cosine and sine transforms are tailored to *boundary-value* problems, because $L\{f^{(n)}(t)\}$ involves values of $f$, $f'$, $\ldots$, $f^{(n-1)}$ at $t = 0$, whereas $F_C\{f^{(n)}(x)\}$ and $F_S\{f^{(n)}(x)\}$ involve values both at $x = 0$ and at $x = \infty$. Whether to use a cosine transform or a sine transform, in solving a differential equation boundary-value problem, depends (as we saw in Example 1) on the type of boundary conditions prescribed at $x = 0$.

Finally, we used the complex exponential form of the Fourier integral to derive the Laplace transform and its inversion formula. Normally, in using the Laplace transform, we have the $t$ domain of interest is $t > 0$. However, we noted that if one were to obtain an inverse Laplace transform by evaluating the inversion integral in (23), for $t < 0$, then one would inevitably find that the inverse function is identically zero for $t < 0$.

**EXERCISES 17.11**

1. Derive the formulas (5a) and (5b) for the Fourier sine transform and its inverse, respectively.
2. Derive (6b), that $F_S\{f'(x)\} = -\omega \check{f}_S(\omega)$.
3. Derive these results:
   (a) $F_C\{f'''(x)\} = \omega^3 \check{f}_C(\omega) + \omega^2 f'(0) - f'''(0)$
   (b) $F_S\{f'''(x)\} = \omega^3 \check{f}_S(\omega) - \omega^3 f(0) + \omega f''(0)$ if $f(x), f'(x), f''(x)$, and $f'''(x)$ all tend to zero as $x \to \infty$.
4. Given the rectangular pulse $f(x) = 50[1 - H(x - 4)]$, evaluate $\check{f}_C(\omega)$ and $F_S(\omega)$.
5. Use computer software, such as the int command on Maple, to evaluate each, where $a > 0$. Show that your result agrees with the corresponding result obtained from Appendix E.

**NOTE:** Be careful, in using tables or software, to be clear on the author's definition of the transform. For instance, in contrast with our definitions (4) and (5) of the Fourier cosine and sine transforms, some authors use the more symmetric versions

$$F_C\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$  (5.1)
8. Modify the quantity using the term, \( u'(0) \). When you give the condition that is (10), and (940) Chapter 17. Fourier Series, Fourier Integral, Fourier Transform

\[ F_C^{-1}\{\hat{f}(\omega)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\omega) \cos \omega x \, d\omega \] (5.2)

and

\[ F_S\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \] (5.3)

\[ F_S^{-1}\{\hat{f}(\omega)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\omega) \sin \omega x \, d\omega. \] (5.4)

6. In Example 1 we avoided the Laplace transform because (10) is of boundary-value type, not of initial-value type. Show that the Laplace transform can be used nevertheless, and does give the solution (15), though not as conveniently. HINT: When you take the transform of \( u'' \) you will be faced with a \( u'(0) \) term, which is not prescribed in (10b). Thus, call that quantity \( C \), say, and evaluate it by imposing on your solution the condition that \( u(\infty) \) be evaluated, at the end.

7. We used a sine transform to solve Example 1. Try to solve (10) using a cosine transform instead, and explain why that method does not work.

8. Modify (10) by changing \( u(0) = u_0 \) to \( u'(0) = u'_0 \), and solve by a cosine or sine transform.

9. Solve, using a cosine or sine transform.

(a) \( u'' - 9u = 50e^{-3x} \), \((0 < x < \infty)\)

\[ u(0) = 0, \quad u(\infty) \text{ bounded} \]

(b) \( u'' - 9u = 50e^{-3x} \), \((0 < x < \infty)\)

\[ u'(0) = 0, \quad u(\infty) \text{ bounded} \]

10. (Convolution Theorem) As for the Fourier and Laplace transforms there are convolution theorems for the Fourier cosine and sine transforms, and these are given in Appendix E.

(a) Prove the Fourier cosine transform convolution theorem (entry 7C), either by showing that the transform of the given integral is \( f_C(\omega)\hat{g}_S(\omega) \) or by showing that the inverse of \( f_C(\omega)\hat{g}_S(\omega) \) is the given integral.

(b) Prove the Fourier sine transform convolution theorem (entry 7S).

(c) Verify entry 7C for the case where \( f(x) = e^{-x} \) and \( g(x) = e^{-3x} \).

(d) Verify entry 7S for the case where \( f(x) = e^{-x} \) and \( g(x) = e^{-3x} \).

Chapter 17 Review

We began with the Fourier series representation

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \quad (-\infty < x < \infty) \] (1)

of any 2\( \pi \)-periodic function \( f \) defined on \(-\infty < x < \infty\), which representation is valid subject to very mild conditions on \( f \). For instance, \( f \) can even have jump discontinuities, whereas to represent \( f \) by a Taylor series \( f \) needs to be infinitely differentiable over the interval under consideration (and even that condition does not quite suffice). In applications, periodic functions arise in a number of ways. For instance an offshore structure is probably subjected to wave forces that are periodic in time, and the temperature distribution around the edge of a circular disk is a \( 2\pi \)-periodic function of the polar angle \( \theta \).

If, instead, the \( x \) domain is finite, say \( 0 < x < L \), then Fourier series can still be employed — by fictitiously extending the domain to \(-\infty < x < \infty\), and extending the definition of \( f \) onto that domain so that \( f_{\text{ext}} \) is periodic. Such extensions can be accomplished in an infinite number of ways, but the four that will be needed, in applications, correspond to extensions that are symmetric or antisymmetric about
the ends $x = 0$ and $x = L$, and these are the half- and quarter-range cosine and sine expansions, which we denote by HRC, HRS, QRC, and QRS, respectively:

$$
\text{HRC: } f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad (0 < x < L) \\
\text{HRS: } f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (0 < x < L) \\
\text{QRC: } f(x) = \sum_{n=1,3,\ldots}^{\infty} a_n \cos \frac{n\pi x}{2L} \quad (0 < x < L) \\
\text{QRS: } f(x) = \sum_{n=1,3,\ldots}^{\infty} b_n \sin \frac{n\pi x}{2L} \quad (0 < x < L)
$$

(For brevity, we do not repeat, here, the formulas for the $a_n$’s and $b_n$’s given in the text.) We emphasized that the choice, as to which of these four expansions to use, will be dictated by the context.

From a vector space viewpoint, (1)–(5) amount to expansions of $f$ in terms of an infinite set of orthogonal base vectors. In (1), for instance, the base vectors are $1, \cos \left(\pi x / \ell \right), \sin \left(\pi x / \ell \right), \cos \left(2\pi x / \ell \right), \sin \left(2\pi x / \ell \right), \ldots$. Such sets of orthogonal base vectors arise as the eigenfunctions of Sturm–Liouville problems, namely, eigenvalue problems of the type

$$
(p y')' + q y + \lambda w y = 0, \quad (a < x < b)
$$

with homogeneous boundary conditions at $x = a$ and $x = b$, with the inner product

$$
\langle u, v \rangle = \int_{a}^{b} u(x)v(x)w(x)\,dx
$$

with weight function $w$. For instance, the base vectors in (1) are the eigenfunctions of the Sturm–Liouville problem

$$
y'' + \lambda y = 0, \quad (-\ell < x < \ell) \\
y(-\ell) - y(\ell) = 0, \quad y'(-\ell) - y'(\ell) = 0,
$$

and the base vectors in (2)–(5) are the eigenfunctions of the Sturm–Liouville problems

$$
y'' + \lambda y = 0, \quad (0 < x < L) \\
\text{HRC: } y'(0) = 0, \quad y'(L) = 0, \\
\text{HRS: } y(0) = 0, \quad y(L) = 0, \\
\text{QRC: } y'(0) = 0, \quad y(L) = 0, \\
\text{QRS: } y(0) = 0, \quad y'(L) = 0,
$$
respectively. The four cases in (9) are examples of regular Sturm–Liouville systems, while the Sturm–Liouville problem in (8) is of periodic type. Singular Sturm–Liouville problems were discussed in Section 17.8, prominent examples involving the Bessel and Legendre equations.

We showed that if we let \( \ell \to \infty \) in (1), then the frequency spectrum \( \{ n\pi/\ell \} \) becomes a continuous spectrum from 0 to \( \infty \) and we obtain, in place of the Fourier series (1), the Fourier integral

\[
f(x) = \int_0^\infty \left[ a(\omega) \cos \omega x + b(\omega) \sin \omega x \right] d\omega. \quad (-\infty < x < \infty)
\]

Expressing (10) in complex exponential form, we obtained the equation pair

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx, \quad (11a)
\]

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega, \quad (11b)
\]

which are equivalent to (10) and which are the Fourier transform and its inverse, respectively. Finally, we obtained the Fourier cosine and sine transforms from (11), for problems defined on the semi-infinite interval \( 0 < x < \infty \), these being analogous to the half-range cosine and sine representations of functions defined on \( 0 < x < L \).
Chapter 18

Diffusion Equation

18.1 Introduction

Chapters 18–20 are about partial differential equations (PDE’s). Two of the most important PDE’s of mathematical physics have already been encountered in Chapter 16. In Example 3 in Section 16.8, we derived the equation

$$\alpha^2 \nabla^2 T = \frac{\partial T}{\partial t}$$

(1)

governing the unsteady diffusion of heat by conduction, where \(\alpha^2\) is a physical constant known as the diffusivity of the material, and \(T(x, y, z, t)\) is the temperature field. The latter is known as the heat equation but is also called the diffusion equation because it governs diffusion process in general. For instance, whereas (1) governs the diffusion of heat, the equation

$$D \nabla^2 c = \frac{\partial c}{\partial t}$$

(2)

governs the unsteady diffusion of material (such as a particular chemical pollutant within a body of water or of an anti-cancer drug within an organ such as the liver), where \(D\), like \(\alpha^2\), is a concentration of the material (i.e., the mass of material per unit volume of medium); (2) is of the same form as (1).

If a steady state is achieved, then \(\partial T/\partial t = 0\) and (1) reduces to the Laplace equation

$$\nabla^2 T = 0.$$  

(3)

In Section 16.10 we found that the Laplace equation also governs the velocity potential \(\Phi(x, y, z)\) for irrotational incompressible flows; other applications of the Laplace equation are discussed in Chapter 20.

Arguably, equations (1) and (3) are two of the three most prominent PDE’s of mathematical physics, the third being the wave equation of the form

$$c^2 \nabla^2 u = \frac{\partial^2 u}{\partial t^2}$$

(4)
where $c^2$ is a constant.

The diffusion, wave, and Laplace equations are the subjects of Chapters 18, 19, and 20, respectively. They could be presented in any order, but we have chosen this sequence for pedagogical reasons that will be explained as we proceed.

Our approach in these three chapters is substantially different from our approach in the early chapters on ODE's (ordinary differential equations). For ODE's we proceeded systematically, beginning with first-order equations and then moving on to equations of second order and higher and developing the general theory—which covered existence, uniqueness, and methods of solution. Though our emphasis was on linear equations, we considered nonlinear equations as well. For PDE's, however, our scope is more limited as we focus almost entirely on the diffusion, wave, and Laplace equations and emphasize solution technique. These three are by no means the only PDE's encountered in applications, but they are extremely important, and the solution methods that we develop can be applied to various other (linear) PDE's as well.

18.2 Preliminary Concepts

18.2.1. Definitions. Recall (from Section 1.2) that a differential equation is a partial differential equation if it contains partial derivatives of the dependent variable with respect to two or more independent variables. For most applications the independent variables are one or more space variables (Cartesian or non-Cartesian), and possibly the time $t$, but the dependent variables encountered are much more varied and include temperature, concentration, deflection of a string or membrane or beam, velocity potential, and electric potential, to name just a few. As for ODE's, there may be more than one dependent variable, and we may have a system of PDE's in two or more unknowns. Here, however, we consider only the case of one equation in one unknown.

Also as for ODE's, we define the order of a PDE as the order of the highest derivative therein, and we say that a function is a solution of a PDE, over a particular domain of the independent variables, if its substitution into the equation reduces that equation to an identity everywhere within that domain. For instance, the PDE

$$u_{xx} - 5u_y + u = xy^2(y - 15) \tag{1}$$

(where subscripts denote partial derivatives) is of second order and admits the solutions

$$u_1(x, y) = 6e^{2x+y} + xy^3 \quad \text{and} \quad u_2(x, y) = -e^{3x+2y} + xy^3$$

over the entire $x, y$ plane (and other solutions as well). For instance, putting $u_1$ into (1) gives

$$24e^{2x+y} - 5(6e^{2x+y} + 3xy^2) + 6e^{2x+y} + xy^3 = xy^2(y - 15),$$

which is indeed an identity for all values of $x$ and $y$. 

It is standard and convenient to use the differential operator notation \( L[ ] \) for PDE’s, as we do for ODE’s. Accordingly, (1) may be written more compactly as

\[
L[u] = f,
\]

where

\[
L = \frac{\partial^2}{\partial x^2} - 5 \frac{\partial}{\partial y} + 1
\]

is the second-order partial differential operator, and \( f(x, y) = xy^2(y - 15) \). Just as the definition of a function is not complete until we specify the domain on which it acts [e.g., the function \( \sin x \) defined on \( 0 \leq x \leq \pi \) is not the same as the function \( \sin x \) defined on \( -\infty < x < \infty \)], likewise the definition of a differential operator is not complete until we specify the domain of functions on which it acts. We do not “figure out” what the domain is; we specify it. For \( L \) given by (3), for instance, we might specify its domain as the set of functions \( u(x, y) \) that are defined on the first quadrant \( (0 < x < \infty, 0 < y < \infty) \) and that are twice differentiable in \( x \) and once in \( y \).

We say that a differential operator, be it an ordinary differential operator or a partial differential operator, is linear if

\[
L[\alpha u + \beta v] = \alpha L[u] + \beta L[v]
\]

for any functions \( u \) and \( v \) in the domain of \( L \) and for any constants \( \alpha \) and \( \beta \); otherwise it is nonlinear.

**EXAMPLE 1.** The operator \( L \) given by (3) is linear because

\[
L[\alpha u + \beta v] = \left( \frac{\partial^2}{\partial x^2} - 5 \frac{\partial}{\partial y} + 1 \right) (\alpha u + \beta v)
= \alpha (\partial (\partial u) - 5 \partial v + u) + \beta (\partial (\partial v) - 5 \partial u + v)
= \alpha L[u] + \beta L[v].
\]

**EXAMPLE 2.** The operator defined by

\[
L[u] = u_{xx} + uu_y
\]

is nonlinear because the difference

\[
L[\alpha u + \beta v] - \alpha L[u] - \beta L[v]
= (\alpha u_{xx} + \beta v_{xx}) + (\alpha u + \beta v)(\alpha u_y + \beta v_y) - \alpha (u_{xx} + uu_y) - \beta (v_{xx} + vv_y)
= (\alpha^2 - \alpha)uu_y + (\beta^2 - \beta)vv_y + \alpha \beta (uu_y + uu_y)
\]

is not identically zero. For instance, if \( u(x, y) = y, v(x, y) = 0, \alpha = 2, \) and \( \beta = 6 \), then the right-hand side is \( 3y \), not zero.

If an operator \( L \) is linear, then it follows immediately from (4) that

\[
L[\alpha_1 u_1 + \cdots + \alpha_k u_k] = \alpha_1 L[u_1] + \cdots + \alpha_k L[u_k]
\]
for any functions \( u_1, \ldots, u_k \) in the domain of \( L \) and for any constants \( \alpha_1, \ldots, \alpha_k \), for any finite \( k \).

Consider a linear differential equation

\[
L[u] = f,
\]

where \( f \) is a prescribed function of the independent variables. If \( f = 0 \) then (6) is homogeneous, and if \( f \neq 0 \) then (6) is nonhomogeneous with \( f \) as a forcing function. Any solution of the full equation \( L[u] = f \) is called a particular solution of (6). The power of the linearity property (5) is that it enables us to build a more robust solution from a collection of individual solutions by superposition. For suppose that \( u_1, \ldots, u_k \) are solutions of the homogeneous version \( L[u] = 0 \) of (6), and that \( u_p \) is a particular solution of (6). Then

\[
u = C_1 u_1 + \cdots + C_k u_k + u_p
\]

is a solution of (6) for any constants \( C_1, \ldots, C_k \) since

\[
L[C_1 u_1 + \cdots + C_k u_k + u_p] = C_1 L[u_1] + \cdots + C_k L[u_k] + L[u_p] = C_1(0) + \cdots + C_k(0) + f = f.
\]

We say that (7) is robust in the sense that it contains \( k \) arbitrary constants, which are available to help \( u \) to satisfy the various boundary conditions that may be prescribed.

However, there is a major difference in the application of the foregoing idea to ODE’s and PDE’s. If (6) is a linear \( k \)th-order ODE and \( u_1, \ldots, u_k \) are linearly independent solutions of the homogeneous equation \( L[u] = 0 \), then (7) is a general solution of (6). For the PDE’s that we study in these chapters we will be able to find even an infinite number of solutions \( u_1, u_2, \ldots \), yet

\[
u = u_p + \sum_{j=1}^{\infty} C_j u_j
\]

may fall short of being a general solution of (6). That is, (6) may admit solutions that cannot be expressed in the form (9) for any choice of the \( C_j \)’s. Though it would be nice to obtain general solutions of our PDE’s (and indeed we will for the wave equation in Chapter 19) our objective is more limited than that. Specifically, we will be content to be able to obtain solutions satisfying specific boundary conditions, and it will turn out that the infinite series solutions that we develop will indeed be robust enough for that, even if they fall short of being general solutions.

18.2.2. Second-order linear equations and their classification. We are especially, though not exclusively, interested in linear second-order PDE’s of the form

\[
Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f,
\]

*We assume that \( A, B, C \) are not all zero for then (10) would not be of second order. Further, we assume that if \( A = B = 0 \) then \( D \neq 0 \), for if \( A = B = D = 0 \) then only \( y \) derivatives appear and (10) would more reasonably be regarded as an ODE rather than a PDE. Similarly, we assume that if \( B = C = 0 \) then \( E \neq 0 \).
where $A, \ldots, F, f$ are prescribed functions of $x$ and $y$ and where the 2 is included for subsequent convenience. Often $A, \ldots, F$ are constants, but not always. The independent variables $x$ and $y$ will be Cartesian space variables, or else $x$ will be a Cartesian space variable and $y$ will be the time – in which case we will use $t$ in place of the generic $y$.

We classify (10) as one of three types, depending on the sign of the discriminant $B^2 - AC$: (10) is

<table>
<thead>
<tr>
<th>Type</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>parabolic</td>
<td>$B^2 - AC = 0$</td>
</tr>
<tr>
<td>hyperbolic</td>
<td>$B^2 - AC &gt; 0$</td>
</tr>
<tr>
<td>elliptic</td>
<td>$B^2 - AC &lt; 0$</td>
</tr>
</tbody>
</table>

in the region under consideration. This terminology is by analogy with the general equation of a conic section, $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$, which gives a parabola, a hyperbola, or an ellipse, according to the sign of the discriminant $b^2 - ac$. Just as parabolas, hyperbolas, and ellipses are governed by distinct geometrical theories, so are parabolic, hyperbolic, and elliptic PDE’s governed by distinct theories. Prototype examples of these three types are as follows:

1. The diffusion equation

$$\alpha^2 u_{xx} = u_t \quad (\alpha^2 = \text{constant}) \quad (12)$$

is parabolic (with $y \to t$) since $B^2 - AC = 0^2 - (\alpha^2)(0) = 0$.

2. The wave equation

$$c^2 u_{xx} = u_{tt} \quad (c^2 = \text{constant}) \quad (13)$$

is hyperbolic (with $y \to t$) since $B^2 - AC = 0^2 - (c^2)(-1) = c^2 > 0$.

3. The Laplace equation

$$u_{xx} + u_{yy} = 0 \quad (14)$$

is elliptic since $B^2 - AC = 0^2 - (1)(1) = -1 < 0$.

Thus, keep in mind in Chapters 18–20 that the diffusion, wave, and Laplace equations are not only of importance in their own right, but that they are also representative of the three equation types – parabolic, hyperbolic, and elliptic, respectively. For instance, the PDE

$$\alpha^2 u_{xx} = u_t + V u_x + H u \quad (15)$$

governs the diffusion of heat in a one-dimensional rod, but differs from the basic diffusion equation (12) by virtue of the $V u_x$ term (which is due to the rod being in motion with constant speed) and the $H u$ term (which is due to heat loss from the lateral surface of the rod to the environment). However, (15) is still parabolic, like (12), since $B^2 - AC = 0 - (\alpha^2)(0) = 0$, so its solution should in fundamental ways
be similar to the solution of (12) that is discussed in this chapter. Such variations from the basic diffusion, wave, or Laplace equation, are considered in a number of the end-of-section exercises.

**EXAMPLE 3.** Since $A, \ldots, F, f$ in (10) may be functions of $x$ and $y$, the discriminant $B^2 - AC$ may be a function of $x$ and $y$. Thus, besides the possibility $B^2 - AC$ is zero, positive, or negative everywhere in the $x, y$ plane, it is also possible that it is zero, positive, or negative in different parts of the $x, y$ plane. To illustrate, consider the Tricomi equation

$$u_{xx} + xu_{yy} = 0,$$  \hspace{1cm} (16)

which arises in the study of the two-dimensional steady transonic flow past a body such as a wing. (Transonic means that flight speed is close to the speed of sound.) Then

$$B^2 - AC = 0 - (1)(x) = -x$$

so the Tricomi equation is elliptic in the right half plane $x > 0$ and hyperbolic in the left half plane $x < 0$. Thus the Tricomi equation is a change-of-type equation, with solutions that are qualitatively different in the two half planes. For ODE’s, an analogous behavior is exhibited by the Airy equation

$$y'' + xy = 0,$$  \hspace{1cm} (17)

which has oscillatory solutions for $x > 0$ and nonoscillatory solutions for $x < 0$.

The moral of this example is that a given example of equation (10) might be of one type in one region and of another type in another region. Such cases are more difficult and are not studied here.

### 18.2.3. Diffusion equation and modeling

In Example 3 of Section 16.8 we derive the heat equation $\alpha^2 \nabla^2 u = u_t$ by considering a heat balance for an arbitrary control volume and using the divergence theorem. (There we used $T$ for the temperature field, but here we use $u$, since we would like to have the letter $T$ available for another purpose.) In slight contrast with the arbitrary control volume approach, engineering textbooks usually consider infinitesimal elements in such derivations. Let us present such a derivation for a one-dimensional rod and, at the same time, include the additional effects of translation of the rod and heat loss to the environment from its lateral surface.

Specifically, consider a uniform rod of cross-sectional area $A$, circumference $s$, mass density $\sigma$ (mass per unit volume), thermal conductivity $k$, and specific heat $c$, and suppose it is in uniform motion with speed $v$ in the positive $x$ direction. Consider an element of the rod between $x$ and $x + \Delta x$, where the $x$ axis is fixed in space and the rod is moving relative to it. Assume, merely for definiteness and with no loss of generality, that the derivative $u_x$ is positive at $x$ and at $x + \Delta x$. Then, according to the Fourier law of heat conduction (Example 3, Section 16.8), heat enters the element through its right-hand face at the rate $kAu_x|_{x+\Delta x}$ and leaves through its left-hand face at the rate $kAu_x|_x$ (Fig. 1).
Additionally, suppose there is a heat loss to the environment from the exposed lateral surface of the element, that is proportional to the surface area \( s \Delta x \) and the temperature difference \( u - u_\infty \), where \( u_\infty \) is the temperature of the environment (which we assume to be constant). If the constant of proportionality is some known heat transfer coefficient \( h \), then by Newton's law of cooling the rate of heat loss is \( hs\Delta x(u - u_\infty) \) (Fig. 1). Finally, there is a transport of heat, in at \( x \) and out at \( x + \Delta x \), because the rod is translating. Recall that the heat contained in a mass \( m \) at (absolute) temperature \( u \) is \( mcu \). In time \( \Delta t \) the mass \( m \) entering at the left and leaving at the right is the dimension \( v \Delta t \) times the area \( A \) times the density \( \sigma \) or, on a per unit time basis, \( vA\sigma \). Thus, the heat in at the left, per unit time, is \( vA\sigma cu \), and the heat out at the right, per unit time, is \( vA\sigma cu_{x+\Delta x} \) (not shown in Fig. 1).

Then the net heat influx into the element, per unit time, is

\[
kA u_x \bigg|_{x+\Delta x} - kA u_x \bigg|_x - hs\Delta x(u - u_\infty) + vA\sigma cu \bigg|_x - vA\sigma cu \bigg|_{x+\Delta x}.
\]

(18)

The latter must equal the rate of change of the heat \( mcu \) contained in the element, where \( m = A\Delta x\sigma \), so

\[
kA \left( u_x \bigg|_{x+\Delta x} - u_x \bigg|_x \right) - hs\Delta x(u - u_\infty) - vA\sigma cu \left( u \bigg|_{x+\Delta x} - u \bigg|_x \right) = \frac{\partial}{\partial t} (A\Delta x\sigma cu).
\]

(19)

Dividing through by \( A\Delta x\sigma \) and letting \( \Delta x \to 0 \) gives the desired field equation

\[
\frac{k}{c\sigma} u_{xx} - \frac{hs}{A\sigma} (u - u_\infty) - v u_x = u_t
\]

(20)

or, more concisely,

\[
\alpha^2 u_{xx} = u_t' + vu'_x + Hu',
\]

(21)

where \( \alpha^2 = k/(c\sigma) \) is the thermal diffusivity of the material, \( v \) is the translational speed, \( H = hs/(A\sigma) \) is proportional to the heat transfer coefficient \( h \), and \( u' = u - u_\infty \) is the temperature at any point in the rod relative to the ambient temperature \( u_\infty \).* If we remember that \( u' \) is the relative temperature we can, for notational

*Some common values of \( \alpha^2 \) (cm\(^2\)/sec) are: silver, 1.70; copper, 1.14; aluminum, 0.86; cast iron, 0.16; brick, 0.0052; glass, 0.0034; and water, 0.0014.
convenience, drop the primes in (21). Of course, (21) is the same as (15), mentioned above.

The derivation given above is only heuristic and is typical of the elemental approach to deriving various field equations, as it is found in engineering science textbooks. But it is not hard to render the derivation rigorous. First, the \( A \Delta x \sigma u \) on the right-hand side of (19) should really have been an integral over the element, but

\[
\frac{\partial}{\partial t} \int_{x}^{x+\Delta x} A \sigma u(\xi, t) \, d\xi = \frac{\partial}{\partial t} A \sigma u(x + \mu \Delta x, t) \Delta x \sim A \Delta x \sigma u_t(x, t) \quad (22)
\]

as \( \Delta x \to 0 \), so we obtain the same result as before. The equality in (22) follows from the mean value theorem of the integral calculus, for which we merely need \( u(\xi, t) \) to be a continuous function of \( x \), where \( \mu \) is some value between 0 and 1. Similarly, the \( h s \Delta x (u - u_\infty) \) term in (19) should actually be an integral over the lateral surface, but once again we can use the mean value theorem and obtain the same result as before.

The \( V u_t \) term in (15) is called a convection term, and the \( H u \) term is called a Newton cooling term: of course, it will be a heating term rather than a cooling term if the environment is hotter than the rod. As a physical application where both terms would be important, consider a hot continuous metal rod being drawn from a furnace and entering an extrusion die at some distance from the furnace (Fig. 2). In designing the facility it is important to predict the temperature of the rod when it reaches the die, as a function of the various design parameters, so that the metal is still sufficiently hot to be extruded.

Let us set \( V = 0 \) and \( H = 0 \) and limit our subsequent attention to the basic one-dimensional diffusion equation (12). As a typical application of (12), consider the heat conduction through the outer wall of a house (Fig. 3), and let \( x, y, z \) be normal, horizontal, and vertical axes, respectively, with \( x > L \) corresponding to the interior of the house and \( x < 0 \) corresponding to the exterior. Actually, we should be working with the three-dimensional diffusion equation

\[
\alpha^2(u_{xx} + u_{yy} + u_{zz}) = u_t, \quad (23)
\]

However, on a cold (or hot) day we expect \( u \) to vary very little with \( y \) and \( z \) compared to its variation with \( x \) (over \( 0 < x < L \)). Thus, as a reasonable approximation we can neglect the \( u_{yy} \) and \( u_{zz} \) terms, in which case (23) simplifies to the one-dimensional equation

\[
\alpha^2 u_{xx} = u_t, \quad (24)
\]

or, in operator form,

\[
L[u] = \left( \alpha^2 \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right)[u] = \alpha^2 u_{xx} - u_t = 0. \quad (25)
\]

To complete the problem formulation it is useful to observe that the domain of interest is the semi-infinite strip \( 0 < x < L \) and \( 0 < t < \infty \) in the \( x, t \) plane.
as shown in Fig. 4. Clearly, (25) admits an infinite number of solutions such as

$$4x, 5x - 30, \sin x \exp(-\alpha^2 t), x + 20 - \cos 3x \exp(-9\alpha^2 t),$$

and so on. Our expectation, if only intuitive at this stage, is that if we append to (25) a suitable set of boundary conditions, then the resulting problem will have a unique solution. To motivate such a set of conditions, observe that (25) is a second-order equation with respect to \(x\), so we expect two \(x\) boundary conditions to be appropriate, one at \(x = 0\) and one at \(x = L\). If we call the outside temperature \(u_1\), and call the inside temperature \(u_2\), which is the temperature setting on our thermostat, then the boundary conditions on \(x\) are

$$
\begin{align*}
    u(0, t) &= u_1, & (0 < t < \infty) \\
    u(L, t) &= u_2, & (0 < t < \infty)
\end{align*}
$$

(26a, b)

Further, (25) is a first-order equation with respect to \(t\), so we expect one \(t\) boundary condition to be appropriate, at \(t = 0\) (i.e., an initial condition). If we prescribe some initial temperature distribution

$$
u(x, 0) = f(x), \quad (0 < x < L)$$

(27)

then the full problem statement consists of (25)–(27) and is summarized in Fig. 5. We solve that problem for \(u(x, t)\) in Section 18.3, and also establish its uniqueness.

Thus, in formulating the problem the idea is to impose boundary and initial conditions that are sufficient to reduce the solution set to a unique solution, but not excessive so that there is no solution. For instance, if we impose not only the conditions shown in Fig. 5 but also the condition \(u_x(0, t) = 0\), then that problem will have no solution. Whereas existence and uniqueness questions were prominent in our study of ODE's, here we are not so concerned about the existence question, because we will generally be able to find a solution. The nagging question that remains, then, is uniqueness. Thus, representative uniqueness theorems are presented in these chapters.

Besides thinking of our problem (25)–(27) as governing heat conduction in a wall, we can think of it as governing heat conduction in a rod that is thermally insulated everywhere on its lateral surface - but not at its two ends, which are subjected to temperatures \(u_1\) and \(u_2\) for all \(0 < t < \infty\) (Fig. 6).

Before closing, let us consider the possible boundary conditions at \(x = 0\) and \(x = L\) more fully. We distinguish three types. Conditions (26a,b) are examples of **boundary conditions of the first kind**, or **Dirichlet boundary conditions**, because they are of the form

$$u \text{ prescribed} \quad (28)$$

on the boundary. In (26a,b) \(u\) is prescribed to be a constant, but Dirichlet boundary conditions need not be constant. For instance, if the time of interest is short compared to a day, then it may well suffice to take \(u(0, t) = u_1\) to be a constant, but if

---

*In fact, even for the wall shown in Fig. 3, any "pencil" of material, parallel to the \(x\) axis and extending from \(x = 0\) to \(x = L\), is essentially an insulated rod, insulated by virtue of the lack of temperature variation with \(y\) and \(z\).*
the time of interest is on the order of a day or longer, then we really need to take
\( u(0, t) = u_1(t) \) to be a function of time.

A boundary condition of the second kind, or Neumann boundary condition, is a derivative boundary condition of the form

\[
\frac{\partial u}{\partial n} \text{ prescribed,}
\]

where \( \frac{\partial u}{\partial n} \) denotes the derivative \( \partial u/\partial n \) of \( u \) normal to the boundary under consideration. In the present example \( \frac{\partial u}{\partial n} \) is \( -u_x \) on the \( x = 0 \) boundary and \( +u_x \) on the \( x = L \) boundary. Physically, (29) amounts to prescribing the heat flux rather than the temperature. For instance, the heat flux \( Q(t) \) crossing the left end of the rod shown in Fig. 6, counted positive if it flows from left to right is, according to Fourier’s law of heat conduction,

\[
Q(t) = -k u_x(0, t).
\]

Thus, if we specify the normal derivative \( u_x(0, t) \) we are, in effect, specifying the heat flux \( Q(t) \). We see from (30) that a homogeneous Neumann boundary condition

\[
u_x(0, t) = 0
\]

[or \( u_x(L, t) = 0 \)] amounts to a stipulation that that end is thermally insulated, for then \( Q(t) = 0 \).

Finally, a boundary condition of the third kind, or Robin boundary condition, occurs when a linear combination of \( u \) and \( u_n \) is prescribed. To illustrate, within the context of the present example, consider the heat flux crossing the end \( x = L \), say. According to the Fourier law of conduction the flux crossing the left is \( -k A u_x(L, t) \), and according to Newton’s law of cooling the flux crossing \( x = L \) into the environment is \( h A [u(L, t) - u_2] \). Since these must be equal we have the boundary condition

\[
-k A u_x(L, t) = h A [u(L, t) - u_2]
\]

or

\[
u_x(L, t) + \frac{h}{k} u(L, t) = -\frac{h}{k} u_2,
\]

which is a boundary condition of Robin type, also called a “mixed” boundary condition. It is useful to nondimensionalize terms in this equation. Nondimensionalizing \( u \) with respect to the reference temperature \( u_2 \) and \( x \) with respect to the reference length \( L \), the nondimensional quantities are

\[
x = x/L \quad \text{and} \quad \bar{u} = u/u_2.
\]

Then (32) becomes

\[
-k A u_2 \left. \frac{\partial u}{\partial x} \right|_{x=1} = h A u_2 \left[ \bar{u} \left|_{x=1} - 1 \right. \right] \quad \text{or}
\]

\[
-\bar{u}_x \left|_{x=1} = \text{Bi} \left[ \bar{u} \left|_{x=1} - 1 \right. \right] ,
\]

where \( \text{Bi} \) is the Biot number.
where the dimensionless parameter

$$\text{Bi} = \frac{hL}{k}$$  \hspace{1cm} (36)

is known in heat transfer as the *Bio number*. If \( \text{Bi} \gg 1 \), then (35) implies that

$$u(x, t) \bigg|_{x=1} - 1 \approx 0 \text{ or, returning to dimensional terms,}$$

$$u(L, t) = u_2,$$  \hspace{1cm} (37)

which was the boundary condition naively adopted in (26b). However, if the values of \( h, L, \) and \( k \) give \( \text{Bi} \ll 1 \), then (35) implies that

$$u_x(L, t) = 0.$$  \hspace{1cm} (38)

If \( \text{Bi} \) is neither very large nor very small, then we should leave the mixed boundary condition (33) intact.

With the foregoing remarks completed we will simply specify boundary conditions such as (37), (38), or (33) without further discussion as we now turn our attention to the solution of such problems.

**Closure.** Keep in mind, as we embark on our study of PDE’s in Chapters 18–20, that we are focusing on the extremely important class of PDE’s of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f,$$  \hspace{1cm} (39)

where \( A, \ldots, F, f \) are prescribed functions of \( x \) and \( y \). Within that class we distinguish three types. Specifically, we say that (39) is parabolic if \( B^2 - AC = 0 \), hyperbolic if \( B^2 - AC > 0 \), and elliptic if \( B^2 - AC < 0 \). Representative of these types are the diffusion, wave, and Laplace equations, respectively, and we will devote one chapter to each of these types.

Since (39) is of second order, appropriate boundary conditions will involve \( u \) and possibly first-order derivatives of \( u \). Specifically, if \( u, u_x, \) or a linear combination of \( u \) and \( u_x \) are prescribed, then we say that the boundary condition is of the first kind (Dirichlet type), second kind (Neumann type), or third kind (Robin type), respectively. Of these, Robin boundary conditions are the most difficult.

**EXERCISES 18.2**

1. Show that (5) follows from (4). HINT: Use mathematical induction.
2. Show whether the equation is linear or nonlinear: \( k, \alpha, \) and \( \beta \) are constants. HINT: See Examples 1 and 2.

(a) Helmholtz equation, \( \nabla^2 u + k^2 u = 0 \)
(b) Korteweg-de Vries (KdV) equation, \( u_t + \alpha u u_x + \beta u_{xxx} = 0 \)
(c) biharmonic equation, \( \nabla^4 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0 \)
(d) Tricomi equation, \( u_{xx} + xu_{yy} = 0 \)
(e) \( u_{xx} + u_{yy} = \epsilon^2 \)
(f) \( x^3 u_{xx} = u_y \)
3. Classify the following PDE's, defined over $-\infty < x < \infty$ and $-\infty < y < \infty$, as elliptic, parabolic, or hyperbolic. If the equation is of mixed type, identify the relevant regions and give the classification within each region.

(a) $u_{xx} + 5u_{xy} - xu = e^x$
(b) $u_{xx} - u_{xy} + yu_{yy} + 3u_y = 1$
(c) $u_{xy} + u_x + 4u_y = 6u$
(d) $xu_{xx} - (\sin^2 y + 1)u_{yy} = x^2 u$
(e) $u_{xx} + u_{xy} + u_{yy} + u_x + u_y + u = 1$
(f) $u_{xx} + (\cos y)u_{yy} = 2xy$
(g) $u_{xx} + u_x + u_y = x^3 u$
(h) $u_{xy} - u_{yy} + e^x u = f(x, y)$

4. In deriving the diffusion equation (21), we assumed that the cross-sectional shape of the rod does not vary with $x$. Reconsider our derivation for the basic case where there is no Newton cooling (i.e., the lateral surface is insulated, $h = 0$) and no translation of the rod ($v = 0$), but allow for the cross-sectional area $A$ to vary with $x$. Show that the revised diffusion equation is

$$\frac{\alpha^2}{A(x)} [A(x) u_{xx}]_x = u_t. \tag{4.1}$$

18.3 Separation of Variables

18.3.1. The method of separation of variables. We explain the method of separation of variables by a sequence of examples.

EXAMPLE 1. Consider the diffusion problem

$$L[u] = \alpha^2 u_{xx} - u_t = 0, \quad (0 < x < L, \ 0 < t < \infty) \tag{1a}$$

$$u(0, t) = u_1, \quad u(L, t) = u_2, \quad (0 < t < \infty) \tag{1b}$$

$$u(x, 0) = f(x), \quad (0 < x < L) \tag{1c}$$

that is derived in Section 18.2 and that governs the temperature field $u(x, t)$ in a rod with insulated lateral surface (or in a wall or slab of thickness $L$); see Fig. 1.

According to the method of separation of variables we begin by seeking solutions of (1a) in the product form

$$u(x, t) = X(x)T(t). \tag{2}$$

Putting (2) into (1a) gives

$$\alpha^2 X''T = XT', \tag{3}$$

where primes denote ordinary differentiation. To separate the variables, divide both sides of (3) by $XT$ and obtain

$$\frac{X''}{X} = \frac{1}{\alpha^2} \frac{T'}{T}. \tag{4}$$

Actually, we divided by $\alpha^2$ as well, but whether we have $1/\alpha^2$ on the right-hand side of (4) or $\alpha^2$ on the left-hand side will not affect the final result. Observe that (4) is of the form

$$F(x) = G(t). \tag{5}$$
Since $x$ and $t$ are independent variables, $G(t)$ does not vary with $x$. But $F(x) = G(t)$, so $F(x)$ does not vary with $x$ either. Hence $F(x)$ is a constant. From (5), $G(t)$ is a constant also, the same constant. Thus,

$$\frac{X''}{X} = \frac{T''}{T} = \text{constant} = -\kappa^2,$$

(6)
say, where we have written $\kappa^2$ for convenience because we will soon need to take the square root of that quantity. Motivation for including the minus sign in (6) is given in the end-of-example comments.

The beauty of the separation procedure is that in place of the partial differential equation $\alpha^2 u_{xx} = u_t$, we now have two ordinary differential equations,

$$\frac{X''}{X} = -\kappa^2 \quad \text{and} \quad \frac{T''}{T} = -\kappa^2,$$

or,

$$X'' + \kappa^2 X = 0,$$

$$T'' + \kappa^2 T = 0.$$

(7a)

(7b)

The separation constant $\kappa$ remains to be determined. Solving (7a,b) gives

$$X = A \cos \kappa x + B \sin \kappa x,$$

(8a)

$$T = C e^{-\kappa^2 x}.$$

(8b)

However, observe that (8a) is the general solution of (7a) only in the event that $\kappa \neq 0$, for if $\kappa = 0$ then the $\sin \kappa x$ term drops out. Since we do not yet know the value(s) of $\kappa$, we must allow for the possibility that the value $\kappa = 0$ will be needed. Setting $\kappa = 0$ in (7a) gives $X'' = 0$, with the general solution $X = D + Ex$, so we replace (8a) by the two-tier statement

$$X = \begin{cases} 
A \cos \kappa x + B \sin \kappa x, & \kappa \neq 0 \\
D + Ex, & \kappa = 0.
\end{cases}$$

(9)

Apparently, we don’t need to revise (8b) the way we revised (8a) because (8b) is a general solution of (7b) whether $\kappa$ is zero or not. However, having already committed ourselves in (9) to the separate treatment of these two cases, we replace (8b) by the two-tier statement

$$T = \begin{cases} 
F e^{-\kappa^2 x}, & \kappa \neq 0 \\
G, & \kappa = 0.
\end{cases}$$

(10)

Thus far we have discussed the product solutions

$$u = XT = (D + Ex)G$$

(11)

An alternative approach that you might prefer is to take $\partial / \partial x$ of (5), which step gives $F''(x) = 0$, so $F(x) = \text{constant}$. From (5), $G(t)$ must also be constant, the same constant. [Or, $\partial / \partial t$ of (5) gives $G'(t) = 0$, so $G(t) = \text{constant}$, and so on.] However, this approach is a bit weaker because it requires an assumption that $X''/X$ is differentiable, an assumption that is not needed.

$F \exp (-\kappa^2 x)$ can be expressed as $F \exp (-t/\tau)$, where $\tau = \frac{1}{(\kappa^2 \alpha^2)}$ is a time constant, namely, the time it takes the exponential to decay by 63% (from $1$ to $e^{-1}$).
corresponding to $\kappa = 0$, and
\[ u = XT = (A \cos \kappa z + B \sin \kappa z) e^{-\kappa^2 \alpha^2 t} \] (12)
for any $\kappa \neq 0$. Since $D, E, G, A, B, F$ are arbitrary constants, we can combine $DG$ as $H$ and $EG$ as $I$ and simplify (11) as
\[ u = H + Ix \]
for $\kappa = 0$, and we can combine $AF$ as $J$ and $BF$ as $K$ and simplify (12) as
\[ u = (J \cos \kappa z + K \sin \kappa z) e^{-\kappa^2 \alpha^2 t} \]
for $\kappa \neq 0$. Since (1a) is linear, the sum of these solutions must also be a solution, so we can write
\[ u = H + Ix + (J \cos \kappa z + K \sin \kappa z) e^{-\kappa^2 \alpha^2 t} \] (13)
where the constants $H, I, J, K$, and $\kappa$ are arbitrary. But it is understood that $\kappa \neq 0$ in (13) because the $H + Ix$ part of (13) already accounts for the $\kappa = 0$ case. By the linearity of (1a) we can superimpose any number of such terms for different values of $\kappa$. With $\kappa = 1, 2,$ and $\sqrt{5}$, for instance, we can write
\[ u(x, t) = (H + Ix) + (J_1 \cos x + K_1 \sin x) e^{-a^2 t} + (J_2 \cos 2x + K_2 \sin 2x) e^{-4a^2 t} + (J_3 \cos \sqrt{5}x + K_3 \sin \sqrt{5}x) e^{-5a^2 t}. \] (14)
The latter expression satisfies (1a) because each term does, and because (1a) is linear. [We urge you to verify that (14) satisfies (1a), by direct substitution.] But since we do not yet know what $\kappa$ values to choose, let us continue with the more compact form (13).

We are ready to apply the boundary conditions (1b) and the initial condition (1c). Beginning with the left end condition, (13) gives
\[ u(0, t) = u_1 = H + Je^{-\kappa^2 \alpha^2 t} \quad (0 < t < \infty) \]
or, in a more suggestive form,
\[ (H - u_1)(1) + J(e^{-\kappa^2 \alpha^2 t}) = 0. \quad (0 < t < \infty) \] (15)
Since the functions 1 and $\exp(-\kappa^2 \alpha^2 t)$ are linearly independent on the $t$ interval, it follows from (15) that we need $H - u_1 = 0$ (so $H = u_1$) and $J = 0$. Updating (13) to incorporate these results we have, thus far,
\[ u(x, t) = u_1 + Ix + K \sin \kappa x e^{-\kappa^2 \alpha^2 t}. \] (16)
Applying the right end condition next, (16) gives
\[ u(L, t) = u_2 = u_1 + IL + K \sin \kappa L e^{-\kappa^2 \alpha^2 t} \]
or
\[ (IL + u_1 - u_2)(1) + K \sin \kappa L e^{-\kappa^2 \alpha^2 t} = 0. \] (17)
*Recall that two functions are linearly dependent if and only if one is a scalar multiple of the other, and neither 1 nor $\exp(-\kappa^2 \alpha^2 t)$ is a scalar multiple of the other.
Again invoking the linear independence of 1 and \( \exp(-\kappa^2 \alpha^2 t) \), it follows from (17) that

\[
IL + u_1 - u_2 = 0
\]  

(18a)

and

\[
K \sin \kappa L = 0.
\]  

(18b)

Equation (18a) gives \( I = (u_2 - u_1)/L \), but (18b) presents a choice: either \( K = 0 \) or \( \sin \kappa L = 0 \) (or both). Here, and at analogous points in examples to follow, the rule is to make that choice so as to maintain as robust a solution as possible [because we still have the initial condition \( u(x,0) = f(x) \) to satisfy, and we will need all the help we can get]. If we choose \( K = 0 \), then we lose the \( \sin \kappa x \exp(-\kappa^2 \alpha^2 t) \) term in (16) and are left with

\[
u(x,t) = u_1 + (u_2 - u_1) \frac{x}{L},
\]  

(19)

which is capable of satisfying the initial condition only in the unlikely event that \( f(x) \) happens to be \( u_1 + (u_2 - u_1)(x/L) \). However, if we choose

\[
\sin \kappa L = 0,
\]  

(20)

then \( K \) need not be zero, we retain the \( \sin \kappa x \exp(-\kappa^2 \alpha^2 t) \) term in (16), and (20) serves to identify the allowable values of \( \kappa \), namely,

\[
\kappa = \frac{n \pi}{L}
\]  

(21)

for \( n = 0, \pm 1, \pm 2, \ldots \). Of these values we discard \( n = 0 \) because it gives \( \kappa = 0 \), whereas it was understood that the \( \kappa \)'s in (16) were to be nonzero. Further, \( n = -1, -2, \ldots \) can be discarded since the \( \sin(n \pi x/L) \exp[-(n \pi \alpha/L)^2 t] \) combination in (16) is insensitive to the sign of \( n \), to within a factor of \( \pm 1 \), which factor can be absorbed by \( K \) anyhow.†

Using superposition as in (14) but for the values \( \kappa = n \pi/L \) \( (n = 1, 2, \ldots) \), (16) gives

\[
u(x,t) = u_1 + (u_2 - u_1) \frac{x}{L} + \sum_{n=1}^{\infty} K_n \sin \frac{n \pi x}{L} e^{-(n \pi \alpha/L)^2 t},
\]  

(22)

Before completing the solution let us review where we stand. The right-hand side of (22) satisfies the boundary conditions (1b) because \( \sin(n \pi x/L) = 0 \) at \( x = 0 \) and at \( x = L \), for each \( n = 1, 2, \ldots \). Further, it appears to satisfy the PDE \( L[u] = 0 \) because it is a linear combination of product solutions

\[
u_1 + (u_2 - u_1) \frac{x}{L}, \quad \sin \frac{n \pi x}{L} e^{-(n \pi \alpha/L)^2 t}, \quad \sin \frac{2n \pi x}{L} e^{-(n \pi \alpha/L)^2 t}, \quad \ldots,
\]  

each of which satisfies \( L[u] = 0 \). That is, if \( L[\phi_1(x,t)] = 0, \ldots, L[\phi_k(x,t)] = 0 \), then \( L[\alpha_1 \phi_1(x,t) + \cdots + \alpha_k \phi_k(x,t)] = 0 \) too because

\[
L[\alpha_1 \phi_1 + \cdots + \alpha_k \phi_k] = \alpha_1 L[\phi_1] + \cdots + \alpha_k L[\phi_k]
\]  

(23)

†Surely, \( \sin z = 0 \) has the roots \( z = n \pi \) on the real axis. In fact, even if we broaden the search and look in the complex \( z \) plane, we find only the roots \( n \pi \) on the real axis.

‡Put differently, observe that only \( \kappa^2 \) appears in (7a) and (7b). Since positive and negative values of \( \kappa \) are therefore indistinguishable in the ODE's, their general solutions for a given positive \( \kappa \), say \( \kappa = \kappa_0 \), must be the same as their general solutions for \( \kappa = -\kappa_0 \).
by virtue of the linearity of $L$. However, $k$ in (23) is finite, whereas $k$ in (22) is infinite, so the step

$$
L \left[ \sum_{n=1}^{\infty} \alpha_n \phi_n \right] = \sum_{n=1}^{\infty} \alpha_n L[\phi_n]
$$

(24)

amounts to an interchange in the order of two limit operations, the derivative in $L$ and the infinite series, and that step needs to be rigorously justified. This point of rigor comes up in Chapter 17 when we use Fourier series solution forms to satisfy ODE’s (e.g., Comment 2 of Example 3 in Section 17.3), and is addressed in the optional Section 17.5. We address it here too, in the optional subsection 18.3.2. Generally, however, in Chapters 18–20, we omit rigorous justification and are satisfied with formal solutions.

Finally, we impose the initial condition (1c) on (22):

$$
u(x, 0) = f(x) = u_1 + (u_2 - u_1) \frac{x}{L} + \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{L}, \quad (0 < x < L)
$$

(25)

Two questions arise: given $f(x)$, is it possible to find coefficients $K_n$ so as to satisfy (25) on $0 < x < L$ and, if it is possible, then how do we determine the $K_n$’s? First, put all the known terms on the left-hand side:

$$f(x) - u_1 - (u_2 - u_1) \frac{x}{L} = \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{L}, \quad (0 < x < L)
$$

(26)

and let us denote $f(x) - u_1 - (u_2 - u_1)(x/L)$ as $F(x)$, for brevity. Observing that

$$F(x) = \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{L}, \quad (0 < x < L)
$$

(27)

is of the form of a half-range sine expansion of $F$, we can conclude that if $F$ is sufficiently well-behaved, then (27) is indeed possible, and (according to (4) in Section 17.4) the $K_n$’s are computed as

$$K_n = \frac{2}{L} \int_0^L F(x) \sin \frac{n\pi x}{L} \, dx.
$$

(28)

Thus, our formal solution consists of (22), with the $K_n$’s computed from (28).

To illustrate, consider a 10 cm-long copper rod held in boiling water until its temperature is $100^\circ$C throughout. At time $t = 0$ it is removed and its ends are quenched with ice for all $t > 0$. (We can either neglect heat loss from its lateral surface or specify that that surface be insulated.) Then $\alpha^2 = 1.14 \text{ cm}^2/\text{sec}$ (for copper), $L = 10 \text{ cm}$, $u_1 = u_2 = 0$, and $F(x) = 100^\circ$C, so (28) gives

$$K_n = \frac{2}{10} \int_0^{10} 100 \sin \frac{n\pi x}{10} \, dx = \begin{cases} 400 \frac{n}{\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}
$$

(29)

*Let $F_{\text{ext}}(x)$ be the extended version of $F(x)$ that is $2L$-periodic and symmetric about $x = 0$ and $x = L$. If $F_{\text{ext}}(x)$ and $F'_{\text{ext}}(x)$ are piecewise continuous on $[-L, L]$, then (27) converges in the sense of Theorem 17.3.1.
18.3. Separation of Variables

and (22) becomes

\[ u(x, t) = \frac{400}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{10} e^{-0.125n^2 t}. \]  

(30)

We have summed the series in (30) at a number of \( x \)'s and at several representative times and have plotted the results in Fig. 2.

Since this is our first example there are numerous points to clarify, and we address them in the following comments.

**Figure 2.** \( u(x, t) \) for the case where \( \alpha^2 = 1.14, L = 10, u_1 = u_2 = 0, \) and \( f(x) = 100. \)

**COMMENT 1.** Notice the physical significance of the terms in (22). As \( t \to \infty \) the exponential terms tend to zero, leaving the **steady-state** solution

\[ \lim_{t \to \infty} u(x, t) = u_*(x) = u_1 + (u_2 - u_1) \frac{x}{L}, \]  

which is a linear variation from \( u_1 \) at \( x = 0 \) to \( u_2 \) at \( x = L \) (Fig. 3). Thus, the summation term in (22) represents the **transient** part of the solution, that links the initial distribution \( u(x, 0) = f(x) \) to the steady-state distribution \( u_*(x) \). For the specific case represented in Fig. 2, the steady-state solution is simply \( u_*(x) = 0 \) because \( u_1 = u_2 = 0. \)

**COMMENT 2.** Notice further that within the transient part of (22) each term dies out exponentially faster (as \( t \to \infty \)) than the preceding term because of the \( n^2 \) in the exponent. To illustrate the significance of that point let us write out (30) at the representative times \( t = 0.2, 1, \) and 12 seconds:

\[
\begin{align*}
  &u(x, 0.2) = 124.5 \sin \frac{\pi x}{10} + 14.5 \sin \frac{3\pi x}{10} + \cdots, \\
  &u(x, 1) = 113.8 \sin \frac{\pi x}{10} + 15.4 \sin \frac{3\pi x}{10} + \cdots, \\
  &u(x, 12) = 33.0 \sin \frac{\pi x}{10} + 0.0002 \sin \frac{3\pi x}{10} + 6 \times 10^{-14} \sin \frac{5\pi x}{10} + \cdots,
\end{align*}
\]

which results are among those shown in Fig. 2. Computationally speaking, then, (30) (more generally, (22)) is an excellent result for "large" times because if \( t \) is large enough then only one or two of the terms in the series are needed for engineering accuracy. Conversely, for "small" times a great many terms in the series may be needed. For instance, at \( t = 0 \) the
terms in (30) die out only like $1/n$ because the exponential factor is unity at $t = 0$.

COMMENT 3. Why did we write $-\kappa^2$ in (16), rather than $+\kappa^2$? That is, how did we anticipate that the separation constant would be negative? The rule of thumb is to choose either sign and then examine the resulting ODE's for clues. Specifically, the minus sign chosen in (6) causes plus signs in (7a) and (7b), both of which look good: the plus sign in (7a) looks good because it results in sine and cosine solutions, which will be needed for the eventual Fourier series expansion of $f(x) - u_0(x)$, and the plus sign in (7b) looks good because it results in exponential decay rather than physically unreasonable exponential growth. (See Exercise 2.)

COMMENT 4. It is essential to apply the boundary conditions before the initial condition. To see why, let us try imposing the initial condition first, instead. Then, (13) gives

$$u(x, 0) = f(x) = H + lx + J \cos \kappa x + K \sin \kappa x,$$

which cannot be satisfied unless $f(x)$ happens to be a linear combination of $1, x, \cos \kappa x$, and $\sin \kappa x$, for some $\kappa$. Applying the boundary conditions first enabled us to obtain the solution form (22), which form was powerful enough to handle any given initial condition $u(x, 0) = f(x)$. This sequencing—boundary conditions first and initial condition second—will be appropriate all through Chapters 18 and 19.

COMMENT 5. As emphasized in Section 18.2.1, we did not find a general solution of the diffusion equation and then apply the boundary and initial conditions. Rather, we used the method of separation of variables to develop a solution that was sufficiently robust to handle the boundary conditions and initial condition.

COMMENT 6. It is interesting that the assumed product form $u(x, t) = X(x)T(t)$ seems a bad choice because it maintains the same shape, $X(x)$, for all time and is merely scaled in magnitude by the time-varying factor $T(t)$. Rather, we would expect the shape of $u(x, t)$ to change with time—as in Fig. 2, for instance, where $u(x, t)$ is initially a constant but approaches a sinusoidal shape as heat diffuses out of the rod at both ends due to the end conditions $u(0, t) = u(L, t) = 0$. However, understand that the superposition of product solutions is not itself a product solution; that is, the final solution (22) is not a function of $x$ times a function of $t$.

COMMENT 7. Still retaining the boundary conditions $u(0, t) = u(L, t) = 0$, what would happen if we changed the initial condition from $u(x, 0) = 100$ to

$$u(x, 0) = \begin{cases} 100, & 0 < x < x_0 \text{ and } x_0 < x < L \\ 5 \times 10^9, & x = x_0 \end{cases}$$

(32)

that is, if we changed the initial temperature at a single point $x_0$ to 5,000,000°? Nothing; the solution would still be given by (30), because a change in the integrand of (28) at a single point, from one finite value to another, cannot change the value of the integral.

COMMENT 8. The solution of (1) is not necessarily an infinite series. For instance, suppose that $u_1 = u_2 = 0$ and

$$u(x, 0) = f(x) = 40 \sin \frac{\pi x}{L}.$$ (33)
Then, application of the initial condition (33) to (22) gives

\[ 40 \sin \frac{\pi x}{L} = \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{L} \]
\[ = K_1 \sin \frac{\pi x}{L} + K_2 \sin \frac{2\pi x}{L} + \cdots, \]

which is satisfied, as can be seen by inspection, by setting \( K_1 = 40 \) and all the other \( K_n \)'s equal to zero. Thus, we obtain the one-term solution

\[ u(x, t) = 40 \sin \frac{\pi x}{L} e^{-(\pi \alpha / L)^2 t}, \tag{34} \]

We can display (34) in two dimensions by plotting \( u \) versus \( x \) at representative times, as we have done in Fig. 4a [and as we did earlier in Fig. 2 for the case where \( f(x) \) was 100], or in three dimensions by plotting the \( u \) surface above the \( x, t \) plane, as we have done in Fig. 4b. Similarly, if

\[ u(x, 0) = f(x) = 30 \sin \frac{2\pi x}{L} - 25 \sin \frac{5\pi x}{L}, \tag{35} \]

then

\[ u(x, t) = 30 \sin \frac{2\pi x}{L} e^{-(2\pi \alpha / L)^2 t} - 25 \sin \frac{5\pi x}{L} e^{-(5\pi \alpha / L)^2 t}, \tag{36} \]

and so on. \[ \]

In dwelling on so many details, in Example 1, we have shown no mercy. However, Example 1 provides the basic model for the application of the method of separation of variables, as it is used here in Chapters 18–20, so careful study of that example is well worth the effort.

**EXAMPLE 2. Different Boundary Conditions.** This time suppose that the left end of the rod is insulated, and the right end is held at a constant temperature 100 for all \( t > 0 \), so the problem is as follows:

\[
\begin{align*}
L[u] &= \alpha^2 u_{xx} - u_t = 0, \quad (0 < x < L, \ 0 < t < \infty) \tag{37a} \\
\alpha_c(0, t) &= 0, \ \ u(L, t) = 100, \quad (0 < t < \infty) \\
u(x, 0) &= f(x), \quad (0 < x < L) \tag{37b}
\end{align*}
\]

and suppose \( f \) is the piecewise constant function

\[ f(x) = \begin{cases} 
60, & 0 < x < L/2 \\
0, & L/2 < x < L 
\end{cases} \tag{38} \]

shown in Fig. 5.

Up until (13) the story is unchanged, so let us begin with

\[ u(x, t) = H + lx + (J \cos \kappa x + K \sin \kappa x)e^{-\kappa^2 \alpha^2 t}. \tag{39} \]
Applying the left-hand boundary condition,

\[ u_x(x, t) = I + (-\kappa J \sin \kappa x + \kappa K \cos \kappa x)e^{-\kappa^2 x^2 t}, \]

so

\[ u_x(0, t) = 0 = I + \kappa K e^{-\kappa^2 x^2 t}, \]

which gives \( I = K = 0 \). Updating (39) accordingly,

\[ u(x, t) = H + J \cos \kappa x e^{-\kappa^2 x^2 t}. \]

Next, the right-hand boundary condition gives

\[ u(L, t) = 100 = H + J \cos \kappa L e^{-\kappa^2 x^2 t}, \]

so \( H = 100 \) and \( J \cos \kappa L = 0 \). We cannot afford to satisfy the latter by setting \( J = 0 \) because if \( J = 0 \) then we lose the \( \cos \kappa L e^{-\kappa^2 x^2 t} \) term in (41), and the latter reduces to \( u(x, t) = H = 100 \), which cannot satisfy the initial condition. Rather, we satisfy \( J \cos \kappa L = 0 \) by setting

\[ \cos \kappa L = 0 \]

and letting \( J \) remain arbitrary. Thus, \( \kappa L = \pi/2, 3\pi/2, 5\pi/2, \ldots \), so

\[ \kappa = \frac{n\pi}{2L}, \quad (n = 1, 3, \ldots) \]

and we have

\[ u(x, t) = 100 + \sum_{n=1,3,\ldots}^{\infty} J_n \cos \frac{n\pi x}{2L} e^{-\left(n\pi \alpha/2L\right)^2 t}. \]

Finally, the initial condition is

\[ u(x, 0) = f(x) = 100 + \sum_{n=1,3,\ldots}^{\infty} J_n \cos \frac{n\pi x}{2L} \]

or

\[ f(x) - 100 = \sum_{n=1,3,\ldots}^{\infty} J_n \cos \frac{n\pi x}{2L}. \quad (0 < x < L) \]

Comparing the form of the right-hand side of (46) with the half- and quarter-range cosine and sine expansion formulas, we see that (46) amounts to a quarter-range cosine expansion of \( f(x) - 100 \), so

\[
J_n = \frac{2}{L} \int_0^L [f(x) - 100] \cos \frac{n\pi x}{2L} \, dx
= \frac{2}{L} \left[ \int_0^{L/2} (-40) \cos \frac{n\pi x}{2L} \, dx + \int_{L/2}^L (-100) \cos \frac{n\pi x}{2L} \, dx \right]
= \frac{80}{n\pi} \left( 3 \sin \frac{n\pi}{4} - 5 \sin \frac{n\pi}{2} \right).
\]
The solution is given by (45) and (47).

To plot the results, let the material be glass (so \( \alpha^2 = 0.0034 \text{cm}^2/\text{sec} \)), and let \( L = 10 \text{ cm} \). In Fig. 6 we plot \( u \) versus \( x \) at representative times.

![Figure 6. Graph of (45) for \( \alpha^2 = 0.0034 \) and \( L = 10 \).](image)

**COMMENT 1.** Notice that each graph of \( u \) is flat at \( x = 0 \), in accord with the boundary condition \( u_x(0, t) = 0 \), and that the steady-state solution \( u_s(x) = 100 \) is approached as \( t \to \infty \). That the initial temperature \( u(x, 0) = f(x) \) also satisfies the boundary condition \( u_x(0, t) = 0 \) is only by coincidence and is not required since we require the boundary conditions to hold only for \( 0 < t < \infty \), not for \( 0 \leq t \leq \infty \).

**COMMENT 2.** You might be tempted to break (46) into two parts,

\[
-40 = \sum_{n=1,3,...}^{\infty} J_n \cos \frac{n\pi x}{2L} \quad \left( 0 < x < \frac{L}{2} \right) \quad (48a)
\]

and

\[
-100 = \sum_{n=1,3,...}^{\infty} J_n \cos \frac{n\pi x}{2L} \quad \left( \frac{L}{2} < x < L \right) \quad (48b)
\]

and to use the quarter-range cosine expansion formulas to solve for the \( J_n \)'s in each case. That procedure would be INCORRECT for two reasons. First, it will give \( J_n \)'s that have different values in the intervals \( 0 < x < L/2 \) and \( L/2 < x < L \), in which case they will be functions of \( x \), not constants — as they were supposed to be. Second, we cannot use the quarter-range cosine formulas to solve (48a) and (48b) for the \( J_n \)'s because \( \sum_{n=1,3,...}^{\infty} J_n \cos \frac{n\pi x}{2L} \) is a quarter-range cosine expansion on \( 0 < x < L \), not on \( 0 < x < L/2 \) or on \( L/2 < x < L \).

In Example 1, \( u(0, t) \) and \( u(L, t) \) are prescribed constants and we end up expanding \( f(x) - u_s(x) \) in a half-range sine series. In Example 2, \( u_x(0, t) \) and \( u_x(L, t) \) are prescribed constants and we use a quarter-range cosine series. You will find that if instead \( u(0, t) \) and \( u_x(L, t) \) are prescribed constants, then a quarter-range sine series will be appropriate. If \( u_x(0, t) \) and \( u_x(L, t) \) are prescribed constants that are equal, then a half-range cosine series will be appropriate, but if they are unequal then the solution is more difficult (Exercise 19).
18.3.2. Verification of solution. (Optional) As stated, we can claim only to have found formal solutions of the diffusion problems in Examples 1 and 2. To illustrate the verification process let us verify that the formal solution (22) of the problem (1), with the \( K_n \)’s given by (28), does satisfy the requirements in (1).

First let us show that (22) satisfies the PDE \( \alpha^2 u_{xx} = u_t \). If we differentiate (22) termwise we obtain

\[
\begin{align*}
  u_x &= \frac{u_0 - u_1}{L} + \sum_{n=1}^{\infty} \frac{n\pi}{L} K_n \cos \frac{n\pi x}{L} e^{-(n\pi\alpha/L)^2 t}, \\
  u_{xx} &= -\sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 K_n \sin \frac{n\pi x}{L} e^{-(n\pi\alpha/L)^2 t}, \\
  u_t &= -\sum_{n=1}^{\infty} \left( \frac{n\pi\alpha}{L} \right)^2 K_n \sin \frac{n\pi x}{L} e^{-(n\pi\alpha/L)^2 t},
\end{align*}
\]

(49a, b, c)

so we see that \( \alpha^2 u_{xx} \) does equal \( u_t \), provided that we can rigorously justify the termwise differentiations that produced (49a, b, c). Theorem 17.5.2 tells us that those steps are permissible if the resulting series [on the right-hand sides of (49a, b, c)] converge uniformly on the problem domain \( 0 < x < L, 0 < t < \infty \), and Theorem 17.5.1 gives us the Weierstrass \( M \)-test as a test for uniform convergence. To apply the latter to the series in (49a), note that

\[
\left| \frac{n\pi}{L} K_n \cos \frac{n\pi x}{L} e^{-(n\pi\alpha/L)^2 t} \right| \leq Q e^{-(n\pi\alpha/L)^2 t_0} \equiv M_n
\]

(50)

for all \( t > t_0 \) and \( 0 < x < L \) because \( \left| \cos \left( n\pi x/L \right) \right| \leq 1 \) and \( K_n \to 0 \) as \( n \to \infty \), so that the \( K_n \)'s must be bounded; thus there must exist a finite positive constant \( Q \) such that \( \pi K_n/L \leq Q \) for all \( n \)'s. It follows easily from the ratio test that \( \sum_{n=1}^{\infty} M_n \) converges, so the series in (49a) does indeed converge uniformly in \( 0 < x < L, t_0 \leq t < \infty \), for arbitrarily small \( t_0 \). Similarly for the series in (49b) and (49c), the only difference being that those series contain a factor of \( n^2 \) rather than \( n \); but the ratio test shows that \( \sum_{n=1}^{\infty} n^2 \exp \left\{ -(n\pi\alpha/L)^2 t_0 \right\} \) converges, just as \( \sum_{n=1}^{\infty} n \exp \left\{ -(n\pi\alpha/L)^2 t_0 \right\} \)

Turning to the boundary conditions (1b) and the initial condition (1c), it might appear that the satisfaction of these conditions does not even need verification. For instance, is not

\[
u(0, t) = u_0 + 0 + \sum_{n=1}^{\infty} 0 = u_1
\]

obviously true? The point to emphasize is that in posing PDE problems we require satisfaction of the PDE in the open region, in this case \( 0 < x < L, 0 < t < \infty \). We do that so that we can allow for not-so-well-behaved boundary and initial data. For if \( \alpha^2 u_{xx} = u_t \) were to be satisfied in the region \( 0 \leq x \leq L, 0 \leq t < \infty \), then we would need \( u(x, 0) = f(x) \) to be twice differentiable, and \( u(0, t) = g(t) \)

*See the last sentence of Exercise 1, Section 17.6.
and \( u(L, t) = h(t) \), say, to be once differentiable, whereas we want to allow for boundary and initial data that are not even continuous, let alone differentiable. In Example 2, for instance, \( f(x) \) is discontinuous. The way we link the solution \( u(x, t) \) in the open domain to the boundary and initial conditions is through limits. That is, by (1b) we mean

\[
\lim_{x \to 0^+} u(x, t) = u_1 \quad \text{and} \quad \lim_{x \to L^-} u(x, t) = u_2, \quad (0 < t < \infty)
\]

(51)

and by (1c) we mean

\[
\lim_{t \to 0^+} u(x, t) = f(x), \quad (0 < x < L)
\]

(52)

To verify that (22) satisfies (51) we can use the following theorem.

---

**THEOREM 18.3.1 Continuity of Sum Function**

If \( \sum_{n=1}^{\infty} a_n(x) \) converges uniformly to \( s(x) \) on some \( x \) interval \( I \) and the \( a_n(x) \)'s are continuous on \( I \), then \( s(x) \) is continuous on \( I \).

---

That is, if the convergence is uniform, then the continuity of the partial sums is passed on to the sum function \( s(x) \).

Applying that result, observe first that the series in (22) converges uniformly on \( 0 \leq x \leq L \) for each \( t \) such that \( 0 < t_0 \leq t < \infty \) because there is a finite constant \( P \) such that

\[
\left| K_n \sin \frac{n\pi x}{L} e^{-(n\pi \alpha/L)^2 t} \right| \leq Pe^{-(n\pi \alpha/L)^2 t_0} \equiv M_n
\]

there, and \( \sum_{n=1}^{\infty} M_n \) is convergent. Thus the right-hand side of (22) is a continuous function of \( x \) on \( 0 \leq x \leq L \), so its values at \( x = 0 \) and at \( x = L \), namely \( u_1 \) and \( u_2 \), respectively, are the same as their limiting values as \( x \to 0^+ \) and as \( x \to L^- \), respectively.

Verification of the initial condition (52) can be accomplished with the help of a theorem of Abel. For that, and a generally more detailed discussion, we refer you to Churchill and Brown.

Finally, there is the question of uniqueness: is (22) the only solution of (1)? A formal proof of uniqueness is outlined in Exercise 25; for detailed discussion we refer you, again, to Churchill and Brown.

---

**18.3.3. Use of Sturm–Liouville theory. (Optional)** In subsection 18.3.1 we found that the final step of the separation-of-variables solution involves the expansion of a given function. For the diffusion equation \( \alpha^2 u_{xx} = u_t \) with Dirichlet or Neumann boundary conditions that expansion was a half- or quarter-range cosine or sinusoidal function.
sine series, so we were guided by our knowledge of such series. More generally, the necessary expansion is an eigenfunction expansion in terms of the orthogonal eigenfunctions of a Sturm–Liouville problem that is "built in." Thus, a more powerful approach is to appeal to the Sturm–Liouville theory, which includes the half- and quarter-range expansions as special cases. Let us illustrate with three examples.

**EXAMPLE 3. Example 1 Reconsidered.** The Sturm–Liouville theory becomes relevant only when we reach the point of needing to carry out the expansion of a given function. Thus, in reconsidering Example 1 in the light of the Sturm–Liouville theory we can begin with equation (26). Our claim is that the \( \sin \left( \frac{n \pi x}{L} \right) \) functions in (26) are the orthogonal eigenfunctions of a Sturm–Liouville problem governing \( X(x) \), namely, the equation (7a) together with the homogeneous boundary conditions \( X(0) = 0 \) and \( X(L) = 0 \):

\[
\begin{align*}
X'' + \kappa^2 X &= 0, \quad (0 < x < L) \\
X(0) &= 0, \quad X(L) = 0,
\end{align*}
\]

which problem is indeed of the Sturm–Liouville form

\[
\begin{align*}
(p y')' + qy + \lambda wy &= 0, \quad (a < x < b) \\
oy(a) + \beta y'(a) &= 0, \quad \gamma y(b) + \delta y'(b) = 0,
\end{align*}
\]

where \( y(x) = X(x) \), \( p(x) = w(x) = 1 \), \( q(x) = 0 \), \( \lambda = \kappa^2 \), \( a = 0 \), \( b = L \), \( \alpha = \gamma = 1 \), and \( \beta = \delta = 0 \). Considering the boundary conditions \( u(0, t) = u_1 \) and \( u(L, t) = u_2 \), where did we get \( X(0) = 0 \) and \( X(L) = 0 \), in (53b)? Recall that the \( u_1 + (u_2 - u_1)x/L \) terms in (22) make up the steady-state solution \( u_s(x) \). The burden of satisfying the nonhomogeneous boundary conditions, \( u(0, t) = u_1 \) and \( u(L, t) = u_2 \), is carried by the steady-state solution \( u_s(x) \), so the transient part, \( \sum_{n=1}^{\infty} K_n \sin \frac{n \pi x}{L} \exp \left[ -\frac{(n \pi \alpha/L)^2}{2}\right] \), must be zero at \( x = 0 \) and \( x = L \). Thus, the \( X(x) = \sin \left( \frac{n \pi x}{L} \right) \) eigenfunctions, contained therein, actually satisfy the homogeneous conditions (53b), as is easily verified by evaluating \( \sin \left( \frac{n \pi x}{L} \right) \) at \( x = 0 \) and \( x = L \).

According to the Sturm–Liouville theory, then, (27) can indeed be satisfied, and the expansion coefficients are given by

\[
K_n = \frac{\langle F(x), \sin \frac{n \pi x}{L} \rangle}{\langle \sin \frac{n \pi x}{L}, \sin \frac{n \pi x}{L} \rangle} = \frac{\int_0^L F(x) \sin \frac{n \pi x}{L} \, dx}{\int_0^L \sin^2 \frac{n \pi x}{L} \, dx} = \frac{2}{L} \int_0^L F(x) \sin \frac{n \pi x}{L} \, dx,
\]

which result is the same as (28). [Recall that the weight function, in the two integrals, in this case is \( w(x) = 1 \).]

Thus, the idea is that satisfaction of the initial condition \( u(x, 0) = f(x) \) requires the expansion of \( F(x) = f(x) - u_s(x) \). The latter will inevitably be an eigenfunction expansion in terms of the eigenfunctions \( \phi_n(x) \) [namely, \( \sin \left( \frac{n \pi x}{L} \right) \)
18.3. Separation of Variables 967

in Example 3} of a Sturm–Liouville problem on \(X(x)\). The Sturm–Liouville theory assures us that the desired expansion is possible, and it tells us how to compute the expansion coefficients — namely, as \(\langle F(x), \phi_n \rangle / \langle \phi_n, \phi_n \rangle\), where the inner product has weight function \(w(x)\).

In Example 3 we were able to use either the half-range sine expansion concept or the Sturm–Liouville theory. In the next example the expansion will not be of half- or quarter-range type, so we will have to use the Sturm–Liouville theory.

**EXAMPLE 4.** This time consider the problem

\[
L[u] = u_{xx} - u_t = 0, \quad (0 < x < 1, 0 < t < \infty) \tag{56a}
\]

\[
u(0, t) - 2u_x(0, t) = 5, \quad u(1, t) = 35, \quad (0 < t < \infty) \tag{56b}
\]

\[
u(x, 0) = f(x), \quad (0 < x < 1) \tag{56c}
\]

where we have set \(\alpha^2 = 1\) and \(L = 1\) for simplicity. Observe that \(u(0, t) - 2u_x(0, t) = 5\) is a Robin boundary condition, a boundary condition of the third kind.

Separating variables as usual, let us begin with equation (13):

\[
u(x, t) = H + Ix + (J \cos \kappa x + K \sin \kappa x)e^{-\kappa^2 t}. \tag{57}
\]

Applying the left end condition gives

\[
u(0, t) - 2u_x(0, t) = 5 = H + Je^{-\kappa^2 t} - 2(I + \kappa Ke^{-\kappa^2 t})
\]

\[
= (H - 2I) + (J - 2\kappa K)e^{-\kappa^2 t}, \quad (0 < t < \infty)
\]

so

\[
H - 2I = 5, \tag{58a}
\]

\[
J - 2\kappa K = 0. \tag{58b}
\]

And the right end condition gives

\[
u(1, t) = 35 = H + I + (J \cos \kappa + K \sin \kappa)e^{-\kappa^2 t}, \quad (0 < t < \infty)
\]

so

\[
H + I = 35, \tag{59a}
\]

\[
J \cos \kappa + K \sin \kappa = 0. \tag{59b}
\]

Equations (58a) and (59a) give \(H = 25\) and \(I = 10\). Equations (58b) and (59b) give the unique trivial solution \(J = K = 0\), unless we choose \(\kappa\) so that the determinant of the coefficient matrix vanishes:

\[
\begin{vmatrix}
1 & -2\kappa \\
\cos \kappa & \sin \kappa
\end{vmatrix} = 0. \tag{60}
\]

We cannot accept the trivial solution \(J = K = 0\) because it reduces (57) to \(\nu(x, t) = 25 + 10x\), which does satisfy the PDE and boundary conditions but which cannot satisfy
the initial condition \( u(x, 0) = f(x) \) unless \( f(x) \) happens to be \( 25 + 10x \). Thus, we impose the determinant condition (60), or

\[
\tan \kappa = -2\kappa, \tag{61}
\]

and designate the positive roots of (61) as \( \kappa_1, \kappa_2, \ldots \). Next, we need to find the corresponding nontrivial solutions of (58b) and (59b) for \( J \) and \( K \). With (61) satisfied, (58b) and (59b) are redundant, so we can drop (59b) and use (58b) to obtain \( J = 2\kappa K \).

With these results, and superposition, (57) gives

\[
u(x, t) = 25 + 10x + \sum_{n=1}^{\infty} K_n \phi_n(x) e^{-\kappa_n^2 t}, \tag{62}
\]

where

\[
\phi_n(x) = 2\kappa_n \cos \kappa_n x + \sin \kappa_n x. \tag{63}
\]

Finally, the initial condition requires that

\[
u(x, 0) = f(x) = 25 + 10x + \sum_{n=1}^{\infty} K_n \phi_n(x),
\]

or,

\[
F(x) = \sum_{n=1}^{\infty} K_n \phi_n(x), \quad (0 < x < 1) \tag{64}
\]

where \( F(x) = f(x) - (25 + 10x) \), \( 25 + 10x \) being the steady-state solution \( u_*(x) \). The \( \phi_n \)'s in (64) are the eigenfunctions generated by the Sturm–Liouville problem

\[
X'' + \kappa^2 X = 0, \quad (0 < x < 1) \tag{65a}
\]

\[
X(0) = 2X'(0) = 0, \quad X(1) = 0, \quad \text{with weight function } w(x) = 1,
\]

so

\[
K_n = \frac{\langle F, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_0^1 F(x)(2\kappa_n \cos \kappa_n x + \sin \kappa_n x) \, dx}{\int_0^1 (2\kappa_n \cos \kappa_n x + \sin \kappa_n x)^2 \, dx}. \tag{66}
\]

In fact, this Sturm–Liouville problem, including the determination of the \( \kappa_n \)'s, is the subject of Example 3 in Section 17.7. \( \square \)

**EXAMPLE 5. Unsteady Conduction in a Disk.** Consider the unsteady conduction of heat in a circular disk such as a coin of radius \( c \), the flat faces of which are insulated (Fig. 7). Then the temperature field \( u(r, t) \) in the disk is governed by the problem

\[
\alpha^2 \nabla^2 u = \alpha^2 \left( u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} \right) = u_t, \quad (0 \leq r < c, \ 0 < t < \infty)
\]

\[
u(r, t) = 100, \quad u(r, 0) = f(r).
\]
That is, the disk is initially at a prescribed temperature \( f(r) \), and then we hold the outer edge (with boiling water, for example) at \( u = 100^\circ \) for all \( t > 0 \). Because the domain is circular, and the initial and boundary temperature are independent of \( \theta \), it appears that the resulting temperature field \( u \) will be independent of \( \theta \) as well, and will be a function only of \( r \) and \( t \). Hence, we can strike out the \( u_{\theta \theta} \) term, and the PDE reduces to

\[
\alpha^2 (u_{rr} + \frac{1}{r} u_r) = u_t. \tag{68}
\]

To solve by separation of variables, seek \( u \) in the product form

\[
u(r, t) = R(r)T(t). \tag{69}
\]

Putting (69) into the PDE and dividing by \( RT \) (and \( \alpha^2 \)) gives

\[
\frac{R'' + \frac{1}{r} R'}{R} = \frac{1}{\alpha^2} \frac{T'}{T} = \text{constant} = -\kappa^2, \tag{70}
\]

and hence the ODE’s

\[
R'' + \frac{1}{r} R' + \kappa^2 R = 0, \tag{71a}
\]

\[
T' + \kappa^2 \alpha^2 T = 0. \tag{71b}
\]

Equation (71a) is the subject of Example 1 in Section 4.6, and its general solution is

\[
R(r) = A J_0(\kappa r) + B Y_0(\kappa r), \tag{72}
\]

where \( J_0, Y_0 \) are the Bessel functions of first and second kind, respectively, of order zero (Fig. 8). Are there any values of \( \kappa \) for which (71) fails to provide a general solution? Yes, for \( \kappa = 0 \) because \( Y_0(0) = -\infty \) does not exist. But for \( \kappa = 0 \) (71a) can be expressed as \( r R' = 0 \), which can be integrated to give \( R(r) = C + D \ln r \). Thus, we distinguish the cases \( \kappa \neq 0 \) and \( \kappa = 0 \), and write

\[
R(r) = \begin{cases} 
A J_0(\kappa r) + B Y_0(\kappa r), & \kappa \neq 0 \\
C + D \ln r, & \kappa = 0
\end{cases} \tag{73}
\]

\[
T(t) = \begin{cases} 
C e^{-\kappa^2 \alpha^2 t}, & \kappa \neq 0 \\
F, & \kappa = 0.
\end{cases} \tag{74}
\]

Thus far, we have

\[
u(r, t) = (C + D \ln r)F + [A J_0(\kappa r) + B Y_0(\kappa r)]e^{-\kappa^2 \alpha^2 t}
\]

\[
= G + H \ln r + [P J_0(\kappa r) + Q Y_0(\kappa r)]e^{-\kappa^2 \alpha^2 t}, \tag{75}
\]

where we have combined \( CF \) as \( G \), \( DF \) as \( H \), and so on.

In the preceding examples, \( X(x) \) is governed by a second-order ODE and there are two \( x \) boundary conditions (at \( x = 0 \) and at \( x = L \)). In the present example, likewise, \( R(r) \) is governed by a second-order ODE, but we find only one \( r \) boundary condition in (67).
\[ u = 100 \text{ at } r = c. \] As a second boundary condition it seems appropriate to require that \( u \)
be bounded at \( r = 0 \).

As in preceding examples, we save the initial condition for last and begin by applying
the boundary conditions. Of the two boundary conditions (\( u \) bounded at \( r = 0 \) and \( u = 100 \)
at \( r = c \)), we recommend applying any boundedness condition first because it gives an
immediate simplification. Specifically, for \( u(0,t) \) to be bounded we need \( H = Q = 0 \) in
(74) because both \( \ln r \) and \( Y_0(\kappa r) \) are unbounded at \( r = 0 \). Then (74) simplifies to
\[ u(r,t) = G + PJ_0(\kappa r)e^{-\kappa^2 \alpha^2 t}. \] (76)

Next,
\[ u(c,t) = 100 = G + PJ_0(\kappa c)e^{-\kappa^2 \alpha^2 t}, \]
or,
\[ (G - 100)(1) + PJ_0(\kappa c)e^{-\kappa^2 \alpha^2 t} = 0, \quad (0 < t < \infty) \] (77)

Since \( 1 \) and \( \exp(-\kappa^2 \alpha^2 t) \) are linearly independent on the \( t \) interval, it follows from (77)
that \( G - 100 = 0 \) and \( PJ_0(\kappa c) = 0 \). The former gives \( G = 100 \) and the latter gives
\( P = 0 \) or \( J_0(\kappa c) = 0 \). We reject the choice \( P = 0 \) because we cannot afford to lose the
\( PJ_0(\kappa r) \exp(-\kappa^2 \alpha^2 t) \) term in (76), and adopt the choice
\[ J_0(\kappa c) = 0 \] (78)

with positive roots \( \kappa_n c = z_n \) for \( n = 1, 2, \ldots \), where the \( z_n \)'s are the (known) zeros of \( J_0 \)
(Fig. 9). With these choices, and the help of superposition, we have
\[ u(r,t) = 100 + \sum_{n=1}^{\infty} P_n J_0 \left( \frac{z_n r}{c} \right) e^{-\left(\frac{z_n}{c}\right)^2 t}, \] (79)

(The \( P_n \)'s are arbitrary constants, not Legendre polynomials.)

Finally, the initial condition requires that
\[ u(r,0) = f(r) = 100 + \sum_{n=1}^{\infty} P_n J_0 \left( \frac{z_n r}{c} \right), \]
or
\[ F(r) = f(r) - 100 = \sum_{n=1}^{\infty} P_n J_0 \left( \frac{z_n r}{c} \right), \quad (0 \leq r < c) \] (80)

The expansion functions \( J_0(z_n r/c) \) are the eigenfunctions of the singular Sturm–Liouville
problem
\[ \left( r R' \right)' + \kappa^2 r R = 0, \quad (0 < r < c) \]
\[ R(0) \text{ bounded, } R(c) = 0, \] (81)

which problem is the subject of Example 2 in Section 17.8. Observe that we multiplied
(71a) through by \( r \) in order to obtain the standard form given in (81). That step is important
because it is only when the equation is in the standard form that we can identify from it
the weight function — that will be needed in our inner product. From (81) we see that the weight function is \( w(r) = r \), so we can write

\[
P_n = \frac{(F(r), J_0(z_n r/c))}{\langle J_0(z_n r/c), J_0(z_n r/c) \rangle} = \frac{\int_0^c F(r) J_0 \left( z_n \frac{r}{c} \right) r \, dr}{\int_0^c \left[ J_0 \left( z_n \frac{r}{c} \right) \right]^2 r \, dr} = \frac{2}{c^2 \lVert J_1(z_n) \rVert^2} \int_0^c F(r) J_0 \left( z_n \frac{r}{c} \right) r \, dr.
\]

(82)

For explanation of the last step, see Example 2 of Section 17.8.

COMMENT 1. As a concrete example, let \( f(r) = 0 \). Then (Exercise 31)

\[
P_n = -\frac{200}{c^2 \lVert J_1(z_n) \rVert^2} \int_0^c J_0 \left( z_n \frac{r}{c} \right) r \, dr = -\frac{200}{z_n J_1(z_n)},
\]

so

\[
a(r, t) = 100 - 200 \sum_{n=1}^{\infty} \frac{1}{z_n J_1(z_n)} J_0 \left( z_n \frac{r}{c} \right) e^{-\left( z_n \alpha/c \right)^2 t}.
\]

(84)

For instance, the temperature history at the center of the disk is

\[
u(0, t) = 100 - 200 \sum_{n=1}^{\infty} \frac{1}{z_n J_1(z_n)} e^{-\left( z_n \alpha/c \right)^2 t},
\]

(85)

which is plotted in Fig. 10 for the case where the material is glass \( \alpha^2 = 0.0034 \text{cm}^2/\text{sec} \) and \( c = 10 \text{ cm} \).

COMMENT 2. Observe that the point \( r = 0 \) is the left endpoint of the \( r \) interval \( 0 \leq r < c \), but, physically, it corresponds to an interior point of the disk, the center of the disk. Thus, the PDE must be satisfied there. In particular, if \( u_r(0, t) \) is to exist, then it must be zero. That is, the \( u \) surface (i.e., the graph of \( u \) above the \( r, \theta \) plane) must be flat at \( r = 0 \) because otherwise the \( u \) surface will be conical there and \( u_r(0, t) \) will not exist. Thus, in place of the condition that \( R(0) \) be bounded we could have used the condition

\[
R'(0) = 0,
\]

(86)

and the latter would have produced the same results.

COMMENT 3. Recall that we argued that \( u \) does not vary with \( \theta \) in this example. Hence, we dropped the \( u_{\theta\theta} \) term in the PDE and solved the reduced equation \( \alpha^2 (u_{rr} + \frac{1}{r} u_r) = u_t \).

If you still have doubts about that step, observe that (79) does satisfy the boundary conditions, the initial condition [if the \( P_n \)'s are computed using (81)], and the full equation \( \alpha^2 \nabla^2 u = u_t \) including the \( u_{\theta\theta} \) term because \( \partial^2 / \partial \theta^2 \) of the right-hand side of (79) is zero.

Closure. This section covers almost all aspects of the method of separation of variables, which method is used in each of Chapters 18—20. The starting point is to assume a procedure form for the solution, for instance \( u(x, t) = X(x)T(t) \) if the
independent variables are \( x \) and \( t \). For the diffusion equation, that step enable us to separate the variables and obtain ODE’s on \( X(x) \) and \( T(t) \). That simplification, from a PDE to two ODE’s, is the point of the method.

The essential ingredients, for the success of the method in a given application, are that the equation needs to be separable, the boundary and initial conditions need to be given on constant coordinate curves, and the PDE needs to be linear—so that a sufficiently robust solution can be built up by the superposition of various product solutions. Only two PDE’s are studied in the foregoing examples, \( \alpha^2 u_{xx} = u_t \) and \( \alpha^2 (u_{rr} + \frac{1}{r} u_r) = u_t \), and these three conditions are met: the equations are separable, conditions are on constant coordinate curves \( (x = 0, x = L, t = 0, r = c, \text{and } r = 0) \), and the equations are linear.

In contrast, an example of a PDE that is not separable is the two-dimensional biharmonic equation in Cartesian coordinates,

\[
\nabla^4 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0
\]

because \( u(x, y) = X(x)Y(y) \) gives

\[
\frac{X''''}{X} + 2 \frac{X''}{X} \frac{Y''}{Y} + \frac{Y''''}{Y} = 0,
\]

which cannot be rearranged in the separated form \( F(x) = G(y) \) because of the \( (X''/X)(Y''/Y) \) term.

And as an example where the boundary conditions are not given on constant coordinate curves consider the problem shown in Fig. 11. There, the temperature \( u_1 \) is so high that the left end of the rod melts and drips away, so that the boundary condition \( u = u_1 \) is applied not on the line \( x = 0 \) but on some curve \( x = a(t) \). The latter is an example of a moving boundary problem. Generally, such problems defy analytical solution, and we resort to numerical solution techniques—such as the finite difference method that is presented in Section 18.6. An interesting application of the problem shown in Fig. 11 is in the design of a space vehicle that reenters the earth’s atmosphere at hypersonic velocity. To protect the craft from the heat thereby generated (remember that meteorites often burn up before reaching the earth’s surface) one can design the nose cone to be long enough so that part of it melts away during reentry. A simple one-dimensional model of the heat conduction in the nose cone would be somewhat like the problem shown in Fig. 11.

In this chapter and the next, the sign of the separation constant is always negative, but in Chapter 20, on the Laplace equation, it is negative or positive, depending on the specific application.

In subsection 18.3.2 we discuss the rigorous verification of a solution. Normally, our solutions will be only formal, in the sense that such verification will not be carried out.

Finally, in subsection 18.3.3 we show how to use the Sturm–Liouville theory to handle the expansion process that is needed to satisfy the initial condition. Using that theory we are not limited to half- and quarter-range cosine and sine expansions, as we were in subsection 18.3.1. In fact, the half- and quarter-range cases are but special cases of the Sturm–Liouville theory. Remember: Liouville, not Louisville.
Computer software. We can use the Maple `sum` command to obtain the infinite series solutions that we generate by separation of variables. As a first illustration, consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

which has the sum \(\ln 2\) because it is the Taylor series of \(\ln (1 + x)\) about \(x = 1\). To sum the series, enter

```
sum ((-1)^n + 1/n, n = 1..infinity);
```

and return. The result is \(\ln 2\). Remarkably, the software not only gives the numerical value, it gives that value in the closed form \(\ln 2\) rather than the open form \(0.693147\ldots\).

Next, consider the solution (30) given in Example 1, namely,

$$u(x, t) = \frac{400}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{10} e^{-0.1125\alpha^2 t}.$$

To illustrate, let us use the sum command to sum this series at \(x = 1\) and \(t = 0.2\). First, change the dummy summation index from \(n\) to \(2n - 1\) so that the new index \(n\) runs continuously from 1 to infinity:

$$u(x, t) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n - 1} \sin \frac{(2n - 1)\pi x}{10} e^{-0.1125(2n-1)^2 t}.$$

If, to compute \(u(1, 0.2)\), we enter

```
sum (400/(Pi*(2*n - 1)))*sin((2*n - 1)*Pi/10)
*exp(-0.1125*(2*n - 1)^2*0.2), n = 1..infinity);
```

the computer is unable to obtain the result in closed form and merely prints the series itself, in a form similar to the right-hand side of (92). Thus, we rerun the command (93) with “infinity” changed to a finite number such as 20. The result, 86.13830281, does not change if we increase the upper summation limit from 20 to 30, say. Thus, it is reasonable to assume that the answer \(u(1, 0.2) = 86.13830281\) is correct to that many significant figures. Of course, in a practical sense it would be foolish to insist on 10 significant figure accuracy in a problem like this if \(\alpha^2 = 0.0034\) and the other data are known only to two or three significant figures.

If we run the sum command from \(n = 1\) to \(n = 20\), say, and the result is merely a printout of the 20 terms rather than their numerical sum, we can use the `evalf()` command to evaluate their sum.
1. Verify by direct substitution that (14) does indeed satisfy the diffusion equation (1a).

2. (On the sign of separation constant) Writing the separation constant as $-\kappa^2$ in (6), rather than as $\kappa^2$, worked out well, and we also give arguments in Comment 3 as to why the separation constant should be negative. In this exercise we explore that point further.

(a) Show that if we use $\kappa^2$ in (6), instead of $-\kappa^2$, then we eventually arrive at the same solution as before, given by (22), although it will be a bit more challenging because complex quantities will arise. HINT: Recall the identity $\sin i\pi x = i \sin x$ if $x$ is a real number. The only roots of the equation $\sin z = 0$ in the complex plane, are the points $z = n\pi$ on the real axis.

(b) Show that if we use $-\kappa^2$ in (6), as we did, then the relevant Sturm–Liouville problem is given by (53). Show that it follows from Theorem 17.7.2 that $\kappa^2$ must be nonnegative.

3. In Comment 6 at the end of Example 1 we note that although we began by seeking $u(\xi, t)$ in product form, the final solution (22) is not itself of that form. Give conditions on $u_1, u_2$ and $f(\xi)$ in (1) such that the final solution will be of product form (so that its graph does not change its shape, with time, though its magnitude may change with time).

4. (Separation) Seeking a solution $u(\xi, t) = X(\xi)T(t)$ for the given PDE, carry out steps analogous to equations (3)–(6), and derive ODE's analogous to (7a,b). Take the separation constant to be $-\kappa^2$, as we do in (6). Obtain general solutions of those ODE's (distinguishing any special $\kappa$ values, as necessary) and use superposition to obtain a solution analogous to the solution (13) of (1a). If the PDE cannot be separated, state that.

(a) $u_{xx} = u_t + 3u$
(b) $u_{xx} + 2u_t = u_t$
\quad HINT: In this case you should find that the value of $\kappa$ that needs to be distinguished (as we distinguished the case $\kappa = 0$ in (9) and (10)) is $\kappa = 1$, not $\kappa = 0$.

(c) $u_{xx} + 4u_t = u_t$
(d) $u_{xx} + 2u_t = u_t$

5. Can we use superposition to conclude from (9) and (10) that

\[ X = A\cos \kappa \xi + B\sin \kappa \xi + D + E\xi, \quad T = F\e^{-\kappa^2 \alpha^2 t} + G, \]

and

\[ u(\xi, t) = (A\cos \kappa \xi + B\sin \kappa \xi + D + E\xi)(F\e^{-\kappa^2 \alpha^2 t} + G)? \]

6. (Continuation of Examples 1 and 2) In each case solve (1), with the boundary conditions (1b) changed as indicated, and for the specified $f(\xi)$. Use a half- or quarter-range cosine or sine expansion, as appropriate. Evaluate the expansion coefficients explicitly, rather than leaving them in integral form. Also, identify the steady-state solution $u_\infty(x)$.

(a) $u(0, t) = 20, \quad u_x(\pi, t) = 3, \quad (i.e., \quad L = \pi, \quad f(x) = 0$
(b) $u(0, t) = 10, \quad u_x(2, t) = -5, \quad f(x) = 10$
(c) $u(0, t) = 0, \quad u_x(2, t) = 0, \quad f(x) = 50\sin (\pi x)^2$
(d) $u(0, t) = 0, \quad u_x(2, t) = 0, \quad f(x) = 5\sin (\pi x)/4 - 12\sin (5\pi x)/4$
(e) $u(0, t) = 25, \quad u_x(4, t) = 0, \quad f(x) = 25$
(f) $u(0, t) = 25, \quad u_x(2, t) = 0, \quad f(x) = 0 \quad \text{for} \quad 0 < x < 1, \quad f(x) = 25 \quad \text{for} \quad 1 < x < 2$
(g) $u_x(0, t) = u_x(\pi, t), \quad f(x) = 300$
(h) $u_x(0, t) = u_x(3\pi, t), \quad f(x) = 0 \quad \text{for} \quad 0 < x < 2\pi, \quad f(x) = 600 \quad \text{for} \quad 2\pi < x < 3\pi$
(i) $u_x(0, t) = u_x(10, t) = 5, \quad f(x) = 45 + 5x$
(j) $u_x(0, t) = u_x(5, t) = 3, \quad f(x) = 2x$
(k) $u(0, t) = 0, \quad u(5, t) = 0, \quad f(x) = \sin \pi x - 3\sin (\pi x)/5 + 6\sin (9\pi x)/5$
(l) $u(0, t) = 0, \quad u(10, t) = 100, \quad f(x) = 0$
(m) $u_x(0, t) = 2, \quad u(6, t) = 12, \quad f(x) = 0$
(n) $u_x(0, t) = 0, \quad u(6, t) = 0, \quad f(x) = \sin x$

7. Use (45) and (47) to compute $u(0, t)$, and plot it versus $t$. At the least, take $t = 1, 000$, $2, 000$, $5, 000$, $10, 000$, $15, 000$, $20, 000$ and $30, 000$. Recall that $\alpha^2 = 0.00034$ and $L = 10.$

8. We stated in Comment 8 that it can be seen by inspection that $K_1 = 40$ and that all the other $K_n$'s are zero. Alternatively, obtain that same result by working out the integral

\[ K_n = \frac{2}{L} \int_0^L \left(40\sin \frac{n\pi x}{L}\right) \sin \frac{n\pi x}{L} \, dx. \]

9. The temperature distribution $u(x, t)$ in a 2-m long brass rod is governed by the problem

\[ \alpha^2 u_{xx} = u_t, \quad (0 < x < 2, \quad 0 < t < \infty) \]

\[ u(0, t) = u(2, t) = 0, \quad (t > 0) \]

\[ u(x, 0) = \begin{cases} 50x, & (0 < x < 1) \\ 100 - 5x, & (1 < x < 2) \end{cases} \]
where \( \alpha^2 = 2.9 \times 10^{-5} \text{ m}^2/\text{sec} \).

(a) Determine the solution for \( u(x, t) \).

(b) Compute the temperature at the midpoint of the rod at the end of 1 hour.

(c) Compute the time it will take for the temperature at that point to diminish to 5\(^\circ\)C.

(d) Compute the time it will take for the temperature at that point to diminish to 1\(^\circ\)C.

10. (Steady-state solution) If the solution \( u(x, t) \) tends to a steady-state solution \( u_s(x) \) as \( t \to \infty \), then we can determine \( u_s(x) \) from \( u(x, t) \) as

\[
\lim_{t \to \infty} u(x, t) = u_s(x). 
\]

However, if we are interested only in \( u_s(x) \) then it is wasteful to first solve for \( u(x, t) \). To solve for \( u_s(x) \) directly, merely set \( u_t = 0 \) in the PDE, which step reduces the PDE to an ODE on \( u_s(x) \). Solve that ODE subject to the boundary conditions (which, we assume here, do not vary with \( t \)).

In Example 1, for instance, \( u_s(x) \) is governed by the problem

\[
\frac{\alpha^2 u_s''}{x^2} = 0, \quad (0 < x < L) \\
u_s(0) = u_1, \quad u_s(L) = u_2, 
\]

which boundary-value problem is readily solved, its solution being \( u_s(x) = u_1 + (u_2 - u_1)x/L \), as obtained in Example 1 by letting \( t \to \infty \) in \( u(x, t) \). Use this method to find \( u_s(x) \) in each case: \( u_1, u_2, Q_1, Q_2, V, H \) are constants, and the initial condition is \( u(x, 0) = f(x) \).

(g) \( \alpha^2 u_{xx} = u_t, \quad u_t(0, t) = u_1, \quad u_x(L, t) = Q_2 \)

(b) \( \alpha^2 u_{xx} = u_t, \quad u_t(0, t) = Q_1, \quad u_x(L, t) = u_2 \)

(c) \( \alpha^2 u_{xx} = u_t, \quad u_x(0, t) = Q_1, \quad u_x(L, t) = Q_2 \)

HINT: Show that \( u_s(x) \) does not exist if \( Q_2 \neq Q_1 \), and explain why that result makes sense in physical terms. Show that if \( Q_2 = Q_1 \equiv Q \), however, then \( u_s(x) \) does exist but contains an undetermined constant, say \( C \). To determine \( C \), integrate the PDE on \( x \), from 0 to \( L \):

\[
\frac{\alpha^2}{L} \int_0^L u_{xx} \, dx = \int_0^L u_t \, dx, 
\]

and show that

\[
\frac{d}{dt} \int_0^L u(x, t) \, dx = 0, 
\]

so that we have the conservation principle

\[
\int_0^L u(x, t) \, dx = \text{constant}. 
\]

Use (10.4) to solve for \( C \), thus completing the solution for \( u_s(x) \).

(d) \( \alpha^2 u_{xx} = u_t + Hu_t, \quad u_t(0, t) = u_1, \quad u_x(L, t) = u_2 \)

(e) \( \alpha^2 u_{xx} = u_t + Hu_t, \quad u_t(0, t) = Q_1, \quad u_x(L, t) = u_2 \)

(f) \( \alpha^2 u_{xx} = u_t + Hu_t, \quad u_t(0, t) = u_1, \quad u_x(L, t) = Q_2 \)

(g) \( \alpha^2 u_{xx} = u_t + Hu_t, \quad u_t(0, t) = Q_1, \quad u_x(L, t) = Q_2 \)

(h) \( \alpha^2 u_{xx} - V u_x = u_1, \quad u(0, t) = u_1, \quad u(L, t) = u_2 \)

(i) \( \alpha^2 u_{xx} - V u_x = u_1, \quad u(0, t) = 0, \quad u(L, t) = u_2 \)

(j) \( \alpha^2 u_{xx} - V u_x = u_1, \quad u(0, t) = 0, \quad u_x(L, t) = 0 \)

(k) \( \alpha^2 u_{xx} - V u_x = u_1, \quad u(0, t) = 0, \quad u_x(L, t) + 2u_x(L, t) = -5 \)

11. (Existence of steady state) For the problem

\[
\alpha^2 u_{xx} = u_t + F(x), \quad (0 < x < L, \ 0 < t < \infty) \\
u_x(0, t) = Q_1, \quad u_x(L, t) = Q_2, \quad u(x, 0) = f(x), 
\]

show that a steady state does not exist unless a certain condition is satisfied by \( Q_1, Q_2 \), and \( F(x) \). Assuming that that condition is satisfied, solve for \( u_s(x) \).

12. (Steady-state extrusion) In Section 18.2 we derive equation (20) governing the temperature distribution \( u(x, t) \) in a heated rod being drawn through an extrusion die, as sketched there in Fig. 2. Actually, (20) holds both inside the furnace \( (x < 0) \) and outside the furnace \( (x > 0) \), but with different \( h \)’s and \( u_\infty \)’s. Let \( h = h_f \) and \( h_o \), and let \( u_\infty = u_f \) and \( u_o \) inside and outside the furnace, respectively. Assuming steady-state operation so that \( u = u_s(x) \), propose a suitable set of boundary conditions and solve for \( u_s(L) \), the rod temperature at the die. HINT: It will be a helpful approximation to consider the rod to extend from \( -\infty \) to \( +\infty \). Over \( -\infty < x < 0 \) use \( h_f \) and \( u_f \) in the ODE, and over \( 0 < x < \infty \) use \( h_o \) and \( u_o \). Solve over \( x < 0 \) and \( x > 0 \) separately, and apply suitable boundary conditions at \( x = -\infty \) and \( x = +\infty \), as well as suitable “matching conditions” at \( x = 0 \).

13. (Diffusion of one gas into another) Consider a cylindrical compressed-gas container of length \( L \), divided in half by a baffle (see sketch). To the left of the baffle is a gas of species \( A \), and to the right of it is a different gas of species \( B \). Suppose they are at the same pressure, so that when the baffle is removed at time \( t = 0 \) the two gases proceed to mix by diffusion.
alone. Considering species $A$, say, its concentration $c_A(x,t)$ moles/cm$^3$ is governed by the problem

$$D \frac{\partial^2 c_A}{\partial x^2} = \frac{\partial c_A}{\partial t}, \quad (0 < x < L, \ 0 < t < \infty) \quad (13.1)$$

$$\frac{\partial c_A}{\partial x}(0,t) = \frac{\partial c_A}{\partial x}(L,t) = 0, \quad (0 < t < \infty) \quad (13.2)$$

$$c_A(x,0) = \begin{cases} c_0, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases} \quad (13.3)$$

where $D$ is the diffusion coefficient and $D$ and $c_0$ are constants.

(a) Solve for $c_A(x,t)$. From $c_A(x,t)$ determine the steady-state solution

$$c_{A_0}(x) = \lim_{t \to \infty} c_A(x,t). \quad (13.4)$$

(b) Integrating equation (13.1) with respect to $x$, from 0 to $L$, show that

$$\int_0^L c_A(x,t) \, dx = \text{constant}, \quad (13.5)$$

which says that the total amount of $A$ is conserved. This result makes sense physically since the container is sealed [note (13.2)] and the gas is being neither created nor destroyed. (The point here is that it may be possible to learn something about the solution, from the PDE and boundary and initial conditions, without actually obtaining the full solution.)

(c) Solve for $c_A(x,t)$ directly, i.e., by solving

$$D c''_A(x) = 0; \quad c'_A(0) = c'_A(L) = 0$$

and using equation (13.5). Your result should, of course, be the same as in part (a).

14. (Conduction in metal ring) Consider the conduction of heat in a circular metal ring, the surface of which is insulated. Actually, whether the shape is a circle or an ellipse or whatever is irrelevant insofar as the heat conduction is concerned. If we measure $x$ along the ring, from some starting point, and denote the length of the ring as $L$, then the temperature $u(x,t)$ is defined on $-\infty < x < \infty$ and is an $L$-periodic function of $x$.

(a) Solve the heat equation $\alpha^2 u_{xx} = u_t$ ($-\infty < x < \infty, \ 0 < t < \infty$) subject to the initial condition $u(x,0) = f(x)$, where $f(x)$ is $L$-periodic on $-\infty < x < \infty$. Letting $t \to \infty$ in your solution, show that the steady-state solution is a constant temperature that is the average value of $f$. HINT: Since $u$ is $L$-periodic in $x$, it must be expressible in the Fourier series form

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos \frac{2n\pi x}{L} + b_n(t) \sin \frac{2n\pi x}{L}, \quad (14.1)$$

where $a_0, a_n, b_n$ vary with time. Putting (14.1) into the PDE and matching the coefficients of $\cos (2n\pi x/L)$, and $\sin (2n\pi x/L)$ on the left- and right-hand sides. derive simple ODE's governing $a_0(t), a_n(t), b_n(t)$. Solve these ODE's and then apply the initial condition $u(x,0) = f(x)$. Give integral formulas for the evaluation of any constants. (Exercise 29 re-examines this problem using the Sturm–Liouville theory.)

(b) Integrating $\alpha^2 u_{xx} = u_t$ from 0 to $L$, derive the conservation principle

$$\int_0^L u(x,t) \, dx = \text{constant}. \quad (14.2)$$

(c) Derive the steady-state solution $u_s(x)$ again, this time by solving

$$\alpha^2 u''_s(x) = 0, \quad (0 < x < L)$$

$$u_s(0) = u_s(L), \quad u'_s(0) = u'_s(L),$$

and using (14.3). Your result should be the same as found in part (a).

15. (Presence of a constant source term) Consider the problem

$$\alpha^2 u_{xx} = u_t - F, \quad (0 < x < L, \ 0 < t < \infty)$$

$$u(0,t) = 0, \quad u(L,t) = 50, \quad (0 < t < \infty)$$

$$u(x,0) = f(x), \quad (0 < x < L)$$

where the source term $F$ is assumed to be a constant. Solve for $u(x,t)$. Expansion coefficients may be left in integral form. HINT: This is the first problem in which the PDE is nonhomogeneous, namely, $L[u] = \alpha^2 u_{xx} - u_t = -F$. Observe that if we seek $u(x,t) = X(x)T(t)$ as in the text examples, and attempt to carry out the separation process, we obtain

$$\frac{X''}{X} = \frac{1}{\alpha^2} \frac{T''}{T} = \frac{F}{\alpha^2}, \quad (15.4)$$

Because of the last term in (15.4), which contains both $x$
and \( t \) dependence, we are unable to complete the separation process successfully. That is, we are unable to get all of the \( x \) dependence on one side of the equation and all of the \( t \) dependence on the other side. Thus, we suggest seeking \( u \) in the form

\[
u(x, t) = u_s(x) + X(x)T(t)
\]  

(15.5)

instead, where \( u_s(x) \) is the steady-state solution. In steady state, \( u_t \to 0 \) and \( u(x, t) \to u_s(x) \), so (15.1) and (15.2) give

\[
a^2 u_s''(x) = -F; \quad u_s(0) = 0, \quad u_s(L) = 50.
\]

(15.6)

Putting (15.5) into (15.1), show that the \( u_s \) term cancels the troublesome \( F \) term, so that this time the separation can be successfully completed to yield the familiar result

\[
\frac{X''}{X} = \frac{1}{a^2} \frac{T'}{T} = \text{constant} = -\kappa^2.
\]

(15.7)

Then solve (15.6) for \( u_s \) and (15.7) for \( X \) and \( T \), and impose the conditions (15.2) and (15.3) on \( u(x, t) = u_s(x) + X(x)T(t) \). Regarding the form of (15.5), observe that, in physical terms, \( u_s \) is the steady-state solution and \( X(T) \) is a transient quantity needed to match the initial condition \( u(x, 0) = f(x) \) with the steady-state solution \( u_s(x) \). In mathematical terms, \( u_s \) is a particular solution (i.e., a solution of the full equation \( \alpha^2 u_{xx} - u_t = -F \)) and \( X(T) \) is a solution of the associated homogeneous equation \( \alpha^2 u_{xx} - u_t = 0 \).

16. Repeat Exercise 15 with (15.2) and (15.3) changed as follows.

(a) \( u(0, t) = u(L, t) = u(x, 0) = 0 \)

(b) \( u_x(0, t) = u_x(L, t) = u(x, 0) = 0 \)

(c) \( u(0, t) = u_x(L, t) = u(x, 0) = 0 \)

(d) \( u(0, t) = 0, \quad u_x(L, t) = -20, \quad u(x, 0) = 0 \)

17. (Presence of nonconstant source term) In Exercise 15 we include a source term \( F \) that is a constant, although the solution method outlined therein would have worked even if \( F \) were a nonconstant function of \( x \). In this exercise we consider the problem

\[
\alpha^2 u_{xx} = u_t - F(x, t), \quad (0 < x < L, \quad 0 < t < \infty)
\]

\[
u(0, t) = u(L, t) = u(x, 0) = 0, \quad (0 < t < \infty)
\]

\[
u(x, 0) = 0, \quad (0 < x < L)
\]

(17.1)

where the source term \( F \) is allowed to be a function of \( x \) and \( t \). To solve, we can use essentially the same eigenvector expansion method that we used in Section 11.3.2 to solve the nonhomogeneous matrix problem \( A x = \lambda x + b \). [In this case the matrix operator \( A \) is analogous to the partial differential operator \( \alpha^2 \partial^2/\partial x^2 - \partial/\partial t \), \( c \) is analogous to \(-F(x, t)\), and \( \lambda = 0 \).]

There, we expand \( x \) and \( c \) in terms of the basis provided by the eigenvectors of \( A \). In the present case that step amounts to expanding \( u(x, t) \) and \( F(x, t) \) in terms of the sine \( \frac{n\pi x}{L} \) eigenuctions provided by the relevant Sturm-Liouville problem. Thus, the problem that we pose is as follows.

(a) Solve for \( u(x, t) \) by seeking

\[
u(x, t) = \sum_{n=1}^{\infty} g_n(t) \sin \frac{n\pi x}{L}
\]

(17.2)

and expanding

\[
F(x, t) = \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi x}{L},
\]

(17.3)

where the

\[
F_n(t) = \frac{2}{L} \int_0^L F(x, t) \sin \frac{n\pi x}{L} dx
\]

(17.4)

coefficients are considered as known (i.e., computable) functions of \( t \). Thus, show that

\[
u(x, t) = \sum_{n=1}^{\infty} \left[ \int_0^t F_n(\tau) e^{i\frac{n\pi x}{L}\tau} d\tau \right] \sin \frac{n\pi x}{L}
\]

(17.5)

(b) Work out the solution (17.5) for the case where \( F(x, t) = e^{-t} \).

(c) Modify the solution procedure described in part (a) if the left end condition is changed from \( u(0, t) = 0 \) to \( u_x(0, t) = 0 \).

18. (Superposition) Show that the solution to the problem

\[
\alpha^2 u_{x x} = u_t + g(x, t), \quad (0 < x < L, \quad 0 < t < \infty)
\]

\[
u(0, t) = p(t), \quad u(L, t) = q(t), \quad u(x, 0) = f(x)
\]

(18.1)

can be expressed as \( u = u_1 + u_2 + u_3 + u_4 \), where \( u_1, \ldots, u_4 \) are solutions of the four problems

\[
\alpha^2 u_{1xx} = u_1 + g(x, t), \quad u_1(0, t) = u_1(L, t) = 0
\]

\[
u_2(0, t) = p(t), \quad u_2(L, t) = u_2(x, 0) = 0,
\]

\[
\alpha^2 u_{3xx} = u_3, \quad u_3(0, t) = \alpha^2 u_{4xx} = u_4, \quad u_4(0, t) = u_4(L, t) = 0
\]

(18.1)
978

Chapter 18. Diffusion Equation

\[ u_0(0, t) = 0, \quad u_3(L, t) = q(t), \quad u_3(x, 0) = 0, \]
\[ \alpha^2 u_{xx} = u_{tt}, \quad (0 < x < L, \ 0 < t < \infty) \]
\[ u_x(0, t) = u_x(L, t) = 0, \quad u(x, 0) = f(x), \]

each on the domain \( 0 < x < L, \ 0 < t < \infty \).

19. (Nonhomogeneous Neumann conditions) Solve the conduction problem

\[ \alpha^2 u_{xx} = u_{tt}, \quad (0 < x < L, \ 0 < t < \infty) \]
\[ u_x(0, t) = -1, \quad u_x(L, t) = 0, \quad u(x, 0) = 0 \]
for \( u(x, t) \). HINT: You should find that the standard separation-of-variables procedure has difficulty coping with the boundary conditions if \( u_x(0, t) \neq u_x(L, t) \), as is true in this case. To proceed successfully, we suggest that you change from \( u(x, t) \) to \( v(x, t) \) according to

\[ u(x, t) = \frac{(x - L)^2}{2L} + v(x, t). \]

Then, show that \( v \) can be split by superposition into \( v = v_1 + v_2 \), where

\[ \alpha^2 v_{1xx} = v_{1tt}, \quad (0 < x < L, \ 0 < t < \infty) \]
\[ v_{1x}(0, t) = v_{1x}(L, t) = 0, \quad v_1(x, 0) = -\frac{(x - L)^2}{2L} \]

and

\[ \alpha^2 v_{2xx} = -\frac{\alpha^2}{L}, \quad (0 < x < L, \ 0 < t < \infty) \]
\[ v_{2x}(0, t) = v_{2x}(L, t) = v_2(x, 0) = 0. \]

The solution for \( v_2 \) can be found easily as a function of \( t \) alone. Solve for \( v_2 \). (But you need not solve for \( v_1 \).)

20. (Variable end conditions) Thus far, our Dirichlet-type boundary conditions have been of constant type, e.g., \( u(0, t) = 50 \). Here, we consider nonconstant conditions. Consider the problem

\[ \alpha^2 u_{xx} = u_{tt}, \quad (0 < x < L, \ 0 < t < \infty) \]
\[ u(0, t) = p(t), \quad u(L, t) = q(t), \quad u(x, 0) = f(x), \]

where \( p(t), q(t), \) and \( f(x) \) are prescribed. Changing dependent variables from \( u(x, t) \) to \( v(x, t) \) according to

\[ u(x, t) = v(x, t) + \left(1 - \frac{x}{L}\right) p(t) + \frac{x}{L} q(t), \]

show that the problem governing \( v \) is precisely of the type treated in Exercise 17. NOTE: Observe how an "input" can be moved from the boundary conditions to the PDE. In the present case, the PDE on \( u \) was homogeneous and the boundary conditions were nonhomogeneous; following the change of variables (20.2), you should find that the PDE on \( v \) is nonhomogeneous and the boundary conditions are homogeneous.

21. (Newton cooling) Consider the conduction of heat in a rod, the lateral surface of which is not insulated. If heat is conducted from the rod to the environment, the PDE governing the temperature \( u(x, t) \) is

\[ \alpha^2 u_{xx} = u_{tt} + h(u - u_\infty), \]

where the constants \( h \) and \( u_\infty \) are the convective heat transfer coefficient and the ambient temperature, respectively. Our interest in (21.1) lies in the Newton cooling term \( h(u - u_\infty) \). Although it is not essential, one normally begins by eliminating the \( u_\infty \) term by setting \( v(x, t) = u(x, t) - u_\infty \) and considering, instead, \( \alpha^2 v_{xx} = v_t + hv \).

(a) Solve the Newton cooling problem

\[ \alpha^2 v_{xx} = v_t + hv, \quad (0 < x < L, \ 0 < t < \infty) \]
\[ v(0, t) = v(L, t) = 50 \quad v(x, 0) = f(x) \]

by separation of variables, leaving expansion coefficients in integral form. HINT: Seeking \( v(x, t) = v_s(x) + X(t) \), where \( v_s(x) \) is the steady-state temperature distribution, show that

\[ \frac{X''}{X} = \frac{1}{\alpha^2} \frac{T' + hT}{T} = \text{constant} = -k^2. \]

(b) Solve (21.2) by omitting the \( v_s(x) \) term and seeking \( v(x, t) = X(x)T(t) \). That is, show that the inclusion of the \( v_s(x) \) term in the solution form is not essential.

22. Show that the change of variables

\[ w(x, t) = e^{ht}[u(x, t) - u_\infty] \]

reduces (21.1), above, to the simpler and more familiar form

\[ \alpha^2 w_{xx} = w_t. \]

23. (Flow of electricity in a cable) The voltage \( v(x, t) \) (volts) and the current \( I(x, t) \) (amperes) in a long underground insulated cable are governed by the PDE's

\[ v_{xx} = LC v_t + (rC + Lg) v_t + r pg, \]
\[ I_{xx} = LCI_t + (rC + Lg) I_t + r gI, \]

where \( L, C, g, r \) are constants.
where $L, C, r, g$ are positive constants: $L$ is the inductance (henries/kilometer), $C$ is the capacitance to ground (farads/kilometer), $r$ is the resistance (ohms/kilometer), and $g$ is the leakage to ground (millhos/kilometer). These PDE's are called the telephone equations and are seen to be of wave (hyperbolic) type. Often, as in telegraph transmission, $L$ and $g$ can be neglected, in which case (23.1) and (23.2) reduce to the telegraph equations
\[
\begin{align*}
\nu_{xx} &= rC\nu_t, \\
I_{xx} &= rCI_t,
\end{align*}
\]
which are of diffusion type. In the present example we suppose only that $L \approx 0$. Considering only the voltage $\nu$, we then have
\[
\nu_{xx} = rC\nu_t + rgv. \tag{23.3}
\]

[Comparing (23.3) with the PDE in Exercise 21(a), we see that the two phenomena are analogous, with the lateral heat loss to the environment corresponding to the voltage loss due to leakage to the ground.] Suppose that the line is of length $L$, the voltage at $x = 0$ is maintained (for a "long time") at 12 volts, the voltage at $x = L$ is maintained (for a "long time") at 6 volts, and then, beginning at $t = 0$, the left end is grounded. Thus,
\[
\begin{align*}
\nu(0, t) &= 0, \quad (0 < t < \infty) \\
\nu(L, t) &= 6, \quad (0 < t < \infty)
\end{align*} \tag{23.4.5}
\]
The initial condition $\nu(x, 0)$ is not given but can be deduced from (23.3) together with the information that the ends have been maintained at 12 and 6 volts, respectively, for a long time.

(a) Determine $\nu(x, 0)$.

(b) Determine the steady-state solution $\nu_0(x)$.

(c) Solve for $\nu(x, t)$ by separation of variables. Fourier expansion coefficients may be left in integral form. HINT: As in Exercise 21, it will be most convenient to seek $\nu(x, t) = \nu_s(x) + X(x)T(t)$.

24. (Conduction in a sphere) Consider the radial conduction of heat within a solid sphere. If the temperature $\nu$ is a function only of the spherical polar coordinate $\rho$ and the time $t$, then $\alpha^2 \nabla^2 \nu = \nu_t$ becomes
\[
\alpha^2 \left( \frac{\partial \nu}{\partial \rho} + \frac{2}{\rho} \frac{\partial \nu}{\partial \rho} \right) = \nu_t. \tag{24.1}
\]
(a) Setting $\nu(\rho, t) = \nu_{s}(\rho) / \rho$, show that $\nu$ needs to satisfy the more familiar PDE
\[
\alpha^2 \nu_{ss} = \nu_t. \tag{24.2}
\]
(b) Use the idea contained in part (a) to solve the problem
\[
\alpha^2 \left( \nu_{ss} + \frac{2}{\rho} \nu_s \right) = \nu_t, \quad \begin{cases} 0 < \rho < a, & 0 < t < \infty \end{cases}
\]
\[
\nu(a, t) = 0, \quad (0 < t < \infty)
\]
\[
\nu(\rho, 0) = f(\rho), \quad (0 < \rho < a)
\]
where $\nu(0, t)$ is bounded. Expansion coefficients may be left in integral form. (Exercise 28 reexamines this problem, using the Sturm–Liouville theory.)

EXERCISES FOR THE OPTIONAL SECTIONS 18.3.2, 18.3.3

25. (Uniqueness) (a) Consider the problem
\[
\alpha^2 \nu_{xx} = \nu_t + f(x, t), \quad \begin{cases} 0 < x < L, & 0 < t < T \end{cases}
\]
\[
\nu(0, t) = p(t), \quad \nu(L, t) = q(t), \quad \nu(x, 0) = r(x),
\]
where $T$ is arbitrarily large. To establish the uniqueness of the solution, suppose that $\nu_1(x, t)$ and $\nu_2(x, t)$ are two solutions, and define
\[
\nu(x, t) = \nu_1(x, t) - \nu_2(x, t). \tag{25.3}
\]
Show that $\nu$ satisfies the "homogenized" problem
\[
\alpha^2 \nu_{xx} = \nu_t, \quad \begin{cases} 0 < x < L, & 0 < t < T \end{cases}
\]
\[
\nu(0, t) = 0, \quad \nu(L, t) = 0, \quad \nu(x, 0) = 0.
\]
Proceeding formally, show that
\[
\frac{d}{dt} \int_0^L \nu^2(x, t) \, dx = 2 \int_0^L w \nu_t \, dx = 2 \alpha^2 \int_0^L w \nu_{xx} \, dx. \tag{25.4}
\]
Integrating the last integral by parts and then integrating both sides of the equation on $t$ from 0 to $t$, show that
\[
\int_0^t \int_0^L w^2(x, \tau) \, dx \, d\tau = -\alpha^2 \int_0^t \int_0^L \frac{d}{d\tau} \left( \frac{d^2}{dx^2} \right) \nu^2 \, dx \, d\tau. \tag{25.5}
\]
Explain why it follows from (25.5) that $w(x, t) = 0$ throughout $0 < x < L, 0 < t < T$. Thus, it must be true that $\nu_1(x, t) = \nu_2(x, t)$ for any solutions $\nu_1$ and $\nu_2$, so that the solution to (25.1) and (25.2) must be unique.

(b) Repeat part (a), but change $\nu(L, t) = q(t)$ in equation (25.2) to $\nu(L, t) = q(t)$. 

(c) Establish the uniqueness of the solution to the problem
\[ \alpha^2 u_{xx} = u_t + f(x, t), \quad (0 < x < L, \ 0 < t < T) \]
\[ u(0, t) = p(t), \quad u(L, t) + \beta u_x(L, t) = q(t), \]
\[ u(x, 0) = r(x), \]
where \( T \) is arbitrarily large and \( \beta > 0 \), following essentially the same lines in part (a).

26. With \( M_n \) defined by (50), use the ratio test to show that \( \sum_{n=1}^{\infty} M_n \) converges, as we claimed.

27. (a)–(n) The problems in Exercise 6 are to be solved using half- or quarter-range expansions. Solve the corresponding problem in Exercise 6 again, this time using the Sturm–Liouville theory.

28. (Conduction in a sphere) In Exercise 24(b) we suggest using the change of variables \( u(\rho, t) = v(\rho, t)/\rho \) to reduce the PDE to the form \( \alpha^2 v_{\rho\rho} = v_t \), that is studied in this section. Here we ask you to solve the problem in Exercise 24(b) directly by seeking \( u(\rho, t) = R(\rho)T(t) \) and using separation of variables. HINT: The ODE governing \( R(\rho) \) can be solved in terms of Bessel functions using the formulas given in Section 4.6.6, and these results can be simplified using the formulas
\[ J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \]
You will need to distinguish the cases \( \kappa \neq 0 \) and \( \kappa = 0 \).

29. (Conduction in a metal ring) Here we reconsider the problem of Exercise 14(a). There we consider the \( x \) domain to be \(-\infty < x < \infty\), so \( u(x, t) \) and \( f(x) \) were \( L \)-periodic in \( x \) and could be expanded in Fourier series. Alternatively, think of the \( x \) domain as finite: \( 0 < x < L \). That step creates the boundaries \( x = 0 \) and \( x = L \), so we need boundary conditions there. Although we don’t know \( u \) or \( u_x \) there, we do know that, physically, the ends \( x = 0 \) and \( x = L \) are abutting, so both the temperature \( u \) and the heat flux (proportional to \( u_x \)) must be continuous there. Thus, we can pose the problem (on the finite interval \( 0 < x < L \)) as
\[ \alpha^2 u_{xx} = u_t, \quad (0 < x < L, \ 0 < t < \infty) \]
\[ u(0, t) = u(L, t), \quad u_x(0, t) = u_x(L, t), \quad u(x, 0) = f(x). \]
Solve that problem by separation of variables, leaving expansion coefficients in integral form. HINT: The Sturm–Liouville problem that arises on \( x \) will have periodic boundary conditions.

30. (g) Solve the problem
\[ u_{xx} - 2u_x = u_t, \quad (0 < x < L, \ 0 < t < \infty) \]
\[ u(0, t) = u(L, t) = 50, \quad u(x, 0) = 0 \]
by separation of variables. HINT: Seeking \( u(x, t) = X(x)T(t) \), obtain
\[ X'' - 2X' + \kappa^2 X = 0, \quad T'' + \kappa^2 T = 0. \]
With \( X(x) = e^{\lambda x} \), obtain \( \lambda = 1 \pm \sqrt{1 - \kappa^2} \), so
\[ X(x) = e^x \left( C_1 e^{\sqrt{1 - \kappa^2} x} + C_2 e^{-\sqrt{1 - \kappa^2} x} \right), \]
\[ T(t) = C_3 e^{-\kappa^2 t}. \]
However, expecting oscillatory functions (for the eventual expansion that will be needed), anticipate that the \( \kappa \)'s will be greater than unity and write \( \lambda = 1 \pm i\sqrt{\kappa^2 - 1} \) instead, so
\[ X(x) = e^x \left( C_3 \cos \sqrt{\kappa^2 - 1} x + C_2 \sin \sqrt{\kappa^2 - 1} x \right). \]
Distinguish the case \( \kappa = 1 \) because if \( \kappa = 1 \) then (30.6) reduces to \( X(x) = C_3 e^x \), which falls short of being a general solution of (30.2). We don’t need to also distinguish the case \( \kappa = 0 \) because if \( \kappa = 0 \) then (30.4) and (30.5) do give the general solutions of (30.2) and (30.3), respectively. However, the case \( \kappa = 0 \) is of special importance because it gives the steady-state part of the solution (since it reduces \( T(t) \) to a constant). Thus, use the “three-tier” solutions
\[ X(x) = \begin{cases} 
\alpha e^{(A \cos \omega x + B \sin \omega x)}, & \kappa \neq 0, 1 \\
\alpha e^{(C + D x)}, & \kappa = 1 \\
E + F e^{\pm \kappa^2 x}, & \kappa = 0 
\end{cases} \]
\[ T(t) = \begin{cases} 
G e^{-\kappa^2 t}, & \kappa \neq 0, 1 \\
H e^{-t}, & \kappa = 1 \\
I, & \kappa = 0 
\end{cases} \]
where \( \omega \equiv \sqrt{\kappa^2 - 1} \), for brevity, and form \( u(x, t) \) as the sum of their respective products.

(b) Same as part (a), but with \( u(0, t) = 0 \) and \( u(L, t) = 50. \) Leave expansion coefficients in integral form.

31. (a) Show that
\[ \int_0^{z_n} x J_0(x) \, dx = z_n J_1(z_n) \]
where \( z_n \) is any root of \( J_0(x) = 0 \). HINT: Integrate the
Bessel equation \((xJ_0')' + xJ_0 = 0\) from 0 to \(z_n\) and use the \((b)\) Then, use (31.1) to verify the last step in (83), relation \(J_0(x) = -J_1(x)\) [from Exercise 4, Section 4.6].

### 18.4 Fourier and Laplace Transforms (Optional)

In Section 18.3 the \(x\) domain is always finite, namely, \(0 < x < L\). Semi-infinite \((0 < x < \infty)\) and infinite \((-\infty < x < \infty)\) domains are also important, and it is these cases that we now address. We organize our discussion around two examples.

**EXAMPLE 1. Heat Conduction in an Infinite Rod.** Let us begin with the problem

\[
\begin{align*}
\alpha^2 u_{xx} &= u_t, & (-\infty < x < \infty, \ 0 < t < \infty) \quad (1a) \\
u(x,0) &= f(x), & (-\infty < x < \infty) \quad (1b)
\end{align*}
\]

summarized in Fig. 1.

If we regard the infinite rod as the limiting case of a finite rod, on \(-L < x < L\), as \(L \to \infty\) and recall that boundary conditions are needed for the finite rod, we might well anticipate that some form of boundary conditions will be needed for the infinite rod at \(x = \pm\infty\). But since a suitable form for those boundary conditions may not yet be apparent, let us defer that issue for the moment, in the hope that the solution process itself may provide a clue.

In selecting a solution technique, remember that if we use the method of separation of variables then we need, in the final step, to expand the \(f(x)\) in \((1b)\) in a Fourier-type series. If the \(x\) domain is finite, then that series will be a half- or quarter-range cosine or sine series, or a generalized Fourier series containing the orthogonal eigenfunctions of a relevant Sturm–Liouville problem. If the \(x\) domain is infinite, then we can still use a separation of variables, provided that \(f(x)\) is periodic with finite period, in which case we can expand \(f(x)\) in a classical Fourier series of cosines and sines.

However, in this example we have an infinite \(x\) domain and are interested in \(f's\) that are not periodic, such as \(e^{-t|x|}\) (Fig. 2a) and the rectangular pulse shown in Fig. 2b. Such \(f's\) can be represented not by Fourier series but by Fourier integrals, which fact suggests seeking a solution for \(u\) in Fourier integral form or, equivalently and more conveniently, using a Fourier transform.

Thus, let us Fourier transform (1a) with respect to \(x\):

\[
\begin{align*}
\mathcal{F}\{\alpha^2 u_{xx}\} &= \mathcal{F}\{u_t\}, \quad (2a) \\
\alpha^2 \mathcal{F}\{u_{xx}\} &= \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\omega x} \, dx, \quad (2b) \\
\alpha^2 (i\omega)^2 \hat{u} &= \frac{d}{dt} \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} \, dx \quad (2c) \\
&= \frac{d\hat{u}}{dt}, \quad (2d)
\end{align*}
\]

![Figure 1. Infinite rod problem.](image1)

![Figure 2. Nonperiodic f's.](image2)
so
\[ \frac{d\hat{u}}{dt} + \alpha^2 \omega^2 \hat{u} = 0, \] (3)
with solution
\[ \hat{u} = A e^{-\alpha^2 \omega^2 t}. \] (4)

In passing from (2a) to (2b) we used the linearity of \( F \{ \} \) on the left, and the definition of the Fourier transform on the right. Next, we used
\[ F \{ u_{xx} \} = (i\omega)^2 F \{ u \} = (i\omega)^2 \hat{u} \] (5)
on the left, and the Leibniz differentiation formula
\[ \frac{d}{dt} \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} \, dx = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\omega x} \, dx \] (6)
on the right. For (5) to hold, we need
\[ u \to 0 \text{ and } u_x \to 0 \text{ as } x \to \pm \infty, \] (7)
so let us adopt (7) as our boundary conditions. For the output \( u \) (and \( u_x \)) to tend to zero as \( x \to \pm \infty \), we expect that we will need to restrict the input \( f \) to die out sufficiently fast, as \( x \to \pm \infty \), as well. However, as suggested in Section 18.3 let us proceed formally to a solution without getting bogged down with such technical points. If we like, we can then rigorously verify that that solution does satisfy the PDE and any boundary and initial conditions (if indeed it does).

Notice carefully that by Fourier transforming (1a) with respect to \( x \) we convert the partial differential equation (1a) on \( u(x,t) \) to the ordinary differential equation (3) on \( \hat{u}(\omega, t) \), in which \( \omega \) appears as a parameter— that is, as a constant.

To evaluate the integration constant \( A \) in (4) we impose the initial condition (1b): \( \hat{u}(\omega, 0) = \hat{f}(\omega) \). Thus,
\[ \left. \hat{u} \right|_{t=0} = \hat{f}(\omega) = (A e^{-\alpha^2 \omega^2 t}) \bigg|_{t=0} = A, \] (8)
so \( A = \hat{f}(\omega) \),* and (4) becomes
\[ \hat{u}(\omega, t) = \hat{f}(\omega)e^{-\alpha^2 \omega^2 t}. \] (9)

Finally, using entry 6 of Appendix D and the Fourier convolution property (entry 21), it follows from (9) that
\[ u(x,t) = f(x) * \frac{1}{2\alpha \sqrt{\pi t}} e^{-x^2/(4\alpha^2 t)} \] (10)
or
\[ u(x,t) = \frac{1}{2\alpha \sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi)e^{-(x-\xi)^2/(4\alpha^2 t)} \, d\xi. \] (11)

*You may be concerned that \( A \) was to be a constant, yet \( \hat{f}(\omega) \) is a function of \( \omega \). The point to keep in mind is that (3) is a differential equation in \( t \), and \( \omega \) is regarded there as a constant.
COMMENT 1. As a general principle it is important to check one's result for any special cases for which the solution is known. The simplest case that comes to mind is the case where

\[ f(x) = \text{constant} = F, \tag{12} \]

for in that case it is obvious that the solution to (1) is simply \( u(x, t) = F \). In fact, if we set \( f(\xi) = F \) in (11) and evaluate the integral we do obtain \( u(x, t) = F \) (Exercise 1). It is striking that (11) gives the correct result for the case where \( f(x) = F \) because in that case \( f(\omega) \) does not even exist, and the solution \( u(x, t) = F \) violates the assumption that \( u \to 0 \) as \( x \to \pm \infty \).

COMMENT 2. As a more interesting special case, let

\[ f(x) = \begin{cases} F, & x > 0 \\ 0, & x < 0 \end{cases} = FH(x), \tag{13} \]

where \( F \) is a constant and \( H(x) \) is the Heaviside function. In this case, putting (13) into (11) gives (Exercise 2)

\[ u(x, t) = \frac{F}{2} \left[ 1 + \text{erf} \left( \frac{x}{2\sigma \sqrt{t}} \right) \right], \tag{14} \]

where

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi \tag{15} \]

is a tabulated function known as the error function. That is, the integral in (15) cannot be evaluated in terms of the so-called elementary functions, so it is given its own name, the "error function," and its properties, tabulated values, and computational formulas for it can be found in the literature. The factor \( 2/\sqrt{\pi} \) is included to "normalize" \( \text{erf}(x) \) so that \( \text{erf}(\infty) = 1 \) because

\[ \int_0^\infty e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2} \tag{16} \]

The greater-than-expected validity of (11) can be traced to the inversion of the order of integration that is inherent in the Fourier convolution step.

The trend in computing has been away from tabulations and toward approximate expressions, not only for \( \text{erf}(x) \) but for the various special functions: Bessel functions, the gamma function, and so on. For instance, the formula

\[ \text{erf}(x) \approx 1 - (a_1 p + a_2 p^2 + a_3 p^3) e^{-x^2}, \]

where

\[ p = \frac{1}{1 + 0.47047x}, \quad a_1 = 0.3480344, \quad a_2 = -0.99993, \quad a_3 = 0.099796, \]

developed by C. Hastings, Jr., is uniformly accurate, over \( 0 \leq x < \infty \), to \( \pm 0.000025 \). The latter formula is an example of a common form of approximation known as rational function approximation because the function \( \exp(x^2)[1 - \text{erf}(x)] \) is being approximated by a rational function of \( x \), namely, the ratio of two polynomials (as can be seen by combining the terms in \( a_1 p + a_2 p^2 + a_3 p^3 \) over a common denominator). For approximations such as this, see M. Abramowitz and I. Stegun (eds.), National Bureau of Standards Applied Math Series, 1964, or Y. L. Luke, Mathematical Functions and Their Approximation (New York: Academic Press, 1975).
derivation of which result is outlined in Exercise 9 of Section 4.5. The graph of \( \text{erf}(x) \) is shown in Fig. 3 for \( x > 0 \); for \( x < 0 \) we rely on the fact that the error function is an odd function (Exercise 4), so that \( \text{erf}(-x) = -\text{erf}(x) \). Besides the integral from 0 to \( x \), in (15), we also encounter the integral from \( x \) to \( \infty \) so frequently that we also define a complementary error function,

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2} d\xi = \text{erf}(\infty) - \text{erf}(x) = 1 - \text{erf}(x).
\]  

(17)

The solution (14) is plotted in Fig. 4 at representative times. Observe carefully how the initially discontinuous temperature distribution smooths out as \( t \) increases. This result illustrates the fact that, in physical terms, diffusion is a smoothing process.

![Figure 3. The error function.](image)

![Figure 4. The solution (14).](image)

COMMENT 3. It is instructive to write (11) in the form

\[
u(x,t) = \int_{-\infty}^\infty f(\xi)K(\xi - x; t) d\xi,
\]  

(18a)

where

\[
K(\xi - x; t) = \frac{e^{-(x-\xi)^2/(4\alpha^2t)}}{2\alpha\sqrt{\pi t}}
\]  

(18b)

is called the kernel. That is, it comprises everything in the integrand other than the input \( f(\xi) \). The kernel \( K(\xi - x; t) \) happens to be a normal (or Gaussian) probability distribution centered at \( \xi = x \), which result correctly suggests that diffusion is an essentially statistical phenomenon. The area between the graph of \( K \) and the \( \xi \) axis is unity for all \( t \) (Exercise 3), and that graph becomes more and more focused as \( t \to 0 \) (Fig. 5). If you studied Section 5.6 you will see that \( K \) appears to approach a delta function at \( x \), \( \delta(\xi - x) \), as \( t \to 0 \) because it has unit area for each \( t \), and it becomes focused at \( x \) as \( t \to 0 \). In fact,

\[\text{Here, } \xi \text{ is the active variable since it is the variable of integration; } x \text{ and } t \text{ are regarded as fixed in (18a). If we write } K(\xi; x, t) \text{ rather than } K(\xi - x; t), \text{ we are emphasizing that } \xi \text{ is the active variable and that } x \text{ and } t \text{ are, at least for the moment, fixed. Even so, we have written } K(\xi - x; t) \text{ instead because only the difference } \xi - x \text{ occurs in (18b). We say that } K(\xi - x; t) \text{ is a difference kernel. The present example is similar to Example 8 of Section 17.10, which we urge you to review when you have finished reading this example.} \]
that observation is confirmed by the initial condition, which is

$$
\lim_{t \to 0} u(x; t) = \lim_{t \to 0} \int_{-\infty}^{\infty} f(\xi)K(\xi - x; t) d\xi = f(x).
$$

(19)

That is, the initial condition (19) is satisfied by virtue of the fact that $K(\xi - x; t)$ tends to a delta function $\delta(\xi - x)$ as $t \to 0$.

COMMENT 4. Since the kernel $K$ is evidently important, we would do well to try to understand its physical significance. To do so, let the input itself be a delta function at some point $x_0$,

$$
f(x) = \delta(x - x_0).
$$

(20)

In a crude way, this case can be conceptualized as corresponding to the application of a welding torch to the rod at $x_0$ for a brief instant. Then (18a) gives

$$
\begin{align*}
  u(x, t) &= \int_{-\infty}^{\infty} \delta(\xi - x_0)K(\xi - x; t) d\xi \\
  &= K(x_0 - x; t) = K(x - x_0; t).
\end{align*}
$$

(21)

The second equality in (21) follows from the fundamental property

$$
\int_{-\infty}^{\infty} \delta(\xi - a)g(\xi) d\xi = g(a)
$$

(22)

of the delta function, and the third equality in (21) follows from the fact that $K$ is an even function of $x_0 - x$ [because it contains $(x_0 - x)^2$]. The upshot is that the kernel $K(x - x_0; t)$ itself is a solution of the heat conduction equation (1a), corresponding to an initial temperature distribution $\delta(x - x_0)$. Thus, $K(\xi - x; t)$, the graph of which is shown in Fig. 5, is the temperature distribution in the rod that results from an initial temperature distribution that is a delta function at $x$. Once again, we see the smoothing nature of the diffusion process, for beginning with the spike-like temperature profile $\delta(\xi - x)$ the temperature distribution $u(x, t)$ smooths out more and more as $t$ increases (Fig. 5). Finally, we can now understand the superposition nature of (18a), for if $K(\xi - x; t)$ is the temperature response to an initial temperature that is a delta function (hence having unit area) at $x$, then the response $du(x, t)$ to the rectangular-pulse initial temperature shown in Fig. 6, having area $\int f(\xi) \, d\xi$, is $K(x - \xi; t)$ scaled by $f(\xi) \, d\xi$.

$$
du(x, t) = K(x - \xi; t)[f(\xi) \, d\xi] = K(\xi - x; t)[f(\xi) \, d\xi].
$$

(23)

Adding such results for all of the rectangular pulses that comprise $f$ gives the integral (18a).

COMMENT 5. As a final observation about the physics, observe that the solution (18) indicates the spreading of information, by diffusion, with an infinite velocity. For instance, in Fig. 4 we see that some of the heat that was initially confined to the interval $0 < x < \infty$ diffuses to the interval $-\infty < x < 0$ over any arbitrarily small time $t$. This spreading can occur only if the speed of propagation is infinite. Insomuch as it is agreed that energy cannot propagate faster than the speed of light, this result evidently reveals a flaw in our diffusion.
equation $\alpha^2 u_{xx} = u_t$, one that is discussed in the literature,* and which is inconsequential in practical applications of the theory.

COMMENT 6. We were able to use a Fourier transform on $x$ to solve (1) because the latter was a boundary value problem on $-\infty < x < \infty$. Alternatively, we could have used a Laplace transform on $t$ because (1) was an initial value problem on $0 < t < \infty$. The latter approach proves to be less attractive because we obtain a nonhomogeneous ODE on $\tilde{u}(x, s)$ rather than the homogeneous ODE (3) on $u(\omega, t)$. (See Exercise 5.)

In Example 1 we illustrate the use of the Fourier transform for a problem on the conduction of heat in an infinite rod ($-\infty < x < \infty$). In Example 2 we illustrate the use of the Laplace transform for a semi-infinite rod ($0 < x < \infty$) problem.

**EXAMPLE 2. Heat Conduction in a Semi-Infinite Rod.** This time, we consider the problem

\begin{align*}
\alpha^2 u_{xx} &= u_t, \quad (0 < x < \infty, \ 0 < t < \infty) \tag{24a} \\
u(x, 0) &= 0, \quad (0 < x < \infty) \tag{24b} \\
u(0, t) &= g(t), \quad (0 < t < \infty) \tag{24c}
\end{align*}

as summarized in Fig. 7. The rod is initially at $0^\circ$ throughout, we subject the left end ($x = 0$) to a prescribed temperature $g(t)$, and we seek the temperature distribution $u(x, t)$ that develops. We will also need a boundary condition at the right end ($x = \infty$), which we take to be $u(\infty, t) = 0$ for all $t$; that is,

$$\lim_{x \to \infty} u(x, t) = 0, \quad (0 < t < \infty) \tag{24d}$$

In this case a Fourier transform is inappropriate because the domain is $0 < x < \infty$ rather than $-\infty < x < \infty$, so let us try a Laplace transform on $t$.* Thus, Laplace transform (24a) with respect to $t$:

$$L\{\alpha^2 u_{xx}\} = L\{u_t\}. \tag{25}$$

Now,

$$L\{\alpha^2 u_{xx}\} = \alpha^2 L\{u_{xx}\} \quad \text{(linearity of } L)$$

$$= \alpha^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-st} \, dt \quad \text{(definition of transform)}$$

$$= \alpha^2 \frac{d^2}{dx^2} \int_0^\infty u(x, t) e^{-st} \, dt \quad \text{(Leibniz rule)}$$

$$= \alpha^2 \frac{d^2}{dx^2} \tilde{u}$$

$$= \alpha^2 \tilde{u}_{xx}(x, s), \tag{26}$$


* Alternatively, we could use a semi-infinite Fourier transform, namely, a Fourier cosine or sine transform. These are studied in the optional Section 17.11.
and
\[ L(u_t) = s\overline{u}(x, s) - u(x, 0) \]
\[ = s\overline{u}(x, s), \quad (27) \]

so (25) becomes \( \alpha^2 \overline{u}_{xx} = s\overline{u} \), or
\[ \overline{u}_{xx} - \frac{s}{\alpha^2} \overline{u} = 0, \quad (28) \]

which is a second-order ODE with respect to \( x \), \( t \) having been eliminated by the Laplace transform process. Solving (27),
\[ \overline{u}(x, s) = A e^{\sqrt{s} \pi/\alpha} + B e^{-\sqrt{s} \pi/\alpha}. \quad (29) \]

We have already used (24a) and (24b). To solve for \( A \) and \( B \) we use the boundary conditions (24c) and (24d), but first we need to express those conditions in terms of \( s \) rather than \( t \). For the condition at \( x = \infty \) observe that
\[ \lim_{x \to \infty} \overline{u}(x, s) = \lim_{x \to \infty} \int_0^\infty u(x, t) e^{-st} \, dt \]
\[ = \int_0^\infty \lim_{x \to \infty} u(x, t) e^{-st} \, dt = \int_0^\infty 0 \, dt = 0; \quad (30) \]

that is, \( u(x, t) \to 0 \) as \( x \to \infty \) implies that
\[ \overline{u}(x, s) \to 0 \text{ as } x \to \infty \quad (31) \]
as well. [Note that the second equality in (30) was carried out only formally inasmuch as we did not rigorously justify the interchange in the order of the two limit processes: the limit \( x \to \infty \) and the limit process that lies behind the Riemann integral.] Applying (31) to (29) gives \( A = 0 \), so
\[ \overline{u}(x, s) = B e^{-\sqrt{s} \pi/\alpha}. \quad (32) \]

To express (24c) in terms of \( s \), take its Laplace transform:
\[ \int_0^\infty u(0, t) e^{-st} \, dt = \int_0^\infty g(t) e^{-st} \, dt, \]
or
\[ \overline{u}(0, s) = \overline{g}(s), \quad (33) \]
and imposing the latter upon (32) gives
\[ \overline{u}(0, s) = \overline{g}(s) = Be^0, \quad (34) \]
so \( B = \overline{g}(s) \) and
\[ \overline{u}(x, s) = \overline{g}(s)e^{-\sqrt{s} \pi/\alpha}. \quad (35) \]

To invert, use entry 21 of Appendix C [with \( a = x/(2\alpha) \)] and the convolution property (entry 28):
\[ u(x, t) = g(t) = \frac{xe^{-x^2/(4\alpha^2 t)}}{2\alpha\sqrt{\pi} t^{3/2}} \]
\[ = \frac{x}{2\alpha\sqrt{\pi}} \int_0^t g(t - \tau) \frac{e^{-x^2/(4\alpha^2 \tau)}}{\tau^{3/2}} \, d\tau, \quad (36) \]
or equivalently,
\[ u(x, t) = \frac{x}{2\alpha \sqrt{\pi}} \int_0^t g(\tau) \frac{e^{-x^2/[4\alpha^2(\tau-t)]}}{(\tau-t)^{3/2}} \, d\tau \]  
(37)

if we prefer.

COMMENT 1. The application of (31) to the solution (29) was easy: the positive exponential grows with \( x \) and the negative exponential dies out, so \( A = 0 \) and \( B \) remained arbitrary. What if we had used the solution form
\[ \tilde{u}(x, s) = C \cosh \frac{\sqrt{s} x}{\alpha} + D \sinh \frac{\sqrt{s} x}{\alpha} \]  
(38)

instead of (29), for both terms on the right-hand side tend to infinity as \( x \to \infty \)? Let us see:
\[ \tilde{u}(x, s) = \frac{C}{2} (e^{\sqrt{s} x/\alpha} + e^{-\sqrt{s} x/\alpha}) + \frac{D}{2} (e^{\sqrt{s} x/\alpha} - e^{-\sqrt{s} x/\alpha}) \]
\[ = \frac{C}{2} + \frac{D}{2} e^{\sqrt{s} x/\alpha} - \frac{D}{2} e^{-\sqrt{s} x/\alpha}. \]  
(39)

To eliminate the positive exponential, set \( D = -C \). Then (39) becomes \( \tilde{u}(x, s) = C \exp(-\sqrt{s} x/\alpha) \), which is equivalent to (32). The point is that either form will work, (29) or (38), but (29) is more convenient insofar as the application of the condition (31).

COMMENT 2. We did not specify the function \( g(t) \). Observe that if \( g(t) \) were specified, it would have been foolish to work out its transform \( \tilde{g}(s) \) because when we apply the convolution theorem to (35) we invert \( \tilde{g}(s) \) and get back the given function \( g(t) \). Thus, it is best to merely call the transform \( \tilde{g}(s) \), as we did.

COMMENT 3. Observe from (37) that \( u(x, t) \) depends upon the boundary data \( g(t) \) only over \( 0 < \tau < t \), not over \( 0 < \tau < \infty \). This result is entirely reasonable since how could the temperature distribution \( u(x, t) \) today depend upon the boundary values \( g(t) \) to be imposed tomorrow?

COMMENT 4. Finally, let us use (37) to determine \( u(x, t) \) for a specific case, say \( g(t) = 100^\circ \). With \( g(t) \) a constant, it is easier to use (36) than (37), and the latter gives (Exercise 13)
\[ u(x, t) = \frac{100x}{2\alpha \sqrt{\pi}} \int_0^t \frac{e^{-x^2/[4\alpha^2 \tau]}}{\tau^{3/2}} \, d\tau \]
\[ = 100 \text{erfc} \left( \frac{x}{2\alpha \sqrt{t}} \right), \]  
(40)

which is plotted in Fig. 8 at representative times.

Closure. In Section 18.3 we learn how to solve the diffusion equation by separation of variables, but the \( x \) domain is always finite and the boundary conditions do not vary with \( t \) (although certain more complicated cases are outlined in the end-of-section exercises). The Fourier and Laplace transforms enable us to deal with
semi-infinite \((0 < x < \infty)\) and infinite \((-\infty < x < \infty)\) domains, as well as nonconstant boundary conditions.

Application of the Laplace and Fourier transform to the diffusion equation
\(\alpha^2 u_{xx} = u_t\) proceeds very much along the same lines as for ordinary differential equations, but instead of producing an algebraic equation it produces an ordinary differential equation. Specifically, Laplace transforming on \(t\) produces the ODE
\(\alpha^2 \tilde{u}_{xx} - s \tilde{u} = -u(x, 0)\) on \(\tilde{u}(x, s)\), and Fourier transforming on \(x\) \((\text{if } -\infty < x < \infty)\) produces the ODE \(\tilde{u}_t + \alpha^2 \omega^2 \tilde{u} = 0\) on \(\tilde{u}(\omega, t)\).

**EXERCISES 18.4**

**NOTE:** Exercises 1–10 relate to Example 1; the remainder relates to Example 2.

1. Show that for the case where \(f(x) = \text{constant } \equiv F\), (11) gives \(u(x, t) = F\). HINT: Make the change of variables \((x - \xi)^2/(4 \alpha^2 t) = \mu^2\) in the integral, and use the known integral \(\int_{-\infty}^{\infty} \exp(-\xi^2) \, d\xi = \sqrt{\pi}\).

2. (a) Show that for the case where \(f(x) = FH(x)\), in (13), (11) gives (14) as the solution.
   (b) Verify, directly, that (14) satisfies the PDE (1a) and the initial condition (1b).

3. Prove the claim, made in Comment 3 of Example 1, that \(\int_{-\infty}^{\infty} K(x - \xi) \, d\xi = 1\) for all \(t\), where \(K\) is given by (18b).
   HINT: Use the change of variables suggested in Exercise 1.

4. Show that erf \((x)\) defined by (15), is an odd function, as is claimed in Comment 2 of Example 1.

5. We used the Fourier transform to solve (1). Use the Laplace transform instead, and obtain the ODE
   \(\alpha^2 \tilde{u}_{xx} - \frac{s}{\alpha^2} \tilde{u} = -\frac{1}{\alpha^2} f(x)\).

6. Verify, by direct substitution, that the kernel \(K\) given by (18b) satisfies the diffusion equation (1a), as was claimed in Example 1.

7. Use (18) to show that if the initial temperature \(f(x)\) is
   (a) a periodic function of \(x\), with period \(\tau\), then so is the solution \(u(x, t)\).
   (b) an odd function of \(x\), then so is the solution \(u(x, t)\).
   (c) an even function of \(x\), then so is the solution \(u(x, t)\).

8. (Inclusion of a source term) In Example 3 of Section 16.8 we find that if there is a heat source distribution \(F(x, t)\) within the medium, then in place of the homogeneous field equation \(L[u] = \alpha^2 u_{xx} - u_t = 0\) we have the nonhomogeneous equation \(L[u] = \alpha^2 u_{xx} - u_t = -F(x, t)\); \(F\) acts as a source where \(F > 0\) and as a sink where \(F < 0\).

   (a) Then the problem
   \[
   \begin{align*}
   \alpha^2 u_{xx} - u_t &= -F(x, t), & (-\infty < x < \infty, 0 < t < \infty) \\
   u(x, 0) &= f(x), & (-\infty < x < \infty)
   \end{align*}
   \]
   together with suitable boundary conditions at \(x = \pm \infty\), has two inputs, the initial temperature \(f(x)\) and the source distribution \(F(x, t)\). By linearity, the response \(u(x, t)\) should be the sum of the responses to these individual inputs. Specifically, show that if \(v(x, t)\) and \(w(x, t)\) are solutions to the problems
   \[
   \begin{align*}
   \alpha^2 v_{xx} - v_t &= 0, & (-\infty < x < \infty, 0 < t < \infty) \\
   v(x, 0) &= f(x), & (-\infty < x < \infty)
   \end{align*}
   \]
   and
   \[
   \begin{align*}
   \alpha^2 w_{xx} - w_t &= -F(x, t), & (-\infty < x < \infty, 0 < t < \infty) \\
   w(x, 0) &= 0, & (-\infty < x < \infty)
   \end{align*}
   \]
   respectively, then \(u(x, t) = v(x, t) + w(x, t)\). Since the \(v\) problem is already solved in Example 1, the remainder of this exercise is devoted to the \(w\) problem.
   (b) Consider the case where \(F = F(t)\) is a function of \(t\) alone. Choosing between the Fourier and Laplace transforms, solve for \(w\). (The answer should be in the form of an integral.) Explain why you selected the transform that you did, and not the other.
   (c) Now consider the case where \(F = F(x)\) is a function of \(x\) alone. Using a Fourier transform, show that

---

8.4. Fourier and Laplace Transforms

989
Chapter 18. Diffusion Equation

\[ w(x,t) = F(x) + F^{-1} \left\{ \frac{1 - e^{-a^2 \omega^2 t}}{a^2 \omega^2} \right\}. \quad (8.4) \]

(d) The inverse needed in (8.4) is not found in our brief table (Appendix D). Nonetheless, show that

\[ F^{-1} \left\{ \frac{1 - e^{-a^2 \omega^2 t}}{a^2 \omega^2} \right\} = \frac{1}{\alpha} \sqrt{\frac{t}{\pi}} e^{-x^2/(4a^2 t)} - \frac{x}{2a^2} \text{erfc} \left( \frac{x}{2a\sqrt{t}} \right), \quad (8.5) \]

so

\[ w(x,t) = \frac{1}{\alpha} \int_{-\infty}^{\infty} F(x - \xi) \left[ \sqrt{\frac{t}{\pi}} e^{-\xi^2/(4a^2 t)} - \frac{\xi}{2a^2} \text{erfc} \left( \frac{\xi}{2a\sqrt{t}} \right) \right] d\xi. \quad (8.6) \]

HINT: Letting \( \tilde{g} = (1 - e^{-a^2 \omega^2 t})/(\alpha^2 \omega^2) \), show that \( \tilde{g}_t = e^{-a^2 \omega^2 t} \tilde{g}_t \), so that \( g_t = \{\exp[-x^2/(4a^2 t)]\}/(2a\sqrt{\pi t}) \), with \( g_{t=0} = 0 \). Thus,

\[ g(x,t) = \int_0^t \frac{e^{-x^2/(4a^2 \tau)}}{2a\sqrt{\pi \tau}} d\tau. \]

Then, use change of variables and integration by parts.

9. (Small-\( t \) solution for finite rod) In Section 18.3 we obtain, by separation of variables, the solution

\[ u(x,t) = \frac{400}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L} e^{-((n\pi a/L)^2 t)} \quad (9.1) \]

to the problem of heat conduction in a rod of length \( L \) initially at a uniform temperature \( u(x,0) = 100^\circ \), with both ends held subsequently at \( 0^\circ \). It was pointed out that (9.1) converges rapidly if \( t \) is large and slowly if \( t \) is small. Our object in this exercise is to show how to use the results obtained in this section to obtain a complementary result: a series solution that converges rapidly for small \( t \) (and slowly for large \( t \)). First, observe that our use of the half-range sine series to solve the stated finite-rod problem, on \( 0 < x < L \), is equivalent to solving the periodically extended infinite-rod problem

\[ \alpha^2 u_{xx} = u_t, \quad (-\infty < x < \infty, \ 0 < t < \infty) \]
\[ u(x,0) = f_{\text{ext}}(x), \]

where \( f_{\text{ext}} \) is the square wave shown here. In effect, the separation-of-variables solution amounts to expanding \( f_{\text{ext}} \) in a Fourier sine series, finding the response due to each sine term, and then adding them. Alternatively, suppose that we expand \( f_{\text{ext}}(x) \) as \( f_{\text{ext}}(x) = f_1(x) + f_2(x) + \cdots \), where the \( f_j \)'s are as shown. That is,

\[ f_{\text{ext}}(x) = \left\{ -100 + 200[H(x) - H(x-L)] \right\} + 200[H(x+2L) - H(x+L)] \]
\[ + H(x-2L) - H(x-3L) + 200[H(x+4L) - H(x+3L)] + H(x-4L) - H(x-5L) + \cdots. \]

(a) Recalling the solution (14), for the case where \( f(x) \) is given by (13), show that the responses to \( f_1, f_2, \ldots \) are

\[ u(t,x) = 100 \left[ \text{erf} \left( \frac{x}{2\alpha \sqrt{t}} \right) - \text{erf} \left( \frac{x-L}{2\alpha \sqrt{t}} \right) - 1 \right] \quad (9.3) \]
and
\[ u_j(x,t) = 100 \left[ \text{erf}\left( \frac{x+2(j-1)L}{2\alpha \sqrt{t}} \right) - \text{erf}\left( \frac{x+(2j-3)L}{2\alpha \sqrt{t}} \right) \right] \\
+ \text{erf}\left( \frac{x-2(j-1)L}{2\alpha \sqrt{t}} \right) - \text{erf}\left( \frac{x-(2j-1)L}{2\alpha \sqrt{t}} \right) \right] ,
\]
for \( j \geq 2 \).

(b) Explain, in simple physical terms, why the series solution
\[ u(x,t) = u_1(x,t) + u_2(x,t) + \cdots \] (9.5)
should converge rapidly for small \( t \).

(c) To illustrate, compute \( u(x,t) \) at \( x = 1 \), \( t = 0.1 \), with \( \alpha^2 = 1.14 \) and \( L = 10 \), using equation (9.1), and again, using equation (9.5). Two-significant-figure accuracy will suffice.

NOTICE: To calculate the error functions, use either the Hastings formula that we gave in a footnote, or use computer software such as the evalf Maple command.

(d) In (b) we stated that (9.5) should converge rapidly for small \( t \). What do we really mean by small \( t \)? \( t < 1 \) sec? \( t < [L/(4\alpha)]^2 \)? Propose some such inequality that can be used as a guarantee that (9.5) will indeed be rapidly convergent.

(e) Obtain a computer plot of the solution at \( t = 0.1, 0.5, \) and 1, using the approximation \( u(x,t) \approx u_1(x,t) \), where \( u_1 \) is given in (9.3). Take \( \alpha^2 = 1.14 \) and \( L = 10 \), as in part (c).

10. (Translating rod) We saw in Section 18.2 that if the rod is translating rightward with constant speed \( v \), then the PDE becomes \( \alpha^2 u_{xx} = u_t + V u_x \), where \( V = v/c \) and \( c \) is the specific heat of the material. Use the Fourier transform to solve the problem.

\[ \alpha^2 u_{xx} = u_t + V u_x, \quad (-\infty < x < \infty, \ 0 < t < \infty) \]
\[ u(x,0) = f(x), \quad (-\infty < x < \infty) \]
where \( u \to 0 \) and \( u_x \to 0 \) as \( x \to \pm \infty \). Of course, your result should reduce to (18) for the case where \( V = 0 \).

11. Rework Example 2 with the initial condition changed to
\[ u(x,0) = \text{constant} = u_0, \] and show that
\[ u(x,t) = u_0 + (u_1 - u_0) \text{erfc} \left( \frac{x}{2\alpha \sqrt{t}} \right). \]

12. (Oscillatory temperature at the left end) If an oscillatory temperature is maintained at the left end of a semi-infinite rod, then we expect the solution \( u(x,t) \) to approach a steady-state oscillation. Specifically, we have

\[ \alpha^2 u_{xx} = u_t, \quad (0 < x < \infty, \ 0 < t < \infty) \]
\[ u(0,t) = u_0 \cos \omega t, \quad (0 < t < \infty) \] (12.1)
\[ u \to 0 \text{ as } x \to \infty, \quad (0 < t < \infty) \]
Derive the steady-state oscillatory solution
\[ u(x,t) = u_0 e^{-r x} \cos (\omega t - r x), \] (12.2)
where \( r = \alpha \sqrt{\omega/2} \). HINT: The simplest approach is to consider, instead, the problem
\[ \alpha^2 v_{xx} = v_t, \quad (0 < x < \infty, \ 0 < t < \infty) \]
\[ v(0,t) = u_0 e^{i\omega t}, \quad (0 < t < \infty) \] (12.3)
\[ v \to 0 \text{ as } x \to \infty, \quad (0 < t < \infty) \]
which can then be solved by seeking \( v \) in the form
\[ v(x,t) = X(x)e^{i\omega t}. \] (12.4)

Then \( u \) is found as the real part of \( v \):
\[ u(x,t) = \text{Re} v(x,t). \] (12.5)

Such a complex function method of solution, for differential equations with oscillating forcing terms, is the subject of Exercise 12 in Section 3.8. Alternatively, we could anticipate the phase shift caused by the \( u_1 \) term in the PDE and seek \( u \), directly, in the form
\[ u(x,t) = A(x) \cos \omega t + B(x) \sin \omega t, \]
but the complex function method is more attractive since it permits us to work with a single quantity \( e^{i\omega t} \) rather than with a cosine and sine. Observe that there is no initial condition \( u(x,0) \) included in (12.1) because we are concerned here only with the steady-state solution. NOTE: Besides the heat conduction problem considered here, the problem (12.1) also arises (with the \( t \)'s changed to \( y \)'s) in fluid mechanics regarding the viscous flow, in the upper half plane \( y > 0 \), that is caused by harmonic oscillation of an infinite flat plate at \( y = 0 \). There, the problem is known as Stokes's second problem and was studied also by Lord Rayleigh.

13. (a) Show that the integral in (40) does give \( u(x,t) = 100 \text{erfc} \left( x/(2\alpha \sqrt{t}) \right) \).
(b) Use computer software to evaluate the latter at \( x = 0, 2, 4, 6, \ldots, 20 \), with \( t = 10 \) and \( \alpha^2 = 1.14 \).

14. (Heat flux at left end) The problem
\[ \alpha^2 u_{xx} = u_t, \quad (0 < x < \infty, \ 0 < t < \infty) \]
\[ u(x, 0) = 0, \quad (0 < x < \infty) \]
\[ u_x(0, t) = -Q, \quad (0 < t < \infty) \]  
(14.1)

with \( u \to 0 \) as \( x \to \infty \), differs from (24) in that the boundary condition at \( x = 0 \) is of Neumann type rather than of Dirichlet type. Physically, \( u_x(0, t) = -Q \) (where \( Q \) is a prescribed constant) corresponds to the maintaining of a constant heat flux into the rod at \( x = 0 \), for all \( t > 0 \) (i.e., into the rod if \( Q > 0 \), out of the rod if \( Q < 0 \)).

(a) Using the Laplace transform, derive the result
\[ u(x, t) = \frac{\alpha Q}{\sqrt{\pi}} \int_0^t e^{-x^2/(4\alpha^2 \tau)} \sqrt{\tau} \, d\tau. \]  
(14.2)

In particular, use (14.2) to solve for the temperature at the left end, \( u(0, t) \), and sketch its graph.

(b) Show that the integral in (14.2) can be evaluated, and that we obtain
\[ u(x, t) = \frac{Q}{\sqrt{\pi}} \left[ 2 \alpha \sqrt{\frac{t}{\pi}} e^{-x^2/(4\alpha^2 t)} - x \text{erfc} \left( \frac{x}{2\alpha \sqrt{t}} \right) \right]. \]  
(14.3)

15. We claimed, in a footnote, that Hastings’s approximate formula for \( \text{erf}(x) \) is uniformly accurate, over \( 0 \leq x < \infty \), to \( \pm 0.000025 \). Check that claim for \( x = 0.5, 1 \), and 2 by using that formula to compute \( \text{erf}(x) \), and comparing those results with results obtained either from computer software (such as the Maple evalf command) or from tables.

18.5 The Method of Images (Optional)

The method of images is a method of fictitiously extending the problem domain so as to satisfy homogeneous boundary conditions by means of symmetries or antisymmetries. In this section we not only illustrate the method for the diffusion equation, we also establish a class of PDE’s for which the method works.

18.5.1. Illustration of the method. To illustrate the method of images, consider the diffusion problem
\[ \alpha^2 u_{xx} = u_t, \quad (0 < x < \infty, \ 0 < t < \infty) \]  
(1a)
\[ u(x, 0) = f(x), \quad (0 < x < \infty) \]  
(1b)
\[ u(0, t) = 0, \quad (0 < t < \infty) \]  
(1c)

where \( u \to 0 \) and \( u_x \to 0 \) as \( x \to \infty \) as depicted in Fig. 1.

The idea behind the image method is to extend the problem domain to \( x = -\infty \) as shown in Fig. 2. In doing so, we also need to extend the initial condition to \( x = -\infty \). Calling the extended function \( f_{\text{ext}} \), we need \( f_{\text{ext}}(x) = f(x) \) for \( x > 0 \), but for \( x < 0 \) we can define \( f_{\text{ext}}(x) \) in any way we choose. Let us choose \( f_{\text{ext}}(x) \) to be the odd extension of \( f(x) \). For instance, if \( f(x) \) is as shown in Fig. 3, then \( f_{\text{ext}}(x) \) is as shown in Fig. 3b. With \( u(x, 0) = f_{\text{ext}}(x) \) being an odd function of \( x \) (i.e., antisymmetric about \( x = 0 \)), we expect \( u(x, t) \) to remain an odd function of \( x \) for all \( t > 0 \), and if \( u(x, t) \) is an odd function of \( x \) for all \( t > 0 \) then \( u(0, t) = 0 \) for all \( t > 0 \). That is, by building antisymmetry about \( x = 0 \) into the extended

*If \( u(x, t) \) is an odd function of \( x \), then \( u(-x, t) = -u(x, t) \), so \( u(0, t) = -u(0, t) \). Hence, \( 2u(0, t) = 0 \), so \( u(0, t) = 0 \).
18.5. The Method of Images

problem we are able to automatically satisfy the \( u(0, t) = 0 \) boundary condition in (1c).

Thus, the solution of the extended problem (Fig. 2) is also the solution of the original problem (Fig. 1). In fact, the extended problem was already solved in Example 1 of Section 18.4, so the desired solution of (1) is given by

\[
 u(x, t) = \int_{-\infty}^{\infty} f_{ext}(\xi) K(\xi - x; t) \, d\xi,
\]

where

\[
 K(\xi - x; t) = \frac{e^{-(\xi-x)^2/4\alpha^2 t}}{2\alpha \sqrt{\pi t}}.
\]

Since we use (2) to compute \( u \) only over the actual domain, \( 0 < x < \infty \), it is preferable (though not necessary) to re-express (2) so as to eliminate all reference to the fictitious extension, which extension is referred to as the image system. To do so, write

\[
 u(x, t) = \int_{-\infty}^{0} f_{ext}(\xi) K(\xi - x; t) \, d\xi + \int_{0}^{\infty} f_{ext}(\xi) K(\xi - x; t) \, d\xi
\]

\[
 = \int_{-\infty}^{0} f_{ext}(-\mu) K(-\mu - x; t) (-\,d\mu) + \int_{0}^{\infty} f_{ext}(\xi) K(\xi - x; t) \, d\xi
\]

\[
 = -\int_{\infty}^{0} [-f_{ext}(\mu)] K(\mu + x; t) \, d\mu + \int_{0}^{\infty} f_{ext}(\xi) K(\xi - x; t) \, d\xi,
\]

where we set \( \xi = -\mu \) in the first integral and used the oddness of \( f_{ext} \) [i.e., \( f_{ext}(-\mu) = -f_{ext}(\mu) \)] and the evenness of \( K \) [i.e., \( K(-\mu - x; t) = K(\mu + x; t) \)] in the fifth integral. Finally, set \( \mu = \xi \) and note that \( f_{ext}(\xi) = f(\xi) \) over \( 0 < \xi < \infty \), and obtain

\[
 u(x, t) = \int_{0}^{\infty} f(\xi) K(\xi - x; t) - K(\xi + x; t) \, d\xi
\]

\[
 = \int_{0}^{\infty} f(\xi) \frac{e^{-(\xi-x)^2/4\alpha^2 t} - e^{-(\xi+x)^2/4\alpha^2 t}}{2\alpha \sqrt{\pi t}} \, d\xi,
\]

which, as we desired, contains no reference to the image system.

Recall that we said that we "expect" \( u(x, t) \) to remain an odd function of \( x \) for all \( t > 0 \) and to thereby satisfy the condition (1c), that \( u(0, t) = 0 \) for all \( t > 0 \).

With (4) in hand, we can now verify that claim since (4) gives

\[
 u(0, t) = \int_{0}^{\infty} f(\xi)(0) \, d\xi = 0.
\]

**EXAMPLE 1.** For instance, let \( f(x) = 100 \) in (4). Then (4) gives (Exercise 1)

\[
 u(x, t) = 100 \text{erf} \left( \frac{x}{2\alpha \sqrt{t}} \right).
\]
Chapter 18. Diffusion Equation

We've plotted (6), at representative times, in Fig. 4, along with the fictitious extension (shown as dashed). Observe how the antisymmetry of \( u \) with respect to \( x \), ensures the satisfaction of the boundary condition \( u(0, t) = 0 \) for all \( t > 0 \).

![Figure 4. Solution for \( f(x) = 100 \).](image)

Suppose that instead of the homogeneous Dirichlet condition (1c) we have the homogeneous Neumann condition

\[
u_x(0, t) = 0. \quad (0 < t < \infty)
\]

In physical terms, instead of applying ice to the left end of the rod we insulate it so there is no heat flux across the face \( x = 0 \). In that case we extend \( f(x) \) so as to be an even function, symmetric about \( x = 0 \). Then, we expect \( u(x, t) \) to be a symmetric function of \( x \) for all \( t > 0 \) so that, by virtue of that symmetry, we will have \( u_x(0, t) = 0 \) for all \( t > 0 \) (Exercise 2).

18.5.2. Mathematical basis for the method. The key point, in applying the method of images is that if

\[
L[u] = \alpha^2 u_{xx} - u_t = 0 \quad (-\infty < x < \infty, \ 0 < t < \infty)
\]

and

\[
u(x, 0) = f(x), \quad (-\infty < x < \infty)
\]

together with suitable boundary conditions at \( x = \pm \infty \), then \( f(x) \) being odd implies that the solution \( u(x, t) \) is an odd function of \( x \) for all \( t > 0 \), and \( f(x) \) being even implies that \( u(x, t) \) is an even function of \( x \) for all \( t > 0 \). Let us explain the mathematical basis for that claim, not just for the diffusion equation but for other PDE’s as well.

We will draw upon the following elementary results, proof of which are left for the exercises.

1. Any function \( F(x) \) can be split into even and odd parts, \( F_e(x) \) and \( F_o(x) \), respectively, as

\[
F(x) = \frac{F(x) + F(-x)}{2} + \frac{F(x) - F(-x)}{2}.
\]

(9)
2. The algebra of even and odd functions is as follows:

\[
\begin{align*}
\text{even} \times \text{even} & = \text{even}, \\
\text{even} \times \text{odd} & = \text{odd}, \\
\text{odd} \times \text{odd} & = \text{even}.
\end{align*}
\] (10)

3. If \( E_1(x) \) and \( E_2(x) \) are even, and \( O_1(x) \) and \( O_2(x) \) are odd, then

\[
E_1(x) + O_1(x) = E_2(x) + O_2(x)
\] (11)

implies that

\[
E_1(x) = E_2(x) \quad \text{and} \quad O_1(x) = O_2(x).
\] (12)

4. If \( F(x) \) is even, then \( F'(0) = 0 \).

5. If \( F(x) \) is odd, then \( F(0) = 0 \).

6. If \( F(x) \) is even, then \( F(x), F''(x), F'''(x), \ldots \) are even and \( F'(x), F''(x), \ldots \) are odd.

7. If \( F(x) \) is odd, then \( F(x), F''(x), F'''(x), \ldots \) are odd and \( F'(x), F''(x), \ldots \) are even.

The foregoing results hold even if \( F \) depends on other variables as well, such as \( t \). For instance, if \( F(x, t) \) is an even function of \( x \), then \( F(x, t), F_x(x, t), F_{xx}(x, t), \ldots \) are even functions of \( x \), and \( F_x(x, t), F_{xx}(x, t), \ldots \) are odd functions of \( x \).

To proceed, it will be useful to consider the problem

\[
L[u] = \alpha^2 u_{xx} - V u_x - u_t = Q(x, t), \quad (|x| < \infty, \; t > 0) \quad (13a)
\]

\[
u(x, 0) = f(x), \quad (|x| < \infty) \quad (13b)
\]

which is more general than (1) by virtue of including the \( Vu_x \) term (associated with translation of the rod at a constant speed) and the \( Q \) term (associated with the presence of a distributed heat source or sink along the rod). Of course, (13) reduces to (1) if we set \( V = 0 \) and \( Q(x, t) = 0 \).

To track the development of the even and odd parts of \( u \), let us break \( u \) into the sum of its even and odd parts, say \( E \) and \( O \), respectively:

\[
u(x, t) = E(x, t) + O(x, t).
\] (14)

Putting (14) into (13a), and also splitting \( Q(x, t) \) and \( f(x) \) into their even and odd parts, gives

\[
\frac{\alpha^2}{\epsilon} E_{xx} + \frac{\alpha^2}{\epsilon} O_{xx} - \frac{V}{\epsilon} E_x - \frac{V}{\epsilon} O_x - \frac{E_t}{\epsilon} - \frac{O_t}{\epsilon} = \frac{Q_E}{\epsilon} + \frac{Q_O}{\epsilon},
\] (15)
Chapter 18. Diffusion Equation

where $e$ or $o$ shown below each term indicates whether that term is even or odd, respectively. For instance, the $\alpha^2O_{xx}$ term is odd according to item (7) above, and the $-VE_x$ term is odd according to item (6). Partial differentiation with respect to $t$ does not alter evenness or oddness (Exercise 7), so $E_t$ is even and $O_t$ is odd. Next, from item (3) it follows from (15) that $\alpha^2E_{xx} - VO_x - E_t = Q_e$, and $\alpha^2O_{xx} - VE_x - O_t = Q_o$. Similarly, it follows from (13b) that $E(x, 0) = f_e(x)$ and $O(x, 0) = f_o(x)$, so (13) can be split into the two problems

$$\alpha^2E_{xx} - E_t = Q_e + VO_x, \quad (-\infty < x < \infty, \ 0 < t < \infty) \ (16a)$$

$$E(x, 0) = f_e(x), \quad (-\infty < x < \infty) \quad (16b)$$

and

$$\alpha^2O_{xx} - O_t = Q_o + VE_x, \quad (-\infty < x < \infty, \ 0 < t < \infty) \ (17a)$$

$$O(x, 0) = f_o(x), \quad (-\infty < x < \infty) \quad (17b)$$

Suppose, first, that $V = 0$. Then (16) and (17) are uncoupled: (16) contains only $E(x, t)$ and (17) contains only $O(x, t)$. If both inputs $f$ and $Q$ are even, then $f_e = 0$ and $Q_e = 0$, the problem (17) on $O$ is homogeneous, and $O(x, t) \equiv 0$. Thus, if the inputs are even, then the solution $u(x, t)$ is even. If, on the other hand, $f$ and $Q$ are odd, then $f_o = 0$ and $Q_o = 0$, the problem on $E$ is homogeneous, and $E(x, t) \equiv 0$. Thus, if the inputs are odd, then the solution $u(x, t)$ is odd. The two italicized results enable us to use the method of images, as we did in Section 18.5.1.

However, if $V \neq 0$ then (16) and (17) are coupled by virtue of the $VO_x$ and $VE_x$ terms. For instance, even if $f$ and $Q$ are even, so $f_o = 0$ and $Q_o = 0$, (17) is still nonhomogeneous due to the $VE_x$, which acts as a source term and causes the development of a nonzero $O(x, t)$. Thus, we cannot use the method of images if $V \neq 0$, which result makes sense physically since translation of the rod (to the right if $V > 0$ and to the left if $V < 0$) will surely destroy any symmetry or antisymmetry in the solution about $x = 0$.

The upshot is that sometimes we can use the method of images and sometimes we cannot, depending on the operator $L$. What we need for the method of images to work is for $L$ to be both linear and even. We say that an operator $L$ is even if $L[u]$ is even whenever $u$ is even and odd whenever $u$ is odd; that is, an even operator preserves evenness and oddness.

Let us limit our attention to the linear operator

$$L = A \frac{\partial^2}{\partial x^2} + B \frac{\partial^2}{\partial t^2} + C \frac{\partial^2}{\partial x \partial t} + D \frac{\partial}{\partial x} + E \frac{\partial}{\partial t} + F. \quad (18)$$

where $A, \ldots, F$ are functions of $x$ and $t$. From the discussion above, we can see that $L$ will be an even operator if $A, B, E, F$ are even functions of $x$ and if $C, D$ are odd functions of $x$. In that case, let us re-express $L$ as

$$L = E_1 \frac{\partial^2}{\partial x^2} + E_2 \frac{\partial^2}{\partial t^2} + O_1 \frac{\partial^2}{\partial x \partial t} + O_2 \frac{\partial}{\partial x} + E_3 \frac{\partial}{\partial t} + E_4. \quad (19)$$
This case is typical rather than exceptional.

**EXAMPLE 2.** For instance, consider the classical one-dimensional wave equation
c^2u_{xx} = u_{tt} (studied in Chapter 19). In this case

\[ L = c^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \tag{20} \]

which is even because \( A = c^2 \) and \( B = -1 \) are even, \( C = D = 0 \) are odd, and \( E = F = 0 \) are even. (Remember that the zero function is both even and odd: it is even because its odd part is zero, and it is odd because its even part is zero.)

**THEOREM 18.5.1 Applicability of the Method of Images**

Let \( L \) be of the form (19) and therefore an even operator. Suppose that the problem

\[
L[u] = Q(x, t), \quad (-\infty < x < \infty, \ 0 < t < \infty) \tag{21a}
\]

\[
u(x, 0) = f(x), \quad (-\infty < x < \infty) \tag{21b}
\]

where \( u \to 0 \) and \( u_x \to 0 \) as \( x \to \pm\infty \), admits a unique solution \( u(x, t) \). If \( f \) and \( Q \) are even functions of \( x \) then so is \( u \), and if \( f \) and \( Q \) are odd functions of \( x \) then so is \( u \).

**Proof:** Let \( u(x, t) = E(x, t) + O(x, t) \), where \( E \) is even and \( O \) is odd. Then, \( L[u] = L[E + O] = L[E] + L[O] \) because \( L \) is linear. Further, \( L[E] \) is even and \( L[O] \) is odd because \( L \) is even. Thus, (21) splits into the problems

\[
L[E] = Q_e(x, t), \quad (-\infty < x < \infty, \ 0 < t < \infty) \tag{22a}
\]

\[
E(x, 0) = f_e(x), \quad (-\infty < x < \infty) \tag{22b}
\]

and

\[
L[O] = Q_o(x, t), \quad (-\infty < x < \infty, \ 0 < t < \infty) \tag{23a}
\]

\[
O(x, 0) = f_o(x), \quad (-\infty < x < \infty) \tag{23b}
\]

where \( E, E_x, O, \) and \( O_x \) tend to zero as \( x \to \pm\infty \). If \( f \) and \( Q \) are even, then \( f_o = 0 \) and \( Q_o = 0 \), and (23) gives the unique solution \( O(x, t) = 0 \), so \( u(x, t) \) is even. If \( f \) and \( Q \) are odd, then \( f_e = 0 \) and \( O_e = 0 \) and (22) gives the unique solution \( E(x, t) = 0 \), so \( u(x, t) \) is odd.

**Closure.** The method of images involves the fictitious extension of the problem domain such that homogeneous boundary conditions, which fall within the interior of the extended domain, are automatically satisfied by virtue of the symmetry or
18.6 Numerical Solution

18.6.1 The finite-difference method. We have seen that a wide variety of diffusion problems can be solved analytically using separation of variables or an integral transform. In more complicated cases one often gives up the hope of obtaining analytical solutions and turns, instead, to a numerical solution technique. For instance, the problem

\[
\alpha^2 u_{xx} = u_t, \quad (0 < x < L, \ 0 < t < \infty) \\
u(0, t) = p(t), \quad (0 < t < \infty)
\]  

antisymmetry of the various inputs. For the method to work, we need the symmetry (or antisymmetry) of the inputs to imply the symmetry (or antisymmetry) of the output \( u \). Such symmetry (or antisymmetry) will, indeed, be passed on to \( u \) if the operator \( L \) is linear and even where, by \( L \) being even, we mean that if \( u \) is even then \( L[u] \) is even and if \( u \) is odd then \( L[u] \) is odd. The linear operator \( L \) in (18) is even if it is of the form (19), where \( E_1(x, t), \ldots, E_4(x, t) \) are even functions of \( x \) and \( \bar{O}_1(x, t), \bar{O}_2(x, t) \) are odd functions of \( x \).
18.6. Numerical Solution

\[ u(L, t) = q(t), \quad (0 < t < \infty) \]  
\[ u(x, 0) = f(x) \quad (0 < x < L) \]

(1)

is readily solved by separation of variables if \( p(t) \) and \( q(t) \) are constants, but is so much more difficult if they are not constants (see Exercise 20, Section 18.3) that we might very well turn to numerical solution instead.

In this section we introduce one of the most important techniques for the numerical solution of PDE’s, the finite-difference method. To explain that method, let us use the representative problem (1).

Our first step is to discretize the problem so that we seek \( u(x, t) \) not over the entire \( x, t \) domain but only at discrete grid points or nodal points, with coordinates \( x_j, t_k \) in the \( x, t \) plane. That is, we divide \( L \) into \( N \) equal parts, of length \( \Delta x = L/N \), and define \( x_j = j\Delta x \), for \( j = 0, 1, \ldots, N \). Further, we choose a time increment \( \Delta t \) and define \( t_k = k\Delta t \), for \( k = 0, 1, 2, \ldots \). The resulting set of grid points, known as the computational grid, is shown in Fig. 1. At the open circle points \( u \) is known, from the initial or boundary conditions, and we wish to solve for \( u \) at the solid circle points.

Next, we seek a finite-difference approximation of the PDE (1a) that will relate \( u \) at the various grid points. For the \( u_t \) term in (1a) we use the difference quotient approximation

\[ u_t(x, t) = \lim_{\Delta t \to 0} \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \]  

(2)

That is, do not take the limit as \( \Delta t \to 0 \); we choose a small \( \Delta t \) and accept the error that results. We can treat the \( u_{xx} \) term in the same manner if we deal with one derivative at a time. Accordingly, write

\[ u_{xx} = (u_x)_x \approx \frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} \]  

(3)

and then approximate each of the derivatives in the numerator in the same way. Thus,

\[ u_{xx}(x, t) \approx \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} - \frac{u(x, t) - u(x - \Delta x, t)}{\Delta x} \]  

(4)

Figure 1. The computational grid.
or
\[ u_{xx}(x, t) \approx \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2}. \]  

Putting (2) and (5) into (1a), with \( x = x_j, x + \Delta x = x_{j+1}, x - \Delta x = x_{j-1}, t = t_k, \) and \( t + \Delta t = t_{k+1}, \) gives
\[ \alpha^2 \frac{u(x_{j+1}, t_k) - 2u(x_j, t_k) + u(x_{j-1}, t_k)}{(\Delta x)^2} \approx \frac{u(x_j, t_{k+1}) - u(x_j, t_k)}{\Delta t} \]

or
\[ \alpha^2 U_{j-1,k} - 2U_{j,k} + U_{j+1,k} \approx \frac{U_{j,k+1} - U_{j,k}}{\Delta t} \]

as our finite-difference approximation of the PDE (1a). Although not essential, we distinguish \( u(x_j, t_k) \) and \( U_{jk} \) as follows: \( u(x_j, t_k) \) is the exact solution of the PDE (1a) at \( x_j, t_k, \) whereas \( U_{jk} \) is the exact solution of the difference equation (7) at \( x_j, t_k. \) Since the two are not (in general) identical, because of the difference quotient approximations (2) and (5), it is useful to denote them by different letters, as we have.

If we solve (7) for \( U_{j,k+1} \) we obtain
\[ U_{j,k+1} = rU_{j-1,k} + (1 - 2r)U_{j,k} + rU_{j+1,k}, \]

where
\[ r \equiv \alpha^2 \frac{\Delta t}{(\Delta x)^2}. \]

Equation (8) enables us to compute \( U \) at a given grid point as a linear combination of \( U \)'s at the preceding time as indicated by the arrows in Fig. 2. Thus, it provides us with a "marching scheme" with which we can march out a solution, one line at a time, subject to the initial and boundary conditions
\[ U_{j,0} = f(j \Delta x) \equiv f_j, \quad (j = 1, 2, \ldots, N - 1) \]
\[ U_{0,k} = p(k \Delta t) \equiv p_k, \quad (k = 1, 2, \ldots) \]
\[ U_{N,k} = q(k \Delta t) \equiv q_k, \quad (k = 1, 2, \ldots) \]

That is, beginning at the initial time \( t = 0 \) we can use (8), together with (10), to compute the \( U \)'s all along the line \( k = 1, \) then along the line \( k = 2, \) and so on.

**EXAMPLE 1.** In (1), let \( \alpha^2 = 1, L = 1, p(t) = 10 + 100t, q(t) = 50, \) and \( f(x) = 20x. \) For purposes of illustration, let it suffice to take \( N = 4 \) (so \( \Delta x = 0.25 \)) and \( \Delta t = 0.01. \) Then the grid and the initial and boundary values are as shown in Fig. 3. Where did

\*We use a forward difference quotient (forward and backward difference quotients are defined in Section 16.4) in (3) and then backward difference quotients in the numerator of the right-hand side of (4) so that the result (5) is a centered formula, i.e., centered about \( x. \) These choices are discussed further in Exercise 1.

\*The finite-difference scheme (8) is generally attributed to E. Schmidt (1924) and L. Binder (1911).
we get the value \( u(0, 0) = 5 \) in the figure? The data is discontinuous at that point because \( u(x, 0) = f(x) = 20x = 0 \) there, whereas \( u(0, t) = 10 + 100t = 10 \) there. Consistent with the fact that diffusion is a smoothing process, it seems reasonable to use the average value \((0 + 10)/2 = 5\) there. Similarly, we use the average value \( u(1, 0) = (20 + 50)/2 = 35\) at the other corner. From (9), \( r = (1)(0.01)/(0.25)^2 = 0.16\), so (8) becomes

\[
U_{j,k+1} = 0.16U_{j-1,k} + 0.68U_{j,k} + 0.16U_{j+1,k}. \tag{11}
\]

Sweeping across the first time line, (11) gives

\[
\begin{align*}
U_{1,1} &= 0.16U_{0,0} + 0.68U_{1,0} + 0.16U_{2,0} \\
&= 0.16(5) + 0.68(5) + 0.16(10) = 5.8, \\
U_{2,1} &= 0.16U_{1,0} + 0.68U_{2,0} + 0.16U_{3,0} \\
&= 0.16(5) + 0.68(10) + 0.16(15) = 10, \\
U_{3,1} &= 0.16U_{2,0} + 0.68U_{3,0} + 0.16U_{4,0} \\
&= 0.16(10) + 0.68(15) + 0.16(35) = 17.4.
\end{align*}
\tag{12}
\]

Moving up to the second time line, (11) gives

\[
\begin{align*}
U_{1,2} &= 0.16(11) + 0.68(5.8) + 0.16(10) = 7.3, \\
U_{2,2} &= 0.16(5.8) + 0.68(10) + 0.16(17.4) = 10.5, \\
U_{3,2} &= 0.16(10) + 0.68(17.4) + 0.16(50) = 21.4,
\end{align*}
\tag{13}
\]

and so on.

**COMMENT.** Since the difference approximations (2) and (5) become exact only as \( \Delta t \to 0 \) and \( \Delta x \to 0 \), respectively, it is clear that we need \( \Delta t \) and \( \Delta x \) to be sufficiently small if we are to expect accurate results. Are \( \Delta t = 0.01 \) and \( \Delta x = 0.25 \) sufficiently small? Remember that smallness (or largeness) is always relative to some reference. Although there is no ready-made reference time with which to compare \( \Delta t \), observe from the PDE (1a) that \( \alpha^2/L^2 \) has the dimensions of \( 1/\text{time} \). Thus, we can use \( T = L^2/\alpha^2 \) as a reference time. And as a reference length we can simply use \( L \). Consequently, for our results to be accurate we need both

\[
\frac{\Delta t}{T} = \frac{\Delta x}{L} \ll 1 \quad \text{and} \quad \frac{\Delta x}{L} \ll 1 \tag{14}
\]

or, since \( \alpha = L = 1 \) in this example,

\[
\Delta t \ll 1 \quad \text{and} \quad \Delta x \ll 1. \tag{15}
\]

Although \( \Delta t = 0.01 \) is much smaller than unity, \( \Delta x = 0.25 \) is not, so we can expect our results to provide only a rough approximation of the exact solution. However, realize that (14) is only a rule of thumb. Typically, one runs the calculation, then reduces \( \Delta t \) and \( \Delta x \) and runs it again, until a sufficient degree of convergence is achieved.

Example 1 illustrates the numerical implementation of (8). In Example 2 let us test the method by choosing \( \Delta x \) and \( \Delta t \) that easily satisfy (15) and comparing
the numerical results with a known exact solution.

**EXAMPLE 2. Test Case.** Let us apply (8) to the case where \( \alpha^2 = 1 \), \( L = 1 \), \( p(t) = q(t) = 0 \), and \( f(x) = 100 \) because this case can be solved exactly by separation of variables [since \( p(t) \) and \( q(t) \) are constants]. Specifically, we have the exact solution

\[
 u(x,t) = \frac{400}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin n\pi x \ e^{-(n\pi)^2 t}
\]  

(16)

with which to compare our numerical solution. With (15) in mind, let us choose \( \Delta x = 0.02 \) (i.e., \( N = 50 \)) and \( \Delta t = 0.00018 \), in which case \( \tau = 0.00018/(0.02)^2 = 0.45 \).

![Figure 4. Application of (8), for the case \( p(t) = q(t) = 0 \), \( f(x) = 100 \), \( L = 1 \), \( \alpha^2 = 1 \), \( \Delta x = 0.02 \), \( \Delta t = 0.00018 \) (\( \tau = 0.45 \)).](image)

In this case our computational grid is quite fine so there are too many calculations to do by hand. However, it is easy to program the calculation (11) with a “do loop” on \( k \) and with the initial and boundary conditions

\[
 U_{j,0} = 100, \quad (j = 1,2,\ldots,49)
\]

(17a)

\[
 U_{0,k} = 0, \quad (k = 1,2,\ldots)
\]

(17b)

\[
 U_{50,k} = 0, \quad (k = 1,2,\ldots)
\]

(17c)

For the corner points (where the data are discontinuous) use average values as in Example 1: \( U_{0,0} = 50 \), \( U_{50,0} = 50 \). The numerical results are shown as dots in Fig. 4, and the solid curves correspond to the exact solution given by (16). We have plotted the results only for \( k = 20,100, \) and \( 300 \) and only over \( 0 < x < 0.5 \) because the solution is symmetric about the midpoint \( x = 0.5 \). Although the exact and numerical results are not identical, they can hardly be distinguished in Fig. 4.

The results of the test case in Example 2 are encouraging. In fact, if we change the boundary temperatures \( p(t) \) and \( q(t) \) from constants to time-varying functions, then the analytical separation of variables solution is made much more difficult.
yet the numerical finite-difference solution based upon equation (8) is unchanged, except for the data on the right-hand sides of (17b) and (17c).

\[ u(x, t) \]

Figure 5. Example 2, with \( \Delta t \) increased to 0.00022.

However, we must qualify our endorsement of the finite difference scheme given by equation (8). For suppose we change \( \Delta t \) even slightly, from 0.00018 to 0.00022. We see from Fig. 5 that the results quickly degenerate: after only 20 time steps a deviation from the exact solution is apparent, and by the time \( k = 40 \) the results are worthless. The error is oscillatory, both spatially (with period \( 2\Delta x \)) and temporally (with period \( 2\Delta t \), although that fact is not observable in the figure because the plots are not for consecutive \( k \) values). Even if we do not rely on the exact solution for comparison (indeed, in real applications we do not know the exact solution) it seems clear that the oscillations are some sort of numerical instability rather than a faithful representation of a physical reality because it would surely be unlikely that the spatial and temporal periodicity of such a physical event would exactly equal \( 2\Delta x \) and \( 2\Delta t \), respectively. To test this assertion we can halve \( \Delta x \), say, and rerun the calculation. Sure enough, an oscillation will result once again, this time with spatial period \( 2\Delta x' \), where \( \Delta x' = \Delta x/2 \) is the new spacing. Furthermore, we know that diffusion is a smoothing process, whereas the numerical results in Fig. 5 reveal quite the opposite tendency. Thus, even if we do not have the exact solution for comparison, it is clear that the oscillations imply some sort of numerical instability.

Short of a detailed analysis, let us briefly explain the breakdown observed in Fig. 5. Recall that we have used different letters to distinguish the exact solution \( u(x, t) \) from the approximate solution \( U_{j,k} \) generated by the finite-difference equation (8). We call the difference \( u(x_j, t_k) - U_{j,k} \) the accumulated truncation error at the \( j, k \) grid point, namely, the error incurred by replacing \( u_t \) and \( u_{xx} \) in (1a) by the finite-difference approximations (2) and (5), respectively. Besides the accumulated truncation error there is an additional error called the accumulated roundoff error, incurred because the computer rounds off numbers after a finite number of significant figures. Thus, if we further distinguish \( U_{j,k} \) as the values computed by a "perfect computer" (one that keeps an infinite number of significant figures), and
As the actual printout of the real computer, then we can express the total error as

\[
\text{total error} = u(x_j, t_k) - U_{j,k}^*
\]

\[
= [u(x_j, t_k) - U_{j,k}] + [U_{j,k} - U_{j,k}^*]
\]

\[
= \text{accumulated truncation error} + \text{accumulated roundoff error}.
\]

Regarding the accumulated truncation error, two closely related questions come to mind.

1. With \( \Delta x \) and \( \Delta t \) fixed, what is the behavior of the accumulated truncation error as \( k \to \infty \)?

2. At the fixed points in the \( x, t \) domain, does the accumulated truncation error tend to zero as the mesh is continually refined?

Let us rephrase the second question, which is crucial: At any chosen fixed point in the \( x, t \) domain, is it possible to reduce the accumulated truncation error to be smaller in magnitude than any prescribed number by sufficiently refining the grid, that is, by sufficiently reducing \( \Delta x \) and \( \Delta t \)? If so, we say that the finite-difference scheme is convegent.

Since the roundoff error enters randomly, we simply ask that the accumulated roundoff error remain small – for instance, that it remain bounded as \( k \to \infty \). If so, we say that the scheme is stable. (Be aware that these definitions are not entirely standard from one text to another. At least, our terminology here is consistent with the terminology used in our analogous discussion for ODE’s in Section 6.5.2.)

Analysis reveals that the finite-difference method (8) is both convergent and stable if \( \Delta x \) and \( \Delta t \) satisfy the criterion

\[
r = \alpha^2 \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2},
\]

and is both divergent and unstable if \( r > \frac{1}{2} \). [Sure enough, the excellent results displayed in Fig. 4 correspond to \( r = (1)(0.00018)/(0.02)^2 = 0.45 \), which satisfies (19), whereas those displayed in Fig. 5 correspond to \( r = (1)(0.00022)/(0.02)^2 = 0.55 \), which does not.] The analysis behind (19) is beyond our present scope, but we do outline the stability part of the analysis in the exercises and urge you to study the latter because it is a typical and powerful application of the matrix eigenvalue problem to the analysis of finite-difference methods.

The restriction (19) may be quite severe, for if \( \Delta x \) is chosen small for the sake of accuracy, then the maximum \( \Delta t \) allowed by (19) may be so small that it necessitates a great many time steps and, consequently, considerable computer time.

---

the optional remainder of this section we show how to modify the method so as to relieve us of the restriction (19).

18.6.2. Implicit methods: Crank–Nicolson, with iterative solution. (Optional)

Equation (8) is by no means the only possible finite-difference method for the diffusion equation (1a). For example, the $x$-wise differencing in (4) is at the initial time $t$, whereas some weighted average over the time interval $(t \rightarrow t + \Delta t)$ should be more accurate, namely,

$$u_{xx}(x, t) \approx (1 - \theta) \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2} + \theta \frac{u(x + \Delta x, t + \Delta t) - 2u(x, t + \Delta t) + u(x - \Delta x, t + \Delta t)}{(\Delta x)^2}, \quad (20)$$

where the number $\theta$ is specified so that $0 \leq \theta \leq 1$. Then, in place of (7) we have

$$\alpha^2 \left[ (1 - \theta) \frac{U_{j-1,k} - 2U_{j,k} + U_{j+1,k}}{(\Delta x)^2} + \theta \frac{U_{j-1,k+1} - 2U_{j,k+1} + U_{j+1,k+1}}{(\Delta x)^2} \right] = \frac{U_{j,k+1} - U_{j,k}}{\Delta t}, \quad (21)$$

which reduces to (7) if $\theta = 0$. It turns out that the parameter $\theta$ gives us the desired control. Specifically, it can be shown that if $\theta \geq \frac{1}{2}$, then (21) is convergent and stable for all $r > 0$; that is, if we use any $\theta$ greater than or equal to $1/2$, then the condition (19) can be discarded. The borderline case $\theta = \frac{1}{2}$ gives the well-known Crank–Nicolson scheme

$$-rU_{j-1,k+1} + 2(1 + r)U_{j,k+1} - rU_{j+1,k+1} = rU_{j-1,k} + 2(1 - r)U_{j,k} + rU_{j+1,k}, \quad (22)$$

for $j = 1, 2, \ldots, N - 1$ and $k = 0, 1, 2, \ldots$, where $r = \alpha^2 \Delta t/(\Delta x)^2$, as before.

Given the initial and boundary values, we can use (22) to compute the first line of unknowns $U_{1,1}, U_{2,1}, \ldots, U_{N-1,1}$, then the second line of unknowns $U_{1,2}, U_{2,2}, \ldots, U_{N-1,2}$, and so on, just as we did using (8). However, (22) presents a difficulty which we illustrate by writing it out for the first line of unknowns, with $r = 1$ and $N = 5$, say, for definiteness.

$$\begin{align*}
j = 1: & \quad -U_{0,1} + 4U_{1,1} - U_{2,1} = U_{0,0} + U_{2,0}, \\
j = 2: & \quad -U_{1,1} + 4U_{2,1} - U_{3,1} = U_{1,0} + U_{3,0}, \\
j = 3: & \quad -U_{2,1} + 4U_{3,1} - U_{4,1} = U_{2,0} + U_{4,0}, \\
j = 4: & \quad -U_{3,1} + 4U_{4,1} - U_{5,1} = U_{3,0} + U_{5,0}.
\end{align*} \quad (23)$$
The two underlined terms are known boundary values, so let us move them to the right with the other known terms. Then (23) becomes

\begin{align}
4U_{1,1} - U_{2,1} &= U_{0,1} + U_{0,0} + U_{2,0}, \\
-U_{1,1} + 4U_{2,1} - U_{3,1} &= U_{1,0} + U_{3,0}, \\
- U_{2,1} + 4U_{3,1} - U_{4,1} &= U_{2,0} + U_{4,0}, \\
- U_{3,1} + 4U_{4,1} &= U_{3,0} + U_{5,0} + U_{3,1}.
\end{align}

(24)

The difficulty, which we can see clearly in (24), is that the equations (24) are coupled. Thus, we need to solve the matrix equation (24) for the first line of unknowns \( U_{1,1}, U_{2,1}, U_{3,1}, U_{4,1} \), then increment \( k \) by 1 in (22) and solve for a similar matrix equation for the second line of unknowns \( U_{1,2}, U_{2,2}, U_{3,2}, U_{4,2} \), and so on. In contrast, the scheme (8) gives uncoupled equations for \( U_{1,k+1}, U_{2,k+1}, \ldots, U_{N-1,k+1} \).

Thus, we call (22) an implicit scheme, whereas (8) is an explicit scheme. Graphically, the Crank–Nicolson scheme (22) corresponds to the computational pattern shown in Fig. 6, which reveals the nearest-neighbor coupling and is in contrast with the pattern shown in Fig. 2 for the scheme (8).

Expressing (22) in matrix form and recalling the initial and boundary conditions given by (10) gives

\[ \begin{bmatrix}
2(1 + r) & -r & \cdots & 0 \\
-r & 2(1 + r) & -r & \\
\vdots & \ddots & \ddots & -r \\
0 & \cdots & -r & 2(1 + r)
\end{bmatrix}
\begin{bmatrix}
U_{1,k+1} \\
U_{2,k+1} \\
\vdots \\
U_{N-2,k+1} \\
U_{N-1,k+1}
\end{bmatrix}
= \begin{bmatrix}
rp_{k+1} + rp_k + 2(1 - r)U_{1,k} + rU_{2,k} \\
rU_{1,k} + 2(1 - r)U_{2,k} + rU_{3,k} \\
\vdots \\
rU_{N-3,k} + 2(1 - r)U_{N-2,k} + rU_{N-1,k} \\
rU_{N-2,k} + 2(1 - r)U_{N-1,k} + rq_{k+1} + rq_k
\end{bmatrix}. \]

(25)

The system (25) is of the matrix form

\[ \mathbf{A}U_{k+1} = \mathbf{c}, \]

(26)

where \( \mathbf{A} \) is tridiagonal (due to the nearest-neighbor coupling) and symmetric. \( \mathbf{A} \) and \( \mathbf{c} \) are known and \( U_{k+1} \) is the unknown. The idea is to set \( k = 0 \) and solve (25) for the entire line of unknowns \( U_{1,1}, U_{2,1}, \ldots, U_{N-1,1} \), then set \( k = 1 \) and solve for the line \( U_{1,2}, U_{2,2}, \ldots, U_{N-1,2} \), and so on.
Notice carefully that the situation is not as bad as it may appear to be because \( A \) is strongly diagonal. That is, the off-diagonal terms \((-r’s\) and \(0’s\)) are small compared to the diagonal terms \([2(1+r)’s]\), so (25) is almost uncoupled. Thus, we say that (25) is only weakly coupled. To take advantage of this circumstance, let us split \( A \) into its diagonal part plus the deviation from that,

\[
A = \begin{bmatrix}
2(1+r) & 0 & \cdots & 0 \\
0 & 2(1+r) & & \vdots \\
& \ddots & \ddots & \ddots \\
0 & \cdots & 2(1+r) & \\
0 & \cdots & 0 & 2(1+r)
\end{bmatrix}

+ \begin{bmatrix}
0 & -r & 0 & \cdots & 0 \\
-2 & 0 & -r & & \vdots \\
& \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & -r & 0 & -r \\
0 & \cdots & 0 & -r & 0
\end{bmatrix}
\]

\[\equiv 2(1+r)I + A',\]  

(27)

where \( I \) is an \((N-1) \times (N-1)\) identity matrix and \( A' \) is the “deviation” matrix. Then (26) becomes

\[ [2(1+r)I + A']U_{k+1} = c \]  

(28)

or

\[ U_{k+1} = \frac{1}{2(1+r)} c - \frac{1}{2(1+r)} A'U_{k+1}. \]  

(29)

Since \( A' \) is small compared to \( 2(1+r)I \), in (28), we could neglect it altogether and obtain, from (29),

\[ U_{k+1} \approx \frac{1}{2(1+r)} c. \]  

(30)

Better yet, we could accept (30) as an initial approximation

\[ U_{k+1}^{(0)} = \frac{1}{2(1+r)} c \]  

(31)

and obtain an improved result as

\[ U_{k+1}^{(1)} = \frac{1}{2(1+r)} c - \frac{1}{2(1+r)} A'U_{k+1}^{(0)}. \]  

(32)

In fact, repeating this procedure, we have the iterative algorithm

\[ U_{k+1}^{(n+1)} = \frac{1}{2(1+r)} \left[ c - A'U_{k+1}^{(n)} \right], \]  

(33)
where $\mathbf{U}^{(n+1)}_{k+1}$ is the $(n+1)$st iterate of $\mathbf{U}_{k+1}$. With $k$ fixed, we carry out (33) for $n = 0, 1, 2, \ldots$ [where $\mathbf{U}^{(0)}_{k+1}$ is given by (31)], until suitable convergence is attained. For example, we might continue the iteration until each element of $\mathbf{U}^{(n+1)}_{k+1}$ and the corresponding element of $\mathbf{U}^{(n)}_{k+1}$ differ in magnitude by less than $10^{-5}$.

The scheme given by (33) and (31) is called Jacobi iteration, and it is shown in the exercises that Jacobi iteration converges to the exact solution of $\mathbf{A}\mathbf{U}_{k+1} = \mathbf{c}$ for all finite values of $r$. In fact, we can improve upon (33) slightly, by using the $(n+1)$st components as soon as they become available. To elaborate, let us write out the Jacobi scheme (33):

\[
\begin{align*}
\mathbf{U}_{1,k+1}^{(n+1)} &= \frac{1}{2(1+r)} \left[ c_1 - a'_{1,1} \mathbf{U}_{1,k+1}^{(n)} - \cdots - a'_{1,N-1} \mathbf{U}_{N-1,k+1}^{(n)} \right] \\
\mathbf{U}_{2,k+1}^{(n+1)} &= \frac{1}{2(1+r)} \left[ c_2 - a'_{2,1} \mathbf{U}_{1,k+1}^{(n)} - a'_{2,2} \mathbf{U}_{2,k+1}^{(n)} - \cdots - a'_{2,N-1} \mathbf{U}_{N-1,k+1}^{(n)} \right] \\
&\vdots \\
\mathbf{U}_{N-1,k+1}^{(n+1)} &= \frac{1}{2(1+r)} \left[ c_{N-1} - a'_{N-1,1} \mathbf{U}_{1,k+1}^{(n)} - \cdots - a'_{N-1,N-2} \mathbf{U}_{N-2,k+1}^{(n)} - a'_{N-1,N-1} \mathbf{U}_{N-1,k+1}^{(n)} \right].
\end{align*}
\]

Having computed $\mathbf{U}_{1,k+1}^{(n+1)}$ from the first equation in (34), let us use that value in place of the less-up-to-date value $\mathbf{U}_{1,k+1}^{(n)}$ that appears in the right-hand side of the second equation. Similarly, let us use the already computed values $\mathbf{U}_{1,k+1}^{(n+1)}$, $\mathbf{U}_{2,k+1}^{(n+1)}$, $\mathbf{U}_{3,k+1}^{(n+1)}$, $\mathbf{U}_{4,k+1}^{(n+1)}$, and so on. This idea produces the Gauss–Seidel scheme

\[
\mathbf{U}_{k+1}^{(n+1)} = \frac{1}{2(1+r)} \left[ \mathbf{c} - \mathbf{L} \mathbf{U}_{k+1}^{(n+1)} - \mathbf{M} \mathbf{U}_{k+1}^{(n)} \right],
\]

where $\mathbf{L}$ and $\mathbf{M}$ are the lower and upper parts, respectively, of $\mathbf{A}'$. That is, $\mathbf{A}' = \mathbf{L} + \mathbf{M}$, where

\[
\mathbf{L} = \begin{bmatrix} 0 & \cdots & 0 \\ -r & 0 & \cdots \\ 0 & -r & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} 0 & -r & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & -r \\ 0 & \cdots & 0 \end{bmatrix}.
\]
Like the Jacobi method, the Gauss–Seidel method converges for all finite \( r \), and approximately twice as fast.

Finally, there is another simple improvement that will further increase the speed of convergence. Re-expressing (35) as

\[
U_{k+1}^{(n+1)} = U_{k+1}^{(n)} + \frac{1}{2(1+r)} \left\{ e - LU_{k+1}^{(n+1)} - [M + 2(1+r)I]U_{k+1}^{(n)} \right\}
\]

we insert a numerical "control parameter" \( \omega \):

\[
U_{k+1}^{(n+1)} = U_{k+1}^{(n)} + \omega \Delta U_{k+1}^{(n)}.
\]

The idea is general and can be applied to any iterative scheme: \( \Delta U_{k+1}^{(n)} \) is a "correction term," and adjusting the size of the correction, by means of \( \omega \), might speed the convergence. Often, one chooses \( \omega \) based on numerical experimentation, but in the present case it can be shown analytically that the optimum \( \omega \) is given by

\[
\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \mu^2}}, \quad \text{where} \quad \mu = \frac{r}{1 + r} \cos \frac{\pi}{N}.
\]

We see from (39) that \( \omega_{\text{opt}} \) lies somewhere between 1 and 2. Since \( \omega_{\text{opt}} > 1 \), the modified Gauss–Seidel scheme (38) is known as **successive overrelaxation**, or SOR for brevity.

**EXAMPLE 3.** To illustrate the Jacobi, Gauss–Seidel, and SOR methods, consider the problem where \( u(0, t) = u(L, t) = 0 \) and \( u(x, 0) = 100 \), and choose \( N = 4 \) and \( r = 1 \). Then the Crank–Nicolson scheme (22) gives these simultaneous equations for the unknown values \( U_{1,1}, U_{2,1}, U_{3,1} \) in the first "time-line":

\[
\begin{align*}
4U_{1,1} - U_{2,1} &= 150, \\
-U_{1,1} + 4U_{2,1} - U_{3,1} &= 200, \\
-U_{2,1} + 4U_{3,1} &= 60.
\end{align*}
\]

The latter is readily solved, and we obtain

\[
U_{1,1} = 55.54, \quad U_{2,1} = 72.14, \quad U_{3,1} = 33.04,
\]

but let us use (40) to illustrate the three methods of iterative solution.

*Jacobi*:

\[
\begin{align*}
U_{1,1}^{(n+1)} &= \frac{1}{4} (U_{2,1}^{(n)} + 150), \\
U_{2,1}^{(n+1)} &= \frac{1}{4} (U_{1,1}^{(n)} + U_{3,1}^{(n)} + 200), \\
U_{3,1}^{(n+1)} &= \frac{1}{4} (U_{2,1}^{(n)} + 60),
\end{align*}
\]
Chapter 18. Diffusion Equation

So

\[ U_{1,1}^{(0)} = \frac{1}{4}(0 + 150) = 37.5, \]
\[ U_{2,1}^{(0)} = \frac{1}{4}(0 + 0 + 200) = 50, \]
\[ U_{3,1}^{(0)} = \frac{1}{4}(0 + 60) = 15, \] (43)
\[ U_{1,1}^{(1)} = \frac{1}{4}(50 + 150) = 50, \]
\[ U_{2,1}^{(1)} = \frac{1}{4}(37.5 + 15 + 200) = 63.13, \] (44)
\[ U_{3,1}^{(1)} = \frac{1}{4}(50 + 60) = 27.5, \]

and so on.

**Gauss–Seidel:**

\[ U_{1,1}^{(n+1)} = \frac{1}{4}(U_{2,1}^{(n)} + 150), \]
\[ U_{2,1}^{(n+1)} = \frac{1}{4}(U_{1,1}^{(n+1)} + U_{3,1}^{(n)} + 200), \]
\[ U_{3,1}^{(n+1)} = \frac{1}{4}(U_{2,1}^{(n+1)} + 60), \] (45)

so

\[ U_{1,1}^{(0)} = \frac{1}{4}(0 + 150) = 37.5, \]
\[ U_{2,1}^{(0)} = \frac{1}{4}(0 + 0 + 200) = 50, \] (46)
\[ U_{3,1}^{(0)} = \frac{1}{4}(0 + 60) = 15, \]
\[ U_{1,1}^{(1)} = \frac{1}{4}(50 + 150) = 50, \]
\[ U_{2,1}^{(1)} = \frac{1}{4}(50 + 15 + 200) = 66.25, \] (47)
\[ U_{3,1}^{(1)} = \frac{1}{4}(66.25 + 60) = 31.56, \]

and so on.

**SOR:** First, we need to compute \( \omega_{opt} \): 
\[ \mu = \frac{1}{2} \cos \frac{\pi}{4} = \frac{\sqrt{2}}{4}, \]  
\[ \omega_{opt} = 2/(1 + \sqrt{1 - \mu^2}) = 1.03. \] As in (43) and (46),

\[ U_{1,1}^{(0)} = 37.5, \quad U_{2,1}^{(0)} = 50, \quad U_{3,1}^{(0)} = 15. \] (48)

Next, make a "tentative" Gauss–Seidel step using (45) (with \( n = 0 \)) and the values given by (48):

\[ U_{1,1}^{(1)} = 50 \quad \text{so} \quad \Delta U_{1,1}^{(0)} = 50 - 37.5 = 12.5, \]
18.6. Numerical Solution

\[ U_{2,1}^{(1)} = 66.25 \quad \text{so} \quad \Delta U_{2,1}^{(0)} = 66.25 - 50 = 16.25, \]
\[ U_{3,1}^{(1)} = 31.56 \quad \text{so} \quad \Delta U_{3,1}^{(0)} = 31.56 - 15 = 16.56. \]

Now make the SOR step:

\[ U_{1,1}^{(1)} = U_{1,1}^{(0)} + \omega \Delta U_{1,1}^{(0)} = 37.5 + 1.03(12.5) = 50.38, \]
\[ U_{2,1}^{(1)} = U_{2,1}^{(0)} + \omega \Delta U_{2,1}^{(0)} = 50 + 1.03(16.25) = 66.74, \]
\[ U_{3,1}^{(1)} = U_{3,1}^{(0)} + \omega \Delta U_{3,1}^{(0)} = 15 + 1.03(16.56) = 32.06. \]

Carrying out one more iteration (Exercise 16) and comparing the successive \( U_{2,1}^{(n)} \) values, which should be representative, gives the results presented in Table 1.

**Table 1. Successive \( U_{2,1}^{(n)} \) values.**

<table>
<thead>
<tr>
<th>( n )</th>
<th>Jacobi</th>
<th>Gauss–Seidel</th>
<th>SOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td>63.13</td>
<td>66.25</td>
<td>66.74</td>
</tr>
<tr>
<td>2</td>
<td>69.38</td>
<td>71.41</td>
<td>71.70</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>Exact</td>
<td>72.14</td>
<td>72.14</td>
<td>72.14</td>
</tr>
</tbody>
</table>

**Closure.** The finite-difference method developed quickly beginning around 1950 when digital computers became widely available, and is one of the most important methods for solving partial differential equations. In this text we discuss it in the present section for the diffusion equation and in Section 20.5 for the Laplace equation. The basic idea is to discretize the problem so that instead of seeking \( u(x, t) \) over the given \( x, t \) domain we seek \( u \) only at a finite set of grid points. The PDE is discretized by replacing the various partial derivations by approximate difference quotients so, in place of the PDE, we end up with linear algebraic equations on the unknown \( U_{j,k} \) values at the grid points. There are many possible finite-difference schemes for a given PDE, depending upon the forms chosen for the difference quotients. The simplest one for our diffusion equation is given by (8) and is called an explicit scheme because it gives \( U_{j,k+1} \) explicitly in terms of known values. However, this scheme is invalid if \( r = \alpha^2 \Delta t / (\Delta x)^2 \) exceeds 1/2, so that if we choose \( \Delta x \) small, for accuracy, then we may need \( \Delta t \) to be so small that a prohibitively large number of time steps may be required. In the optional second part of this section we turn to implicit schemes to eliminate the \( r \leq 1/2 \) restriction. The price that we pay for this improvement is that at each time step we need to solve an \( (N - 1) \times (N - 1) \) matrix equation for the \( N - 1 \) values of \( U \) along that line. Fortunately, the matrix is strongly diagonal, so that efficient iterative methods can be used for the solution.
EXERCISES 18.6

1. We use a forward difference quotient in (3) and then backward difference quotients in the numerator of the right-hand side of (4).

(a) Show that if we use forward difference quotients in (3) and (4), then we obtain the scheme

\[ U_{j,k+1} = (1 + r)U_{j,k} - 2rU_{j+1,k} + rU_{j+2,k} \]  

(1.1)
in place of (8).

(b) Show that if we use backward difference quotients in (3) and (4), then we obtain the scheme

\[ U_{j,k+1} = (1 + r)U_{j,k} - 2rU_{j-1,k} + rU_{j-2,k} \]  

(1.2)

(c) Discuss any advantages or disadvantages that occur to you for the schemes (1.1) and (1.2) in comparison with (8).

2. Continue the hand calculation begun in Example 1. Specifically, determine \(U_{1,3}, U_{2,3}, U_{3,3}\), and \(U_{1,4}, U_{2,4}, U_{3,4}\).

3. Consider the problem \(\alpha^2 u_{xx} = u_t\) \((0 < x < 10, 0 < t < \infty)\) with the boundary and initial conditions \(u(0,t) = 100, u(t,0) = 100\), and \(u(x,0) = 0\). With \(\Delta x = 2.5, \Delta t = 2\), and \(\alpha^2 = 1\), use (8) to compute the first three “lines” of \(U_{j,k}\)’s: the nine values \(U_{1,1}\) through \(U_{3,3}\).

4. Show that if the PDE (1.1) is modified to include a Newton cooling term \(H u\) and a heat source distribution term \(F(x,t)\), as

\[ \alpha^2 u_{xx} = u_t + Hu - F(x,t), \quad (0 < x < 9, 0 < t < \infty) \]  

(4.1)

then in place of (8) we obtain

\[ U_{j,k+1} = u_{j+1,k} + \alpha^2 \frac{1}{2} (U_{j,k} - U_{j-1,k}) + rU_{j,k} + F(x,t) \Delta t \]  

(4.2)

where \(F_{j,k}\) denotes \(F(x_j, t_k)\) and \(H\) is a constant.

5. In Exercise 4, let \(\alpha^2 = 1\), \(L = 1\), \(u(0,t) = 0\), \(u(x,0) = 0\), \(u(1,t) = 0\), \(H = 0\), and \(F(x,t) = 10\). With \(\Delta t = 0.02\) and \(\Delta x = 0.25\), use (4.2) to compute the first three “lines” of \(U_{j,k}\)’s, i.e., the nine values \(U_{1,1}\) through \(U_{3,3}\).

6. Repeat Exercise 5, with these changes: \(u(0,t) = 100\), \(F(x,t) = 10\sin \pi x\).

7. Repeat Exercise 5, with these changes: \(u(x,0) = 100\), \(H = 4\), \(F(x,t) = 0\).

8. To see the smoothing nature of the diffusion process in a simple numerical example, consider the problem

\[ \alpha^2 u_{xx} = u_t, \quad (-\infty < x < \infty, 0 < t < \infty) \]

\[ u(x,0) = 100H(x), \]

where \(H(x)\) is the Heaviside function, and the diffusivity is \(\alpha^2 = 0.2\). Use (8), with \(\Delta t = 0.5\) and \(\Delta x = 1\), to compute the \(U_{j,k}\) values for the first three lines; i.e., through \(k = 3\). Plot \(U_{j,k}\) versus \(j\), for \(k = 0, 1, 2, 3\).

9. (a) To see the effect of the condition (19) in a simple calculation, use (8) for the problem

\[ u_{xx} = u_t, \quad (-\infty < x < \infty, 0 < t < \infty) \]

\[ u(x,0) = 100H(x), \]

where \(H(x)\) is the Heaviside function. With \(\Delta t = 0.2\) and \(\Delta x = 1\), compute the \(U_{j,k}\) values for the first five lines; i.e., through \(k = 5\). Keeping \(\Delta x = 1\), repeat the calculation with \(\Delta t = 0.4, 0.6, \text{and} 0.8\).

(b) Plot your results and discuss them.

(c) For the four different cases (\(\Delta t = 0.2, 0.4, 0.6, 0.8\)), plot your final temperature distribution (i.e., at \(k = 5\)) and also the exact solution (which can be found in Section 18.4).

10. In (1) let \(L = 12, p(t) = 100, q(t) = 0\), and \(f(x) = 0\), and suppose the rod is comprised of two different materials, with \(\alpha^2 = 1.8\) over \(0 < x < 6\) and \(\alpha^2 = 0.2\) over \(6 < x < 12\). With \(\Delta t = 0.5\) and \(\Delta x = 2\), use (8) to compute the \(U_{j,k}\) values for the first four lines; i.e., through \(k = 4\). Note that (8) holds over \(x < 6\) and over \(x > 6\) but not at the junction \(x = 6\). There, use the fact that the heat flux crossing \(x = 6\) from the left must equal the heat flux crossing \(x = 6\) toward the right or, from the Fourier law of heat conduction,

\[ K_L \frac{\partial u}{\partial x} \bigg|_{x=6-} = K_R \frac{\partial u}{\partial x} \bigg|_{x=6+}, \]

(10.1)

where \(K_L = 25\) is the thermal conductivity of the material to the left of \(x = 6\) and \(K_R = 3\) is the thermal conductivity of the material to the right of \(x = 6\). You will need to express (10.1) in finite-difference form.

11. The problem

\[ u_{xx} = u_t, \quad (0 < x < 1, 0 < t < \infty) \]

\[ u(0,t) = u(1,t) = 0, \quad u(x,0) = 100 \sin \pi x \]

(11.1)
admits the exact one-term solution \( u(x, t) = 100(\sin \pi x) \exp(-\pi^2 t) \).

(g) Use (8), with \( \Delta t = 0.02 \) and \( \Delta x = 0.25 \) to compute the first three lines of \( U_{1,k}^0 \)’s (i.e., through \( k = 3 \)) and compare your results with the exact values.

(b) Use (8) to evaluate \( u(0.5, 0.06) \), with \( \Delta t \) and \( \Delta x \) sufficiently small so that your value of \( u(0.5, 0.06) \) is correct to four significant figures. (Use a computer.)

(c) Use (8) to evaluate (by computer) the \( U_{1,k}^0 \)’s through \( k = 45 \), with \( \Delta t = 0.0010 \) and \( \Delta x = 0.05 \). Plot both your computed solution and the exact solution at \( t = 0.045 \) (i.e., at \( k = 45 \)).

(d) Use (8) to evaluate (by computer) the \( U_{1,k}^0 \)’s through \( k = 30 \), and \( \Delta t = 0.0015 \) and \( \Delta x = 0.05 \). Plot both your computed solution and the exact solution at \( t = 0.045 \) (i.e., at \( k = 30 \)). Interpret your results in the light of your results to parts (c) and (d) if, indeed, you worked those parts.

12. (Use of Taylor series) To derive the finite-difference approximation (5) we used the classical difference quotient definition of the derivative. Derive (5) using Taylor series instead.

HINT: Expand \( u(x+\Delta x, t) \) about \( x \):

\[
u(x+\Delta x, t) = u(x, t) + u_x(x, t)\Delta x + \frac{1}{2}u_{xx}(x, t)(\Delta x)^2 + \ldots.
\]

Similarly, expand \( u(x-\Delta x, t) \) about \( x \). Add those two formulas, cut off the series on the right-hand side after the first couple of terms (as an approximation), and solve for \( u_{xx}(x, t) \).

13. (Deriving the stability criterion (19)) We stated that the finite-difference scheme (8) is both convergent and stable if \( r \leq \frac{1}{2} \), and is both divergent and unstable if \( r > \frac{1}{2} \). Here, we outline a proof of the stability part of that claim and ask you to write out the steps and to supply any missing steps or reasoning. To begin, show that (8) can be expressed in matrix form as

\[
\begin{bmatrix}
    U_{1,k+1} & \cdots & U_{N-1,k+1}
\end{bmatrix}^T
\begin{bmatrix}
    1-2r & r & 0 & \cdots & 0 \\
    r & 1-2r & r & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & r & 1-2r & r \\
    0 & \cdots & 0 & r & 1-2r
\end{bmatrix}
\begin{bmatrix}
    U_{1,k} \\
    U_{2,k} \\
    \vdots \\
    U_{N-2,k} \\
    U_{N-1,k}
\end{bmatrix}
+ [rU_{0,k}, 0, \ldots, 0, rU_{N,k}]^T,
\]

where we use the transpose notation to save vertical space. More compactly, express (13.1) as

\[
U_{k+1} = AU_k + c_k,
\]

In stability analyses it is commonly assumed that roundoff errors occur only along the first line \((k = 0)\), say, and then to see whether the errors remain bounded as \( k \) increases. Thus, in place of the exact values \( U_0, c_0 \), the initial roundoff errors result in the actual values \( U_0^*, c_0^* \). The machine proceeds to compute values \( U_1^*, U_2^*, \ldots \) according to

\[
U_{k+1}^* = AU_k^* + c_k^*,
\]

rather than exact values \( U_1, U_2, \ldots \) according to (13.2). Show that the roundoff error \( e_k = \|U_k - U_k^*\| \) propagates according to

\[
e_{k+1} = A e_k + b_k,
\]

where \( b_k = c_k - c_k^* = [r e_{0,k}, 0, \ldots, 0, r e_{N,k}]^T \). From (13.4) show that

\[
e_k = \lambda^k e_0 + \lambda^{k-1} e_0.
\]

Show that the \((N-1) \times (N-1)\) matrix \( A \) must have \( N-1 \) orthogonal eigenvectors, say \( \Phi_1, \ldots, \Phi_{N-1} \), so that \( e_0 \) and \( b_0 \) can be expressed in the form

\[
e_0 = \alpha_1 \Phi_1 + \cdots + \alpha_{N-1} \Phi_{N-1},
\]

\[
b_0 = \beta_1 \Phi_1 + \cdots + \beta_{N-1} \Phi_{N-1}.
\]

Putting these into (13.5), show that

\[
e_k = (\alpha_1 \lambda_1^k + \beta_1 \lambda_1^{k-1}) \Phi_1 + \cdots + (\alpha_{N-1} \lambda_{N-1}^k + \beta_{N-1} \lambda_{N-1}^{k-1}) \Phi_{N-1},
\]

where \( \lambda_1, \ldots, \lambda_{N-1} \) are the eigenvalues of \( A \). Explain why it follows from (13.8) that for stability it is necessary and sufficient that

\[
|\lambda_1| \leq 1, \ldots, |\lambda_{N-1}| \leq 1.
\]

Since \( A \) is a function of \( r \) its \( \lambda \)’s are too. Hence, the most restrictive of the conditions (13.9) should give (19). Show that (13.9) does, indeed, give \( r \leq \frac{1}{2} \) or, more precisely,
\[ r \leq \frac{1}{1 - \cos \left( \frac{N - 1}{N} \pi \right)} \]  

(13.10)

For large \( N \) the right-hand side is approximately \( \frac{1}{2} \); e.g., for \( N = 50 \) (13.10) gives \( r \approx 0.50049 \). HINT: The \( A \) matrix is tridiagonal, of the type shown in Exercise 7 of Section 11.2 with \( \alpha = r, \beta = 1 - 2r, \) and \( \gamma = r \). Thus, its eigenvalues are, according to equation (7.1) of that exercise,

\[ \lambda_n = 1 - 2r + 2r \cos \frac{n\pi}{N} \]  

(13.11)

for \( n = 1, 2, \ldots, N - 1 \). Further, note that each inequality \( |\lambda_j| \leq 1 \) in (13.9) amounts to the statement \(-1 \leq \lambda_j \leq 1 \), hence the two inequalities \(-1 \leq \lambda_j \) and \( \lambda_j \leq 1 \). Of the \( 2N - 2 \) inequalities in (13.9), you should find that the most restrictive condition on \( r \) is given by (13.10).

14. (Stability of implicit scheme) The stability of the implicit scheme (21) for all \( r \) (if \( \theta \leq \frac{1}{2} \)) is a striking result. Proof of that result would follow the same lines as the proof that is outlined in Exercise 13 for the explicit scheme. However, for pedagogical purposes it might be better to consider the simpler case of an ordinary differential equation. Specifically, consider the simple test equation

\[ u_t = -Au, \]  

(14.1)

where \( A \) is a prescribed positive constant, with the implicit finite-difference approximation

\[ \frac{U_{k+1} - U_k}{\Delta t} = -(1 - \theta)AU_k - \theta AU_{k+1}. \]  

(14.2)

Following the same lines as in Exercise 13, show that the roundoff error \( e_k \equiv U_k - U^*_k \) propagates according to

\[ e_{k+1} = Ke_k, \]  

(14.3)

where

\[ K = \frac{1 - A(1 - \theta)\Delta t}{1 + \theta A\Delta t}. \]  

(14.4)

Thus, show that the scheme is stable if and only if

\[ A(1 - 2\theta)\Delta t \leq 2. \]  

(14.5)

NOTE: Suppose that \( \theta = 0 \) so the scheme (14.2) is explicit. Then (14.5) tells us that for stability we must choose \( \Delta t \leq 2/A \). In some problems, such as occur in the study of chemical kinetics, \( A \) can be extremely large, so that \( \Delta t \leq 2/A \) forces us to use extremely small time steps. However, observe that if we use the implicit scheme (14.2) with \( \theta \geq \frac{1}{2} \), then (14.5) is satisfied with no restriction on \( \Delta t \).

15. Use the Crank–Nicolson scheme (22) to solve the problem (11.1) in Exercise 11 through the first three lines of \( U_j,k \)'s (i.e., through \( k = 3 \)), with \( \Delta t = 0.1 \) and \( \Delta x = 0.25 \), using computer software (such as the Maple linsolve command) to solve the matrix equation obtained at each time step. Compare your results at \( t = 0.3 \) with the exact solution.

16. In Example 3 we used the Jacobi, Gauss–Seidel, and SOR methods to work out the iterates \( U_{1,1}^{(0)}, U_{2,1}^{(0)}, U_{3,1}^{(0)} \) and \( U_{1,1}^{(1)}, U_{2,1}^{(1)}, U_{3,1}^{(1)} \). Continuing the calculation, work out \( U_{1,1}^{(2)}, U_{2,1}^{(2)}, U_{3,1}^{(2)} \). NOTE: For \( U_{2,1}^{(2)} \) your three values should agree with the values 69.38, 71.41, 71.70 given in Table 1.

17. In (28) let \( r = 2 \) and \( c = [1, 1, 1, 1]^T \). (Thus, \( N = 5 \).)

(a) Solve for \( U_{k+1} \) using Jacobi iteration, terminating the iterations when \( |U_{j,k+1}^{(n+1)} - U_{j,k+1}^{(n)}| < 0.0001 \), say, for each \( j \) \( (j = 1, 2, 3, 4) \). Record the number of iterations needed to attain that accuracy.

(b) Same as part (a) but using Gauss–Seidel iteration instead.

(c) Same as part (a) but using the SOR method. Use \( \omega = 0.9, 1.0, 1.1, \) and 1.2, and compare the optimum \( \omega \) with that predicted by (39).

18. (Convergence of Jacobi iteration) It is stated below (33) that the Jacobi iteration (33) converges to the solution of (26) for all finite values of \( r \). Prove that claim. HINT: Denote the eigenvalues and eigenvectors of \( A \) as \( \lambda_1, \ldots, \lambda_N \) and \( \Phi_1, \ldots, \Phi_N \). Expanding \( c \) as

\[ c = \sum_{j=1}^{N-1} c_j \Phi_j, \]  

(18.1)

(31) and (33) give

\[ U_{k+1}^{(0)} = \beta \sum_{j=1}^{N-1} c_j \Phi_j, \quad \left( \beta \equiv \frac{1}{2(1 + r)} \right) \]

\[ U_{k+1}^{(1)} = \beta \sum_{j=1}^{N-1} c_j \Phi_j - \beta^2 \sum_{j=1}^{N-1} c_j \lambda_j \Phi_j \]

\[ = \beta \sum_{j=1}^{N-1} (1 - \beta \lambda_j) c_j \Phi_j, \]

\[ U_{k+1}^{(2)} = \beta \sum_{j=1}^{N-1} (1 - \beta \lambda_j + \beta^2 \lambda_j^2) c_j \Phi_j, \]
The central problem of this chapter is the one-dimensional diffusion equation

\[ \alpha^2 u_{xx} = u_t \] (1)

on a finite \( x \) interval, with constant Dirichlet (\( u \) given) or Neumann (\( u_x \) given) boundary conditions. The problem can be solved by the method of separation of variables, the key point of which is the reduction of the PDE to two ODE's governing the factors \( X(x) \) and \( T(t) \). We emphasize that the boundary conditions are to be applied before the initial condition and we show how to use Fourier series to satisfy the initial condition. Our approach is only formal, but the issue of rigorous justification is addressed in the brief optional Section 18.3.2.

In other cases Fourier series (i.e., the half- and quarter-range formulas and the full Fourier series of a periodic function) may not suffice – for instance, if we have Robin boundary conditions (a linear combination of \( u \) and \( u_x \) given) or if we have axisymmetric heat conduction in a circular disk or cylinder, governed by the PDE

\[ \alpha^2 \left( u_{rr} + \frac{1}{r} u_r \right) = u_t. \] (2)

These cases can be handled using the more powerful Sturm–Liouville theory, as discussed in the optional Section 18.3.3, and the series expansion required for satisfaction of the initial condition is in terms of the eigenfunctions of the relevant Sturm–Liouville problem.

Additional generalizations of (1) are contained in the exercises for Section 18.3, such as the inclusion of one or more of the terms \( V u_x \) (due to axial convection of the medium), \( Hu \) (a Newton cooling term due to lateral heat loss), and \( F(x, t) \) (due to a distributed source within the medium) in the equation

\[ \alpha^2 u_{xx} = u_t + V u_x + Hu - F(x, t). \] (3)

The Fourier and Laplace transforms enable us to handle problems for which the basic separation of variable method fails or is awkward – for instance, problems on a semi-infinite \( (0 < x < \infty) \) or infinite \( (-\infty < x < \infty) \) \( x \) domain, or with nonconstant boundary conditions or a distributed source term. These cases are discussed in Section 18.4.

The optional Section 18.5 explains a useful method known as the method of images, which is based on the idea of satisfying homogeneous boundary conditions
by fictitiously extending the problem so as to build in symmetry or antisymmetry about the boundary in question. The method is used again in Chapters 19 and 20, but the idea is simple enough so that Section 18.5 is not a prerequisite for those chapters.

The final section, 18.6, explains the numerical solution of the diffusion equation by the method of finite differences and gives a glimpse of the power of numerical simulation. We find that the simple explicit method is limited by the condition that

$$r = \frac{\alpha^2 \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$  \hspace{1cm} (4)

for convergence and stability. To remove the condition (4) we can use an implicit method instead, such as the Crank–Nicolson method, but the price that we pay is that to generate each time line of \( U_{j,k} \) values we need to solve an \((N - 1) \times (N - 1)\) matrix equation, where \( N \) is the number of divisions of \( L \) (i.e., \( \Delta x = L/N \)). Fortunately, the matrix is strongly diagonal, so the equation can be solved efficiently by iteration. The mathematics of the finite-difference method is linear algebra since one is replacing a linear PDE by a large system of linear algebraic equations. Questions of convergence, stability, and efficiency are best dealt with using linear algebra methods such as are covered here in Chapters 8–12.

Important general features of the diffusion equation are as follows:

1. The diffusion equation is an initial-value problem in \( t \), as is especially clear in Section 18.6 where we see from the “marching” scheme

\[
U_{j,k+1} = rU_{j-1,k} + (1 - 2r)U_{j,k} + rU_{j+1,k}
\]  \hspace{1cm} (5)

that the solution along the line \( t = (k + 1)\Delta t \) is implied by the solution along the preceding line \( t = k\Delta t \).

2. Diffusion is a smoothing process, as seen from our various solution plots and also from the finite-difference formula (5), since \( U_{j,k+1} \) is thereby given as a weighted average of the preceding values \( U_{j-1,k}, U_{j,k}, U_{j+1,k} \), “average” because the coefficients \( r, 1 - 2r, r \) in (5) sum to unity.

Recall from Section 18.2 that the linear PDE

\[
Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + Fu = f
\]  \hspace{1cm} (6)

is parabolic if \( B^2 - AC = 0 \), and that the diffusion equation (1) is a simple or canonical form of the general second-order parabolic PDE (with the \( y \)'s changed to \( t \)'s). Thus, understand (1) to be representative of that entire class of PDE’s.
Chapter 19

Wave Equation

19.1 Introduction

The wave equation

$$c^2 \nabla^2 u = u_{tt}$$

(1)

governs a wide variety of wave phenomena such as electromagnetic waves, water waves, supersonic flow, pulsatile blood flow, acoustics, elastic waves in solids, and vibrating strings and membranes. In this introductory section we derive the wave equations governing the vibrating string and vibrating membrane and outline, in the exercises, several other such cases leading to wave equations.

By a **vibrating string** we mean a taut string, such as a guitar string, undergoing a planar vibratory motion. The problem of the vibrating string is of historical importance since it was studied extensively by the great mathematicians Leonhard Euler (1707–1783), Jean Le Rond d’Alembert (1717–1783), Daniel Bernoulli (1700–1782), and Joseph-Louis Lagrange (1736–1813). That work gave birth to the subject of partial differential equations. For example, it was in his study of the vibrating string that d’Alembert developed the method of separation of variables used in Chapter 18 and which is now a standard tool in the solution of PDE’s. Likewise, he found it necessary to represent the initial shape of the string by what is now known as a Fourier series. Whether such representation is possible, for any given initial shape of the string, was the subject of heated debate, a debate that continued well into the nineteenth century. The theory of Fourier series that emerged is the subject of Chapter 17.

Beginning our derivation, consider a flexible string stretched under tension $\tau$ newtons between fixed endpoints at $x = 0$ and $x = L$ on a horizontal $x$ axis (Fig. 1). Let the mass per unit length of the string be $\sigma$, a constant, and suppose that the string supports a distributed load $f(x, t)$ newtons per unit $x$ length, counted as positive if it acts downward. Considering the plane vertical motion of the string (in the $x, y$ plane), the desired unknown is the displacement $y(x, t)$. 

1017
We assume that

1. the slope $\partial y/\partial x$ is uniformly small over the length of the string (i.e., $|\partial y/\partial x| \ll 1$);

2. acting on each cross section of the string is the tangentially oriented tension force $\tau$, but no shear force or bending moment as is explained in Example 2 of Section 1.3.

The relevant physical principle is Newton's second law of motion, which we apply to the vertical motion of an element of the string between $x$ and $x + \Delta x$ (Fig. 2):

$$\tau \sin \theta(x + \Delta x, t) - \tau \sin \theta(x, t) = f(x + \alpha \Delta x, t) \Delta x = \sigma \Delta s \frac{\partial^2 y}{\partial t^2}(x + \beta \Delta x, t),$$

where $\Delta s$ is the arc length. That is, the sum of the vertical forces [on the left-hand side of (2)] equals the mass $\sigma \Delta s$ times the vertical acceleration of the mass center at $x + \beta \Delta x$ (for some $\beta$ such that $0 \leq \beta \leq 1$). We have assumed that $f$ is a continuous function of $x$ (although that condition could be relaxed) so that there is a point $x + \alpha \Delta x$ (for some $\alpha$ between 0 and 1) at which $f$ takes on its average value over the $(x, x + \Delta x)$ interval.

According to Assumption 1, $\theta$ is small so we have these approximations from the Taylor series of $\sin \theta$ and $\tan \theta$:

$$\sin \theta = \theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \cdots \approx \theta$$

(3)

and

$$\tan \theta = \theta + \frac{1}{3!} \theta^3 + \frac{2}{15} \theta^5 + \cdots \approx \theta.$$  

(4)
Hence, for small $\theta$ we have $\sin \theta \approx \tan \theta$, the truth of which can also be seen graphically in Fig. 3. But $\tan \theta$ is the slope $\partial y / \partial x$. It also follows from assumption 1 that $\cos \theta = 1 - \frac{1}{2} \theta^2 + \cdots \approx 1$, so $\Delta s = \Delta x / \cos \theta \approx \Delta x$. Thus, we can eliminate the temporary variables $\theta$ and $s$ in favor of $y$ and $x$ and re-express (2) as

$$
\frac{\partial y}{\partial x}(x + \Delta x, t) - \frac{\partial y}{\partial x}(x, t) = f(x + \alpha \Delta x, t) \frac{\partial^2 y}{\partial t^2}(x + \beta \Delta x, t). \tag{5}
$$

Finally, letting $\Delta x \to 0$ in (5) gives the desired PDE

$$
\tau \frac{\partial^2 y}{\partial x^2}(x, t) - f(x, t) = \sigma \frac{\partial^2 y}{\partial t^2}(x, t) \tag{6}
$$

governing $y(x, t)$. If $f$ is simply the gravitational force on the string (i.e., the weight force per unit length of the string), then $f(x, t) = \sigma g = \text{constant}$, where $g$ is the acceleration due to gravity, and (6) becomes

$$
\tau \frac{\partial^2 y}{\partial x^2} = \sigma \frac{\partial^2 y}{\partial t^2} + \sigma g. \tag{7}
$$

If the gravitational term is negligible, as is surely the case for a guitar string,* then (7) reduces to

$$
\tau \frac{\partial^2 y}{\partial x^2} = \sigma \frac{\partial^2 y}{\partial t^2} \quad \tag{8}
$$

or, setting $c \equiv \sqrt{\tau / \sigma}$ and using the more compact subscript notation for partial derivatives,

$$
c^2 y_{xx} = y_{tt}. \tag{9}
$$

* A guitar sounds the same if it is played horizontally [so the $\sigma g$ term is present in (7)] or vertically [so the $\sigma g$ term is not present in (7)]. Applying dimensional reasoning to (7), we find that the $\sigma g$ term can be neglected if $\sigma g \ll \tau / L$. 

**Figure 2.** String element.

**Figure 3.** $\sin \theta \approx \theta \approx \tan \theta$ for small $\theta$. 
which is the classical one-dimensional wave equation governing \( y(x, t) \).

It is always important to tie together the mathematics and the physics. Accordingly, how are we to understand the terms in (8)? Essentially, (8) is of the form force = mass \( \times \) acceleration. The \( \tau y_{xx} \) term is the net vertical force on the element due to the tension \( \tau \). To understand its form, recall from the calculus that the local radius of curvature \( R \) and curvature \( \kappa \) for a plane curve \( y(x) \) (Fig. 4) are given by

\[
\kappa = \frac{1}{R} = \frac{y''}{(1 + y'^2)^{3/2}}.
\]

(10)

In our case \( |y'| \ll 1 \), so (10) becomes \( \kappa \approx y'' \). Thus, within our assumption of small deflection and small slope the \( \tau y_{xx} \) term in (8) is the product of the tension and the curvature. That result makes sense physically because it is through the curvature that the two tension forces in Fig. 2 are misaligned and therefore have a nonzero vertical resultant.

We close this section with an introduction to the two-dimensional wave equation. Specifically, consider the two-dimensional version of a vibrating string, a vibrating membrane such as a drumhead. We assume that the membrane is stretched uniformly under a tension \( \tau \) per unit length. That is, at each point of the membrane the tension per unit length along any straight line through that point, independent of the orientation of the line, is \( \tau \). If, for example, we stretch a rectangular membrane horizontally (Fig. 5) and then clamp the four edges, then the membrane would not be stretched uniformly, for the tension per unit length along any vertical line would be \( \tau \) whereas along any horizontal line it would be zero.

Denoting the displacement of the membrane out of the \( x, y \) plane as \( w(x, y, t) \), we proceed essentially as before. Thus, we assume that the slopes \( \partial w/\partial x \) and \( \partial w/\partial y \) are uniformly small over the domain (i.e., \( |\partial w/\partial x| \ll 1 \) and \( |\partial w/\partial y| \ll 1 \), and that the membrane is perfectly flexible, so that only the tangential tensile force \( \tau \) acts. Then, applying Newton's second law to a membrane element lying between \( x \) and \( x + \Delta x \) and between \( y \) and \( y + \Delta y \) (Fig. 6), gives

\[
\tau \Delta y \sin \theta \bigg|_{x+\Delta x}^{x} - \tau \Delta y \sin \theta \bigg|_{x} + \tau \Delta x \sin \phi \bigg|_{y+\Delta y}^{y} - \tau \Delta x \sin \phi \bigg|_{y} - f \Delta x \Delta y = \sigma \Delta A \frac{\partial^2 w}{\partial y^2},
\]

(11)

where \( \sigma \) is the mass per unit area of the membrane, \( \Delta A \) is the surface area of the element under consideration, and \( f(x, y, t) \) is a distributed load counted as positive if it acts downward. By virtue of the stated assumptions, \( \sin \theta \approx \theta \approx \tan \theta = \partial w/\partial x \), \( \sin \phi \approx \phi \approx \tan \phi = \partial w/\partial y \), and \( \Delta A \approx \Delta x \Delta y \), so (11) becomes

\[
\tau \left. \frac{w_x|_{x+\Delta x} - w_x|_x}{\Delta x} \right|_{x} + \tau \left. \frac{w_y|_{y+\Delta y} - w_y|_y}{\Delta y} \right|_{y} - f = \sigma w_{tt},
\]

(12)

*More precisely, (8) is on a per unit \( x \)-length basis: the vertical force per unit \( x \) length is equal to the mass per unit \( x \) length times the vertical acceleration.*
and letting $\Delta x \to 0$ and $\Delta y \to 0$ gives the PDE

$$\tau(w_{xx} + w_{yy}) - f(x, y, t) = \sigma w_{tt}. \quad (13)$$

If $f = 0$ and if we define $c \equiv \sqrt{\tau/\sigma}$, then (13) becomes the classical two-dimensional wave equation

$$c^2(w_{xx} + w_{yy}) = w_{tt}. \quad (14)$$

governing $w(x, y, t)$.

In the next two sections we solve the vibrating string and vibrating membrane equations by separation of variables.

**Closure.** In this section we derive the one- and two-dimensional wave equations (9) and (14) governing vibrating strings and vibrating membranes, respectively, subject to the assumptions that the string or membrane is flexible and that the slopes are small. Several additional problems governed by wave equations are given in the exercises.

**EXERCISES 19.1**

1. (Hanging chain) Let a flexible chain hang from the ceiling. Measure $x$ downward from the ceiling and let $y(x, t)$ be its lateral displacement. Modifying our derivation of (9) as appropriate, show that the PDE governing $y(x, t)$ is

$$g[(L - x)y_x]_x = y_{tt}, \quad (1.1)$$

where $g$ is the acceleration due to gravity and $L$ is the length.
of the chain.

2. (Longitudinal waves in a rod) Consider a uniform metal bar, of cross-sectional area \( A \) and mass per unit length \( \sigma \), with a stress distribution \( s(x, t) \) and a resulting longitudinal displacement \( u(x, t) \). Applying Newton’s second law to an element of the rod between \( x \) and \( x + \Delta x \), shown in the figure, show that

\[
A \frac{\partial s}{\partial t} = \sigma \frac{\partial^2 u}{\partial t^2}.
\]

(2.1)

Suppose that the material admits a linear stress-strain relationship \( s = E\varepsilon \), where the constant of proportionality \( E \) is Young’s modulus, and the strain \( \varepsilon \) is defined as the “stretch per unit length,” so that \( \varepsilon = \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} \). From these relations, show that \( s \) and \( u \) both satisfy one-dimensional wave equations

\[
c^2 u_{xx} = u_{tt} \quad \text{and} \quad c^2 s_{xx} = s_{tt},
\]

(2.2)

where \( c \equiv \sqrt{EA/\sigma} \).

3. (Electromagnetic waves) (a) Electromagnetic fields in free space (i.e., in a vacuum) are governed by the famous Maxwell’s equations:

\[
\nabla \times \mathbf{E} = -\varepsilon_0 \frac{\partial \mathbf{H}}{\partial t},
\]

\[
\nabla \times \mathbf{H} = -\mu_0 \frac{\partial \mathbf{E}}{\partial t},
\]

\[
\nabla \cdot \mathbf{E} = 0,
\]

\[
\nabla \cdot \mathbf{H} = 0,
\]

(3.1,2,3,4)

where \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic field intensities, respectively, and where \( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and permeability of free space, respectively. Show that it follows from (3.1)–(3.4) that \( \mathbf{E} \) and \( \mathbf{H} \) (which are functions of \( x, y, z \) and the time \( t \)) satisfy the wave equations

\[
c^2 \nabla^2 \mathbf{H} = \mathbf{H}_{tt}
\]

and

\[
c^2 \nabla^2 \mathbf{E} = \mathbf{E}_{tt},
\]

(3.5)

where \( c \equiv 1/\sqrt{\varepsilon_0 \mu_0} \). HINT: Recall the identity \( \nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} \) from Section 16.6.

(b) Write the \( x, y, z \) components of the vector wave equations (3.5).

4. (Current and magnetic field) Suppose that the “string” is actually a flexible wire of mass per unit length \( \sigma \), under tension \( \tau \), carrying a current \( I \) in the presence of a uniform magnetic field of magnetic flux density \( \mathbf{B} = B_1 \mathbf{i} + B_2 \mathbf{j} + B_3 \mathbf{k} \). Since a charge \( Q \) moving with velocity \( \mathbf{U} \) in such a field experiences a force \( \mathbf{F} = \mathbf{QU} \times \mathbf{B} \), we may need to revise the vibrating string equation to include lateral magnetic forcing terms.

(a) Show that within the usual vibrating string approximations, the magnetic force exerted on an element of the wire is

\[
\Delta \mathbf{F} = q \mathbf{A} \Delta \mathbf{x} \left[ (U \mathbf{t} + (U_y z + y_t) \mathbf{j} + (U_z x + z_t) \mathbf{k}) \times \mathbf{B} \right],
\]

(4.1)

where the constants \( q, A, U \) are the charge per unit volume within the wire, cross-sectional area of the wire, and velocity of the charges within the wire, respectively. HINT: The \( x, y, z \) velocity components of an element of charge within the wire are

\[
\frac{dx}{dt} = U, \quad \frac{dy}{dt} = U y_t + y_l, \quad \frac{dz}{dt} = U z_t + z_l.
\]

(b) Working out the cross product in (4.1) and noting that \( qAU \) is the current \( I \), show that the equations of (the not necessarily planar) motion of the wire are

\[
\tau y_{xx} - IB_3 + (I z_x + qA z_l) B_1 = \sigma y_{tt},
\]

\[
\tau z_{xx} + IB_2 - (I y_x + qA y_l) B_1 = \sigma z_{tt},
\]

(4.2)

where \( x, y, z \) are right-handed Cartesian coordinates. Notice that these equations are coupled due to the \( z_x, z_l, y_x, \) and \( y_l \) terms. Naturally, in a particular application one or more of the additional magnetic-field terms might be negligible.

5. (Water waves) Consider plane water waves in water of depth \( h(x) \) as shown in the figure. If the wavelength is much greater
than $h$ (as is true for ocean tides and certain waves in shallow water), the governing equations are found to be

\begin{align}
  u_t + uu_x &= -g \eta_x, \\
  [u(\eta + h)]_x &= -\eta_t,
\end{align}

where $u$ is the $x$ velocity [which is approximately constant with respect to $y$ so $u = u(x, t)$], $\eta(x, t)$ is the free-surface elevation relative to the undisturbed water level, and $g$ is the acceleration due to gravity. If we restrict our attention to small amplitude motions — so that the “second-order” term $uu_x$ can be neglected relative to the “first-order” terms $u_t$ and $g\eta_x$, and $\eta$ can be neglected relative to $h$ — then show that $\eta$ satisfies the equation $g(h\eta_x)_x = \eta_{tt}$ or, if $h(x)$ is a constant,

\begin{equation}
  c^2 \eta_{xx} = \eta_{tt},
\end{equation}

where $c \equiv \sqrt{gh}$.

## 19.2 Separation of Variables; Vibrating String

### 19.2.1 Solution by separation of variables. In Section 19.1 we derive the wave equation governing the motion of a vibrating string. In the present section we complete the formulation by appending boundary and initial conditions and then show how to solve for $y(x, t)$ by the method of separation of variables.

Specifically, let us consider a finite string extending over $0 < x < L$, tied at its ends, and having initial displacement $y(x, 0) = f(x)$ and initial velocity $y_t(x, 0) = g(x)$, where $f$ and $g$ are prescribed. Thus, the complete problem statement is

\begin{align}
  c^2 y_{xx} &= y_{tt}, \quad (0 < x < L, \ 0 < t < \infty) \quad & (1a) \\
  y(0, t) &= 0, \ y(L, t) = 0, \quad (0 < t < \infty) \quad & (1b) \\
  y(x, 0) &= f(x), \ y_t(x, 0) = g(x) \quad (0 < x < L) \quad & (1c)
\end{align}

and is summarized in Fig. 1.

Notice carefully that for the diffusion equation $\alpha^2 u_{xx} = u_t$ (Chapter 18) we prescribe only $u$ initially, whereas for the wave equation we prescribe both $y$ and $y_t$ initially. Intuitively, it certainly seems reasonable that to predict $y(x, t)$ we will need to know how the string is set in motion, namely, both the initial displacement $y(x, 0)$ and the initial velocity $y_t(x, 0)$. From a mathematical point of view we are guided by the fact that the wave equation (1a) is a second-order equation with respect to $t$ (whereas the diffusion equation is of first order), so we expect to need two initial conditions. Verification that the problem statement (1) *uniquely* determines $y(x, t)$ is left for the exercises. Here, we limit our attention to **finding** the solution.

Using separation of variables, seek

\begin{equation}
  y(x, t) = X(x)T(t). \tag{2}
\end{equation}

Putting (2) into (1a) and separating the variables gives

\begin{equation}
  \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T}. \tag{3}
\end{equation}
Since the left-hand side of (3) is a function of $x$ alone and the right-hand side is a function of $t$ alone, it follows from (3) (as discussed in Section 18.3) that both sides are constants, say $-\kappa^2$, so

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = \text{constant} = -\kappa^2. \quad (4)$$

Thus,

$$X'' + \kappa^2 X = 0, \quad \text{(5a)}$$

$$T'' + \kappa^2 c^2 T = 0, \quad \text{(5b)}$$

and

$$X = \begin{cases} 
A + Bx, & \kappa = 0 \\
D \cos \kappa x + E \sin \kappa x, & \kappa \neq 0
\end{cases} \quad \text{(6)}$$

$$T = \begin{cases} 
H + It, & \kappa = 0 \\
J \cos \kappa ct + K \sin \kappa ct, & \kappa \neq 0.
\end{cases} \quad \text{(7)}$$

Our motivation for writing the minus sign in (4) is that the resulting ODE (5b) admits $\cos \kappa ct$ and $\sin \kappa ct$ solutions, which look correct since we anticipate a vibratory motion; if we write $\kappa^2$ in (4) instead of $-\kappa^2$, then we obtain $T'' - \kappa^2 c^2 T = 0$, instead of (5b), with nonvibratory exponential solutions.\(^\ast\)

Thus, we have found product solutions of the form $(A + Bx)(H + It)$ and $(D \cos \kappa x + E \sin \kappa x)(J \cos \kappa ct + K \sin \kappa ct)$ and, relying on the linearity of \(L[y] = c^2 y_{xx} - y_{tt} = 0\), we can use superposition and write

$$y(x,t) = \left( A + Bx \right)(H + It) \quad \text{(8)}$$

As in Chapter 18, we apply the boundary conditions before the initial conditions. Accordingly,

$$y(0, t) = 0 = A(H + It) + D(J \cos \kappa ct + K \sin \kappa ct). \quad \text{(9)}$$

Since the right-hand side of (9) is a linear combination of the linearly independent functions $1, t, \cos \kappa ct, \sin \kappa ct$, it follows from (9) that we must have either $A = 0$ or $H = I = 0$, and either $D = 0$ or $J = K = 0$. We choose $A = 0$ and $D = 0$ so as to be left with as robust a solution as possible, namely,

$$y(x,t) = Bx(H + It) + E \sin \kappa x(J \cos \kappa ct + K \sin \kappa ct) \quad \text{(10)}$$

or, combining $BH$ as $P$, $BI$ as $Q$, $EJ$ as $R$, and $EK$ as $S$, for brevity,

$$y(x,t) = x(P + Qt) + \sin \kappa x(R \cos \kappa ct + S \sin \kappa ct). \quad \text{(11)}$$

\(^\ast\)As discussed in Exercise 2 of Section 18.3, we can survive writing $\kappa^2$ instead of $-\kappa^2$, but the choice is less convenient because $\kappa$ will then end up being purely imaginary rather than real.
In case it is not clear how the choice \( A = D = 0 \) gives "as robust a solution as possible," let us return to (9) and focus on the \( D(J \cos \kappa t + K \sin \kappa t) \) term. If we choose \( J = K = 0 \), then we lose the entire \( (D \cos \kappa x + E \sin \kappa x)(J \cos \kappa t + K \sin \kappa t) \) term in (8), whereas if we choose \( D = 0 \) as, indeed, we did, then we are left with \( E \sin \kappa x(J \cos \kappa t + K \sin \kappa t) \). Similarly, if we infer from (9) that \( H = J = 0 \), then we lose the entire \( (A + Bx)(H + It) \) term in (8), whereas if we infer that \( A = 0 \), then we are left with \( Bx(H + It) \).

Next,

\[
y(L, t) = 0 = L(P + Qt) + \sin \kappa L(R \cos \kappa t + S \sin \kappa t)
\]

so we need

\[
LP = 0 \quad \text{and} \quad LQ = 0
\]

as well as

\[
R \sin \kappa L = 0 \quad \text{and} \quad S \sin \kappa L = 0.
\]

Since \( L \neq 0 \), (13) gives \( P = Q = 0 \). And if we are to avoid having \( R = S = 0 \), we must choose

\[
\sin \kappa L = 0.
\]

Thus, \( \kappa = n\pi/L \) for \( n = 1, 2, \ldots \). Putting these results into (11) and using superposition gives

\[
y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( R_n \cos \frac{n\pi t}{L} + S_n \sin \frac{n\pi t}{L} \right).
\]

[If any steps in deriving (16) are unclear, we urge you to review Example 1 in Section 18.3]

Our expectation is that the \( R's \) and \( S's \) can now be determined from the initial conditions (1c). Imposing those conditions gives

\[
y(x, 0) = f(x) = \sum_{n=1}^{\infty} R_n \sin \frac{n\pi x}{L} \quad (0 < x < L)
\]

and

\[
y_t(x, 0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi t}{L} S_n \sin \frac{n\pi x}{L}. \quad (0 < x < L)
\]

We can identify each of (17a,b) as a half-range sine series so

\[
R_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx
\]

and

\[
\frac{n\pi t}{L} S_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx
\]
or

$$S_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx.$$  \hspace{1cm} (18b)

Hence, the solution of (1) is given by (16) with the $R_n$'s and $S_n$'s computed according to (18a) and (18b).

To illustrate, let $f(x)$ be as shown in Fig. 2 and let $g(x) = 0$. That is, we pull the string up at its midpoint and then release it from rest. Then (18a,b) give

$$R_n = \frac{8f_0}{n\pi^2} \sin \frac{n\pi}{2} \quad \text{and} \quad S_n = 0,$$  \hspace{1cm} (19)

so

$$g(x, t) = \frac{8f_0}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}.$$  \hspace{1cm} (20)

The right-hand side of (20) is a superposition of distinct modes of vibration $\sin (n\pi x/L) \cos (n\pi ct/L)$, each of which is a standing wave of spatial frequency $n\pi/L$ and temporal frequency $n\pi c/L$. The first three mode shapes are depicted in Fig. 3. The points $x = 0$ and $x = L$ (heavy dots in Fig. 3) are called nodal points of the first mode because $y = 0$ there for all $t$; $x = 0$, $L/2$, and $L$ are nodal points of the second mode, and so on. Further, we say that the modes are orthogonal inasmuch as their shapes satisfy the orthogonality relation

$$\int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} \, dx = 0$$  \hspace{1cm} (21)

for any pair of integers $m$ and $n$ with $m \neq n$ [equation (24b), Section 17.3].

It is interesting to see what (20) can tell us about the musical quality of a violin string plucked in the manner shown in Fig. 2. Suppose, for definiteness, that we tune the string so that its fundamental frequency $\pi c/L$ corresponds to the lowest $A$ on a piano, say $A_0$. Since $A_0$'s frequency is 27.5 cycles/sec, we accomplish that tuning by adjusting the tension $\tau$ so that

$$\frac{\pi c}{L} = \frac{\pi}{L} \sqrt{\frac{\tau}{\sigma}} \sec = \left(27.5 \text{ cycles/sec}\right) \left(\frac{2\pi \text{ rad}}{\text{cycle}}\right),$$  \hspace{1cm} (22)

or

$$\tau = (55L)^2 \sigma.$$  \hspace{1cm} (23)

Then the first several terms in (20) correspond to the combination of notes shown in Table 1, where $A_1$ is an $A$ one octave higher than $A_0$, and so on, and $C^\#_3$ is $C$ sharp in the third octave above the lowest. Observe that the overtones (with nonzero amplitude), $E_2, C^\#_3, \ldots$, do not occur at octaves above $A_0$. Thus, the sound is not a pristine $A_0$ with octave overtones but it is, nonetheless, fairly “clean” because of the relatively small amplitudes of the $E_2, C^\#_3, \ldots$ contributions. The mix of frequencies and amplitudes is different for different instruments and an $A_0$ played on
19.2. Separation of Variables; Vibrating String

Table 1. The first five notes in (20).

| Frequency $n \frac{\pi c}{L}$ (cycles/sec) | Relative Amplitude $\frac{1}{n^2} \left| \sin \frac{n \pi}{2} \right|$ | Musical Note |
|------------------------------------------|---------------------------------|---------------|
| 1                                        | 27.5                            | $A_0$         |
| 2                                        | 55.0                            | $A_1$         |
| 3                                        | 82.5                            | $E_2$         |
| 4                                        | 110.0                           | $A_2$         |
| 5                                        | 137.5                           | $C\#_3$       |

A violin sounds different from an $A_0$ played on a tuba.*

19.2.2. Traveling wave interpretation. We have seen that if $y(x, 0) = f(x)$ is prescribed and $y_t(x, 0) = 0$, then $y(x, t)$ is given by

$$y(x, t) = \sum_{n=1}^{\infty} R_n \sin \frac{n \pi x}{L} \cos \frac{n \pi c t}{L}, \quad (0 < x < L, \ 0 < t < \infty) \quad (24)$$

where the $R_n$'s are computed from the initial condition

$$y(x, 0) = f(x) = \sum_{n=1}^{\infty} R_n \sin \frac{n \pi x}{L} \quad (0 < x < L) \quad (25)$$

as

$$R_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx. \quad (26)$$

If we wish to plot $y(x, t)$ at various times, we can sum the series (24) at a number of different $x$'s and $t$'s. However, we can do much better as we will now show. First, use the trigonometric identity

$$\sin A \cos B = \frac{1}{2} [\sin (A - B) + \sin (A + B)] \quad (27)$$

to re-express (24) as

$$y(x, t) = \frac{1}{2} \sum_{n=1}^{\infty} R_n \left[ \sin \frac{n \pi}{L} (x - ct) + \sin \frac{n \pi}{L} (x + ct) \right]$$

*Actually, the sound of a violin is due very little to the air being set in motion by the vibrating string. Rather, the string drives the sounding board: through its connection at the bridge, and it is the vibrating sounding board that sets the adjacent air in motion and creates an audible sound. Thus, a serious investigation of violin mechanics would lead immediately to a much more difficult analysis of the vibrating sounding board.
Thus, we have the following series expansion for $f(x)$:

$$f(x) = f(0) + f'(0)x + \sum_{n=1}^{\infty} f^{(n)}(0) \frac{x^n}{n!} = f(0) + f'(0)x + \sum_{n=1}^{\infty} f^{(n)}(0) \frac{x^n}{n!}.$$  

Finally, comparing (28) with (25) we see that the two series in (28) can be summed into closed form in terms of $f$ as

$$y(x, t) = \frac{1}{2} [f(x - ct) + f(x + ct)].$$  

(29)

There is a difficulty associated with (29): for a given $x$ in the interval $(0, L)$ and a given $t$ in the $t$ interval $(0, \infty)$, one or both of the arguments $x - ct$ and $x + ct$ in (29) may lie outside the domain of definition of $f$, $0 \leq x \leq L$, in which event (29) is meaningless. However, recall from our study of half-range sine series (Section 17.4) that the half-range sine expansion of $f(x)$ on $0 < x < L$, given by (25), is also the Fourier series expansion of the extended function $f_{\text{ext}}(x)$ that is $2L$-periodic and antisymmetric about $x = 0$ and $x = L$. For example, if $f(x)$ is the function shown in Fig. 4a, then $f_{\text{ext}}(x)$ is the function shown in Fig. 4b. Thus, the right-hand side of (28) actually sums to

$$y(x, t) = \frac{1}{2} [f_{\text{ext}}(x - ct) + f_{\text{ext}}(x + ct)],$$  

(30)

for any values of the arguments $x - ct$ and $x + ct$.

To illustrate the use of (30), let $L = 10$ and $c = 12$, let $f$ be as shown in Fig. 2 with $f_0 = 1$, and let us compute $y$ at $x = 2$ and $t = 3$. Using (30) and the fact that $f_{\text{ext}}$ is periodic with period $2L = 20$ and odd, we have

$$y(2, 3) = \frac{1}{2} [f_{\text{ext}}(2 - 36) + f_{\text{ext}}(2 + 36)]$$

$$= \frac{1}{2} [f_{\text{ext}}(-34) + f_{\text{ext}}(38)]$$

$$= \frac{1}{2} [f_{\text{ext}}(-14) + f_{\text{ext}}(18)]$$

(periodicity of $f_{\text{ext}}$)

$$= \frac{1}{2} [f_{\text{ext}}(6) + f_{\text{ext}}(-2)]$$

(definition of $F$ in Fig. 2)

$$= \frac{1}{2} [f(6) - f(2)]$$

(on $0 < x < 10$)

$$= \frac{1}{2} \left( \frac{4}{5} - \frac{2}{5} \right)$$

= $\frac{1}{5}$.

Besides using (30) to compute $y$ at any specific $x$ and $t$, we can use it to obtain the graph of $y(x, t)$ over the entire interval $0 < x < L$, at any given $t$, by observing that the graph of $f_{\text{ext}}(x - ct)$ plotted as a function of $x$ is simply the graph of $f_{\text{ext}}(x)$ translated to the right through a distance $ct$. Similarly, the graph of $f_{\text{ext}}(x + ct)$ is the graph of $f_{\text{ext}}(x)$ translated to the left through a distance $ct$. Thus, if $f$ is as shown in Fig. 2, then $f_{\text{ext}}(x)$ is as shown in Fig. 5a, $f_{\text{ext}}(x - ct)$ and $f_{\text{ext}}(x + ct)$ are as shown in Fig. 5b, and $y(x, t) = \frac{1}{2} [f_{\text{ext}}(x - ct) + f_{\text{ext}}(x + ct)]$ is (by addition of the two graphs in Fig. 5b and scaling by 1/2) as shown in Fig. 5c.
Carrying out this graphical procedure at a number of different times over one complete cycle yields the solution sequence shown in Fig. 6. Consider the results shown in Fig. 6 in terms of the physics. At \( t = 0 \) the string segments \( AB \) and \( BC \) are straight, so each string element is in static equilibrium – except at \( B \), which point is driven downward (Fig. 7). The plateau \( DE \) (Fig. 6) moves downward at constant velocity (i.e., with no acceleration) since there is no net vertical force on the elements between \( D \) and \( E \). (Remember that we have neglected the effects of gravity.)

We can now explain why \( c^2 y_{xx} = y_{tt} \) is called the "wave" equation for we see from (24) that \( y(x, t) \) can be expressed as a superposition of standing waves of shape \( \sin (n \pi x / L) \) and temporal frequency \( n \pi c / L \) rad/sec. The sum of these waves is also a standing wave, of temporal frequency \( \pi c / L \). Alternatively, (28) expresses \( y(x, t) \) as a superposition of traveling waves traveling to the right and left with speed \( c \). Thus, the parameter \( c \) in the wave equation \( c^2 y_{xx} = y_{tt} \) is now seen to be the wave speed – that is, the speed of propagation of the traveling waves. Recalling that \( c = \sqrt{\tau / \sigma} \), it does seem reasonable physically that the wave speed should increase with the tension \( \tau \) and decrease with the linear mass density \( \sigma \). To reiterate, (28) expresses \( y \) as a superposition of sinusoidal traveling waves crisscrossing leftward and rightward with speed \( c \). Over the physical interval \((0, L)\) these waves sum to a standing wave with nodes at \( x = 0 \) and \( x = L \).

**19.2.3. Using Sturm–Liouville theory.** (Optional) If we rely on the Sturm–Liouville theory, then we understand (17a) and (17b) as eigenfunction expansions of \( f \) and \( g \) in terms of the orthogonal eigenfunctions \( \sin (n \pi x / L) \) of the relevant
Sturm–Liouville problem, namely,

\[ X'' + \kappa^2 X = 0, \quad (0 < x < L) \]
\[ X(0) = 0, \quad X(L) = 0. \]  \hspace{1cm} (31a)  

Then the weight function in the inner product is 1, and

\[ R_n = \frac{\langle F(x), \sin \frac{n\pi x}{L} \rangle}{\langle \sin \frac{n\pi x}{L}, \sin \frac{n\pi x}{L} \rangle} = \frac{\int_0^L F(x) \sin \frac{n\pi x}{L} \, dx}{\int_0^L \sin^2 \frac{n\pi x}{L} \, dx} = \frac{2}{L} \int_0^L F(x) \sin \frac{n\pi x}{L} \, dx \]  \hspace{1cm} (32)

as in (18a), and similarly for \( S_n \).

**Closure.** Like the problem of heat conduction in a finite rod (Section 18.3), the vibrating finite string problem (1) is defined on a semi-infinite strip in the \( x, t \) plane, with boundary conditions at \( x = 0 \) and \( x = L \). However, the PDE is of second order with respect to \( t \), so two initial conditions are appropriate rather than one. Solution by separation of variables proceeds in essentially the same manner as in Section 18.3 and, as for the heat equation, it is necessary to apply the boundary conditions before the initial conditions. The solution (24) is in the form of a superposition of orthogonal modes, each one being a standing wave. Alternatively, the trigonometric identity (27) enables us to re-express the solution in the form (28), namely, a superposition of left- and right-running traveling waves with wave speed \( c \). More simply, (30) gives the solution as the sum of two traveling waves, one left-running and one right-running.

From Fig. 6 we see that the initially “kinky” deflection \( y(x, t) \) does not smooth out as \( t \) increases, the way an initially kinky or discontinuous temperature distribution \( u(x, t) \) does, and this is a major difference between the wave and diffusion processes. Rather, kinks and discontinuities in the initial conditions propagate into the \( x, t \) domain.

---

**EXERCISES 19.2**

1. Let \( L = 10, \ c = 12, \) and \( f_0 = 1 \) in (20). Use (20) to compute \( y \) at the specified values of \( x \) and \( t \), to two significant figures. Then use (30) to compute \( y \) and show that your two results agree.

   \( (a) \ y(5, 1) \quad (b) \ y(5, 2) \quad (c) \ y(5, 3) \quad (d) \ y(5, 4) \quad (e) \ y(5, 6) \quad (f) \ y(5, 10) \quad (g) \ y(5, 20) \quad (h) \ y(3, 1) \quad (i) \ y(3, 10) \quad (j) \ y(3, 40) \quad (k) \ y(1, 2) \quad (l) \ y(1, 10) \)

2. Solve (1) for \( y(x, t) \) for the case where \( f(x) = 0 \) and

   \( (a) \ y(x) = 50 \sin \left( \frac{\pi x}{L} \right) \)
   \( (b) \ y(x) = 3 \sin \left( \frac{\pi x}{L} \right) - 5 \sin \left( \frac{4\pi x}{L} \right) \)
   \( (c) \ y(x) = \sin \left( 2\pi x/L \right) + \sin \left( 3\pi x/L \right) + 4 \sin \left( 8\pi x/L \right) \)

3. Using the solution technique illustrated in Fig. 5, obtain the graphs shown in Fig. 6, for the given values of \( t \), and label any key values.
4. Construct neat labeled sketches of the graph of $y(x,t)$ versus $t$ for the oscillation depicted in Fig. 6, for $x = L/2$ and also for $x = L/4$.

5. If, instead of the string being tied at $x = L$ such that $y(L,t) = 0$, the string is looped around a vertical frictionless wire (as shown in the figure), then in place of $y(L,t) = 0$ the boundary condition becomes

$$\frac{dy}{dx}(L,t) = 0.$$ (5.1)

(a) Explain why (5.1) is true.
(b) Solve (1a) for $y(x,t)$ by separation of variables, with these boundary and initial conditions:

$$y(0,t) = 0, \quad y_x(L,t) = 0,$$

$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$

leaving expansion coefficients in integral form.

(c) Solve (1a) for $y(x,t)$ by separation of variables, with these boundary and initial conditions:

$$y_x(0,t) = 0, \quad y_x(L,t) = 0,$$

$$y(x,0) = 0, \quad y_t(x,0) = V,$$

where $V$ is a constant.

(d) Solve (1a) for $y(x,t)$ by separation of variables, with these boundary and initial conditions:

$$y_x(0,t) = 0, \quad y(L,t) = 0,$$

$$y(x,0) = 0, \quad y_t(x,0) = g(x),$$

leaving expansion coefficients in integral form.

6. (Inclusion of damping) Unless the string vibrates in a vacuum, there will be some damping due to the movement of the string through the fluid (be it air, water, or whatever). If the damping force is proportional to the velocity $y_t$, the modified equation of motion becomes

$$c^2 y_{xx} = y_{tt} + ay_t,$$ (6.1)

where $a$ is a known constant. For definiteness, suppose that $0 < a < 2\pi c/L$. Solve (6.1) by separation of variables, subject to the conditions.

$$y(0,t) = 0, \quad y(L,t) = 0,$$

$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$

leaving expansion coefficients in integral form. Summarize, in words, the effect(s) of the damping term $ay_t$.

7. (Inclusion of lateral spring) If, as shown in the figure, a lateral distributed spring is included, then the modified equation of motion for the vibrating string is $\tau y_{xx} - ky = \sigma y_{tt}$, where $k$ is the spring stiffness per unit length (newtons per meter per meter) or

$$c^2 y_{xx} - by = y_{tt}.$$ (7.1)

Solve (7.1) by separation of variables, subject to the conditions

$$y(0,t) = 0, \quad y(L,t) = 0,$$

$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$

leaving expansion coefficients in integral form. Summarize, in words, the effect(s) of the spring term $by$ in (7.1).

8. (Constant forcing function) We saw in Section 19.1 that if the effects of gravity are included, then the governing PDE is

$$c^2 y_{xx} = y_{tt} + g.$$ (8.1)

That is, the PDE $L[y] = c^2 y_{xx} - y_{tt} - g$ is nonhomogeneous. Solve (8.1) subject to the conditions

$$y(0,t) = 0, \quad y(L,t) = 0,$$

$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$
leaving expansion coefficients in integral form. HINT: The form \( y(x, t) = X(x)T(t) \) gives

\[
\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} + \frac{g}{c^2 XT}.
\] (8.2)

Because of the last term in (8.2), which contains both \( x \) and \( t \) dependence, we are unable to successfully complete the separation process (i.e., we are unable to get all of the \( x \) dependence on one side of the equation and all of the \( t \) dependence on the other). Thus, we suggest seeking \( y \) in the form

\[
y(x, t) = y_p(x) + X(x)T(t)
\] (8.3)

instead. Putting (8.3) into (8.1), obtain

\[
c^2 y_p'' + c^2 X''T = XT'' + g.
\] (8.4)

Thus, we can remove the unwelcome \( g \) term by setting \( c^2 y_p'' + g = 0 \). Then we can complete the separation of variable in (8.4) as usual. Mathematically, \( y_p(x) \) is a particular solution of the nonhomogeneous equation (8.1) since it satisfies the full equation (7.1), and \( XT \) is a solution of the associated homogeneous equation \( c^2 y'' = g \). But in physical terms you will find that it is simply the "static sag" of the string due to gravity, satisfying the problem

\[
c^2 y_p''(x) = g, \quad y_p(0) = 0, \quad y_p(L) = 0.
\] (8.5)

9. (Nonconstant forcing function) In Exercise 8 we included a forcing term that was a constant. The suggested solution technique would have worked even if the forcing term was a nonconstant function of \( x \). But in this exercise we allow for \( t \) dependence as well. Thus, consider the problem

\[
c^2 y_{xx} = g_t + F(x, t),
\]

\[
y(0, t) = 0, \quad y(L, t) = 0,
\]

\[
y(x, 0) = f(x), \quad y_t (x, 0) = 0.
\] (9.1)

To solve, we can use essentially the same eigenvector expansion method that is used in Section 11.3.2 to solve the nonhomogeneous matrix problem \( A \mathbf{x} = \lambda \mathbf{x} + \mathbf{c} \), and again in Exercise 17 of Section 18.3 to solve the nonhomogeneous diffusion equation \( \alpha^2 u_{xx} = u_t - F(x, t) \).

(a) Accordingly, solve (9.1) by seeking

\[
y(x, t) = \sum_{n=1}^{\infty} h_n(t) \sin \frac{n\pi x}{L}
\] (9.2)

and expanding

\[
F(x, t) = \sum_{n=1}^{\infty} F_n(t) \sin \frac{n\pi x}{L}
\] (9.3)

and

\[
f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{L},
\]

where the coefficients

\[
F_n(t) = \frac{2}{L} \int_0^L F(x, t) \sin \frac{n\pi x}{L} \, dx
\] (9.4)

and

\[
f_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx
\] (9.5)

are considered as known [i.e., computable from \( F(x, t) \) and \( f(x) \)]. With \( \omega_n = n\pi c/L \), show that

\[
y(x, t) = \sum_{n=1}^{\infty} \left[ f_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t F_n(\tau) \sin \omega_n (\tau - t) \, d\tau \right] \sin \frac{n\pi x}{L},
\] (9.6)

(b) With the help of the Leibniz rule formally verify that (9.6) satisfies (9.1).

(c) Work out the solution (9.6) for the case where \( F(x, t) = F_0 \sin \Omega t \) and \( f(x) = 0 \), assuming that the driving frequency does not equal any of the natural frequencies \( \omega_n \).

(d) Same as (c), but where \( \Omega \) equals one of the natural frequencies, say \( \omega_k \).

10. (Variable end conditions) Thus far our boundary conditions have been constant in time. Here, we consider nonconstant conditions. Consider the problem

\[
c^2 y_{xx} = g_t,
\]

\[
y(0, t) = p(t), \quad y(L, t) = q(t),
\]

\[
y(x, 0) = f(x), \quad y_t (x, 0) = 0.
\] (10.1)

Changing dependent variables from \( y(x, t) \) to \( z(x, t) \) according to

\[
y(x, t) = z(x, t) + \left(1 - \frac{x}{L}\right) p(t) + \frac{x}{L} q(t),
\] (10.2)

show that the problem governing \( z(x, t) \) is of the type treated in Exercise 9. NOTE: Notice how an "input" can be moved from the boundary conditions to a forcing term in the PDE. In the present case the PDE on \( y \) was homogeneous and the boundary conditions were nonhomogeneous. Following the
change of variables you should find that the PDE on $z$ is non-

homogeneous and the boundary conditions are homogeneous.

11. The voltage $v(x,t)$ in an underground cable is governed

by a PDE of the form

$$v_{xx} = Av_{tt} + Bv_t + Cv,$$

(11.1)

where $A, B, C$ are constants. It is proposed that if $v_1$ is

a solution of $v_{xx} = Av_{tt} + Bv_t$ and $v_2$ is a solution of

$v_{xx} = Av_{tt} + Cv$, then $v = v_1 + v_2$ is, by superposition,

a solution of (10.1). Give a critical evaluation of that proposal.

12. (Longitudinal waves in a rod) First, read Exercise 2 of

Section 19.1. Consider a rod of length $L$, cross-sectional area

$A$, Young's modulus $E$, and mass per unit length $\sigma$. At $x = 0$

the rod is attached to a rigid wall and at $x = L$ the rod is

free. Prior to time $t = 0$ we pull on the free end with a force

$F_0$, so the rod is in static equilibrium, with a uniform stress

$s_0 = F_0/A$. At $t = 0$ we remove the force.

(a) Show that the problem governing the displacement is

$$c^2 u_{xx} = u_{tt},$$

$$u(0,t) = 0, \quad u_x(L,t) = 0,$$

$$u(x,0) = \frac{s_0}{E} x, \quad u_t(x,0) = 0.$$  

(12.1)

(b) Solve (12.1) for $u(x,t)$.

(c) Determine the stress at the wall, $s(0,t)$.

13. (Uniqueness) In this section and in the preceding exercises

we have developed solution techniques for wave problems,

most of which are of the form

$$c^2 y_{xx} = y_{tt} + F(x,t),$$

$$y(0,t) = p(t), \quad y(L,t) = q(t),$$

$$y(x,0) = f(x), \quad y_t(x,0) = g(x).$$  

(13.1)

Show that the solution to (13.1) is unique. HINT: As usual

in uniqueness proofs, let and $y_1(x,t)$ and $y_2(x,t)$ be two solutions

and consider the difference $w(x,t) = y_1(x,t) - y_2(x,t)$. Show

that $w$ satisfies the homogeneous version of (13.1),

$$c^2 w_{xx} = w_{tt},$$

$$w(0,t) = 0, \quad w(L,t) = 0,$$

$$w(x,0) = 0, \quad w_t(x,0) = 0.$$  

(13.2)

Considering the integral

$$I(t) = \int_0^L \left( w_t^2 + c^2 w_{xx}^2 \right) \, dx,$$  

(13.3)

show, with the help of Leibniz differentiation and (13.2), that

$$\frac{dI}{dt} = 0.$$  

Thus, show that $I(t) = 0$ for all $t \geq 0$ and hence

that $w(x,t) = 0$ for all $0 < x < L$ and $0 < t < \infty$. Since

$w(x,t) = y_1(x,t) - y_2(x,t) = 0$, $y_1$ and $y_2$ are necessarily
identical, so there exists at most one solution to (13.1).

14. (Vibrating beam) It is known from mechanics that the free

vibration of a uniform beam is governed by the fourth-order PDE

$$yxxxx + \frac{\sigma}{EI} ytt = 0,$$  

(14.1)

where $y(x,t)$ is the deflection, $\sigma$ is the mass per unit length,

$E$ is Young's modulus of the material, and $I$ is the moment of

inertia of the cross section about its neutral axis.

(a) If the beam is "cantilevered" (see figure), then the boundary

conditions are

$$y(0,t) = 0, \quad y_x(0,t) = 0,$$

$$y_{xx}(L,t) = 0, \quad (\text{no bending moment at free end})$$

$$y_{xxx}(L,t) = 0, \quad (\text{no shear force at free end})$$

and if the beam is initially deflected and at rest, then

$$y(x,0) = f(x), \quad y_t(x,0) = 0.$$  

(14.3)

Seeking $y(x,t) = X(x)T(t)$, derive the solution form

$$y(x,t) = \sum_{n=1}^{\infty} A_n X_n(x) \cos \omega_n t,$$  

(14.4)

where the mode shapes are given by

$$X_n(x) = \sin \frac{z_n x}{L} - \sinh \frac{z_n x}{L}$$

$$+ \left( \frac{\cos z_n + \cosh z_n}{\sin z_n - \sinh z_n} \right) \left( \cos \frac{z_n x}{L} - \cosh \frac{z_n x}{L} \right),$$  

(14.5)

where the frequencies are

\begin{itemize}
\[ \omega_n = \sqrt{\frac{EI}{\sigma}} \left( \frac{z_n}{L} \right)^2, \quad (14.6) \]

and where the \( z_n \)'s are the solutions of the transcendental equation \( \cos z \cosh z + 1 = 0 \). NOTE: The initial condition

\[ y(x,0) = f(x) = \sum_{n=1}^{\infty} A_n X_n(x) \quad (14.7) \]

can then be used to evaluate the \( A_n \)'s, but we do not ask you to go that far. Indeed, the eigenvalue problem on \( X(x) \), which yields the eigenfunctions \( X_n(x) \), is not of Sturm–Liouville type, so we do not find (in this text) the theoretical foundation needed to guide us through the evaluation of the \( A_n \)'s in (14.7). Let it suffice to observe that (14.4) gives \( y(x,t) \) as the superposition of infinitely many modes, having shape \( X_n(x) \) and frequency \( \omega_n \).

(b) If, instead of being cantilevered the beam is pinned at both ends, then the boundary conditions are

\[ y(0,t) = 0, \quad y(L,t) = 0, \quad y_x(0,t) = 0, \quad y_x(L,t) = 0, \quad (14.8) \]
in place of (14.2). Find the solution form analogous to the one cited in part (a), corresponding to the boundary conditions (14.8).

15. (Lumped-parameter model) It is sometimes useful to model a continuous system, such as the string in our vibrating string problem, approximately, by a discrete system. To illustrate, let us divide the string into four equal parts and focus the mass \( \sigma L/4 \) of each segment at the center of that segment. (Such a system is called a lumped-parameter system.) Thus, we have four "beads" of mass \( \sigma L/4 \) connected by massless string under tension \( \tau \).

(a) Applying Newton's second law of motion to the first bead, show that

\[ \frac{\sigma L}{4} \ddot{y}_1 \approx \tau \left( \frac{y_2 - y_1}{L/4} - \frac{y_1}{L/8} \right), \quad (15.1) \]

where dots denote time derivatives. Doing the same for the remaining three masses, show that the resulting ODE's can be expressed in matrix form as

\[ \ddot{y} + A \dot{y} = 0, \quad (15.2) \]

where \( y = [y_1(t), \ldots, y_4(t)]^T \) and

\[ A = \frac{16\tau}{\sigma L^2} \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}, \quad (15.3) \]

(b) More generally, with \( N \) beads one obtains (15.2), where \( y = [y_1(t), \ldots, y_N(t)]^T \) and

\[ A = \left( \frac{N}{L} \right)^2 \frac{\tau}{\sigma} \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 3 \end{bmatrix}, \quad (15.4) \]

(You need not derive this result.) Seeking a solution in the form

\[ y_1(t) = \eta_1 \sin(\omega t + \phi) \\
\vdots \\
y_N(t) = \eta_N \sin(\omega t + \phi) \]

or \( y(t) = \eta \sin(\omega t + \phi) \), show that

\[ B \eta = \lambda \eta, \quad (15.6) \]

where \( B \) is the matrix in (15.4), without the \( (N/L)^2(\tau/\sigma) \) factor, and \( \lambda = (\sigma/\tau)(L/N)^2 \omega^2 \).

(c) If, for a chosen value of \( N \), one solves the eigenvalue problem (15.6), then one obtains approximations of the first \( N \) orthogonal mode shapes and eigenfrequencies of the continuous string: the eigenvectors \( \eta_1, \ldots, \eta_N \) give the approximate mode shapes and the eigenvalues \( \lambda_1, \ldots, \lambda_N \) give the approximate eigenfrequencies \( \omega_1 = (N/L) \sqrt{\tau/\sigma \lambda_1}, \ldots, \omega_N = (N/L) \sqrt{\tau/\sigma \lambda_N} \). Here is the problem: Use computer software to solve (15.6) for the case where \( N = 4 \). Compare the computed eigenfrequencies with the exact values \( \omega_1 = \pi c/L = (\pi/L) \sqrt{\tau/\sigma}, \ldots, \omega_N = N\pi c/L = (N\pi/L) \sqrt{\tau/\sigma} \), and compare the \( m \) computed mode shapes with the exact shapes \( \sin(\pi x/L), \ldots, \sin(4\pi x/L) \).

(d) Same as (c), with \( N = 10 \).
19.3 Separation of Variables; Vibrating Membrane

In Section 19.1 we derive not only the equation \( c^2 y_{xx} = y_{tt} \) governing the vibrating string but also the equation

\[
\square{c^2(w_{xx} + w_{yy}) = w_{tt}} \quad \text{(1a)}
\]

governing the vibrating membrane; \( w(x, y, t) \) is the membrane deflection normal to the \( x, y \) plane, and \( c^2 = \tau/\sigma \) where \( \tau \) is the (uniform) tension per unit length and \( \sigma \) is the (uniform) mass per unit area. Let us consider the domain to be the rectangle \( 0 < x < a \) and \( 0 < y < b \) (Fig. 1), and let us solve (1) subject to the boundary conditions

\[
w(0, y, t) = w(a, y, t) = w(x, 0, t) = w(x, b, t) = 0 \quad \text{(1b)}
\]

and the initial conditions

\[
w(x, y, 0) = f(x, y), \quad w_t(x, y, 0) = 0, \quad \text{(1c)}
\]

where \( w(x, y, 0) = f(x, y) \) is the initial deflection and \( w_t(x, y, 0) \) is the initial velocity. That is, the membrane is initially at rest.

The reason that we isolate this problem in a separate section is that it is our first problem with more than two independent variables – namely \( x, y \), and \( t \). Our approach here is intended to serve as a model for such cases.

To solve by separation of variables, seek

\[
w(x, y, t) = X(x)Y(y)T(t) \quad \text{(2)}
\]

Putting (2) into (1a) gives

\[
c^2(X''YT + XY''T) = XYT''
\]

or, dividing by \( c^2 X Y T \),

\[
\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2} \frac{T''}{T}. \quad \text{(3)}
\]

The left-hand side is a function of \( x \) and \( y \), and the right-hand side is a function of \( t \). Since \( x, y, t \) are independent variables, it follows in the usual way that each side must be a constant. Thus,

\[
\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2} \frac{T''}{T} = \text{constant} = -\kappa^2, \quad \text{(4)}
\]

with the minus sign included so that the \( T \) equation

\[
T'' + \kappa^2 c^2 T = 0 \quad \text{(5)}
\]
Chapter 19. Wave Equation

gives oscillatory solutions \((\cos \kappa t \text{ and } \sin \kappa t)\), since we anticipate a vibratory motion. Next, we separate the \(x\) and \(y\) dependence in (4) by writing
\[
\frac{X''}{X} = \frac{Y''}{Y} - \kappa^2.
\]
The left-hand side is a function of \(x\) alone and the right-hand side is a function of \(y\) alone so it follows, as usual, that
\[
\frac{X''}{X} = -\frac{Y''}{Y} - \kappa^2 = \text{constant} \equiv -\alpha^2.
\]
Hence,
\[
X'' + \alpha^2 X = 0, \quad Y'' + (\kappa^2 - \alpha^2)Y = 0.
\]
From (5), (7), and (8), we obtain
\[
X = A \cos \alpha x + B \sin \alpha x, \quad (9a)
Y = D \cos \sqrt{\kappa^2 - \alpha^2} y + E \sin \sqrt{\kappa^2 - \alpha^2} y, \quad (9b)
T = F \cos \kappa t + G \sin \kappa t. \quad (9c)
\]
The right-hand side of (9c) is the general solution of (5) only if \(\kappa \neq 0\), for if \(\kappa = 0\) then the sine term drops out. Consistent with the strategy that we have used until now, we should write, in place of (9c),
\[
T = \begin{cases} 
F \cos \kappa t + G \sin \kappa t, & \kappa \neq 0 \\
H + It, & \kappa = 0.
\end{cases} \quad (10)
\]
Similarly for (9a) for the case \(\alpha = 0\) and for (9b) for the case \(\kappa = \alpha\). However, in this problem we anticipate that the \(I\) in (10) will be found to be zero when we apply the boundary and initial conditions because it gives a linear variation in \(t\), whereas we expect an oscillatory motion. Similarly, we do not expect to need the linear terms in \(x\) and \(y\) (corresponding to the special cases \(\alpha = 0\) and \(\kappa = \alpha\)) because \(w = 0\) on the boundary. Thus, let us proceed with the solution forms (9a,b,c) rather than carry extra terms, terms that we know will drop out later.

Next, we put (9a,b,c) into (2) and apply the boundary and initial conditions. However, it is more efficient to observe from the boundary condition
\[
w(0, y, t) = 0 = X(0)Y(y)T(t) \quad (11)
\]
that \(X(0) = 0\), and to observe from the other boundary conditions in (1b) that \(X(a) = 0, Y(0) = 0, \text{ and } Y(b) = 0\).* Applying the boundary condition \(X(0) = 0\)

*Alternatively, (11) is satisfied by \(Y(y) = 0\) or \(T(t) = 0\), but we cannot tolerate these choices because they give \(w(x, y, t) = 0\), which will not satisfy the initial condition \(w(x, y, 0) = f(x, y)\). That is, as usual, we make such choices so as to maintain as robust a solution as possible.
to (9a) gives \(A = 0\), so \(X(x) = B \sin \alpha x\). Then, \(X(a) = 0 = B \sin \alpha a\). We cannot afford the choice \(B = 0\) for then \(X(x) = 0\) and \(w(x, y, z) = 0\), so we require that \(\sin \alpha a = 0\). Hence,

\[
\alpha = \frac{m\pi}{a}
\]

for any integer \(m = 1, 2, \ldots\). Similarly, \(Y(0) = 0\) gives \(D = 0\) and \(Y(b) = 0\) gives \(\sqrt{n^2 - \alpha^2} = n\pi/b\) for any integer \(n = 1, 2, \ldots\), or because of (12),

\[
\kappa = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.
\]

Further, the initial condition

\[
w_t(x, y, 0) = 0 = X(x)Y(y)T'(0)
\]

gives \(T'(0) = 0\), and application of this condition to (9c) gives \(G = 0\).

Putting these results into (9a,b,c) and then putting the latter into (2) gives the product solution

\[
w(x, y, t) = BEF \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} t
\]

for any \(m = 1, 2, \ldots, n = 1, 2, \ldots\), where \(BEF \equiv H\), say, is arbitrary. Or, with the help of superposition,

\[
w(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \omega_{mn} t,
\]

where

\[
\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.
\]

Finally, the initial condition \(w(x, y, 0) = f(x, y)\) requires that

\[
f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

on the rectangular domain. The latter series is an example of a double series; more specifically, it is a double Fourier series. Suppose that \(f(x, y)\) can, indeed, be

\[\text{Recall that for "single series" we say that } \sum_{n=1}^{\infty} a_n \text{ converges to } s \text{ if to each } \epsilon > 0 \text{ no matter how small there corresponds an integer } N_\epsilon \text{ such that} \]

\[|s - \sum_{n=1}^{\infty} a_n| < \epsilon\]
Chapter 19. Wave Equation

represented by such a series. Then we can compute the $H_{mn}$ coefficients formally as follows.

Re-express (17) as

$$f(x, y) = \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} H_{mn} \sin \frac{m\pi x}{a} \right) \sin \frac{n\pi y}{b}$$

$$= \sum_{n=1}^{\infty} R_n(x) \sin \frac{n\pi y}{b}. \quad (18)$$

For fixed $x$ ($a < x < b$) the latter is a half-range sine expansion of $f(x, y)$ on $0 < y < b$, so

$$R_n(x) = \frac{2}{b} \int_{0}^{b} f(x, y) \sin \frac{n\pi y}{b} \, dy. \quad (19)$$

In turn,

$$R_n(x) = \sum_{m=1}^{\infty} H_{mn} \sin \frac{m\pi x}{a} \quad (20)$$

is a half-range sine expansion of $R_n(x)$ on $0 < x < a$, so

$$H_{mn} = \frac{2}{a} \int_{0}^{a} R_n(x) \sin \frac{m\pi x}{a} \, dx. \quad (21)$$

Putting (19) into (21) gives the formula

$$H_{mn} = \frac{4}{ab} \int_{0}^{b} \int_{0}^{a} f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \quad (22)$$

for the evaluation of the Fourier coefficients $H_{mn}$ in (17). Thus, the solution of (1) is given by (16) with the coefficients calculated according to (22). With these results in hand let us comment on their derivation and physical interpretation.

**COMMENT 1.** Our choice of the minus sign in (4) was dictated by our understanding that the motion is, indeed, a vibration; the minus sign in (4) led to a plus sign whenever $N > N_0$. Other definitions are used occasionally; this one is called *ordinary convergence* and we say that $\sum_{n=1}^{\infty} a_n$ converges to $s$ in the sense of ordinary convergence. If to each $\epsilon > 0$ there does not exist such an then the series is said to diverge. Analogously, the *double series* $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}$ is said to converge to $s$, in the sense of *Pringsheim convergence*, if to each $\epsilon > 0$ (no matter how small) there correspond integers $M_0(\epsilon)$ and $N_0(\epsilon)$ such that

$$\left| s - \sum_{n=1}^{N} \sum_{m=1}^{M} a_{mn} \right| < \epsilon$$

whenever $M > M_0$ and $N > N_0$; otherwise the series is said to diverge. For further discussion, we refer the interested reader to P. W. Berg and J. L. McGregor, *Elementary Partial Differential Equations*, prelim. ed. (San Francisco: Holden-Day, 1964), Sec. 10.3.
in (5) and the oscillatory solutions \( \cos kct \) and \( \sin kct \). But what motivated us to choose the minus in front of the \( \alpha^2 \) in (6)? Once again, we need to look ahead. Specifically, the eventual Fourier series expansion of \( f(x, y) \) dictates the need for cosine and sine solutions of (7) and (8). The \(-\alpha^2 \) in (7), which results from the \(-\alpha^2 \) in (6), does give \( \cos \alpha x \) and \( \sin \alpha x \) solutions. Further, observe that we wrote the \( Y \) equation (8) in the form \( Y'' + (\kappa^2 - \alpha^2)Y = 0 \) rather than in the equally correct form \( Y'' - (\alpha^2 - \kappa^2)Y = 0 \) in order to force the cosine and sine solutions given in (9b) (see Exercise 1).

**COMMENT 2.** What do the individual modes look like? The shape of the \( m, n \) mode is given by \( \sin(m\pi x/a)\sin(n\pi y/b) \), which is modulated periodically in time by the \( \cos\omega_{mn}t \) factor. For \( m, n = 1, 1 \) and \( 2, 1 \), for instance, the mode shapes are as indicated schematically in Fig. 2. In the case of the \( 2, 1 \) mode \( w = 0 \) all along the line \( x = a/2 \) because the \( \sin(2\pi x/a) \) factor is zero on that line. We call that line, or any curve within the domain and along which \( w = 0 \) for all time, a **nodal line**, and show it as solid in Fig. 2a. If we simply show the nodal lines and indicate the positive and negative deflections by +’s and −’s, then the first several mode patterns are as shown in Fig. 3. Of course, the +’s and −’s alternate in time due to the \( \cos \omega_{mn}t \) factor.

**COMMENT 3.** Concerning the temporal frequencies, observe that whereas for the vibrating string the frequencies \( \omega_n = n\pi c/L \) are integer multiples of a fundamental frequency \( \pi c/L \), the frequencies

\[
\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \text{ rad sec} \tag{23}
\]

of the vibrating membrane are not. To illustrate, let \( a = b \) and let us examine the musical notes corresponding to the various modes. For comparison with the analogous results for a violin string (Table 1, Section 19.2), let us tune our “square drum” so that its fundamental frequency is 27.5 cycles/sec, corresponding to \( A_0 \), the lowest \( A \) on a piano. That is, adjust the tension \( \tau \) so that

\[
\omega_{1,1} = \left( \pi c \sqrt{\frac{1^2}{a^2} + \frac{1^2}{a^2}} \text{ rad sec} \right) \frac{1 \text{ cycle}}{2\pi \text{ rad}} = \frac{c}{\sqrt{2}a} \text{ cycles sec} = 27.5,
\]

where \( c = \sqrt{\tau/\sigma} \). Then the frequencies are

\[
\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{a^2}} \text{ rad sec} \frac{1 \text{ cycle}}{2\pi \text{ rad}} = \frac{c}{2a} \sqrt{m^2 + n^2} \text{ cycles sec}
\]

and these are listed in ascending order in Fig. 4, along with the corresponding musical note. From the tabulation in Fig. 4 we can see why square drums are not prized as musical instruments for, beginning with \( A_1 \), virtually every note is

![Figure 3. The first nine modes.](image-url)
present. This profusion of notes is due to the dense values of \( \sqrt{m^2 + n^2} \). The result would be somewhat like playing the piano with our forearms rather than with our fingers. Circular drums are better and are discussed in the exercises.

**COMMENT 4.** Alternative to the product form (6) we can seek \( w \) in the product form

\[
w(x, y, t) = W(x, y)T(t).
\]

Putting (4) into (1a) and separating gives

\[
c^2 \left( W_{xx} + W_{yy} \right) T = W T'' ,
\]

\[
\frac{W_{xx} + W_{yy}}{W} = \frac{1}{c^2} \frac{T''}{T} = -\kappa^2,
\]

and the separated equations

\[
W_{xx} + W_{yy} + \kappa^2 W = 0
\]

on \( W \), and \( T'' + \kappa^2 c^2 T = 0 \) on \( T \), as before. Then we can seek

\[
W(x, y) = X(x)Y(y)
\]

in (26). The resulting ODE’s on \( X, Y, T \) are the same as before, but we mention this option for two reasons. First, some people prefer this method, and it is often used in textbooks. Second, (26) is a well known PDE, the Helmholz equation, named after Hermann von Helmholtz (1821–1894) and studied by him in connection with acoustics.

**EXAMPLE 1.** Let the initial deflection be

\[
w(x, y, 0) = f(x, y) = 0.2 \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} - 0.1 \sin \frac{5\pi x}{a} \sin \frac{3\pi y}{b}.
\]

We could put this \( f \) into (22) and integrate, to determine the \( H_{mn} \)’s, but it is much simpler to “match” terms in (17):

\[
0.2 \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} - 0.1 \sin \frac{5\pi x}{a} \sin \frac{3\pi y}{b} = H_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} + H_{12} \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} + H_{21} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b} + \ldots.
\]

Thus, \( H_{12} = 0.2 \), \( H_{53} = -0.1 \), and all other \( H_{mn} \)’s are zero. Then, \( \omega_{12} \) and \( \omega_{53} \) are obtained from (16b), so the solution is

\[
w(x, y, t) = 0.2 \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} \cos \left( \frac{\pi c}{a^2 + \frac{4}{b^2} t} \right) - 0.1 \sin \frac{5\pi x}{a} \sin \frac{3\pi y}{b} \cos \left( \frac{\pi c}{a^2 + \frac{9}{b^2} t} \right).
\]
In this example only two modes are present, and these have nodal lines as indicated in Fig. 5. What, if any, are the nodal lines of the solution \( w(x, y, t) \), lines along which \( w = 0 \) for all \( t \)? Since the two cosine functions in (30) are linearly independent, \( w = 0 \) for all \( t \) requires that both \( \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b} \) and \( \sin \frac{5\pi x}{a} \sin \frac{3\pi y}{b} \) be zero. But the curves on which these products are zero are disjoint (Fig. 5), so the solution \( w(x, y, t) \) has no nodal lines. In other cases \( w(x, y, t) \) can, indeed, have nodal lines and these lines are not necessarily straight, as discussed in the exercises.

**EXAMPLE 2.** Using the mks system of units (meters, kilograms, seconds, and newtons), let \( a = 1 \) m, \( b = 0.5 \) m, \( \tau = 8 \) m/s, \( \sigma = 0.02 \) kg/m², and let the initial displacement be

\[
w(x, y, 0) = f(x, y) = 0.02(x - x^2)(y - 2y^2) \text{ meters.}
\]  

Then \( c = \sqrt{\tau/\sigma} = 20 \) and, from (22),

\[
H_{mn} = 8 \int_0^{0.5} \int_0^1 0.02(x - x^2)(y - 2y^2) \sin m\pi x \sin 2n\pi y \, dx \, dy
\]

\[
= 0.16 \int_0^1 (x - x^2) \sin m\pi x \, dx \int_0^{0.5} (y - 2y^2) \sin 2n\pi y \, dy
\]

\[
= \frac{0.64}{m^2n^2 \pi^6}
\]  

(32)

if \( m \) and \( n \) are odd, and zero if \( m \) and/or \( n \) is even. Hence, the solution is

\[
w(x, y, t) = \frac{0.64}{\pi^6} \sum_{n=1,3,...} \sum_{m=1,3,...} \frac{1}{m^2n^2} \sin m\pi x \sin 2n\pi y \cos \left( 20\pi \sqrt{m^2 + 4n^2} \right) t.
\]  

(33)

**COMMENT.** Observe that the terms diminish rapidly with \( m \) and \( n \) due to the \( 1/(m^2n^2) \) factor, so that even the first term alone gives a reasonable approximation of the solution. That result is a consequence of the fact that the \((x - x^2)(y - 2y^2) \) product in (31) is approximated well by a scalar multiple of the first mode \( \sin \pi x \sin 2\pi y \) (Fig. 6).

**Closure.** To solve the wave equation in three independent variables \((x, y, t)\) by separation of variables we seek \( w(x, y, t) \) in the form \( X(x)Y(y)T(t) \). The separation process is successful, and we obtain ODE’s on \( X, Y, \) and \( T \), and two separation constants rather than one. The chief complication is that the solution is a double series rather a single series, with the coefficients given by a double integral rather than a single integral.

**Computer software.** Nodal lines are not always identified as readily as those in Fig. 3 but can be obtained using computer software. For instance, if \( a = \pi, b = 2\pi, \) \( c = 1, \) and

\[
w(x, y, t) = (2 \sin 3x \sin y - \sin x \sin 3y) \cos \left( \sqrt{10} t \right),
\]  

(34)

then the nodal lines, if any, are given implicitly by the relation

\[
2 \sin 3x \sin y - \sin x \sin 3y = 0.
\]  

(35)
EXERCISES 19.3

1. Obtain the solution to the initial-value problem (1) for the given $f(x, y)$, $a$, and $b$, and give the relations that define the nodal lines of $w$, if $c = 1$ in each case.

(a) $f(x, y) = 8 \sin 2 \pi x \sin 2y$, $a = \pi, b = 2\pi$
(b) $f(x, y) = -5 \sin 2 \pi x \sin 4y$, $a = 2\pi, b = \pi$
(c) $f(x, y) = \sin 3 \pi x \sin y - \sin x \sin 3y$, $a = b = \pi$
(d) $f(x, y) = \sin 3 \pi x \sin y - \sin x \sin 3y$, $a = \pi, b = 2\pi$
(e) $f(x, y) = 1.05 \sin 3 \pi x \sin y - \sin x \sin 3y$, $a = \pi, b = 2\pi$
(f) $f(x, y) = 10 \sin 3 \pi x \sin y - \sin x \sin 3y$, $a = \pi, b = 2\pi$
(g) $f(x, y) = 8 \sin \pi x \sin 7y + \sin 5 \pi x \sin 5y$, $a = b = \pi$
(h) $f(x, y) = 8 \sin 2 \pi x \sin 7y + \sin 5 \pi x \sin 5y$, $a = b = \pi$
(i) $f(x, y) = 6 \sin \pi x \sin 7y + \sin 5 \pi x \sin 5y$, $a = b = 1$
(j) $f(x, y) = \sin \pi x \sin 4 \pi y - \sin 3 \pi x \sin 5 \pi y$, $a = 1, b = 2$

2. Obtain the solution to the initial-value problem (1) for the case where $f(x, y) = 20 \sin 3 \pi x \sin 4 \pi y - 8 \sin 5 \pi x \sin 12 \pi y$ and $a = b = c = 1$, and determine the period of the motion.

3. Evaluate the following claim and reasoning and indicate whether it is correct or incorrect. Claim: Whereas the solution to the vibrating string problem [given by (16) in Section 19.2] is periodic in time (for any choice of the $R_n$'s and $S_n$'s), the solution to the vibrating membrane problem (1) [given by (16a) in this section] is not, in general, a periodic function of $t$. Reasoning: It will suffice to consider two terms in (16a) so that $w(x, y)$ is of the form $A \cos \omega_{kl} t + B \cos \omega_{mn} t$ for some choice of the integers $k, l, m, n$ and the constants $A$ and $B$. (Of course, $A$ and $B$ depend on $x$ and $y$, but we can consider $x$ and $y$ as fixed here.) If the period is $T$, then there must be integers $M$ and $N$ such that $\omega_{kl} T = (M)(2\pi)$ and $\omega_{mn} T = (N)(2\pi)$. Division gives $\omega_{kl} / \omega_{mn} = M/N$. That is, $\omega_{kl} / \omega_{mn}$ must be a rational number. However, we see from (16b) that $\omega_{kl} / \omega_{mn}$ is, except for certain choices of $k, l, m$, and $n$, irrational.

4. (Nonzero initial velocity) We used $w_1(x, y, 0) = 0$ in (1c) merely for brevity.

(a) With $w(x, y, 0) = 0$, $w_2(x, y, 0) = g(x, y)$
in place of (1c), rederive the solution and obtain results analogous to (16) and (22).

(b) With $w(x, y, 0) = f(x, y)$, $w_t(x, y, 0) = g(x, y)$
in place of (1c), re-derive the solution and obtain results analogous to (16) and (22).

5. (Circular drum) If the membrane is stretched over the circular disk $r < a$, then it is best to use the polar coordinates $r, \theta$ rather than the Cartesian coordinates $x, y$. Then the wave equation $c^2 \nabla^2 w = w_{tt}$ on $w(r, \theta, t)$ becomes

$$c^2 \left( w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} \right) = w_{tt}.$$ (5.1)

Let us consider only axisymmetric motions (i.e., we consider $w$ to be independent of $\theta$). Then the $w_{\theta\theta}$ term in (5.1) is zero, and (5.1) reduces to

$$c^2 \left( w_{rr} + \frac{1}{r} w_r \right) = w_{tt}.$$ (5.2)
on $w(r, t)$.

(a) Seeking $w(r, t) = R(r)T(t)$, obtain

$$r R'' + R' + \kappa^2 R = 0,$n \quad T'' + \kappa^2 c^2 T = 0$$

with (plots):

implicitplot $(2 \sin (3 \times x) \sin (5 \times y) - \sin (x) \sin (3 \times y) = 0$,
$x = 0..P_i, y = 2 \times P_i, \text{grid} = [100, 200]$);

The result is shown in Fig. 7. The "grid = [100, 200]" part is an option that we have used to set the plotting grid at 100 divisions of the $x$ interval and 200 divisions of the $y$ interval since $w$ the default grid. [25, 25], is too coarse in this case.
and hence the solution form

\[ w(r,t) = (A + B \ln r)(D + Et) + [F J_0(\kappa r) + G Y_0(\kappa r)](H \cos \kappa ct + I \sin \kappa ct). \]  

(5.3)

(b) If \( w(a,t) = 0 \) and \( w(r,t) \) is to be bounded on \( r < a \) (in particular at \( r = 0 \)), show that we obtain

\[ w(r,t) = \sum_{n=1}^{\infty} J_0 \left( \frac{r}{a} \right) (H_n \cos \omega_n t + I_n \sin \omega_n t), \]  

(5.4)

where the \( z_n \)'s \( (n = 1, 2, \ldots) \) are the (known) zeros of \( J_0 \) [i.e., \( J_0(z_n) = 0 \)] and \( \omega_n = z_n c/\alpha \).

(c) Let the initial conditions be

\[ w(r,0) = f(r), \quad w_t(r,0) = g(r). \]  

(5.5)

Imposing these conditions on (5.4), show that

\[ H_n = \frac{2}{\alpha^2} \left| J_1(z_n) \right|^2 \int_0^a f(r) J_0 \left( \frac{r}{a} \right) r \, dr, \]

\[ I_n = \frac{2}{\omega_n \alpha^2} \left| J_1(z_n) \right|^2 \int_0^a g(r) J_0 \left( \frac{r}{a} \right) r \, dr. \]  

(5.6a,b)

HINT: This problem is similar to Example 5 in Section 18.3.

(d) Show that if we seek \( w(r,\theta,t) = R(r)\Theta(\theta)T(t) \) in the full equation (5.1), then we obtain the ODE's

\[ r^2 R'' + r R' + (\kappa^2 r^2 - \alpha^2) R = 0, \]

\[ \Theta'' + \alpha^2 \Theta = 0, \]

\[ T'' + \kappa^2 \epsilon^2 T = 0, \]  

(6.1a,b,c)

where \( \kappa \) and \( \alpha \) are separation constants.

6. (Two-dimensional diffusion) In Chapter 18 we study one-dimensional diffusion phenomena such as the unsteady conduction of heat in a rod. Using the methods discussed in this section we can now return to Chapter 18 and solve two-dimensional problems as well. Specifically, consider the temperature field \( u(x,y,t) \) in a rectangular plate \((0 < x < a, 0 < y < b)\), governed by the problem

\[ \alpha^2 (u_{xx} + u_{yy}) = u_t, \]

\[ u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0, \]

\[ u(x,y,0) = f(x,y). \]  

(6.2)

(a) Derive the solution

\[ u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-\kappa^2_{mn} \alpha^2 t}, \]  

(6.3)

where

\[ \kappa_{mn}^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \]  

and

\[ A_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy. \]  

(6.4)

(b) Verify, formally, that (6.2)--(6.4) does satisfy (6.1).

(c) Evaluate the \( A_{mn} \)'s for the case where \( f(x,y) = 100 \).

19.4 Vibrating String; d’Alembert’s Solution

19.4.1. d’Alembert’s solution. For the wave equation

\[ c^2 y_{xx} = y_{tt} \]  

(1)

there exists a striking solution form that is due to Jean Le Rond d’Alembert (1717–1783). D’Alembert’s method is based on a change of independent variables, from \( x \) and \( t \) to \( \xi \) and \( \eta \), say, according to the simple relations

\[ x = \frac{1}{2}(\xi + \eta), \quad t = \frac{1}{2c}(\xi - \eta). \]  

(2)
To re-express (1) in terms of \( \xi \) and \( \eta \), the needed “building blocks” are the \( \partial/\partial x \) and \( \partial/\partial t \) operators. According to the chain rule (Sections 13.4, 13.6.4),

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta},
\]

and

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} = -c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta},
\]

so (1) becomes

\[
e^2 \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) y = \left( -c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta} \right) \left( -c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta} \right) y
\]
or

\[
\left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) (\xi + \eta) = \left( -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) (-\xi + \eta)
\]
or

\[
y\xi + y\eta + y\xi + y\eta = y\xi - y\xi - y\eta + y\eta.
\]

Assuming that \( y \) is well enough behaved so that \( y\xi y\eta = 0 \), (5) simplifies to \( 4y\xi y\eta = 0 \), or

\[
y\xi y\eta = 0.
\]

The point, then, is that when expressed in terms of \( \xi \) and \( \eta \) the wave equation (1) simplifies dramatically to the form (6), and we say that (6) is the canonical, or simplest, form of (1). It is simple because it can be solved directly by integration. First, integrating (6) with respect to \( \eta \) gives

\[
y\xi = \int 0 \partial \eta = 0 + A(\xi) = A(\xi),
\]

where the “constant of integration” \( A \) is allowed to be an arbitrary function of \( \xi \) since \( \xi \) was held fixed during the integration on \( \eta \). Next, integration of (7) with respect to \( \xi \) gives \( y = \int A(\xi) \partial \xi \) plus an arbitrary function of \( \eta \), say \( G(\eta) \). Since \( A \) is arbitrary we might as well simply write \( F(\xi) \) in place of \( \int A(\xi) \partial \xi \). Thus

\[
y = F(\xi) + G(\eta)
\]
or, returning to the original variables,

\[
y(x, t) = F(x - ct) + G(x + ct),
\]

*See Theorem 13.3.1.
where $F$ and $G$ are arbitrary functions [although they do need to be twice differentiable if (9) is to satisfy (1)]. For example, each of the three functions

$$
6e^{x-ct}, \quad \sin(x - ct) + 5\sin[3(x + ct)], \\
100 - 5(x - ct)^{3} - 8e^{4(x+ct)^{2}}
$$

(10)
is of the form (9) and it is readily verified that each one satisfies (1). To verify (9), in general, we can use the chain rule:

$$
y_t = F'(x - ct) \frac{\partial(x - ct)}{\partial t} + G'(x + ct) \frac{\partial(x + ct)}{\partial t} \\
y_{tt} = -eF'(x - ct) + eG'(x + ct)
$$

(11a)

Similarly, we obtain

$$
y_{xx} = F''(x - ct) + G''(x + ct),
$$

(11b)

so we see from (11) that (1) is satisfied. Remember that primes are standard notation for the derivative of a function of a single variable with respect to that variable; for functions of more than one variable we use the partial derivative notation. Thus, by $F'(x - ct)$, for example, we mean the ordinary derivative $\frac{dF(x-ct)}{dx}$ of $F$ with respect to its single argument $x - ct$. To illustrate, if

$$
F(x - ct) = 3 \sin(x - ct)^{2},
$$

then

$$
F'(x - ct) = 6(x - ct) \cos(x - ct)^{2}.
$$

We say that (9) is the **general solution** of the wave equation (1). There is no analog of (9) in Chapter 18; nowhere in Chapter 18 did we find the general solution of the diffusion equation. Even the infinite series solutions that we found by separation of variables, which contained an infinite number of arbitrary constants, are not general solutions; they are simply comprehensive enough so as to be capable of satisfying the initial condition.

It is interesting to compare the form of (9) with the form $y(x) = C_{1}y_{1}(x) + C_{2}y_{2}(x)$ of the general solution to a linear homogeneous second-order ordinary differential equation; in place of an arbitrary constant $C_{1} times y_{1}(x)$ we have an arbitrary function of $x - ct$, and in place of an arbitrary constant $C_{2} times y_{2}(x)$ we have an arbitrary function of $x + ct$.

To illustrate the use of (9), consider the infinite string problem

$$
c^{2}y_{xx} = y_{tt}, \quad ( -\infty < x < \infty, \quad 0 < t < \infty) \\
y(x, 0) = f(x), \quad y_{t}(x, 0) = g(x), \quad ( -\infty < x < \infty)
$$

(12a)
Since we already have in (9) the general solution of (12a), we can bypass (12a) and immediately impose the initial conditions (12b) in order to determine the functions $F$ and $G$ in (9). Doing so gives

\[
\begin{align*}
y(x,0) &= f(x) = F(x) + G(x), \\
y_t(x,0) &= g(x) = -cF'(x) + cG'(x)
\end{align*}
\]

as two equations on $F$ and $G$. Integrating (13b) from any fixed point, say 0, to $x$ gives

\[
\int_0^x g(\xi) \, d\xi = -cF(x) + cF(0) + cG(x) - cG(0),
\]

and solving (13a) and (14) for $F(x)$ and $G(x)$ gives

\[
\begin{align*}
F(x) &= \frac{f(x)}{2} - \frac{1}{2c} \int_0^x g(\xi) \, d\xi + \frac{F(0) - G(0)}{2}, \\
G(x) &= \frac{f(x)}{2} + \frac{1}{2c} \int_0^x g(\xi) \, d\xi - \frac{F(0) - G(0)}{2}.
\end{align*}
\]

Since $F(x)$ is given by the right-hand side of (15a) $F(x - ct)$ is likewise given by the right-hand side of (15a), but with each of the two $x$'s changed to $x - ct$. Similarly, $G(x + ct)$ is obtained by changing each of the $x$'s in (15b) to $x + ct$. Doing so, we obtain

\[
\begin{align*}
y(x,t) &= F(x - ct) + G(x + ct) \\
&= \frac{f(x - ct)}{2} - \frac{1}{2c} \int_0^{x-ct} g(\xi) \, d\xi + \frac{F(0) - G(0)}{2} \\
&\quad + \frac{f(x + ct)}{2} + \frac{1}{2c} \int_0^{x+ct} g(\xi) \, d\xi - \frac{F(0) - G(0)}{2},
\end{align*}
\]

or

\[
y(x,t) = \left[ \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) \, d\xi \right].
\]

Let us interpret (17) in the $x, t$ plane. Since $x, t$ is the specific point at which $y$ is being evaluated, let us use dummy variables $\xi, \eta$ for the axes (Fig. 1). In words, (17) tells us that $y$ at $P$ is the average of the $f$ values at $A$ and $B$ plus $1/2c$ times the integral of $g$ from $A$ to $B$. Thus, the value of $y$ at $P$ depends on initial data only on the interval $AB$. Similarly, the value of $y$ at $x_0, t_0$ depends on initial data only on the interval $CD$, and so on. Thus, we call the triangular region $ABP$ the domain of influence of the interval $AB$; initial data on $AB$ determine the solution within the triangle $ABP$.

This result for the wave equation is in contrast with the result that the solution $u(x,t)$ of the diffusion equation depends on the initial data $u(x,0) = f(x)$ all
along the axis, as can be seen from (11) in Section 18.4 where the integration is from \( \xi = -\infty \) to \( \xi = +\infty \).

**EXAMPLE 1.** *The Case Where* \( y_t(x, 0) = 0 \). To illustrate (17), suppose we release the string from rest in the configuration \( y(x, 0) = f(x) \), where \( f \) is the triangular pulse shown in Fig. 2. Then \( g(x) = 0 \) and (17) gives

\[
y(x, t) = \frac{f(x - ct) + f(x + ct)}{2}.
\]

At \( t = 0 \), (18) becomes \( y(x, 0) = f(x)/2 + f(x)/2 \), so we may regard the initial pulse \( f(x) \) as the sum of two "half-pulses," \( f(x)/2 \) and \( f(x)/2 \). For \( t > 0 \) the \( f(x - ct)/2 \) term in (18) amounts to one of the half-pulses translated to the right through a distance \( ct \), and the \( f(x + ct)/2 \) term amounts to the other half-pulse translated to the left through a distance \( ct \), as depicted in Fig. 3. Evidently, the speed of these right- and left-running waves is \( c \). Thus, the \( c = \sqrt{\tau/\sigma} \) that appears in (1) is the wave speed.

Besides plotting \( y \) versus \( x \) as we have in Fig. 3, it is illuminating to display these results in the \( x, t \) plane as we have in Fig. 4. Naturally, plotting \( y(x, t) \) above the \( x, t \) plane calls for three-dimensional graphics, but it is simpler to merely "lay the solution curves down" in the plane of the paper as we have in the figure. Since the \( f(x - ct)/2 \) term is constant along \( x - ct = \) constant lines, the values of \( f(x - ct)/2 \) propagate along these lines without change. Similarly, the values of \( f(x + ct)/2 \) propagate along \( x + ct = \) constant lines.

These special families of curves, the lines \( x - ct = \) constant and \( x + ct = \) constant, are known as characteristics, and it is along the characteristics that information (i.e., va-
ues) propagates. In Fig. 4 we have drawn only four characteristics, those that define the "channels" I and II in which the half-pulses propagate, but there are actually an infinite number of characteristics, all of the curves $\xi = x - ct = \text{constant}$ and all of the curves $\eta = x + ct = \text{constant}$ as suggested more fully in Fig. 5.

![Figure 5. The two families of characteristics.](image)

The upshot is that the initial $f$ pulse breaks into two half-pulses that travel outward, without change in shape, in the channels I and II that are bounded by characteristics. Outside of these channels $y(x, t) = 0$.

**EXAMPLE 2. The Case Where $y(x, 0) = 0$.** (This example can be omitted if you have not read the optional Section 5.6 on the Dirac delta function.) In Example 1 we took $g(x) = 0$ and $f(x)$ to be a simple triangular pulse in order to gain understanding of the $[f(x - ct) + f(x + ct)]/2$ term in (17). Now, to study the integral term let the initial displacement be $g(x, 0) = f(x) = 0$ and let the initial velocity be

$$y_t(x, 0) = g(x) = \delta(x - x_0),$$  

(19)

a delta function at some point $x_0$. These initial conditions are similar to those to which a piano wire is subjected, for the initial displacement is zero but a localized velocity is imparted at time $t = 0$ when the string is struck by a narrow hammer.

![Figure 6. Response to hammer blow.](image)

With $f(x) = 0$ and $g(x) = \delta(x - x_0)$, (17) gives

$$y(x, t) = 0 + \frac{1}{2c} \int_{x - ct}^{x + ct} \delta(\xi - x_0) \, d\xi,$$  

(20)
From the definition of the delta function [see (13) in Section 5.6], the integral is 1 if \( x - ct < x_0 < x + ct \) or, equivalently, if \( x_0 - ct < x < x_0 + ct \). Hence, the solution (20) is the rectangular pulse shown in Fig. 6; in the \( x,t \) plane the solution is as shown in Fig. 7.

19.4.2. Use of images. (NOTE: The optional Section 18.5 on the method of images is not a prerequisite for this section.) Remember that the solution form (17) is for infinite strings, on \(-\infty < x < \infty\). If we have a semi-infinite or finite string, then we have boundary conditions to deal with besides the two initial conditions. To illustrate, consider the following semi-infinite string problem:

\[
\begin{align*}
\frac{\partial^2 y}{\partial t^2} &= y_{tt}, \quad (0 < x < \infty, \ 0 < t < \infty) \\
y(0, t) &= 0, \quad (0 < t < \infty) \\
y(x, 0) &= f(x), \quad y_t(x, 0) = 0, \quad (0 < x < \infty)
\end{align*}
\]  

where \( f \) is the triangular pulse shown in Fig. 4. Observe that the solution shown in Fig. 4 does indeed satisfy (21) but only up until time \( T \), at which time the left-running wave in channel I reaches the end point \( x = 0 \) and upsets the boundary condition (21b).

We can overcome this difficulty by an artifice known as the **method of images**. Namely, consider an infinite string \((-\infty < x < \infty)\) with the initial conditions

\[
y(x, 0) = f_{\text{ext}}(x), \quad y_t(x, 0) = 0, \quad (-\infty < x < \infty)
\]  

where \( f_{\text{ext}}(x) \) is identical to \( f(x) \) on \( 0 < x < \infty \) and is an antisymmetric extension of \( f(x) \) for \(-\infty < x < 0\) as shown in Fig. 8. The solution of the extended problem is

![Figure 8. Image system.](image-url)
Chapter 19. Wave Equation

Figure 9. Cancellation at $x = 0$.

Figure 10. $f(x)$ in (24).

Figure 11. $f_{\text{ext}}(x)$, antisymmetric about $x = 0$ and $x = L$.

\[
y(x, 0) = \frac{f_{\text{ext}}(x - ct) + f_{\text{ext}}(x + ct)}{2}
\]

(23)

and the latter is comprised of two positive half-pulses in channels I and II plus the two negative half-pulses in channels III and IV. The point is that over the time interval $T_1 < t < T_3$ the waves in channels I and IV pass over each other, so that the negative wave in channel IV cancels the positive wave in channel I along the $t$ axis. At $t = T_2$, for instance, the situation is as shown in Fig. 9, where the sum of the two waves is indicated by the solid line. That part of the diagram that is in the second quadrant ($-\infty < x < 0, 0 < t < \infty$) is called the image system and is fictitious, serving only to automatically satisfy the boundary condition (21b) by virtue of the antisymmetry about $x = 0$. It can now be ignored, or even discarded, since the wave system in the first quadrant fully satisfies conditions (21a,b,c) and is the desired solution.

Since the image system is, after all, fictitious, to apprehend the physical event we look only at the first quadrant. We see that the initial pulse breaks into two half-pulses. One travels rightward indefinitely, while the left-running wave is both reflected (into channel IV) and inverted when it encounters the left end of the string, which is tied at $x = 0$.

Having studied the infinite string and semi-infinite string, in that order, we can finally return to the finite string problem

\[
c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}, \quad (0 < x < L, \ 0 < t < \infty) \tag{24a}
\]

\[
y(0, t) = 0, \ y(L, t) = 0, \quad (0 < t < \infty) \tag{24b}
\]

\[
y(x, 0) = f(x), \ y_t(x, 0) = 0 \quad (0 < x < L) \tag{24c}
\]

that is solved by separation of variables in Section 19.2, and solve it by d’Alembert’s method - with help from the method of images. Let $f(x)$ be our usual generic triangular pulse (Fig. 10). Using the method of images, we consider instead the infinite string problem with initial conditions

\[
y(x, 0) = f_{\text{ext}}(x), \ y_t(x, 0) = 0, \quad (-\infty < x < \infty) \tag{25}
\]

where $f_{\text{ext}}(x)$ is identical to $f(x)$ on $0 < x < L$ and is defined on $x < 0$ and on $x > L$ so as to be antisymmetric about both $x = 0$ and $x = L$. Thus, $f_{\text{ext}}(x)$ is the $2L$-periodic function shown in Fig. 11. Since the extended initial conditions (25) are antisymmetric about $x = 0$ and $x = L$, the solution $y(x, t)$ will be too. Hence, $y$ will be zero all along the $x = 0$ and $x = L$ lines in the $x, t$ plane. That is, the image system is designed so as to satisfy the boundary conditions (24b).

With $f_{\text{ext}}(x)$ as defined in Fig. 11, the d’Alembert solution (17) gives the closed form solution

\[
y(x, t) = \frac{f_{\text{ext}}(x - ct) + f_{\text{ext}}(x + ct)}{2} \tag{26}
\]

To relate (26) to the separation of variables solution obtained in Section 19.2, expand the periodic function $f_{\text{ext}}(x)$ in a Fourier series as

\[
f_{\text{ext}}(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \tag{27}
\]
where

$$b_n = \frac{2}{L} \int_0^L f_{\text{ext}}(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

since $f_{\text{ext}}(x) = f(x)$ on $0 < x < L$. Then (26) becomes

$$y(x, t) = \frac{1}{2} \sum_{n=1}^{\infty} b_n \left[ \sin \frac{n\pi}{L}(x - ct) + \sin \frac{n\pi}{L}(x + ct) \right]$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L},$$

(28)

which is identical to the result that we obtained by separation of variables.

19.4.3. Solution by integral transforms. (Optional) Following d’Alembert, we derived the solution (17) of the infinite string problem (12) by using the general solution $y(x, t) = F(x - ct) + G(x + ct)$. Alternatively, we could solve (12) by Laplace transforming with respect to the $t$ variable or by Fourier transforming with respect to the $x$ variable.

Let us try the Laplace transform. Transforming (12a) gives

$c^2 \bar{y}_{xx} = s^2 \bar{y} - sy(x, 0) - y_t(x, 0)$

$$= s^2 \bar{y} - sf(x) - g(x)$$

or

$$\bar{y}_{xx} - \frac{s^2}{c^2} \bar{y} = -\frac{1}{c^2} [sf(x) + g(x)].$$

(29)

The homogeneous solution of (29) is simple but obtaining a particular solution (e.g., by the method of variation of parameters) is messy, so let us see whether the Fourier transform is more convenient. Fourier transforming (12a) gives

$$c^2 (i\omega)^2 \bar{y} = \bar{y}_{tt}$$

or

$$\bar{y}_{tt} + \omega^2 c^2 \bar{y} = 0,$$

(30)

which is simpler than (29) because it is homogeneous, so let us continue. From (30),

$$\bar{y} = A \cos \omega ct + B \sin \omega ct.$$

(31)

To solve for $A$ and $B$ we impose Fourier transformed versions of (12b):

$$\bar{y} \bigg|_{t=0} = \hat{f}(\omega) = A,$$

(32a)

$$\bar{y}_t \bigg|_{t=0} = \hat{g}(\omega) = \omega c B.$$

(32b)
so \( A = \hat{f}(\omega) \) and \( B = \hat{g}(\omega)/\omega c \) and

\[
\hat{y} = \hat{f}(\omega) \cos \omega ct + \frac{\hat{g}(\omega) \sin \omega ct}{\omega c}.
\]

(33)

To invert the two terms on the right, use entries 15, 9, and 21 of Appendix D and obtain

\[
\begin{align*}
F^{-1} \left\{ \hat{f}(\omega) \cos \omega ct \right\} &= \frac{f(x - ct) + f(x + ct)}{2}, \\
F^{-1} \left\{ \hat{g}(\omega) \frac{\sin \omega ct}{\omega c} \right\} &= g(x) * \frac{1}{2c} [H(x + ct) - H(x - ct)] \\
&= \frac{1}{2c} \int_{-\infty}^{\infty} g(x - \xi) [H(\xi + ct) - H(\xi - ct)] \, d\xi \\
&= \frac{1}{2c} \int_{-ct}^{ct} g(x - \xi) \, d\xi \quad (x - \xi = \mu) \\
&= \frac{1}{2c} \int_{x-ct}^{x+ct} g(\mu) \, d\mu,
\end{align*}
\]

(34a)

(34b)

where the third equality in (34b) follows from the fact that \( H(x + ct) - H(x - ct) \) is zero for \( x < -ct \), unity for \( -ct < x < ct \), and zero for \( x > ct \). Thus,

\[
y(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\mu) \, d\mu,
\]

which result is the same as (17).

**Closure.** Following d’Alembert, we change the independent variables from \( x, t \) to \( \xi, \eta \) according to the relations \( \xi = x - ct \) and \( \eta = x + ct \). That change reduces the wave equation \( c^2 y_{xx} = y_{tt} \) to its canonical (i.e., simplest) form \( y_{\xi\eta} = 0 \), which can be solved by integration to give the general solution of the wave equation

\[
y(x, t) = F(x - ct) + G(x + ct),
\]

where \( F \) and \( G \) are arbitrary twice differentiable functions (Exercise 15). To illustrate the use of this general solution we solve an infinite string problem with prescribed initial displacement \( f(x) \) and velocity \( g(x) \) and obtain the solution (17).

The \( \xi, \eta \) variables are the most “natural” independent variables and the \( \xi = \) constant and \( \eta = \) constant lines constitute two families of lines (Fig. 5) called characteristics. Rather than kinks and discontinuities in the initial conditions smoothing out, as occurs in the process of diffusion, they propagate into the solution domain along characteristics. And whereas information spreads to the left and right at an infinite speed for the diffusion equation, it spreads with a finite wave speed \( c \) for the wave equation.

Remember that, while extremely important in their own right, the diffusion (\( \alpha^2 u_{xx} = u_t \)) and wave (\( c^2 y_{xx} = y_{tt} \)) equations also serve to represent the parabolic
and hyperbolic types of PDE's, respectively, and the main features of these two equations are shared by other PDE's of their type. Thus, every hyperbolic PDE has its own families of characteristics, which may be curved rather than straight and along which information propagates.

In Section 19.4.2 we use the infinite string solution (17) together with the method of images to solve semi-infinite and finite string problems. If you studied Section 18.5 you will understand that the method of images works, here, because the \( L = c^2 \partial^2 / \partial x^2 - \partial^2 / \partial t^2 \) operator is linear and even, but it is not necessary to understand why the method works in these cases inasmuch as we can see that it does lead to satisfaction of the homogeneous boundary conditions.

Finally, in Section 19.4.3 we rederive the solution (17), this time using a Fourier transform on \( x \).

**EXERCISES 19.4**

1. Verify that (17) does satisfy (12a) and (12b).

2. Show that, like the wave equation, the given PDE is hyperbolic and find its general solution by introducing the suggested change of variables.

   (g) \( u_{xx} + 4u_{xy} + 3u_{yy} = 0 \): \( \xi = x - y, \ \eta = 3x - y \)
   (b) \( u_{xx} - 4u_{xy} - 5u_{yy} = 0 \): \( \xi = x - y, \ \eta = 5x + y \)
   (c) \( u_{xx} + 6u_{xy} + 11u_{yy} = 0 \): \( \xi = x - y, \ \eta = 2x - y \)
   (d) \( u_{xx} + 4u_{xy} - 5u_{yy} = 0 \): \( \xi = x + y, \ \eta = 5x - y \)
   (e) \( u_{xx} + 2u_{xy} - 3u_{yy} = 0 \): \( \xi = 3x - y, \ \eta = x + y \)

3. Show that, like the wave equation, the given PDE is hyperbolic and find its general solution by introducing a suitable change of variables of the form \( \xi = ax + by, \ \eta = cx + dy \).

   (g) \( u_{xx} + 8u_{xy} + 12u_{yy} = 0 \)
   (b) \( u_{xx} - 2u_{xy} - 3u_{yy} = 0 \)
   (c) \( u_{xx} - 10u_{xy} + 9u_{yy} = 0 \)
   (d) \( u_{xx} + 2u_{xy} - 8u_{yy} = 0 \)

4. Find the general solution of the first-order PDE \( u_t + cu_x = 0 \), where \( c \) is a constant, by introducing the change of variables \( \xi = x - ct, \ \eta = t \), and then use that general solution to solve the problem.

   \[ u_t + cu_x = 0, \quad (-\infty < x < \infty, \ 0 < t < \infty) \]
   \[ u(x,0) = f(x). \]

5. In Figs. 3 and 4 we show \( y(x,t) \) at times \( t \) which were large enough so that the two half-pulses had completely separated (i.e., they do not overlap). In this exercise we examine the case in which they do overlap. Letting

   \[ f(t) = \begin{cases} 
   0, & x < 1 \\
   2x - 2, & 1 < x < 1.5 \\
   4 - 2x, & 1.5 < x < 2 \\
   0, & x > 2,
   \end{cases} \]

   \( g(x) = 0, \text{ and } c = 20 \), give labeled graphs of \( y(x,t) \) at \( t = 0.005 \) and at \( t = 0.02 \).

6. With \( c = 100 \), sketch the solution to (12) at \( t = 0.02 \) and \( t = 0.04 \) using an \( x,t \) plane display similar to those in Figs. 4 and 7, and labeling all key values. As usual, \( H \) is the Heaviside function.

   (a) \( f(x) = H(x + 1) - H(x - 1), \ g(x) = 0 \)
   (b) \( f(x) = H(x), \ g(x) = 0 \)
   (c) \( f(x) = 0, \ g(x) = H(x + 1) - H(x - 1) \)
   (d) \( f(x) = 0, \ g(x) = H(x) \)

7. (a) Obtain the general solution for the nonhomogeneous wave equation \( c^2 y_{xx} = y_{tt} - K \), where \( K \) is a constant.

   (b) Use the general solution obtained in part (a) to solve the problem

   \[ c^2 y_{xx} = y_{tt} - K, \quad (-\infty < x < \infty, \ 0 < t < \infty) \]
   \[ y(x,0) = p(x), \ y_t(x,0) = q(x). \]

8. Use the general solution (9) to solve the problem.

   \[ c^2 y_{xx} = y_{tt}, \quad (0 < x < \infty, \ 0 < t < \infty) \]
   \[ y(x,0) = y_t(x,0) = 0, \quad (0 < x < \infty) \]
   \[ y(0,t) = h(t), \quad (0 < t < \infty) \]
where \( h(t) \) is prescribed. [We can imagine taking the left end of the string between two fingers and, beginning at \( t = 0 \), "jiggling" it according to \( y(0, t) = h(t) \).]

9. Same as Exercise 8, but solve by using the Laplace transform rather than (9). HINT: You may assume the truth of the statement

\[
\lim_{t \to \infty} \bar{y}(x, s) = \lim_{x \to \infty} \int_0^\infty y(x, t)e^{-st} \, dt = \int_0^\infty \lim_{x \to \infty} y(x, t)e^{-st} \, dt = \int_0^\infty 0 \, dt = 0.
\]

10. If spherical symmetry is present so that \( u \) depends only on \( \rho \) and \( t \) (where \( \rho, \phi, \theta \) are the spherical polar coordinates), then the wave equation \( c^2 \nabla^2 u = \ddot{u} \) becomes

\[
\frac{\partial^2 u}{\partial \rho^2} \left( \frac{\partial}{\partial \rho} \right) + \frac{\rho}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} = \frac{\partial^2 u}{\partial t^2}.
\]

Show that (10.1) may be re-expressed as

\[
\frac{\partial^2 u}{\partial \rho^2} (\rho u) = \frac{\partial^2}{\partial t^2} (\rho u) \tag{10.2}
\]

and thus derive the general solution

\[
u(\rho, t) = \frac{1}{\rho} [F(\rho - ct) + G(\rho + ct)] \tag{10.3}
\]

of (10.1), where \( F \) and \( G \) are arbitrary twice-differentiable functions. NOTE: Observe from the \( 1/\rho \) factor in (10.3) that in the case of spherical waves the wave amplitude tends to zero as \( \rho \to \infty \).

11. Show whether traveling wave solutions of the form \( F(x - at) \) are possible for the diffusion problem

\[
\sigma^2 u_{xx} = u_t, \quad (-\infty < x < \infty, 0 < t < \infty)
\]

\[
u(x, 0) = f(x), \quad (-\infty < x < \infty)
\]

where \( u(x, t) \) is bounded, for all \( t \), as \( x \to \pm \infty \).

12. (Infinite string with density discontinuity) Consider a string stretched over \(-\infty < x < \infty \) under a tension \( \tau \), where the density \( \sigma(x) \) has a step discontinuity at \( x = 0 \); i.e., \( \sigma = \sigma_1 \) for \( x < 0 \) and \( \sigma = \sigma_2 \) for \( x > 0 \). Suppose a rightward-running wave \( y(x, t) = F(x - ct) \) such as a triangular pulse, is initiated in the \( x < 0 \) part of the string, where \( c_1 = \sqrt{\tau / \sigma_1} \) is the wave speed for \( x < 0 \) and \( c_2 = \sqrt{\tau / \sigma_2} \) is the wave speed for \( x > 0 \). What happens when this wave encounters the density discontinuity (see the accompanying figure)? That is, determine the solution \( y(x, t) \). HINT: Use the general solution \( y(x, t) = G(x - ct) + H(x + ct) \) for \( x > 0 \), and

\[
g(x, t) = I(x - ct) + J(x + ct) \quad \text{for} \quad x > 0.
\]

13. Use (17) to show that if \( y(x, 0) = f(x) \) and \( y_t(x, 0) = g(x) \) are even functions of \( x \), then \( y(x, t) \) remains an even function for all \( t > 0 \), and that if \( f(x) \) and \( g(x) \) are odd functions of \( x \), then \( y(x, t) \) remains an odd function for all \( t > 0 \).

14. We solve (21) by the method of images, where \( f(x) \) is the triangular pulse shown in Fig. 4. Repeat that solution, with (21b) changed to \( y_t(0, t) = 0 \), and obtain an \( x, t \) plane diagram analogous to that in Fig. 8. Is there a reflection and an inversion as in Fig. 8? Explain.

15. (Breakdowns at kinks) We have emphasized that kinks and discontinuities in the initial conditions \( y(x, 0) = f(x) \) and \( y_t(x, 0) = g(x) \) propagate into the solution domain as in Fig. 4, for instance, where the kinks in \( f \) propagate along both right- and left-running characteristics. However, it must be confessed that at each point along those characteristics the PDE \( c^2 u_{xx} = y_{tt} \) is not satisfied because both \( y_{xx} \) and \( y_{tt} \) fail to exist. (That \( y_{xx} \) does not exist there should be evident from Fig. 4. Do you see why \( y_{tt} \) fails to exist there as well?) Explain why results such as those displayed in Fig. 4 are acceptable nonetheless.

16. (Finite-difference method) In Section 18.6 we discretize the problem by means of the computational grid shown there in Fig. 1, and use difference quotient approximations of \( u_{xx} \) and \( u_t \) to obtain the computational formula

\[
U_{j+1,k} = rU_{j-1,k} + (1 - 2r)U_{j,k} + rU_{j+1,k},
\]

where \( U_{j,k} \) denotes the exact solution of the difference equation.

(a) Proceeding in the same manner, derive the scheme

\[
Y_{i,j+1} = r^2 Y_{i-1,j} + 2(1 - r^2)Y_{i,j} + r^2 Y_{i+1,j} - Y_{i,j-1} \quad (16.1)
\]

for the wave equation problem

\[
c^2 u_{xx} = y_{tt},
\]

\[
y(0, t) = p(t), \quad y(L, t) = q(t),
\]

\[
y(x, 0) = f(x), \quad y_t(x, 0) = g(x)
\]

on \( 0 < x < L, 0 < t < \infty \), where \( r = c \Delta t / \Delta x \).
(b) Letting \( c = 10, L = 1, \Delta x = 0.25, \Delta t = 0.02,\)
\( p(t) = q(t) = f(x) = 0, \) and \( g(x) = \sin \pi x, \) use (16.1) to
evaluate the \( Y_{i,j} \)'s along the first two time lines, i.e., at
\( (i, j) = (1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2). \) Compare
your results with the exact solution. HINT: For points on the
first time (16.1) gives
\[
\begin{align*}
Y_{1,1} &= r^2 Y_{0,0} + 2(1 - r^2) Y_{1,0} + r^2 Y_{2,0} - Y_{1,-1}, \\
Y_{2,1} &= r^2 Y_{1,0} + 2(1 - r^2) Y_{2,0} + r^2 Y_{3,0} - Y_{2,-1}, \\
Y_{3,1} &= r^2 Y_{2,0} + 2(1 - r^2) Y_{3,0} + r^2 Y_{4,0} - Y_{3,-1},
\end{align*}
\]
but how are we to deal with the \( Y_{1,-1}, Y_{2,-1} \) and \( Y_{3,-1} \) terms
when the points \( (i, j) = (1, -1), (2, -1) \), and \( (3, -1) \) do not
lie on the computational grid? Use a finite-difference version
of the initial condition \( y_t(x, 0) = g(x). \)
(c) Same as (b) but with \( p(t) = 20t. \)
(d) Same as (b) but with \( q(t) = 1000(1 - \cos t). \)
(e) Same as (b) but with \( f(x) = 10 \sin \pi x \) and \( g(x) = 0. \)
(f) It can be shown that for the stability and convergence of
the scheme (16.1) we need \( r \leq 1. \) You need not derive this result;
we merely ask you to interpret it graphically in terms of the
concept of the domain of influence.

---

**Chapter 19 Review**

The wave and diffusion equations are similar in several ways:

1. For both the one-dimensional wave and diffusion equations the solution
domain is, typically, a semi-infinite strip \( 0 < x < L, 0 < t < \infty, \) the quarter
plane \( 0 < x < \infty, 0 < t < \infty, \) or the half plane \( -\infty < x < \infty, 0 < t < \infty. \)

2. For both equations we can use the method of separation of variables, a Laplace
transform on the \( t \) variable, or a Fourier transform on the \( x \) variable if the \( x \)
domain is \( -\infty < x < \infty. \)

3. When solving by separation of variables, the boundary conditions are to be
applied before the initial conditions.

But the wave and diffusion equations also differ in some ways:

1. A single initial condition \( u(x, 0) = f(x) \) is appropriate for the diffusion
equation \( \alpha^2 u_{xx} = u_t, \) whereas the two initial conditions \( y(x, 0) = f(x) \) and
\( y_t(x, 0) = g(x) \) are appropriate for the wave equation \( c^2 u_{xx} = y_{tt}. \)

2. Only for the wave equation do we find a general solution, namely,
\[
y(x, t) = F(x - ct) + G(x + ct),
\]
where \( F \) and \( G \) are arbitrary (twice differentiable) functions. The graphs of
\( F(x - ct) \) and \( G(x + ct), \) plotted versus \( x, \) translate rightward and leftward,
respectively, with speed \( c. \) Of course, \( F \) and \( G \) need not be single terms. For
example, in the solution
\[
y(x, t) = \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} (x - ct) + \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} (x + ct)
\]
of the problem

\[ c^2 y_{xx} = y_{tt}, \quad (-\infty < x < \infty, \ 0 < t < \infty) \]  

\[ y(x, 0) = f(x), \ y_t(x) = 0, \quad (-\infty < x < \infty) \]  

where

\[ a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \]

each of the functions

\[ F(x - ct) = \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} (x - ct), \]  

\[ G(x + ct) = \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} (x + ct) \]

is a superposition of individual traveling waves. In this case the trigonometric identities \( \sin (A \pm B) = \sin A \cos B \pm \sin B \cos A \) reduce (2) to the form

\[ y(x, t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}, \]  

which is a superposition of standing waves. Thus, the wave equation inevitably gives traveling waves, by virtue of (1), and in some cases these traveling waves sum to standing waves. The variables \( \xi = x - ct \) and \( \eta = x + ct \) give two families of characteristics in the \( x, t \) plane, the \( \xi = \) constant and \( \eta = \) constant lines, along which information propagates as can be seen from (1) because \( F(x - ct) \) is constant along \( x - ct = \) constant lines and \( G(x + ct) \) is constant along \( x + ct = \) constant lines.

3. Whereas diffusion is a smoothing process, we see in Chapter 19 that kinks and discontinuities in the initial conditions propagate into the \( x, t \) solution domain indefinitely, along characteristics.

The two-dimensional wave equation

\[ c^2 (w_{xx} + w_{yy}) = w_{tt}, \]  

such as governs the displacement \( w(x, y, t) \) of a vibrating drumhead, can be solved by separation of variables by seeking

\[ w(x, y, t) = W(x, y)T(t) \]

and obtaining

\[ W_{xx} + W_{yy} + \kappa^2 W = 0, \]  

\[ T'' + \kappa^2 c^2 T = 0, \]
where (8a) is the two-dimensional Helmholtz equation. In turn, (8a) can be
separated by seeking
\[ W(x, y) = X(x)Y(y), \]
or we could seek \( w(x, y, t) = X(x)Y(y)T(t) \)
right from the start. The result is a double series, and each initial condition \( w(x, y, 0) = f(x, y), \ w_3(x, y, 0) = g(x, y) \) leads to a double Fourier series expansion. The
same method can be applied to the two-dimensional diffusion equation
\[ \alpha^2 (u_{xx} + u_{yy}) = u_t, \]
the chief difference being that for the wave equation the \( T \) equation (8b)
gives the oscillatory solutions \( \cos \kappa t \) and \( \sin \kappa t \), whereas for the diffusion equation the \( T \) equation \( T'' + \kappa^2 \alpha^2 T = 0 \) gives the exponential decay \( \exp(-\kappa^2 \alpha^2 t) \).

Finally, note that in this chapter on the wave equation there is no section on
numerical solution analogous to Section 18.6 on the numerical solution of the
diffusion equation by the method of finite differences. The finite-difference method can
indeed be applied to the wave equation, but if the initial conditions \( y(x, 0) = f(x) \)
and \( y_t(x, 0) = g(x) \) are not smooth functions, then inaccuracies result from the
discontinuities that propagate into the solution domain. In that case it is better to
use the method of characteristics, which uses the more natural characteristic variables \( \xi \) and \( \eta \) in place of \( x \) and \( t \). Essentially, the calculation is carried out along
the characteristics.*

---

*For an introduction to the method of characteristics see G. D. Smith, *Numerical Solution of
Partial Differential Equations* (New York: Oxford University Press, 1965) or, for a more complete
discussion, see E. Zauderer, *Partial Differential Equations of Applied Mathematics* (New York: Wi-
ley, 1983).
Chapter 20

Laplace Equation

20.1 Introduction

We have already encountered the Laplace equation

\[ \nabla^2 u = 0 \]  \hspace{1cm} (1)

in Chapter 16, as well as its nonhomogeneous version, the Poisson equation

\[ \nabla^2 u = f, \]  \hspace{1cm} (2)

where \( f \) is a "source" function that is prescribed over the region in question. Recall, for example, the unsteady diffusion equation

\[ \alpha^2 \nabla^2 u = u_t - F(x, y, z, t) \]  \hspace{1cm} (3)

governing the temperature field \( u(x, y, z, t) \), in which \( F(x, y, z, t) \) is a heat source distribution. If \( F \) does not vary with \( t \) and if there exists a steady-state solution \( u(x, y, z) \), then the latter satisfies the Poisson equation \( \nabla^2 u = -F(x, y, z)/\alpha^2 \). If there is no heat source distribution [i.e., if \( F(x, y, z) = 0 \)], then the steady-state temperature distribution \( u(x, y, z) \) satisfies the Laplace equation (1).

As a second example observe that the electric potential (i.e., the voltage) field \( \Phi(x, y, z) \) is governed by the Poisson equation

\[ \nabla^2 \Phi = -\frac{1}{\epsilon} q(x, y, z), \]  \hspace{1cm} (4)

where \( q(x, y, z) \) is the charge density distribution (which serves as a "source" for the electric potential) and \( \epsilon \) is a physical constant known as the permittivity of the medium. If \( q(x, y, z) = 0 \) in the region, then \( \Phi \) satisfies the Laplace equation \( \nabla^2 \Phi = 0 \). Since the presence of an electric field is solely attributable to the presence of charges, how can there be anything other than the trivial solution \( \Phi(x, y, z) = 0 \) in the event that \( q(x, y, z) = 0 \)? The answer is that there may be charges outside of the region under consideration or on its boundary.
Finally, recall from Example 3 of Section 16.10 that the velocity potential \( \Phi(x, y, z) \) for any irrotational incompressible fluid flow is governed by the Laplace equation

\[ \nabla^2 \Phi = 0, \]

where \( \Phi \) is related to the velocity field \( \mathbf{v}(x, y, z) \) by the formula \( \mathbf{v} = \nabla \Phi \).

Emphasis in this chapter is on the Laplace equation, with the Poisson equation considered only within the exercises. Like Chapters 18 and 19, Chapter 20 is organized according to the various methods of solution. In Sections 20.2 and 20.3 we study the solution of the Laplace equation by separation of variables – using Cartesian coordinates in Section 20.2 and non-Cartesian coordinates in Section 20.3. Solution of certain problems by the Fourier transform is covered in Section 20.4, and the numerical finite difference method is the subject of Section 20.5.

### 20.2 Separation of Variables; Cartesian Coordinates

We limit our attention in this chapter to two-dimensional problems, so the domain \( \mathcal{D} \) is some part of the \( x, y \) plane. For the method of separation of variables to work, \( \mathcal{D} \) must be bounded by constant-coordinate curves, so if we use the Cartesian coordinates \( x, y \), then the generic domain is a rectangle, bounded by constant-\( x \) and constant-\( y \) lines.

#### EXAMPLE 1. Dirichlet Problem for Rectangle.

Consider the boundary-value problem

\[ \begin{align*}
\nabla^2 u &= u_{xx} + u_{yy} = 0 & \text{in } \mathcal{D}, \\
u(0, y) &= 0, \quad (0 < y < b) \\
u(a, y) &= f(y), \quad (0 < y < b) \\
u(x, 0) &= u(x, b) = 0, \quad (0 < x < a)
\end{align*} \]

where \( \mathcal{D} \) is the rectangle shown in Fig. 1. Since all of the boundary conditions are of Dirichlet type (i.e., where \( u \) is given), we call (1) a **Dirichlet problem**.

To solve by separation of variables, seek

\[ u(x, y) = X(x)Y(y). \]

Putting (2) into (1a) and separating the variables gives

\[ \frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} = \kappa^2, \]

so

\[ \begin{align*}
X'' - \kappa^2 X &= 0, \\
Y'' + \kappa^2 Y &= 0,
\end{align*} \]

**Figure 1.** The Dirichlet problem (1).
Chap. 20. Laplace Equation

and

\[
X(x) = \begin{cases} 
A + Bx, & \kappa = 0 \\
C \cosh \kappa x + D \sinh \kappa x, & \kappa \neq 0 
\end{cases} \\
Y(y) = \begin{cases} 
E + Fy, & \kappa = 0 \\
G \cos \kappa y + H \sin \kappa y, & \kappa \neq 0 
\end{cases}
\]  

(5a, 5b)

Why did we choose the constant in (3) as \(+\kappa^2\) rather than as \(-\kappa^2\)? Because, looking ahead to the application of the boundary conditions we anticipate the eventual Fourier series expansion of \(f(y)\). The choice \(+\kappa^2\) does indeed lead to the \(\cos \kappa y\) and \(\sin \kappa y\) solutions in (5b) that will be needed for that Fourier series expansion; \(-\kappa^2\) would have led to \(\cosh \kappa y\) and \(\sinh \kappa y\) instead.

Because the Laplace equation \(\nabla^2 \phi = 0\) is linear, we can use superposition and combine the \(\kappa = 0\) and \(\kappa \neq 0\) product solutions as

\[
u(x, y) = (A + Bx)(E + Fy) + (C \cosh \kappa x + D \sinh \kappa x)(G \cos \kappa y + H \sin \kappa y).
\]

(6)

Saving the boundary condition at \(x = a\) for last, we first apply the boundary conditions on the edges \(y = 0, y = b\), and \(x = 0\):

\[u(x, 0) = 0 = (A + Bx)E + (C \cosh \kappa x + D \sinh \kappa x)G,
\]

so (to retain as robust a solution as possible) set \(E = 0\) (rather than \(A = 0\) and \(B = 0\)) and \(G = 0\) (rather than \(C = 0\) and \(D = 0\)). Then (6) becomes

\[
u(x, y) = (I + Jx)y + (P \cosh \kappa x + Q \sinh \kappa x) \sin \kappa y,
\]

(7)

where we have combined \(AF\) as \(I\), \(BF\) as \(J\), \(CH\) as \(P\), and \(DH\) as \(Q\) for brevity. Next,

\[u(x, b) = 0 = (I + Jx)b + (P \cosh \kappa x + Q \sinh \kappa x) \sin \kappa b
\]

(8)

gives \(I = 0, J = 0, and \sin \kappa b = 0\). Hence,

\[
k = n\pi/b \quad (n = 1, 2, \ldots)
\]

so, with the help of superposition, we can update (7) as

\[
u(x, y) = \sum_{n=1}^{\infty} \left( P_n \cosh \frac{n\pi x}{b} + Q_n \sinh \frac{n\pi x}{b} \right) \sin \frac{n\pi y}{b}.
\]

(9)

Next, the “western” boundary condition (i.e., at \(x = 0\)) gives

\[u(0, y) = 0 = \sum_{n=1}^{\infty} P_n \sin \frac{n\pi y}{b}, \quad (0 < y < b)
\]

(10)

which is satisfied by setting \(P_n = 0\) for \(n = 1, 2, \ldots\). Thus, (9) becomes

\[
u(x, y) = \sum_{n=1}^{\infty} Q_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}.
\]

(11)
Finally, the eastern boundary condition
\[ u(x, y) = f(y) = \sum_{n=1}^{\infty} Q_n \sin \left( \frac{n \pi x}{b} \right) \sin \left( \frac{n \pi y}{b} \right) \quad (0 < y < b) \] (12)
is seen to be a half-range sine series so
\[ Q_n \sin \left( \frac{n \pi x}{b} \right) = \frac{2}{b} \int_0^b f(y) \sin \left( \frac{n \pi y}{b} \right) \, dy, \]
or
\[ Q_n = \frac{2}{b \sinh \left( \frac{n \pi x}{b} \right)} \frac{1}{1} \int_0^b f(y) \sin \left( \frac{n \pi y}{b} \right) \, dy. \] (13)
The solution to (1) is given by (11) and (13).

COMMENT 1. We stated that (10) is satisfied by setting \( P_n = 0 \) for \( n = 1, 2, \ldots \). Obviously that is true, but there is a subtle question here: must the \( P_n \)'s be zero? That is, might the \( \sin(n \pi y/b) \) terms cancel to zero on \( 0 < y < b \) without all the \( P_n \)'s being zero? Surely the \( \sin(n \pi y/b) \) terms are linearly independent on \( 0 < y < b \), and if a finite linear combination of linearly independent functions is zero, then each coefficient must be known. However, the sum in (10) is an infinite series, not a linear combination of a finite number of terms. The easiest way to handle (10) is to see that it is actually a half-range sine expansion of the function \( u(0, y) = 0 \). Thus,
\[ P_n = \frac{2}{b} \int_0^b (0) \sin \left( \frac{n \pi y}{b} \right) \, dy = 0, \]
as stated.

COMMENT 2. To make our results more concrete, consider a specific \( f(y) \). For simplicity, let \( f(y) \) be a constant, say
\[ f(y) = 100. \]
Then (13) gives
\[ Q_n = \frac{400}{n \pi \sinh \left( \frac{n \pi x}{b} \right)} \]
for \( n = 1, 3, \ldots \) and 0 for \( n = 2, 4, \ldots \), so (11) becomes
\[ u(x, y) = \frac{400}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{1} \sinh \left( \frac{n \pi x}{b} \right) \sin \left( \frac{n \pi y}{b} \right). \] (14)
One useful way of presenting such two-dimensional results is to plot a number of \( u = \) constant curves, isothermal curves if we consider (1) in the context of steady-state heat conduction. We have done so in Fig. 2 for the case where \( b = a \). The \( u = 0 \) isotherm is the northern, western, and southern boundary; the \( u = 100 \) isotherm is the eastern edge; and all other isotherms spring from the corners \((a, 0)\) and \((a, a)\). As implied by Fig. 2, all \( u \) values within the rectangle are between the minimum and maximum values of \( u \) on the boundary, namely, 0 and 100, respectively. This result illustrates the important and
fundamental maximum principle of potential theory (i.e., the theory associated with the Laplace equation), which states that if the Laplace equation $\nabla^2 u = 0$ holds on a domain $\mathcal{D}$, then the maximum (and minimum) values of $u$ occur on the boundary of $\mathcal{D}$, not in its interior. This result is proved in the next section.

**COMMENT 3.** We have also plotted the isotherms corresponding to (14) for the high-aspect-ratio case where $b = 10a$, again with $f(y) = 100$ (Fig. 3). Observe that over $2a < y < 8a$, say, the problem is essentially one-dimensional, with $u$ varying with $x$ but hardly at all with $y$. If, accordingly, we neglect the $\partial^2 u / \partial y^2$ term in the Laplace equation, then we have the problem

$$\frac{\partial^2 u}{\partial x^2} \approx 0; \quad u \bigg|_{y=0} = 0, \quad u \bigg|_{y=a} = 100,$$

with solution

$$u(x, y) \approx 100 \frac{x}{a},$$

the isotherms of which are the lines $x = \text{constant}$ in Fig. 3. We see that (16) is an excellent approximation to $u(x, y)$ except near the ends, that is, except within one or two widths (the width being $a$) of the ends $y = 0$ and $y = 10a$. Indeed, if we are interested in the solution only within $2a < y < 8a$, say, then the simpler one-dimensional model and its solution (16) may well suffice in place of the two-dimensional model and the more cumbersome solution (14). Of course, (16) does not satisfy the boundary conditions $u(x, 0) = 0$ and $u(x, 10a) = 0$, so there are regions of adjustment near those two ends, where the approximate solution (16) needs to be blended with the boundary conditions $u(x, 0) = 0$ and $u(x, 10a) = 0$ in a way that satisfies the Laplace equation.

The results found in this example hold in general; namely, we can expect end effects to be significant only within one or two widths of the end. 

Remember that in Chapters 18 and 19 we always chose the separation constant to be $-\kappa^2$, and that we always applied the boundary conditions before the initial condition. For the Laplace equation, however, the sign of the separation constant and the sequencing of the boundary conditions needs to be decided on a case-by-case basis. We offer this rule of thumb as guidance:

*Anticipating the edge along which the eventual Fourier series will take place, choose the $+\kappa^2$ or $-\kappa^2$ so as to obtain oscillatory functions along that edge. Then, apply the boundary conditions adjacent to that edge first.*

For instance, in Example 1 we can anticipate that it is the boundary condition $u(a, y) = f(y)$ that will require the Fourier series, so we choose $+\kappa^2$ in (3) so as to obtain the oscillatory solutions $Y(y) = \cos \kappa y$ and $\sin \kappa y$ in the $y$ variable. Then, we apply the boundary conditions on the northern and southern edges, which are adjacent to that edge. Similarly in Chapters 18 and 19. There, the Fourier series is always along the southern edge in the $x, t$ plane (i.e., along the edge $t = 0$) so we chose $-\kappa^2$ in order to obtain $\cos \kappa x$ and $\sin \kappa x$; we applied the boundary conditions on the adjacent edges $x = 0$ and $x = L$ first, and on the edge $t = 0$ last.

What if the boundary conditions are, in place of those in Fig. 1, $u(0, y) = p(y)$, $u(a, y) = f(y)$, $u(x, 0) = q(x)$, and $u(x, b) = g(x)$? Evidently each edge
will require a Fourier series expansion, so how can we apply the adjacent boundary conditions first if they require Fourier series expansions themselves? This difficulty can be resolved by using the concepts of linearity and superposition to break the problem into four simpler ones as indicated schematically in Fig. 4. The idea is to solve each of the four problems on the right along the lines indicated in Example 1 (using $+\kappa^2$ for the first and third problems and $-\kappa^2$ for the second and fourth) and then obtain $u$ as the sum

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y).$$  \hspace{1cm} (17)

![Figure 4. Use of superposition.](image)

To see that (17) is true, add the equations $\nabla^2 u_1 = 0$, $\nabla^2 u_2 = 0$, $\nabla^2 u_3 = 0$, and $\nabla^2 u_4 = 0$, and obtain

$$\nabla^2 u_1 + \nabla^2 u_2 + \nabla^2 u_3 + \nabla^2 u_4 = 0.$$ \hspace{1cm} (18)

But since the operator $\nabla^2$ is linear, it follows from (18) that

$$\nabla^2 (u_1 + u_2 + u_3 + u_4) = 0,$$

so $u = u_1 + u_2 + u_3 + u_4$ does satisfy the Laplace equation, as required. Turning to the boundary conditions, consider the eastern condition. From (17) and the conditions imposed on $u_1, u_2, u_3$, and $u_4$,

$$u(a, y) = u_1(a, y) + u_2(a, y) + u_3(a, y) + u_4(a, y)$$

$$= f(y) + 0 + 0 + 0$$

$$= f(y),$$

as required. Similarly for the other boundary conditions.

**EXAMPLE 2.** Semi-Infinite Strip. Consider the Dirichlet problem

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \text{ in } \mathcal{D},$$ \hspace{1cm} (19a)

$$u(0, y) = 20, \quad u(5, y) = 50, \quad (0 < y < \infty)$$ \hspace{1cm} (19b)

$$u(x, 0) = f(x), \quad (0 < x < 5)$$ \hspace{1cm} (19c)

$$u(x, y) \text{ bounded as } y \to \infty.$$ \hspace{1cm} (19d)

where $\mathcal{D}$ is the semi-infinite strip $0 < x < 5$, $0 < y < \infty$ (Fig. 5).

Since physical objects are of finite size, why might we be interested in a semi-infinite strip that extends to $y = \infty$? The actual physical object might, for instance, be a slender

![Figure 5. Semi-infinite strip.](image)
"fin" attached to a thick base as shown in Fig. 6. Then the problem is actually defined on a complicated three-dimensional domain. However, suppose that our interest is not in finding the temperature field everywhere in the object but only near the end of the fin, within the rectangle \(ABCE\). And suppose that we know the boundary conditions \(u = 20\) along \(AE\), \(u = 50\) along \(BC\), and \(u = f(x)\) along \(EC\). We could take our domain \(D\) to be the rectangle \(ABCE\), but we do not know a boundary condition along \(AB\). To within a good approximation, we render the problem tractable by letting \(D\) be the entire semi-infinite strip shown in Fig. 5, with the boundedness condition (19d) as our missing boundary condition at \(y = \infty\). Based on Comment 3 of Example 1, we expect the difference between the actual temperature field and the one defined by (19) to be very small within the region of interest, \(ABCE\).

We anticipate that the eventual Fourier series expansion will be a half- or quarter-range expansion on the edge \(y = 0\). Thus, to obtain oscillatory functions of \(x\) (namely \(\cos \kappa x \) and \(\sin \kappa x \)) rather than of \(y\), write

\[
\frac{X''}{X} = -\frac{Y''}{Y} = -\kappa^2
\]

in place of (3). Solving the resulting ODE’s on \(X\) and \(Y\) and superimposing the \(\kappa = 0\) and \(\kappa \neq 0\) solutions gives

\[
u(x, y) = (A + Bx)(E + Fy) + (C \cos \kappa x + D \sin \kappa x)(Ge^{\kappa y} + He^{-\kappa y}).
\]

(21)

Apply the boundedness condition (19d) first. Since the \(y\) and \(e^{\kappa y}\) terms in (21) grow unboundedly as \(y \to \infty\), we must set \(F = 0\) and \(G = 0\) to eliminate those terms. Then (21) becomes

\[
u(x, y) = I + Jx + (P \cos \kappa x + Q \sin \kappa x)e^{-\kappa y},
\]

(22)

where we have combined \(AE\) as \(I\), \(BE\) as \(J\), \(CH\) as \(P\), and \(DH\) as \(Q\). Since we anticipate the Fourier expansion to be on the \(y = 0\) edge, we save that boundary condition for last. Next,

\[
u(0, y) = 20 = I + Pe^{-\kappa y},
\]

(23)

and matching the coefficients of the (linearly independent) constant and \(e^{-\kappa y}\) terms on both sides of (23) gives \(20 = I\) and \(0 = P\). Using these results to update (22) gives

\[
u(x, y) = 20 + Jx + Q \sin \kappa x e^{-\kappa y}.
\]

(24)

Next,

\[
u(5, y) = 50 = 20 + 5J + Q \sin 5\kappa e^{-\kappa y},
\]

(25)

so \(50 = 20 + 5J\) and \(\sin 5\kappa = 0\). Thus, \(J = 6\) and \(\kappa = n\pi/5\) \((n = 1, 2, \ldots)\). Putting these results into (24) we have, with the help of superposition,

\[
u(x, y) = 20 + 6x + \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi x}{5} e^{-n\pi y/5}.
\]

(26)

Finally,

\[
u(x, 0) = f(x) = 20 + 6x + \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi x}{5}.
\]

(27)
or, moving the (known) $20 + 6x$ terms to the left-hand side,

$$f(x) - 20 - 6x = \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi x}{5}, \quad (0 < x < 5) \quad (28)$$

The latter is a half-range sine expansion of $f(x) - 20 - 6x$, so we can compute the $Q_n$'s from

$$Q_n = \frac{2}{5} \int_{0}^{5} [f(x) - 20 - 6x] \sin \frac{n\pi x}{5} \, dx. \quad (29)$$

Thus, the solution to (19) is given by (26), with the $Q_n$'s computed according to (29).

**COMMENT 1.** The $\kappa = 0$ term $(A + Bx)(E + Fy)$ in (21) contributes the $20 + 6x$ part of the final solution (26). The graph of that part of the solution is like a ramp, from $u = 20$ along the $x = 0$ edge to $u = 50$ along the $x = 5$ edge. In the language of Chapter 18 we can think of $20 + 6x$ as the “steady-state” solution, the solution that is approached as $y \to \infty$ (analogous to the limit $t \to \infty$ in Chapter 18), and we can think of the series in (26) as the “transient” that blends the “steady-state” $20 + 6x$ with the “initial condition” $u(x, 0) = f(x)$. We see from (26) that the transient part, or end effect, dies out exponentially with $y$; with $n = 1$, the $\exp (-n\pi y/5)$ factor is merely 0.043 at $y = 5$ and 0.0019 at $y = 10$, and with $n = 2, 3, \ldots$ it is even smaller. Only if $f(x)$ happens to equal $20 + 6x$ does the end effect vanish entirely, for then all the $Q_n$'s are zero.

**COMMENT 2.** In (21) we wrote $G e^{\kappa y} + He^{-\kappa y}$ but could have written $R \cosh \kappa y + S \sinh \kappa y$, say, instead. The choice is immaterial since the two forms are equivalent, but the former is more convenient regarding the application of the boundedness condition, for it is clear that $e^{\kappa y}$ is a “bad” term (unbounded) and that $e^{-\kappa y}$ is a “good” term (bounded) so we set $G = 0$. Working with $\cosh \kappa y$ and $\sinh \kappa y$ instead would be awkward because both are unbounded (Fig. 7). However,

$$R \cosh \kappa y + S \sinh \kappa y = \frac{R}{2} (e^{\kappa y} + e^{-\kappa y}) + \frac{S}{2} (e^{\kappa y} - e^{-\kappa y}) = \frac{R + S}{2} e^{\kappa y} + \frac{R - S}{2} e^{-\kappa y}, \quad (30)$$

so we can arrange for the unbounded parts of the $\cosh \kappa y$ and $\sinh \kappa y$ to cancel by choosing

$$S = -R,$$

in which case (30) reduces to $Re^{-\kappa y}$. Thus, we arrive at the same place but the trip is more arduous.

**COMMENT 3.** Since we had three nonzero boundary conditions (Fig. 5), why did we not break the problem into three sub-problems along the lines indicated in Fig. 4? We could have but did not need to because the boundary conditions along the edges $x = 0$ and $x = 5$ are merely constants and can therefore be handled by the $(A + Bx)(E + Fy)$ ramp term in the solution.

**COMMENT 4.** In these first two examples the Fourier expansions happened to be half-range sine series, but that will not always be the case and will depend on the boundary conditions. For instance, if we change the Dirichlet boundary condition $u(0, y) = 20$ to a Neumann boundary condition such as $u_x(0, y) = 3$, then in place of the half-range sine expansion (28) we would have a quarter-range cosine expansion. \[\text{Figure 7. } \cosh \kappa y \text{ and } \sinh \kappa y.\]
Closure. In this section we study the separation-of-variable solution of the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ on domains bounded by constant-$x$ and constant-$y$ lines. Unlike Chapters 18 and 19, where we always take the separation constant to be $-\kappa^2$ and apply the boundary conditions at $x = 0$ and $x = L$ before the initial condition(s) at $t = 0$, here we need to choose the sign of $\kappa^2$ and the sequence of application of the boundary conditions on a case-by-case basis. The rule of thumb is to choose the separation constant as $+\kappa^2$ or $-\kappa^2$ so as to give oscillatory solutions (cosines and sines) along the edge where the eventual Fourier expansion will take place, and then to be sure to apply the boundary conditions adjacent to that edge first. (For example, if the Fourier expansion is on the eastern edge, then by the “adjacent” edges we mean the northern and southern ones.)

Consider the general Dirichlet problem on $u$ shown in Fig. 4. If all of the functions $p(y), g(x), f(y),$ and $q(x)$ are nonconstant, then we can break the problem into four sub-problems, as shown in the figure, and solve separately for $u_1(x,y), u_2(x,y), u_3(x,y),$ and $u_4(x,y)$. For the $u_1$ and $u_3$ problems, set $X''/X = -Y''/Y = +\kappa^2$, so as to obtain cosine and sine solutions in the $y$ variable, and apply the southern and northern boundary conditions before attempting the nonhomogeneous boundary condition (eastern in the $u_1$ problem, western in the $u_3$ problem). In fact, you will find that if the boundary conditions on any two opposite boundaries are constants, then it is not necessary to break the problem down as we do in Fig. 4 (although we can if we wish). For instance, suppose that both $p(y)$ and $f(y)$ are constants. Then set $X''/X = -Y''/Y = -\kappa^2$ to obtain cosine and sine solutions in the $x$ variable, and apply the western and eastern boundary conditions first.

Similar statements apply if boundary conditions of Neumann or Robin type are present on one or more edges, but be careful because if all four edges have Neumann boundary conditions, then there may be no solution or a nonunique solution (Exercises 17 and 18).

In Example 2 we consider the domain to be an idealized semi-infinite strip and adopt a boundedness condition in lieu of the missing boundary condition at $y = \infty$. The Fourier expansion is necessarily on the finite edge (the edge $y = 0$ in Example 2). Given the choice between expressing $Y$ in terms of $\cosh \kappa y$ and $\sinh \kappa y$ or in terms of $e^{\kappa y}$ and $e^{-\kappa y}$, we choose the latter because of convenience in regard to the application of the boundedness condition.

Finally, and very important, is the fact that the Laplace and Poisson equations arise in the context of boundary-value problems; that is, a boundary condition is supplied on each of the four edges. In contrast, the diffusion and wave problems studied in Chapters 18 and 19 are of initial-value type (with respect to the $t$ variable) since conditions are given at $t = 0$ but not at a final time or at $t = \infty$. This boundary-value nature will be felt more acutely when we use the finite-difference method in Section 20.5.
EXERCISES 20.2

1. Solve \( u_{xx} + u_{yy} = 0 \) in the rectangle \( 0 < x < 3, 0 < y < 2 \) by separation of variables, subject to the given boundary conditions. \( H \) denotes the Heaviside function.

(a) \( u(0,y) = u(x, 2) = u(3,y) = 0, \ u(x,0) = 50 \sin(\pi x/3) \)
(b) \( u(0,y) = u(x, 0) = u(3,y) = 0, \ u(x,2) = 10 \sin(\pi x/3) - 4 \sin \pi x \)
(c) \( u(x,0) = u(3,y) = u(x, 2) = 0, \ u(0,y) = 5 \sin \pi y + 4 \sin 2\pi y - \sin 3\pi y \)
(d) \( u(0,y) = u(x, 2) = u(3,y) = 0, \ u(x,0) = 50 H(x-2) \)
(e) \( u_0(0,y) = u(x, 2) = u(3,y) = 0, \ u(x,0) = 50 H(x-2) \)
(f) \( u(0,y) = u(x, 2) = u_0(3,y) = 0, \ u(x,0) = 50 H(x-2) \)
(g) \( u_2(0,y) = u(x, 2) = u_2(3,y) = 0, \ u(x,0) = 50 H(x-2) \)
(h) \( u_4(0,y) = u(x, 2) = u(3,y) = 0, \ u(x,0) = H(y-1) \)
(i) \( u(0,y) = u(x, 0) = u(3,y) = 0, \ u(x,2) = 5x \)
(j) \( u(0,y) = u(x, 2) = u(3,y) = 0, \ u(x,0) = 5[ H(x-1) - H(x-2)] \)
(k) \( u(x, 2) = u(3,y) = u(x,0) = 0, \ u_4(0,y) = 5 \sin 3\pi y \)
(l) \( u(x, 2) = u(3,y) = u(x,0) = 0, \ u_4(0,y) = 20 \)
(m) \( u_2(x, 2) = u(3,y) = u_2(x,0) = 0, \ u_2(x,0) = 20 \)
(n) \( u_2(x, 2) = u(3,y) = u_2(x,0) = 0, \ u_2(x,0) = 20 \)

2. (a) The solution to Exercise 1(d) is given in the Answers to Selected Exercises. Using that solution and computer software, evaluate \( u(2.5,1), u(2.5,0.5), v(2.5,0.2), \) and \( v(2.5,0.1) \), correct to two decimal places. In each case, tell how many terms must be summed to achieve that accuracy. Explain why more terms are needed as the point approaches the x-axis.

(b) The same as part (a), but using the solution to Exercise 1(h).

(c) In Exercise 1(e) evaluate \( u(1,1) \) to three significant figures.
(d) In Exercise 1(e) evaluate \( u(0,y) \) at \( y = 0.25, 0.5, 0.75, \ldots, 1.75 \), to three significant figures, and plot \( u(0,y) \) versus \( y \), by hand or by computer.

3. Solve \( u_{xx} + u_{yy} = 0 \) in the square \( 0 < x < 2, \ 0 < y < 2 \) by separation of variables, subject to the given boundary conditions. (You should be able to obtain the solution in closed form.) Then, obtain a computer plot of the \( u = 10, 20, 30, \ldots, 90 \) isotherms using software such as the Maple implicitplot command. NOTE: We urge you to try sketching the isotherms even before you solve the problem.

(a) \( u(0,y) = u(x, 2) = u(2,y) = 0, \ u(x,0) = 100 \sin(\pi x/2) \)
(b) \( u(0,y) = u(x, 2) = u(2,y) = 0, \ u(x,0) = 100 \sin(\pi x/2) \)
(c) \( u(0,y) = u(x, 2) = 0, \ u(2,y) = 100 \sin(\pi y/2) \)
(d) \( u(0,y) = u(x, 2) = 0, \ u_2(2,y) = 0, \ u(0,x) = 100 \sin(\pi x/4) \)
(e) \( u(0,y) = u(x, 2) = 0, \ u(0,x) = 100 \sin(\pi x/2) \)
(f) \( u(0,y) = u(2,y) = 0, \ u(x,0) = 100 \sin(\pi x/2) \)
(g) \( u(0,y) = u(2,y) = 100 \sin(\pi y/2), \ u(x,0) = u(2, x) = 100 \sin(\pi y/2) \)

4. Solve \( u_{xx} + u_{yy} = 0 \) in the rectangle \( 0 < x < a, \ 0 < y < b \) subject to the boundary conditions \( u(0,y) = p(y) \), \( u(x,b) = u_2, \ u(a,y) = f(y), \ u(x,0) = u_1 \) without breaking the problem into sub-problems; \( p(y) \) and \( f(y) \) are prescribed functions and \( u_1 \) and \( u_2 \) are prescribed constants. HINT: Read the second paragraph of the closure.

5. Same as Exercise 4, but with these boundary conditions:

(a) \( u(0,y) = u_1, \ u(x,b) = p(x), \ u(a,y) = u_2, \ u(x,0) = f(x) \)
(b) \( u_2(0,y) = p(y), \ u(x,b) = u_1, \ u(a,y) = f(y), \ u(x,0) = u_2 \)
(c) \( u_2(0,y) = p(y), \ u(x,b) = u_1, \ u_2(x,y) = f(y), \ u(x,0) = u_2 \)

6. Solve \( u_{xx} + u_{yy} = 0 \) in the rectangle \( 0 < x < a, \ 0 < y < b \) subject to the boundary conditions \( u(0,x) = u_1, \ u(x,b) = u_2, \ u(0,y) = u_3, \ u(a,y) = u_4 \), where \( u_1, \ldots, u_4 \) are constants. Do not break the problem into subproblems; you don’t need to. Rather, choose \( X''/X = -Y''/Y = \pm k^2 \) and apply the southern and northern boundary conditions first. (You may leave expansion coefficients in integral form.) Next, solve the problem again, this time choosing \( X''/X = -Y''/Y = -k^2 \) and applying the western and eastern boundary conditions first. Your two solutions will look different but will merely be two different representations of the same function. If \( b = 10a \), which of the two solution forms would you prefer—why for purposes of calculation? Explain your reasoning.

7. Let \( f(y) = 100 \) in (1c), and let \( b = a \). Without solving the problem (or using the solution in the text), show that \( u(a/2, a/2) = 25 \). HINT: Let \( p(y) = g(x) = f(y) = q(x) = 100 \) in Fig. 4.

8. To promote physical “intuition,” we ask you to draw a neat, labeled sketch of representative isotherms for the problem con-
sisting of the Laplace equation on the square $0 < x < a$, $0 < y < a$ with the given boundary conditions.

(a) $u(0, y) = u(a, y) = u(x, 0) = 100 \sin \left(\pi x / a\right)$
(b) $u(0, y) = u(x, a) = u_x(a, y) = 0$, $u(x, 0) = 100 \sin \left(\pi y / 2a\right)$
(c) $u(0, y) = a(x, 0) = 0$, $u(a, y) = a(x, a) = 20$
(d) $u(0, y) = u(x, 0) = u(y, 1) = 0$, $u(x, a) = 100$ for $0 < x < a / 2$ and $0$ for $a / 2 < x < a$
(e) $u(0, y) = u(x, 0) = 0$, $u(x, a) = 0$, $u(a, y) = 20$

9. Show that the solution (11) of the problem (1) can be expressed in the form of an integration over the boundary data, namely,

$$u(x, y) = \int_0^{\alpha} K(\eta; x, y)f(\eta) d\eta,$$

and give an expression for the kernel $K(\eta; x, y)$; it will be in the form of an infinite series.

10. Solve $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $0 < x < \infty$, $0 < y < 1$ subject to the given boundary conditions plus the condition that $u$ is bounded as $x \to \infty$.

(a) $u(0, y) = 0$, $u(x, 0) = 10$, $u_y(x, 1) = 0$
(b) $u(0, y) = 100$, $u_y(x, 0) = u_y(x, 1) = 0$
(c) $u_x(0, y) = 5$, $u_x(0, 0) = u(x, 1) = 0$
(d) $u(0, y) = 0$, $u(x, 0) = 50$, $u(x, 1) = 10$
(e) $u(0, y) = 10y$, $u(x, 0) = 20$, $u(x, 1) = 50$

11. (a)-(c) Give a labeled sketch of representative isotherms for the corresponding problem in Exercise 10.

12. The problem

$$u_{xx} + u_{yy} = 0,$$
$$u(x, 0) = 0, \quad u(x, b) = 50e^{-(x / 10b)^2}$$
on $-\infty < x < \infty$, $0 < y < b$ admits a simple and accurate approximate solution, which we ask you to find. HINT: $e^{-(x / 10b)^2}$ is a slowly varying function of $x$.

13. In Example 2 we apply a bounded condition on $u$ at $y = \infty$. Dropping that condition, put forward two or three solutions that are unbounded on the semi-infinite strip.

14. Consider the Laplace equation $u_{xx} + u_{yy} = 0$ on the parallelogram $D$ shown, bounded by the lines $y = 0$, $y = 1$.

$$y = 2x, \quad y = 2x - 2,$$
with boundary conditions given on the four edges. There is no future in seeking $u(x, y) = X(x)Y(y)$ and using separation of variables because the boundary is not comprised of constant coordinate curves. Specifically, the left and right edges are neither constant-$x$ nor constant-$y$ lines.

One possibility seems to be a change of variables from $x, y$ to $\xi, \eta$ according to

$$\xi = y - 2x, \quad \eta = y$$

so that the new domain, in the $\xi, \eta$ plane, will be a rectangle bounded by constant-$\xi$ and constant-$\eta$ lines.

(a) Show that new domain in a labeled sketch.
(b) Show that in terms of $\xi$ and $\eta$ the Laplace equation becomes

$$5u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta} = 0$$

and that our plan fails because (14.2) is not separable. NOTE: Nonetheless, the idea is a good one; we simply need to figure out how to design a change of variables $\xi = F(x, y)$ so as to simplify the domain without at the same time complicating the PDE. How to do this, for the two-dimensional Laplace equation, is the subject of Chapter 22 on conformal mapping.

15. (Poisson equation) Consider the Poisson problem

$$u_{xx} + u_{yy} = f(x, y),$$
$$u(0, y) = u(x, b) = u(a, y) = u(x, 0) = 0$$
on the rectangle $0 < x < a$, $0 < y < b$, where the source function $f(x, y)$ is prescribed.

(a) Solve (15.1) by separation of variables for the case where $f(x, y) = \text{constant} \equiv f$, leaving expansion coefficients in integral form. HINT: Noticing that $f x^2 / 2$ is a simple particular solution of (15.1a), seek $u$ in the form

$$u(x, y) = \frac{f}{2} x^2 + U(x, y).$$

Show that the "homogeneous solution" $U$ satisfies the problem

$$U_{xx} + U_{yy} = 0,$$
$$U(0, y) = 0, \quad U(a, y) = -f a^2 / 2, \quad U(x, 0) = U(x, b) = -f x^2 / 2,$$

(15.3a,b,c)
and solve for \( U \) by separation of variables. NOTE: \( f x^2/2 \) is not the only particular solution that could be used; for instance, we could use \( f y^2/2 \), \( f x^2/2 + 3Tx - 5y + 6 \), and so on, but \( f x^2/2 \) (or \( f y^2/2 \)) seems a simple and natural choice. (b) Observe that the method proposed in (a) will work not only when \( f(x,y) \) is a constant, but also when it is a function only of \( x \) or only of \( y \). Here, we ask you to solve (15.1) for the case where \( f(x,y) \) is not necessarily of that form. HINT: Use the eigenvector expansion method in very much the same manner as we did in Exercise 17 of Section 18.3. Essentially, we can use either the eigenfunctions \( \sin n\pi x/a \) in the \( x \) variable or the eigenfunctions \( \sin n\pi y/b \) in the \( y \) variable to expand “everything in sight.” Specifically, expanding

\[
f(x, y) = \sum_{n=1}^{\infty} f_n(x) \sin \frac{n\pi y}{b},
\]

where

\[
f_n(x) = \frac{2}{b} \int_{0}^{b} f(x, y) \sin \frac{n\pi y}{b} \, dy,
\]

and seeking

\[
u(x, y) = \sum_{n=1}^{\infty} g_n(x) \sin \frac{n\pi y}{b},
\]

show that the \( g_n \)'s are found by solving the problems

\[
g''_n - \frac{n^2}{a^2} g_n = f_n(x): \quad g_n(0) = g_n(a) = 0
\]

for \( n = 1, 2, \ldots \). (c) Implement the method of part (b) for the case where \( f(x, y) = xy \), and solve for \( u(x, y) \).

16. (Three-dimensional case) Consider the three-dimensional problem

\[
\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0
\]

in the rectangular prism \( 0 < x < a, 0 < y < b, 0 < z < c \), where \( u = 0 \) on each of the six faces except for the face \( z = c \), on which \( u \) is a prescribed function \( f(x, y) \).

\[
u(x, y, c) = f(x, y), \quad (0 < x < a, 0 < y < b)
\]

(a) Use separation of variables to derive the solution

\[
u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} D_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh \omega_m z,
\]

where

\[
\omega_{mn} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}},
\]

\[
D_{mn} = \frac{4}{\sinh \omega_m b} \int_{0}^{b} \int_{0}^{a} f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy.
\]

(b) For the case where \( c = b = a \) and \( f(x, y) = 100 \), use (16.3) and (16.4) to evaluate \( u(a/2, a/2, a/2) \).

17. (A necessary condition for existence) Consider the Poisson problem

\[
\nabla^2 u = f(x, y, z)
\]

in some three-dimensional domain \( \mathcal{D} \) with surface \( \mathcal{S} \). Integrating (17.1) over \( \mathcal{D} \), show that

\[
\int_{\mathcal{S}} \frac{\partial u}{\partial n} \, dA = \int_{\mathcal{D}} f \, dV.
\]

NOTE: Thus, the boundary values of \( \partial u/\partial n \) (whether they are specified or not) need to be consistent with the source \( f \) in the sense of (17.2) if a solution to (17.1) is to exist. For instance, suppose a homogeneous Neumann condition is appended to (17.1), that \( \partial u/\partial n = 0 \) everywhere on \( \mathcal{S} \). Then, (17.2) tells us that for a solution to exist the net source must be zero: \( \int_{\mathcal{D}} f \, dV = 0 \). That result makes sense physically because if the integral were positive, say, then the average temperature within \( \mathcal{D} \) would be an increasing function of time, whereas (17.1) is based on steady-state conduction.

18. (Uniqueness) Suppose that \( u(x, y, z) \) is \( C^2 \) and satisfies the Poisson equation (17.1) throughout a domain \( \mathcal{D} \), together with a Dirichlet boundary condition \( u = g(x, y, z) \) on the (piecewise smooth orientable) surface \( \mathcal{S} \) of \( \mathcal{D} \).

(a) Show that that solution is unique. HINT: Suppose that there are two such solutions, say \( u_1 \) and \( u_2 \). With \( w = u_1 - u_2 \), show that \( \nabla^2 w = 0 \) in \( \mathcal{D} \) and \( w = 0 \) on \( \mathcal{S} \). With “\( u \) = “\( v \) = \( w \) in Green’s first identity, show that

\[
\int_{\mathcal{D}} (w_x^2 + w_y^2 + w_z^2) \, dV = 0,
\]

and conclude that \( w_x = w_y = w_z \) so \( w \) is at most a constant. Show that the constant must be zero, so \( u_1 = u_2 \) in \( \mathcal{D} \).

(b) Repeat (a) with the Dirichlet condition replaced by the Neumann condition \( \partial u/\partial n = y \). This time, show that the solution is unique only to within an arbitrary additive constant.
(c) Repeat (a) with the Dirichlet condition replaced by a mixed boundary condition whereby \( u = g \) over part of \( S \) and \( \partial u / \partial n = h \) over the rest of \( S \).

19. (Application of Sturm–Liouville theory) (a) Solve the problem

\[
\begin{align*}
    u_{xx} + u_{yy} &= 0, \\
    u(0, y) &= u(x, 0) = u(x, 3) + 5y(x, 3) = 0, \\
    u(4, y) &= 100
\end{align*}
\]

(19.1a,b,c)
on the rectangle \( 0 < x < 4, 0 < y < 3 \) by separation of variables, with the help of the Sturm–Liouville theory. Show that the values of the separation constant \( \kappa \) are the (nonzero) roots of the equation

\[
\tan 3\kappa = -5\kappa, 
\]

and denote those roots as \( \kappa_n \) (\( n = 1, 2, \ldots \)). Use computer software to evaluate \( \kappa_1 \) through \( \kappa_5 \) and use the first five terms of your series solution to estimate \( u(2, 1) \).

(b) Same as (a), with (19.1c) changed to \( u(2, y) = 100 \).

### 20.3 Separation of Variables; Non-Cartesian Coordinates

#### 20.3.1. Plane polar coordinates.

Let \( r, \theta \) be the usual plane polar coordinates with \( x = r \cos \theta \) and \( y = r \sin \theta \). If the problem domain is bounded by constant \( r \) and constant \( \theta \) curves, then we must use \( r \) and \( \theta \) as our independent variables because, for the separation of variable method to work, we need the boundary conditions to be given on the constant coordinate curves. Since we will need to express the Laplace equation in terms of \( r \) and \( \theta \), we recall from (24) in Section 16.7 that for plane polar coordinates the Laplacian is

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.
\]

**EXAMPLE 1.** Consider the Dirichlet problem

\[
\begin{align*}
    \nabla^2 u &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, & (a < r < b, & \ 0 < \theta < \alpha) \\
    u(r, 0) &= u_1, & (a < r < b) \\
    u(r, \alpha) &= u_2, & (a < r < b) \\
    u(a, \theta) &= 0, & (0 < \theta < \alpha) \\
    u(b, \theta) &= f(\theta), & (0 < \theta < \alpha)
\end{align*}
\]

shown in Fig. 1. Observe that this problem is similar to the basic Cartesian coordinate version (Fig. 1 of Section 20.2), with some “distortion,” so we expect our solution steps to be similar as well.

According to the method of separation of variables, we seek \( u \) in the product form

\[
u(r, \theta) = R(r)\Theta(\theta),
\]

Putting (3) into (2a) gives

\[
R'' \Theta + \frac{1}{r} R' \Theta + \frac{1}{r^2} R \Theta'' = 0.
\]
and if we multiply by \(r^2\) and divide by \(R\Theta\) to separate the variables we obtain

\[
\frac{r^2 R'' + r R'}{R} - \frac{\Omega''}{\Omega} = \text{constant} = \kappa^2.
\]  

(4)

Here, we choose \(\pm \kappa^2\) so as to obtain \(\cos \kappa \theta\) and \(\sin \kappa \theta\) solutions for \(\Theta\) (rather than \(\cosh \kappa \theta\) and \(\sinh \kappa \theta\)) since we anticipate that to satisfy the \(u(b, \theta) = f(\theta)\) boundary condition we will need to expand \(f(\theta)\) in a Fourier series.

Proceeding, the \(R\) and \(\Theta\) equations are

\[
r^2 R'' + r R' - \kappa^2 R = 0,
\]

(5)

\[
\Theta'' + \kappa^2 \Theta = 0.
\]

(6)

Although (5) has nonconstant coefficients (so the solution form \(R = e^{\lambda r}\) will not work), it is elementary because it is of Cauchy–Euler form. Accordingly, seek \(R = r^\lambda\) and obtain \(\lambda (\lambda - 1) + \lambda - \kappa^2 = 0\) or \(\lambda = \pm \kappa\). Thus, (5) admits two linearly independent solutions, \(R = r^\kappa\) and \(R = r^{-\kappa}\), unless \(\kappa = 0\) in which case the two solutions coalesce into the one solution \(R = \text{constant}\). To find the missing solution, for the case \(\kappa = 0\), put \(\kappa = 0\) into (5) and obtain \(r^2 R'' + r R' = 0\). The latter can be reduced to the first-order equation \(r \frac{dp}{d\theta} + p = 0\) by the substitution \(R' = p\) and integrated to give \(p(r) = C_1/r\). Thus, \(R(r) = \int p \, dr = C_1 \ln r + C_2\), so we have these general solutions for \(R\) and \(\Theta\):

\[
R(r) = \begin{cases} 
A + B \ln r, & \kappa = 0 \\
Cr^\kappa + Dr^{-\kappa}, & \kappa \neq 0
\end{cases}
\]  

(7)

\[
\Theta(\theta) = \begin{cases} 
E + F \theta, & \kappa = 0 \\
G \cos \kappa \theta + H \sin \kappa \theta, & \kappa \neq 0
\end{cases}
\]  

(8)

Then, with the help of superposition, we have

\[
u(r, \theta) = (A + B \ln r)(E + F \theta) + (Cr^\kappa + Dr^{-\kappa})(G \cos \kappa \theta + H \sin \kappa \theta).
\]

(9)

Saving the boundary condition on \(r = b\) for last, we first apply the boundary conditions on the adjacent edges \(\theta = 0\) and \(\theta = \alpha\):

\[
u(r, 0) = u_1 = (A + B \ln r)E + (Cr^\kappa + Dr^{-\kappa})G,
\]

(10)

so we set \(AE = u_1\), \(B = 0\), and \(G = 0\). Updating (9) accordingly,

\[
u(r, \theta) = u_1 + I \theta + (Pr^\kappa + Qr^{-\kappa}) \sin \kappa \theta,
\]

(11)

where we have combined \(AE\) as \(I\), \(CH\) as \(P\), and \(DH\) as \(Q\). Next,

\[
u(r, \alpha) = u_2 = u_1 + I \alpha + (Pr^\kappa + Qr^{-\kappa}) \sin \kappa \alpha,
\]

(12)

so \(u_1 + I \alpha = u_2\) and \(\sin \kappa \alpha = 0\), hence \(I = (u_2 - u_1)/\alpha\) and \(\kappa = n \pi/\alpha\) \((n = 1, 2, \ldots)\).

Thus,

\[
u(r, \theta) = u_1 + (u_2 - u_1) \frac{\theta}{\alpha} - \sum_{n=1}^{\infty} \left(Pr^{n \pi/\alpha} + Qr^{-n \pi/\alpha}\right) \sin \frac{n \pi \theta}{\alpha}.
\]

(13)
Then
\[ u(a, \theta) = 0 = u_1 + (u_2 - u_1) \frac{\theta}{\alpha} + \sum_{n=1}^{\infty} \left( P_n a^{n\pi/\alpha} + Q_n a^{-n\pi/\alpha} \right) \sin \frac{n\pi\theta}{\alpha}. \]

or, moving the known terms on the right to the left-hand side,
\[ -u_1 - (u_2 - u_1) \frac{\theta}{\alpha} = \sum_{n=1}^{\infty} \left( P_n a^{n\pi/\alpha} + Q_n a^{-n\pi/\alpha} \right) \sin \frac{n\pi\theta}{\alpha}. \]

The latter is a half-range sine expansion of \(-u_1 - (u_2 - u_1)(\theta/\alpha)\) so we can compute the coefficients \(P_n a^{n\pi/\alpha} + Q_n a^{-n\pi/\alpha}\) as
\[ P_n a^{n\pi/\alpha} + Q_n a^{-n\pi/\alpha} = \frac{2}{\alpha} \int_0^{\alpha} \left( -u_1 - (u_2 - u_1) \frac{\theta}{\alpha} \right) \sin \frac{n\pi\theta}{\alpha} \, d\theta. \]  

Finally,
\[ u(b, \theta) = f(\theta) = u_1 + (u_2 - u_1) \frac{\theta}{\alpha} + \sum_{n=1}^{\infty} \left( P_n b^{n\pi/\alpha} + Q_n b^{-n\pi/\alpha} \right) \sin \frac{n\pi\theta}{\alpha}, \]

or
\[ f(\theta) - u_1 - (u_2 - u_1) \frac{\theta}{\alpha} = \sum_{n=1}^{\infty} \left( P_n b^{n\pi/\alpha} + Q_n b^{-n\pi/\alpha} \right) \sin \frac{n\pi\theta}{\alpha}, \]

so
\[ P_n b^{n\pi/\alpha} + Q_n b^{-n\pi/\alpha} = \frac{2}{\alpha} \int_0^{\alpha} \left[ f(\theta) - u_1 - (u_2 - u_1) \frac{\theta}{\alpha} \right] \sin \frac{n\pi\theta}{\alpha} \, d\theta. \]

We can evaluate the integrals in (15) and (17), once \(f(\theta)\) is specified, so (15) and (17) amount to two linear algebraic equations in the unknown \(P_n\)’s and \(Q_n\)’s. They have a unique solution because the determinant of the coefficient matrix is
\[ \begin{vmatrix} a^{n\pi/\alpha} & a^{-n\pi/\alpha} \\ b^{n\pi/\alpha} & b^{-n\pi/\alpha} \end{vmatrix} = \left( \frac{a}{b} \right)^{n\pi/\alpha} - \left( \frac{b}{a} \right)^{-n\pi/\alpha} \neq 0 \]  

since \(b \neq a\). Thus, the solution to (2) is given by (13), with \(P_n\) and \(Q_n\) determined from (15) and (17). For instance, if \(u_1 = u_2 = 0\) and \(f(\theta) = 100\), then we obtain (Exercise 1)
\[ u(r, \theta) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{r}{a} \right)^{n\pi/\alpha} - \left( \frac{a}{r} \right)^{n\pi/\alpha} \sin \frac{n\pi\theta}{\alpha}. \]

COMMENT 1. The Laplace equation in polar coordinates, (2a), did indeed prove to be separable. That is, putting \(u(r, \theta) = \tilde{u}(r)\tilde{\Theta}(\theta)\) into (2a) we were able to get all of the \(r\) dependence on one side of the equation and all of the \(\theta\) dependence on the other [in (4)], and hence to infer the ODE’s (5) and (6) on \(\tilde{r}\) and \(\tilde{\Theta}\). In fact, the diffusion, wave, and Laplace equations are all separable in Cartesian, polar, cylindrical, and spherical coordinates, for
which we can be grateful because not all PDE's are separable; see, for instance, Exercise 14 in Section 20.2.

COMMENT 2. Given that the Fourier expansion will take place on the \( r = b \) edge, there is no extra difficulty in admitting \( u = \) constant boundary conditions on the adjacent edges, namely, the conditions \( u = u_3 \) on \( \theta = 0 \) and \( u = u_2 \) on \( \theta = \alpha \). The reason is that the \( A(E + F\theta) \) part of the solution (9), which comes from \( \kappa = 0 \), is able to handle those boundary conditions; it gives the \( u_1 + (u_2 - u_1)\theta/\alpha \) part of the final solution (13). If we spoke of the analogous \( A + Bx(E + Fy) \) term in Section 20.2 as being a "ramp" function (if \( B \) or \( F \) is zero), we might call the \( A(E + F\theta) \) term a "fan" function since its graph "fans" from the value \( u_1 \) at one value of \( \theta \) to \( u_2 \) at another value of \( \theta \) (Fig. 2).

COMMENT 3. If the boundary condition along either radial edge (\( \theta = 0 \) or \( \theta = \alpha \)) is nonconstant, so that a Fourier expansion is needed along that edge, then the solution is more difficult because the needed expansion turns out not to be a familiar half- or quarter-range Fourier series but rather a Sturm–Liouville eigenfunction expansion. The solution for that case is outlined in Exercise 13.

COMMENT 4. The boundary conditions (2b)–(2e) are of Dirichlet type. What if they were of Neumann type (\( \partial u/\partial n \) prescribed)? For example, if (2b) were replaced by the Neumann condition

\[
\frac{\partial u}{\partial n}(r, 0) = g(r), \quad (a < r < b)
\]

where \( g(r) \) is prescribed, then what is \( \partial u/\partial n \) in terms of \( r \) and \( \theta \)? The key is to use the directional derivative formula \( du/da = \nabla u \cdot \hat{s} \) in Section 16.4, which gives

\[
\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n} = \left( \frac{\partial u}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial u}{\partial \theta} \hat{e}_\theta \right) \cdot (-\hat{e}_\theta) = -\frac{1}{r} \frac{\partial u}{\partial \theta}
\]

on the \( \theta = 0 \) edge, so (20) can be expressed in terms of \( r, \theta \) as \(-1/r\)\( \partial u/\partial \theta = g(r) \) or

\[
\frac{\partial u}{\partial \theta}(r, 0) = -rg(r). \quad (a < r < b)
\]

Similarly, a Neumann condition

\[
\frac{\partial u}{\partial n}(a, \theta) = h(\theta) \quad (0 < \theta < \alpha)
\]

becomes

\[
\frac{\partial u}{\partial r}(a, \theta) = -h(\theta) \quad (0 < \theta < \alpha)
\]

because

\[
\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n} = \left( \frac{\partial u}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial u}{\partial \theta} \hat{e}_\theta \right) \cdot (-\hat{e}_r) = -\frac{\partial u}{\partial r}
\]

on \( r = a \). Physically, remember that if \( u \) is a temperature field then \( \partial u/\partial n \) is proportional to the heat flux across that boundary.

EXAMPLE 2. Dirichlet Problem for Circular Disk. Next, consider the Dirichlet problem

\[
\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0. \quad (0 \leq r < b)
\]

(24a)
shown in Fig. 3. We begin with (9) and write

\[ u(r, \theta) = f(\theta) \quad (-\infty < \theta < \infty) \quad (24b) \]

but comparing (24) with (2) we seem to be missing some boundary conditions. After all, the PDE is of second order in \( r \) and in \( \theta \), so we expect to need two \( r \) boundary conditions and two \( \theta \) boundary conditions, as were present in (2b)–(2e). For insight, it is useful to imagine obtaining the disk in Fig. 3 as the limiting case of the region in Fig. 1 as \( a \to 0 \) and \( \alpha \to 2\pi \). Letting \( a \to 0 \) first, observe that the \( r = a \) boundary curve shrinks to a point; when that happens we lose that boundary curve and corresponding boundary condition and have a pie-shaped region. Next, let \( \alpha \to 2\pi \). The moment \( \alpha \) becomes \( 2\pi \) we lose the two boundary conditions on the edges \( \theta = 0 \) and \( \theta = \alpha \) because those edges disappear as boundary edges and become interior to the region.

To compensate for these losses we do the following. First, we adjoin to (24a,b), in lieu of the missing \( r \) boundary condition, a boundedness condition at \( r = 0 \), namely,

\[ u(r, \theta) \text{ bounded as } r \to 0. \quad (26) \]

To apply this condition, observe that both the \( \ln r \) and \( r^{-\kappa} \) terms in (25) are unbounded as \( r \to 0 \); \( \ln r \to -\infty \) and \( r^{-\kappa} \to \infty \) as \( r \to 0 \). Thus, (26) requires us to remove those terms by setting \( B = D = 0 \), in which case (25) reduces to

\[ u(r, \theta) = I + J\theta + r^{\kappa}(P \cos \kappa \theta + Q \sin \kappa \theta). \quad (27) \]

To remedy the situation regarding the missing \( \theta \) boundary condition we begin by observing a key difference between Example 1 and the present example: the \( \theta \) domain in Example 1 was finite \((0 < \theta < \alpha)\) whereas here it is infinite \((-\infty < \theta < \infty)\); for we see from Fig. 3 that there is nothing to prevent the representative point from encircling the origin repeatedly, clockwise or counterclockwise. Thus, if \( u(r, \theta) \) is to be a single-valued function of \( \theta \) and hence uniquely defined at each point within the disk, then it needs to be \( 2\pi \)-periodic in \( \theta \):

\[ u(r, \theta + 2\pi) = u(r, \theta). \quad (28) \]

This periodicity will compensate for the two missing \( \theta \) boundary conditions (see also Exercise 8) so the full problem is given by (24a), (24b), (26), and (28).

Let us impose (28) on each term in (27). First, \( I \) is a constant and is therefore \( 2\pi \)-periodic; hence, retain that term. Next, \( J\theta \) is \emph{not} periodic (as can be seen from its linear graph), so we must set \( J = 0 \) to remove that term. Finally, the \( \cos \kappa \theta \) and \( \sin \kappa \theta \) terms are periodic, but we need to determine the allowable \( \kappa \)'s so that they are \( 2\pi \)-periodic. According to the definition of periodicity, we need

\[ \cos \kappa(\theta + 2\pi) = \cos \kappa \theta \]

for all \( \theta \) (and similarly for the sine term) or, since \( \cos (A + B) = \cos A \cos B - \sin A \sin B \), we need

\[ \cos \kappa \theta \cos 2\pi \kappa - \sin \kappa \theta \sin 2\pi \kappa = \cos \kappa \theta. \]
Equating coefficients of the linearly independent \( \cos \kappa \theta \) and \( \sin \kappa \theta \) terms gives \( \cos 2\pi \kappa = 1 \) and \( \sin 2\pi \kappa = 0 \), with the roots:

\[
\begin{align*}
\cos 2\pi \kappa &= 1 : \quad \kappa = 1, 2, 3, \ldots, \\
\sin 2\pi \kappa &= 0 : \quad \kappa = 1, 2, 3, \ldots.
\end{align*}
\]

(30a) (30b)

Since both conditions need to hold, we accept only \( \kappa \)'s that are in both lists, namely, \( \kappa = 1, 2, 3, \ldots \). The same result is obtained when we enforce the \( 2\pi \)-periodicity of the \( \sin \kappa \theta \) term.

Thus, with \( J = 0 \), \( \kappa = n \), and with the help of superposition, (26) gives

\[
\sum
\]

(31)

We are ready for the final boundary condition:

\[
\sum
\]

(32)

which holds on \( -\infty < \theta < \infty \). Notice carefully that whereas (14) and (16) are half-range sine expansions on the finite interval \( 0 < \theta < \alpha \), (32) is the full Fourier series expansion of the \( 2\pi \)-periodic function \( f(\theta) \) on \( -\infty < \theta < \infty \). Accordingly (see (5) in Section 17.3, with \( \ell = \pi \)),

\[
I = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta, \quad P_n = \frac{1}{\pi b^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta, \quad Q_n = \frac{1}{\pi b^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta,
\]

(33)

and the solution is given by (31) and (33).

To illustrate, let the boundary temperature \( f(\theta) \) be 100 on the upper half of the circle and 0 on the lower half, in which case \( f \) is actually the \( 2\pi \)-periodic square wave shown in Fig. 4. Using (33), the result is

\[
u(r, \theta) = 50 + \frac{200}{\pi} \sum_{n=1,3,\ldots}^{\infty} \left( \frac{r}{b} \right)^n \sin n\theta \frac{n}{n},
\]

(34)

and representative isotherms are shown in Fig. 5.

COMMENT 1. Setting \( r = 0 \) in (31), observe that \( u \) at the center of the disk equals \( I \) and \( I \), according to (33), is the average value of the boundary temperature. For the example shown in Fig. 5, for instance, \( f(\theta) \) is 100 on the upper half of the circumference and 0 on

---

*As usual, we exclude negative \( \kappa \) values since they contribute nothing new. For instance, if we change \( \kappa \) to \( -\kappa \) in the last term in (25) then that term takes the equivalent form \( (Cr^{-\kappa} + Dr^{-\kappa})(G \cos \kappa \theta - H \sin \kappa \theta) \). This result is not surprising since the \( \kappa^2 \) in (4) cannot distinguish between positive \( \kappa \)'s and negative \( \kappa \)'s. Further, we disallow \( \kappa = 0 \) in (30) because the \( \kappa = 0 \) case is handled separately, in (25), by the \( (A + B ln r)(E + F \theta) \) term.
the lower half so the average value is 50. Sure enough, the isotherm \( u = 50 \) does pass through the origin in Fig. 5.

**COMMENT 2.** Let us put (33) into (31). First changing the dummy variable of integration to \( \theta \), say, to avoid confusion with the \( \theta \)'s in (31), the result is

\[
u(r, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{r}{b} \right)^n \cos n(\theta - \theta) \right] f(\theta) \, d\theta,
\]

(35)

where we have formally interchanged the order of integration and summation. It is striking that the infinite series in (35) can be summed, that is, gotten into closed form. The result (Exercise 9) is

\[
u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{b^2 - r^2}{b^2 - 2br \cos (\theta - \theta) + r^2} f(\theta) \, d\theta
\]

\[
\equiv \int_{-\pi}^{\pi} P'(r, \theta - \theta) f(\theta) \, d\theta;
\]

(36)

is the **Poisson integral formula for the circular disk** and \( P'(r, \theta - \theta) \) is the corresponding Poisson kernel.  

![Figure 6. Average value theorem.](image)

In Comment 1, above, we wrote that the temperature \( u \) at the center of the disk equals the average of the boundary temperatures \( f(\theta) \) around the circumference. This result can be generalized as follows. Within an arbitrary domain \( D \), not necessarily circular (Fig. 6), consider any point \( P' \) and any circular domain \( D' \) that is centered at \( P' \) and that lies entirely within \( D \), and suppose that \( \nabla^2 u = 0 \) in \( D \). Then, whatever the temperatures are on the boundary \( \partial D' \) of \( D' \), we can consider them as boundary conditions for the sub-problem \( \nabla^2 u = 0 \) in \( D' \), which problem was the subject of Example 2. From the average value result found in Example 2 we know that \( u \) at \( P' \) is the average of the \( u \) values around \( \partial D' \). Thus, if \( \nabla^2 u = 0 \) in a two-dimensional domain \( D \), then the temperature \( u \) at any point \( P' \) within \( D \) is equal to the average temperature around any circle centered at \( P' \) and lying within \( D \). This result, known as the **average value property** of the Laplace equation, is said to be a "local" result since it holds for an arbitrarily small circle \( \partial C' \) and is insensitive to the shape of \( D \). It holds in one dimension (Exercise 10) and in three dimensions as well (as we will see in Example 5).

The average value property enables us to prove the **maximum principle** for the Laplace equation, which is as follows: *Let \( u \) be the steady-state temperature field within a two-dimensional domain \( D \), so \( u \) satisfies the Laplace equation \( \nabla^2 u = 0 \). Then \( u \) cannot attain its maximum value in \( D \) (unless \( u \) is a constant everywhere); it must attain its maximum on the boundary of \( D \).* For suppose that \( u \) does have a maximum value \( M \), say, at a point \( P \) within \( D \). Since \( u \) at \( P \) equals the average value of \( u \) around any circle centered at \( P \), \( u \) must achieve values less than \( M \) and greater than \( M \) at points within \( D \) if it is not simply a constant everywhere. But that result contradicts our assumption that the maximum value of \( u \) is \( M \), hence \( u \) cannot have a maximum value within \( D \). By virtually the same argument we can show...
that \( u \) cannot attain its minimum value in \( D \), and we obtain the \textbf{minimum principle}, which is identical to the maximum principle (italicized above) with the two "maximums" changed to "minimums." It follows from these maximum and minimum principles that the values of \( u \) within \( D \) necessarily lie between the minimum and maximum values of \( u \) on the boundary of \( D \).

These results make sense physically, for suppose that the temperature maintained on the boundary of \( D \) lies between 50°C and 70°C and let \( u = 2,000°C \) at some point inside \( D \). Surely that "hot spot" will cool down, with time, and the surrounding material will heat up. But that is impossible for we have assumed steady-state heat conduction; that is, the Laplace equation \( \nabla^2 u = 0 \) is the steady-state version of the heat conduction equation \( \alpha^2 \nabla^2 u = u_t \).

\subsection*{20.3.2. Cylindrical coordinates. (Optional)} The steady-state temperature field within a cylindrical rod (Fig. 7) is governed by the Laplace equation in cylindrical coordinates,

\[ \nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} + u_{zz} = 0. \]  
(37)

Suppose there is axisymmetry so that \( u \) does not vary with \( \theta \). Then (37) reduces to

\[ \nabla^2 u = u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \]  
(38)

which case we consider here. Specifically, we consider the problem shown in the left-hand member of Fig. 8, and we begin by breaking it into the two problems shown in the figure. We will solve the \( u_1 \) problem as Example 3 and the \( u_2 \) problem as Example 4.

\[ \begin{align*}
\nabla^2 u &= 0 \\
0 &= 0 \\
0 &= 0 \\
0 &= 0
\end{align*} \]

\begin{align*}
\nabla^2 u_1 &= 0 \\
\nabla^2 u_2 &= 0 \\
0 &= 0 \\
0 &= 0
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cylindrical_rod.png}
\caption{Cylindrical rod.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cylindrical_breakdown.png}
\caption{Breakdown by superposition.}
\end{figure}

\textbf{EXAMPLE 3.} \textit{The \( u_1 \) Problem.} To solve (38), seek

\[ u_1(r, z) = R(r)Z(z). \]  
(39)

Putting the latter into (38) and separating variables gives

\[ \frac{R'' + \frac{1}{r} R'}{R} = \frac{Z''}{Z} = \text{constant} = \kappa^2 \]  
(40)

and the ODE's

\[ \begin{align*}
R'' + \frac{1}{r} R' - \kappa^2 R &= 0, \\
Z'' + \kappa^2 Z &= 0.
\end{align*} \]  
(41, 42)
We chose the $+\kappa^2$ in (40) so as to obtain $\cos \kappa z$ and $\sin \kappa z$ (rather than $\cosh \kappa z$ and $\sinh \kappa z$) solutions of the $Z$ equation since we look ahead to a Fourier series expansion of $h(z)$. Distinguishing the cases $\kappa = 0$ and $\kappa \neq 0$, the general solutions of (41) and (42) are (Exercise 14):

$$
R = \begin{cases} 
A + B \ln r, & \kappa = 0 \\
C I_0(\kappa r) + D K_0(\kappa r), & \kappa \neq 0
\end{cases} \quad (43)
$$

$$
Z = \begin{cases} 
E + Fz, & \kappa = 0 \\
G \cos \kappa z + H \sin \kappa z, & \kappa \neq 0
\end{cases} \quad (44)
$$

where $I_0, K_0$ are modified Bessel functions of the first and second kind, respectively, of order zero. Thus,

$$
u_1(r, z) = (A + B \ln r)(E + Fz) + [C I_0(\kappa r) + D K_0(\kappa r)](G \cos \kappa z + H \sin \kappa z). \quad (45)
$$

In the $z$ variable we have the two boundary conditions at $z = 0$ and $z = L$, but in $r$ we have only the boundary condition at $r = b$ so in lieu of a second $r$ boundary condition we require that $u$ be bounded as $r \to 0$ (i.e., all along the $z$ axis). Since $\ln r \to -\infty$ we set $B = 0$, and since $K_0(\kappa r) \sim -\ln r \to \infty$ as $r \to 0$ (Fig. 9) we set $D = 0$, so (45) becomes

$$
u_1(r, z) = P + Q z + I_0(\kappa r)(S \cos \kappa z + T \sin \kappa z), \quad (46)
$$

where we have combined $AE$ as $P$, $AF$ as $Q$, $CG$ as $S$, and $CH$ as $T$, for brevity.

Next, we apply the conditions at $z = 0$, $z = L$, and $r = b$, in turn:

$$
u_1(r, 0) = 0 = P + I_0(\kappa r)S, \quad (47)
$$

so $P = 0$ and $S = 0$. Updating (46) accordingly,

$$
u_1(r, z) = Q z + T I_0(\kappa r) \sin \kappa z. \quad (48)
$$

Next,

$$
u_1(r, L) = 0 = Q L + T I_0(\kappa r) \sin \kappa L, \quad (49)
$$

so $Q = 0$ and $\kappa = n\pi/L$ for $n = 1, 2, \ldots$ Updating (48),

$$
u_1(r, z) = \sum_{n=1}^{\infty} T_n I_0 \left( \frac{n\pi}{L} \right) \sin \frac{n\pi z}{L}. \quad (50)
$$

Finally,

$$
u_1(b, z) = h(z) = \sum_{n=1}^{\infty} T_n I_0 \left( \frac{n\pi b}{L} \right) \sin \frac{n\pi z}{L}, \quad (0 < z < L) \quad (51)
$$

which is a half-range sine series so

$$
T_n I_0 \left( \frac{n\pi b}{L} \right) = \frac{2}{L} \int_0^L h(z) \sin \frac{n\pi z}{L} \, dz,
$$

or

$$
T_n = \frac{2}{L I_0(n\pi b/L)} \int_0^L h(z) \sin \frac{n\pi z}{L} \, dz. \quad (52)
$$
The desired solution is given by (51) and (52).

**COMMENT.** As in Example 2 we applied the boundedness condition first. Indeed, whenever there is a boundedness condition we suggest that you apply it first because it will eliminate one or more terms, thereby giving an immediate simplification of the solution form.

**EXAMPLE 4.** The \( u_2 \) Problem. To solve for \( u_2 \) (see Fig. 8) we again seek

\[
u_2(r, z) = R(r)Z(z)
\]

and obtain

\[
\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = \text{constant} = -\kappa^2
\]

and the ODE's

\[
R'' + \frac{1}{r}R' + \kappa^2 R = 0,
\]
\[
Z'' - \kappa^2 Z = 0,
\]

where this time we chose \(-\kappa^2\), so as to obtain oscillatory solutions of the \( R \) equation (rather than the \( I_0, K_0 \) pair obtained in Example 3). Specifically (Exercise 15),

\[
R = \begin{cases} 
A + B \ln r, & \kappa = 0 \\
C J_0(\kappa r) + D Y_0(\kappa r), & \kappa \neq 0
\end{cases}
\]
\[
Z = \begin{cases} 
E + Fz, & \kappa = 0 \\
G \cosh \kappa z + H \sinh \kappa z, & \kappa \neq 0
\end{cases}
\]

where \( J_0, Y_0 \) are Bessel functions of the first and second kind, respectively, of order zero. Thus,

\[
u_2(r, z) = (A + B \ln r)(E + Fz) + [C J_0(\kappa r) + D Y_0(\kappa r)](G \cosh \kappa z + H \sinh \kappa z).
\]

Boundedness as \( r \to 0 \) requires that \( B = 0 \) and \( D = 0 \), since \( Y_0(\kappa r) \sim (2/\pi) \ln r \to -\infty \) as \( r \to 0 \) (Fig. 10), so (59) reduces to

\[
u_2(r, z) = P + Qz + J_0(\kappa r)(S \cosh \kappa z + T \sinh \kappa z).
\]

Since we look ahead to expanding \( f(r) \) and \( g(r) \), we must apply both \( r \) boundary conditions first, before attempting either of the end conditions at \( z = 0 \) and \( L \). Having already applied the boundedness condition at \( r = 0 \), we next write

\[
u_2(b, z) = 0 = P + Qz + J_0(\kappa b)(S \cosh \kappa z + T \sinh \kappa z),
\]

which requires that \( P = 0, Q = 0, \) and

\[
J_0(\kappa b) = 0
\]
Chapter 20. Laplace Equation

[since we cannot afford to set \( S = T = 0 \) and lose the entire \( J_0(\lambda r)(S \cosh \kappa z + T \sinh \kappa z) \) term in (60)]. Denoting the roots of \( J_0(z) = 0 \) as \( z_1, z_2, \ldots \), because they give the zeros of \( J_0 \), we learn from (61) that \( \kappa = z_n/b \) (\( n = 1, 2, \ldots \)). Updating (60) accordingly,

\[
    u_2(r, z) = \sum_{n=1}^{\infty} J_0 \left( z_n \frac{r}{b} \right) \left[ S_n \cosh \left( z_n \frac{z}{b} \right) + T_n \sinh \left( z_n \frac{z}{b} \right) \right].
\]  

(62)

Finally, the end conditions give

\[
    u_2(r, 0) = f(r) = \sum_{n=1}^{\infty} S_n J_0 \left( z_n \frac{r}{b} \right),
\]  

(63)

and

\[
    u_2(r, L) = g(r) = \sum_{n=1}^{\infty} \left[ S_n \cosh \left( z_n \frac{L}{b} \right) + T_n \sinh \left( z_n \frac{L}{b} \right) \right] J_0 \left( z_n \frac{r}{b} \right).
\]  

(64)

on \( 0 < r < b \). How are we to solve (63) and (64) for \( S_n \) and \( T_n \)? There are two questions raised by (63): first, is it possible to expand a given function \( f(r) \) on the interval \( 0 < r < b \) in the form of an infinite linear combination of \( J_0(z_n r/b) \) terms and, second, if so, how do we compute the \( S_n \) coefficients? Similarly for (64). Both questions are answered by the Sturm–Liouville theory, for the problem

\[
    (r R')' + \kappa^2 r R = 0, \quad (0 < r < b)
\]  

(65a)

\[
    R(0) \text{ bounded,} \quad R(b) = 0
\]  

(65b)

governing \( R \) is a Sturm–Liouville problem, where \( \lambda = \kappa^2 \). This problem is studied in Example 2 of Section 17.8 so, referring you to that example for the details, we can conclude from (63) that

\[
    S_n = \frac{2}{b^2 [J_1(z_n)]^2} \int_0^b f(r) J_0 \left( z_n \frac{r}{b} \right) r \, dr,
\]  

(66)

and from (64) that

\[
    S_n \cosh \left( z_n \frac{L}{b} \right) + T_n \sinh \left( z_n \frac{L}{b} \right) = \frac{2}{b^2 [J_1(z_n)]^2} \int_0^b g(r) J_0 \left( z_n \frac{r}{b} \right) r \, dr.
\]  

(67)

Thus, the solution is obtained by solving (66) and (67) for \( S_n \) and \( T_n \) (once \( f \) and \( g \) are specified), and putting these values into (62).

COMMENT. Observe how the problem is “self-contained”: how to carry out the necessary expansions (63) and (64) is fully explained by the Sturm–Liouville problem on \( R \) that is “built right in.” Likewise in Example 3, although we did not mention it because we merely noticed that (51) is a half-range sine expansion. Alternatively, we could have used the Sturm–Liouville theory there too. Specifically, the relevant Sturm–Liouville problem there is

\[
    Z'' + \kappa^2 Z = 0, \quad (0 < z < L)
\]  

(68a)

\[
    Z(0) = 0, \quad Z(L) = 0,
\]  

(68b)
with the eigenfunctions $\sin \left( \frac{n\pi z}{L} \right)$, so (51) gives

$$
T_n I_0 \left( \frac{n\pi h}{L} \right) = \frac{\langle h(z), \sin \frac{n\pi z}{L} \rangle}{\langle \sin \frac{n\pi z}{L}, \sin \frac{n\pi z}{L} \rangle} = \frac{\int_0^L h(z) \sin \frac{n\pi z}{L} \, dz}{\int_0^L \sin^2 \frac{n\pi z}{L} \, dz} = \frac{2}{L} \int_0^L h(z) \sin \frac{n\pi z}{L} \, dz,
$$

which result agrees with (52). Note that the inner product weight function is $r$ in (65) and 1 in (68).

Likewise in Example 1. We deduced (15) from (14) and (17) from (16) by noticing that (14) and (16) were half-range sine expansions of their left-hand sides, over $0 < \theta < \alpha$. Alternatively, we could have used the Sturm–Liouville theory. Since the expansions were in the $\theta$ variable, look to the $\Theta$ problem for the Sturm–Liouville problem, namely,

$$
\Theta'' + \kappa^2 \Theta = 0, \quad (0 < \theta < \alpha) \quad (69a)
$$

$$
\Theta(0) = 0, \quad \Theta(\alpha) = 0, \quad (69b)
$$

with eigenfunctions $\sin \left( \frac{n\pi \theta}{\alpha} \right)$ and weight function 1. Where do we get the homogeneous boundary conditions (69b) from, considering that $u(r,0) = u_1$ and $u(r,\alpha) = u_2$ are, in general, nonzero? If we retrace the solution steps, beginning with (9), we find that the $G \cos \kappa \theta + H \sin \kappa \theta$ factor in (9), which contributes the $\sin \left( \frac{n\pi \theta}{\alpha} \right)$ eigenfunctions, satisfies the homogeneous boundary conditions (69b), with the $(A + B \ln r)(E + F \theta)$ term handling the $u_1$ and $u_2$ values. And, of course, we can see directly that $\sin \left( \frac{n\pi \theta}{\alpha} \right)$ does indeed vanish at $\theta = 0$ and at $\theta = \alpha$.

Then the solution to the problem on $u$ that is shown in the left-hand part of Fig. 8 is $u(r,z) = u_1(r,z) + u_2(r,z)$.

20.3.3. **Spherical coordinates. (Optional)** If the domain under consideration is bounded by constant $\rho, \phi, \theta$ surfaces, then we need to work with the Laplace equation in spherical coordinates,

$$
\nabla^2 u = \frac{1}{\rho^2} \left[ \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2} \right] = 0. \quad (70)
$$

Let us restrict our attention to cases where $u$ is axisymmetric about the polar axis $z$, that is, where $u$ does not vary with $\theta$. Then the $u_{\theta\theta}$ term in (70) is zero and (70) reduces to the PDE

$$
\nabla^2 u = u_{\rho\rho} + \frac{2}{\rho} u_{\rho} + \frac{1}{\rho^2} u_{\phi\phi} + \frac{\cot \phi}{\rho^2} u_{\phi} = 0 \quad (71)
$$

on $u(\rho, \phi)$. 
EXAMPLE 5. Dirichlet Problem for Sphere. Consider the Dirichlet problem consisting of the PDE (71) in the sphere \( 0 \leq \rho < c \), with the boundary condition

\[
u(c, \phi) = f(\phi) \quad (0 \leq \phi \leq \pi)
\]

and the stipulation that \( u \) be bounded in the given domain (Fig. 11). This problem is the three-dimensional analog of the Dirichlet problem for a circular disk, which was the subject of Example 2.

To solve, seek

\[
u(\rho, \phi) = R(\rho)\Phi(\phi)
\]

and obtain, from (71),

\[
\frac{\rho^2 R'' + 2\rho R'}{R} = -\frac{\Phi'' + \cot \phi \Phi'}{\Phi} = \text{constant} = \kappa^2,
\]

and the ODE's

\[
\begin{align*}
\rho^2 R'' + 2\rho R' - \kappa^2 R &= 0, \\
\Phi'' + \cot \phi \Phi' + \kappa^2 \Phi &= 0.
\end{align*}
\]

The change of variables

\[
\mu = \cos \phi
\]

in (76) reduces that equation (Exercise 17) to the Legendre equation

\[
(1 - \mu^2) \frac{d^2 \Phi}{d\mu^2} - 2\mu \frac{d\Phi}{d\mu} + \kappa^2 \Phi = 0.
\]

From our study of the Legendre equation in Section 4.4, we know that to obtain solutions of (77) that are bounded on \(-1 \leq \mu \leq 1\) we need

\[
\kappa^2 = n(n + 1), \quad (n = 0, 1, 2, \ldots)
\]

in which case the corresponding bounded solutions are the Legendre polynomials

\[
\Phi = P_n(\mu) = P_n(\cos \phi).
\]

In terms of the physical domain, \( \mu = 1 \) corresponds to \( \phi = 0 \) so that unboundedness of \( \Phi \) at \( \mu = 1 \) would mean unboundedness of the solution \( u \) all along the \( z \) axis from the “north pole” to the origin. Similarly, unboundedness of \( \Phi \) at \( \mu = -1 \) would mean unboundedness of \( u \) all along the \( z \) axis from the origin to the “south pole.”

Turning to the \( R \) equation, with \( \kappa^2 = n(n + 1) \), the general solution of (75), which is a Cauchy–Euler equation, is

\[
R(\rho) = A\rho^n + \frac{B}{\rho^{n+1}}, \quad (n = 0, 1, 2, \ldots)
\]

Here, the stipulated boundedness of \( u \) requires that \( B = 0 \). Putting (79) and (80) together and using superposition we have, thus far,

\[
u(\rho, \phi) = \sum_{n=0}^{\infty} A_n \rho^n P_n(\cos \phi).
\]
Finally, the boundary condition (72) requires that

\[ u(c, \phi) = f(\phi) = \sum_{n=0}^{\infty} A_n c^n P_n(\cos \phi), \quad (0 \leq \phi \leq \pi) \]  (82)

Since (82) involves an expansion in the \( \phi \) variable, let us examine the boundary-value problem on \( \Phi \), namely:

\[ ((1 - \mu^2) \Phi')' + \kappa^2 \Phi = 0, \quad (-1 \leq \mu \leq 1) \]  (83a)

\[ \Phi(-1) \text{ and } \Phi(1) \text{ finite.} \]  (83b)

Thus, we see that (82) amounts to a Fourier–Legendre expansion of the given function \( f \) in terms of the orthogonal eigenfunctions \( P_n(\mu) \) or \( P_n(\cos \phi) \), as was illustrated in Example 3 of Section 17.8. Accordingly,

\[ A_n c^n = \frac{\int_{-1}^{1} f P_n d\mu}{\int_{-1}^{1} P_n^2 d\mu} = \frac{2n + 1}{2} \int_{-1}^{1} f P_n d\mu, \]  (84)

or

\[ A_n = \frac{2n + 1}{2c^n} \int_{0}^{\pi} f(\phi) P_n(\cos \phi) \sin \phi d\phi. \]  (85)

Hence, the solution is given by (81) and (85).

**COMMENT.** Observe that the value of \( u \) at the center of the sphere is

\[ u(0, \phi) = A_0 \quad \text{[from (81)]} \]

\[ = \frac{1}{2} \int_{-1}^{1} f d\mu, \quad \text{[from (84)]} \]

which is the average value of the boundary temperature \( f \). Thus, the average value property of the Laplace equation, discussed above, holds in three dimensions as well as two, and similarly for the maximum principle.

**Closure.** We see, in Section 20.3.1, that the Laplace equation in plane polar coordinates is successfully separated and that although the ODE on \( R(r) \) has non-constant coefficients it is nevertheless an elementary equation, a Cauchy–Euler equation. We derive, as a result of Example 2, the average value property of the Laplace equation in two dimensions and the maximum principle as well. In Sections 20.3.2 and 20.3.3 we find that the Laplace equation can be separated in cylindrical and spherical coordinates as well, but that not all of the resulting ODE’s are elementary: in cylindrical coordinates the \( R(r) \) equation gives Bessel functions, and in spherical coordinates the \( \Phi(\phi) \) equation gives Legendre polynomials.

*Strictly speaking, we should use a new name, such as \( \Phi(\phi) = \Phi(\phi(\mu)) \equiv \Psi(\mu) \), say, but for economy of notation we use \( \Phi \) whether the independent variable is \( \phi \) or \( \mu \).
Expansions in Examples 4 and 5 involve the Bessel and Legendre functions, and they are carried out by relying on the Sturm–Liouville theory. If the expansion is on the \( r \) variable, for example, then we bring to light the Sturm–Liouville problem on \( R(r) \) and use that problem and the Sturm–Liouville theory to guide our expansion.

Until now, our rule of thumb for choosing the sign of the \( \kappa^2 \) separation constant has been to choose the sign that makes oscillatory functions available for the eventual Fourier series expansion. If you are familiar with the Sturm–Liouville theory it would be helpful to make that rule more explicit as follows. Choose the sign of \( \kappa^2 \) so that the sign of the last term in the Sturm–Liouville ODE is positive because in the Sturm–Liouville equation

\[
(py')' + qy + \lambda ry = 0
\]

the eigenvalues \( \lambda \) are generally nonnegative. For instance, in Example 5 we anticipate (from the boundary conditions) that the expansion will be on the \( \phi \) variable, so the Sturm–Liouville ODE is (76), not (75). Thus, we choose the +\( \kappa^2 \) in (74), so that the last term on the left side of (76) is +\( \kappa^2 \Phi \), not \( -\kappa^2 \Phi \).

---

**EXERCISES 20.3**

1. Show that if \( u_1 = u_2 = 0 \) and \( f(\theta) = 100 \), then (13), (15), and (17) give the result (19).

2. Solve for \( u(r, \theta) \) and sketch, based on intuition, the \( u = 25, 50, 75 \) isoliners: \( \nabla^2 u = 0 \)

   (a) in \( 1 < r < 2, \ 0 < \theta < \pi; \ u(r, \theta) = u(2, \theta) = 0, \ u(r, \pi) = u(1, \theta) = 100 \)

   (b) in \( 1 < r < 2, \ 0 < \theta < \pi; \ u_\theta(r, \theta) = u(2, \theta) = 0, \ u(r, \pi) = u(1, \theta) = 100 \)

   (c) in \( 1 < r < 2, \ 0 < \theta < \pi; \ u(r, \pi) = 100, \ u(r, 0) = u(r, \theta) = 0 \)

   (d) in \( 1 < r < 2, -\infty < \theta < \infty; \ u(1, \theta) = 0, u(2, \theta) = 100 \)

   (e) in \( 1 < r < 2, \ 0 < \theta < \pi; \ u(1, \theta) = u_\theta(r, 0) = u_\theta(r, \pi) = 0, u(2, \theta) = 100 \)

   (f) in \( 0 < r < 3, \ 0 < \theta < 3\pi/2; \ u(r, 0) = u(r, 3\pi/2) = 100, u(3, \theta) = 0, u \text{ bounded} \)

   (g) in \( 0 < r < 3, \ 0 < \theta < 3\pi/2; \ u_\theta(r, 0) = u(3, \theta) = 0, u(r, 3\pi/2) = 100, u \text{ bounded} \)

   (h) in \( 0 < r < 3, \ 0 < \theta < 3\pi/2; \ u(r, 0) = u(r, 3\pi/2) = 100, u \text{ bounded} \)

   (i) in \( 0 < r < 3, \ 0 < \theta < 3\pi/2; \ u(r, 0) = u(r, 3\pi/2) = 0, u(2, \theta) = 100 \text{ on } 0 < \theta < \pi/2 \text{ and } 0 \text{ on } \pi/2 < \theta < 3\pi/2, u \text{ bounded} \)

   (j) in \( 3 < r < \infty, \ 0 < \theta < \pi/2; \ u(r, 0) = u(r, \pi/2) = 0, u(3, \theta) = 100, u \text{ bounded} \)

3. Solve for \( u(r, \theta) \) and give a labeled plot (by computer if necessary) of representative isotherms, as many as it takes to give a clear picture of the temperature field: \( \nabla^2 u = 0 \) in \( r < 1, u \text{ bounded}, u(1, 0) = f(\theta) \)

   (a) \( f(\theta) = 50 + 20 \cos \theta \)

   (b) \( f(\theta) = 50 + 50(\cos \theta + \sin \theta) \)

   (c) \( f(\theta) = 20 \cos 2\theta \)

   (d) \( f(\theta) = 20 \sin 2\theta \)

   (e) \( f(\theta) = 20 \cos 3\theta \)

   (f) \( f(\theta) = 20 \sin 3\theta \)

   (g) \( f(\theta) = 20 \cos 4\theta \)

   (h) \( f(\theta) = 20 \cos 5\theta \)

4. Consider a thin flat circular plate of radius \( b \), that is thermally insulated on its two flat faces. With a hacksaw we make a radial cut along \( \theta = 0 \), say, from \( r = b \) to \( r = 0 \). The small gap, due to the cut, may be approximated as a thermal insulator, so that \( \partial u/\partial n = 0 \) on the edges \( \theta = 0 \) and \( \theta = 2\pi \). If the circumference of the plate is held at the temperature \( 50(1 + \sin \theta) \) for a long time, the steady-state temperature field
20.3. Separation of Variables: Non-Cartesian Coordinates

\( u(r, \theta) \) is governed by the boundary-value problem

\[
\begin{align*}
\nabla^2 u &= 0, & (0 < r < b, \ 0 < \theta < 2\pi) \\
\frac{\partial u}{\partial n}(r, 0) &= \frac{\partial u}{\partial n}(r, 2\pi) &= 0, \\
u(b, \theta) &= 50(1 + \sin \theta), & u \text{ bounded.}
\end{align*}
\]

Solve for, and plot, the temperature distributions \( u(r, 0) \) and \( u(r, 2\pi) \) along the two edges of the cut. Does the answer depend in any way on whether the plate is steel or brass or whatever? Explain.

5. (Plane with circular hole) As a sort of “inverse” of Example 2, consider the domain to be the whole plane, with a circular hole of radius \( a \):

\[
\begin{align*}
\nabla^2 u &= 0, & (a < r < \infty) \\
u(a, \theta) &= f(\theta), & u \text{ bounded as } r \to \infty.
\end{align*}
\]

Solve for \( u(r, \theta) \), leaving expansion coefficients in integral form. What is the value of \( u \) at \( r = \infty \)?

6. (Plane flow over a circular bump) First, read Example 3 of Section 16.10.

(a) Then, solve (38) in Section 16.10 and derive the solution

\[
\Phi(r, \theta) = U \left( r + \frac{a^2}{r} \right) \cos \theta + C,
\]

where \( C \) is an arbitrary constant that can be set equal to zero without loss. NOTE: The velocity field \( v \) is then available as \( v = \nabla \Phi \). Knowing \( v \), one could use the Bernoulli equation of fluid mechanics (which is derived in Exercise 12 of Section 16.10) to determine the pressure field and, in particular, the resulting aerodynamic force on the semicircular bump (which might, for instance, be the roof of a building). Observe that in this problem a boundedness condition on \( \Phi \) would be inappropriate since \( \Phi \sim Ur \cos \theta \) as \( r \to \infty \). Nevertheless, the physical quantity \( v = \nabla \Phi \) is bounded: \( v \sim U I \) as \( r \to \infty \). Finally, observe that (6.1) can be expressed as the superposition \( \Phi = \Phi_1 + \Phi_2 \), where \( \Phi_1 = Ur \cos \theta = Ur \) is the potential of the “free stream,” and \( \Phi_2 = U (a^2/r) \cos \theta \) accounts for the disturbance caused by the presence of the bump (indeed, \( \Phi_2 \to 0 \) as \( a \to 0 \)).

(b) By way of graphics, the most interesting display is not a display of constant \( \Phi \) curves but a display of representative streamlines, as in Fig. 7 of Section 16.10. By streamlines we mean the constant \( \Psi \) curves, where \( \Psi \) is the stream function introduced in Exercise 8 of Section 16.10. From (8.3) therein, \( \Psi \) is related to \( \Phi \) according to

\[
\frac{\partial \Psi}{\partial x} = -\frac{\partial \Phi}{\partial y}, \quad \frac{\partial \Psi}{\partial y} = \frac{\partial \Phi}{\partial x}.
\]

Use (6.2) and (6.1) to derive the result

\[
\Psi(x, y) = U \left( y - \frac{a^2 y}{x^2 + y^2} \right).
\]

(c) Then (with \( U = a = 1 \), say) use computer software such as the Maple implicitplot command to generate the streamline pattern that we sketched in Fig. 7 (Section 16.10). Choose the streamlines through the points \((x, y) = (4, 0,2), (4, 0,8), (4, 1,4), \) and \((4, 2), \) say.

7. (Flow past a circular cylinder: nonuniqueness) In Exercise 6 we consider the flow of a free stream past a semicircular bump. Here, we consider the flow of a free stream past a circular cylinder. The boundary-value problem is

\[
\begin{align*}
\nabla^2 \Phi &= \Phi_{rr} + \frac{1}{r} \Phi_r + \frac{1}{r^2} \Phi_{\theta \theta} = 0, \\
\Phi_n(a, \theta) &= 0, \\
\nabla \Phi \bigg|_{\theta=0} &= \nabla \Phi \bigg|_{\theta=2\pi} \text{ over } a < r < \infty, \\
\Phi \sim U r \cos \theta \text{ as } r \to \infty.
\end{align*}
\]

That is, if we specify that \( 0 < \theta < 2\pi \), the radial lines \( \theta = 0 \) and \( \theta = 2\pi \) \((a < r < \infty)\) become part of the boundary (see the accompanying sketch), so that boundary conditions are needed along these lines. The appropriate boundary condition is that physical velocity \( v \) be the same at \( \theta = 0 \) and \( \theta = 2\pi \) \((a < r < \infty)\), which condition is expressed as (7.1c) or

\[
\begin{align*}
\Phi_r(r, 0) &= \Phi_r(r, 2\pi), & (a < r < \infty) \\
\frac{1}{r} \Phi_\theta(r, 0) &= \frac{1}{r} \Phi_\theta(r, 2\pi). & (a < r < \infty)
\end{align*}
\]

Integrating (7.1c) with respect to \( r \) and cancelling \( r \)'s in (7.1f), we obtain
\[ \Phi(r,0) = \Phi(r,2\pi) + \Gamma, \quad \Phi_\theta(r,0) = \Phi_\theta(r,2\pi) \quad (7.1g,h) \]
on a = r < \infty, where \( \Gamma \) is an arbitrary constant.

(a) Solve the resulting boundary-value problem (7.1a,b,g,h,d) and show that
\[ \Phi(r, \theta) = U \left( r + \frac{\alpha^2}{r^2} \right) \cos \theta - \frac{\Gamma}{2\pi} \theta, \quad (7.2) \]
which solution differs from (7.1) (with \( C = 0 \), say) only by the \(-\Gamma \theta / 2\pi \) term. \( \text{NOTE: In physical terms, the} \ U(r + \alpha^2 r^{-1}) \cos \theta \text{ term in (7.2) corresponds to a flow that is symmetric about the} \ x \text{ axis (as sketched in the figure below), and which has stagnation points (i.e., where} \ v = 0 \text{) at} \ r = a \text{ and} \ \theta = 0, \pi. \text{ The} \ -\Gamma \theta / 2\pi \text{ term contributes the additional velocity} \]
\[ v = \nabla \left( -\frac{\Gamma}{2\pi} \theta \right) = -\frac{\Gamma}{2\pi} \hat{r}, \]
which is a clockwise circular vortex flow (see Exercise 3 of Section 16.5) induced by a fictitious clockwise vortex of strength \( \Gamma \), at the origin—“fictitious” because there is no fluid inside the circle \( r = a \). The vector superposition of these two velocity contributions gives a flow somewhat as we have sketched in the next figure, namely, the symmetric flow (the preceding figure) plus some clockwise “swirl” that is proportional to \( \Gamma \).

The two flows add on the upper part of the cylinder and subtract on the lower part, so there are higher velocities on the upper surface of the cylinder and lower velocities on the lower surface. Since Bernoulli’s equation \( \sigma v^2/2 + p = \text{constant} \) (\( \sigma = \text{mass density}, \ p = \text{pressure} \) tells us that the higher the velocity the lower the pressure, and vice versa, it follows that a lift force is generated on the cylinder, \( L = \sigma U \Gamma \) force per unit length of the cylinder (i.e., per unit z length).

(b) Show that the two stagnation points (in the preceding figure) are located on the cylinder surface by the equation \( \sin \theta = -\Gamma/(4\pi U a) \); e.g., if \( \Gamma = 0 \), then \( \theta = \pi \) and \( 2\pi \), as in the symmetric-flow figure. What happens regarding the existence and location of stagnation points if \( \Gamma > 4\pi U a \)? Explain.

(c) From (7.2), \( v = \nabla \Phi \), and the Bernoulli \( U \) equation, obtain the pressure distribution on the cylinder, integrate it, and thus derive the famous Kutta–Joukowski lift formula
\[ L = \sigma U \Gamma \quad (7.3) \]

stated above.

8. (Alternative approach to Example 2) In Example 2 we observe that \(-\infty < \theta < \infty\) and impose the periodicity condition (28). Alternatively, we can consider that \( 0 < \theta < 2\pi \), in which case the lines \( \theta = 0 \) and \( \theta = 2\pi \) occur as boundary edges of the domain. That is, we make an infinitely thin slit in the region as shown in the figure below. Since \( 0 < \theta < 2\pi \), we discard the \( 2\pi \)-periodicity condition, but we now have two artificially created boundaries, \( \theta = 0 \) and \( \theta = 2\pi \), along which to specify boundary conditions. In particular, we impose the two conditions
\[ u(r,0) = u(r,2\pi), \quad u_\theta(r,0) = u_\theta(r,2\pi) \quad (8.1,2) \]

over \( 0 < r < b \), so that both the temperature and heat flux are continuous across the slit. With the problem reformulated in this manner, solve for \( u(r,\theta) \) and show that the solution obtained is the same as in Example 2. \( \text{NOTE: The boundedness condition at} \ r = 0 \text{ is still needed.} \)

9. Derive the result
\[ \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{r}{b} \right)^n \cos n(\theta - \theta) \]
\[ = \frac{1}{2} \frac{b^2 - r^2}{b^2 - 2br \cos(\theta - \theta) + r^2} \quad (9.1) \]

stated in Comment 2 of Example 2. \( \text{HINT: Write} \)
20.3. Separation of Variables: Non-Cartesian Coordinates

\[
\sum_{n=1}^{\infty} \left( \frac{r}{b} \right)^n \cos(n(\vartheta - \theta)) = \text{Re} \left( \sum_{n=1}^{\infty} \left( \frac{r}{b} \right)^n e^{in(\vartheta - \theta)} \right), \quad (9.2)
\]

and use the geometric series

\[
\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z},
\]

(9.3)

which holds even if \(z\) is complex, provided that \(|z| < 1\). As usual, \(R\) denotes the real part of the quantity.

10. Below Example 2 we show that the average value property of the Laplace equation holds in two dimensions, and at the end of Example 5 we showed that it holds in three dimensions. Here, we ask you to show that it holds for the one-dimensional Laplace equation \(d^2u/dx^2 = 0\) as well.

11. Use the maximum and minimum principles to show, for the Laplace equation (in two or three dimensions), that if we change the boundary values only slightly, then the solution values (i.e., within the solution domain) change only slightly too. Specifically, show that if \(\nabla^2 u_1 = 0\) in \(D\) with boundary values \(u_1 = f_1\), and \(\nabla^2 u_2 = 0\) in \(D\) with boundary values \(u_2 = f_2\), then \(\min(f_1 - f_2) \leq u_1 - u_2 \leq \max(f_1 - f_2)\).

12. (Delta function behavior of Poisson kernel) With \(b = 1\) and \(\theta = 1\), say, obtain computer plots of the Poisson kernel \(P(r, \vartheta - \theta)\) versus \(\theta\), from \(\vartheta = -\pi\) to \(\vartheta = +\pi\), for these \(r\) values: \(r = 0, 0.5, 0.8,\) and \(0.9\). Using Maple, for instance, you can use the plot command, which can be accessed by first typing the with(plots); command. NOTE: Surely, if the boundary temperature in (36) is \(f(\theta) = \text{constant} = 1\), then the solution will be \(u(r, \theta) = \text{constant} = 1\) as well. It follows that \(\frac{1}{\pi} \int_{-\pi}^{\pi} P(r, \vartheta - \theta) d\theta = 1\) for all \(0 \leq r < b\). Thus, each of your plots, for different \(r\) values, will have unit area. Further, they become increasingly focused at \(\vartheta = \theta\) as \(r \to b\). Thus, it appears that \(P(r, \vartheta - \theta)\) is a delta sequence at \(\theta\) as \(r \to b\). In fact, the boundary condition requires of (36) that

\[
f(\theta) = \lim_{r \to b} \int_{-\pi}^{\pi} P(r, \vartheta - \theta) f(\theta) d\theta,
\]

which result confirms our suspicion: as \(r \to b\), the Poisson kernel becomes a delta function at \(\vartheta = \theta\), which picks out the value \(f(\theta)\) and satisfies the boundary condition. This result is typical of linear PDE's: The solution due to Dirichlet boundary data \(f\) can be expressed as an integration, over the boundary, of a kernel times the boundary values. As a boundary point is approached from within the domain, the kernel becomes a delta function and picks out the value of \(f\) at that boundary point, thereby satisfying the boundary condition.

13. Consider the Dirichlet problem

\[
\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0
\]

(13.1)
in \(a < r < b\), \(0 < \theta < \alpha\), with boundary conditions

\[
u(r, 0) = u(a, \theta) = u(b, \theta) = 0
\]

(13.2.3)

(a) Seeking \(u(r, \theta) = R(r)\Theta(\theta)\) and anticipating the Fourier expansion along the \(\theta = \alpha\) edge, obtain

\[
u(r, \theta) = (A + B \ln r)(E + F \theta) + G \cos(\kappa \ln r) + H \sin(\kappa \ln r)(G \cos \kappa \theta + H \sin \kappa \theta).
\]

(b) Applying the boundary conditions (13.2), arrive at

\[
u(r, \theta) = \sum_{n=1}^{\infty} I_n \sin(\kappa_n \ln \frac{r}{a}) \sin \kappa_n \theta,
\]

(13.4)

where

\[
\kappa_n = \frac{n\pi}{\ln \frac{b}{a}}.
\]

(13.5)

(c) Applying the boundary condition (13.3), show that

\[
I_n = \frac{1}{\sinh \kappa_n \alpha} \left[ \int_{a}^{b} f(r) \phi_n(r) \frac{1}{r} dr - \int_{a}^{b} \phi_n(r) \frac{1}{r} dr \right] \left( \frac{2}{\sinh \kappa_n \alpha} \right)
\]

\[
= \frac{2}{\sinh \kappa_n \alpha} \int_{a}^{b} f(r) \phi_n(r) \frac{1}{r} dr,
\]

(13.6)

where \(\phi_n(r) = \sinh[\kappa_n \ln(r/a)]\). HINT: The \(\phi_n\)'s are the eigenfunctions of the Sturm–Liouville problem on \(R(r)\). Identify that problem (i.e., the ODE, the \(r\) interval, the boundary conditions on \(R\), and the weight function).

EXERCISES FOR THE OPTIONAL SECTIONS 20.3.2, 20.3.3

14. Derive the \(\kappa \neq 0\) part of the general solution (43) using equation (50) in Section 4.6.

15. Derive the \(\kappa \neq 0\) part of the general solution (57) using equation (50) in Section 4.6.

16. Solve by separation of variables, leaving expansion coefficients in integral form: \(u\) is to be bounded, and

\[
\nabla^2 u = u_{rr} + \frac{1}{r} u_r + u_{zz} = 0
\]
20.4 Fourier Transform (Optional)

In Section 18.4 we studied the solution of diffusion problems by the Fourier and Laplace transforms. Laplace transforming on the $t$ variable is always an option for the diffusion equation because $0 < t < \infty$, and the problem is of initial-value type with respect to $t$. However, it is of boundary-value type with respect to $x$ (i.e., there is a boundary condition at each end) so, alternatively, we can use a Fourier transform on $x$ if the $x$ domain is $-\infty < x < \infty$ or a Fourier cosine or sine transform on $x$ if the domain is $0 < x < \infty$.

In contrast, the Laplace equation is of boundary-value type in both independent variables, so the Laplace transform is not helpful. Still, if the domain is infinite in one of the independent variables then we can employ a Fourier transform on that variable; if it is semi-infinite then we can employ a Fourier cosine or sine transform.

**EXAMPLE 1. Dirichlet Problem for Half Plane.** Consider the half-plane problem

\[
\begin{align*}
\nabla^2 u &= u_{xx} + u_{yy} = 0, & -\infty < x < \infty, \ 0 < y < \infty \\
\quad u(x,0) &= f(x), & -\infty < x < \infty
\end{align*}
\]

\(1a\)  \(1b\)
depicted in Fig. 1, realizing that we may be stipulating additional boundary conditions as we proceed.

Fourier transform (1a) with respect to $x$:

$$F\{u_{xx} + u_{yy}\} = F\{0\},$$  \hspace{1cm} (2a)

$$F\{u_{xx}\} + F\{u_{yy}\} = 0,$$  \hspace{1cm} (2b)

$$(i\omega)^2 \hat{u} + \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2} e^{-i\omega x} \, dx = 0,$$  \hspace{1cm} (2c)

$$-\omega^2 \hat{u} + \int_{-\infty}^{\infty} u(x, y)e^{-i\omega x} \, dx = 0,$$  \hspace{1cm} (2d)

or

$$\frac{d^2 \hat{u}}{dy^2} - \omega^2 \hat{u} = 0,$$  \hspace{1cm} (3)

with general solution

$$\hat{u}(\omega, y) = A e^{i|\omega|y} + B e^{-i|\omega|y}.$$  \hspace{1cm} (4)

(Let us defer discussion of the absolute value signs to Comment 1 below.) Recall from our study of the Fourier transform that for $F\{u_{xx}\}$ to equal $(i\omega)^2 \hat{u}$ we need

$$u \to 0 \text{ and } u_x \to 0 \text{ as } x \to \pm \infty,$$  \hspace{1cm} (5)

so let us suppose that $u$ does satisfy the boundary conditions (5) to the east ($x \to +\infty$) and to the west ($x \to -\infty$). Condition (1b) is our southern boundary condition, but we are still lacking a second $y$ boundary condition, to the north as $y \to \infty$. If we assume that

$$u(x, y) \to 0 \text{ as } y \to \infty,$$  \hspace{1cm} (6)

then we formally obtain

$$\lim_{y \to \infty} \hat{u}(\omega, y) = \lim_{y \to \infty} \int_{-\infty}^{\infty} u(x, y)e^{-i\omega x} \, dx$$

$$= \int_{-\infty}^{\infty} \left[ \lim_{y \to \infty} u(x, y) \right] e^{-i\omega x} \, dx$$

$$= \int_{-\infty}^{\infty} 0 e^{-i\omega x} \, dx = 0.$$  \hspace{1cm} (7)

Applying the latter result to (4) reveals that we need $A = 0$, so

$$\hat{u}(\omega, y) = B e^{-i|\omega|y}.$$  \hspace{1cm} (8)

To evaluate $B$ take the transform of (1b),

$$\left. \hat{u} \right|_{y=0} = \hat{f}(\omega),$$  \hspace{1cm} (9)

and impose that condition on (8):

$$\left. \hat{u} \right|_{y=0} = \hat{f}(\omega) = B.$$  \hspace{1cm} (10)
Thus,
\[ \tilde{u}(\omega, y) = \tilde{f}(\omega)e^{-|\omega|y}. \]

If we use entry 1 in Appendix D, along with the Fourier convolution property (entry 21), we obtain the final result
\[ u(x, y) = f(x) \ast \frac{y}{\pi} \frac{1}{x^2 + y^2}, \tag{11} \]
or
\[ u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x - \xi)^2 + y^2} d\xi \equiv \int_{-\infty}^{\infty} P(\xi - x, y) f(\xi) d\xi; \tag{12} \]
(12) is the Poisson integral formula for the half plane, and \( P(\xi - x, y) \) is the corresponding Poisson kernel. The analogous formula for the circular disk is given in (36) of Section 20.3.

**COMMENT 1.** Why did we express the solution of (3) as \( A e^{\omega y} + B e^{-|\omega|y} \) rather than as \( U(\omega, y) = Ce^{\omega y} + De^{-\omega y} \)?

The forms (4) and (13) are indeed equivalent, but (4) is more convenient for applying the northern boundary condition (7) (namely, that \( \tilde{u} \to 0 \) as \( y \to \infty \)). For remember that the Fourier inversion formula involves an integral on \( \omega \) from \( \omega = -\infty \) to \( \omega = +\infty \). Thus, we need to allow for \( \omega > 0 \) and \( \omega < 0 \) in (13) and conclude, from (7), that \( C(\omega) = 0 \) for \( \omega > 0 \) and \( D(\omega) = 0 \) for \( \omega < 0 \), which story is more complicated than observing, in (4), that \( e^{\omega y} \) is the "bad" term and \( e^{-|\omega|y} \) is the "good" term so that we need to set \( A = 0 \).

**COMMENT 2.** Let us focus our attention on the kernel \( P \). Observe that \( P \) has unit area, for each \( y > 0 \), since
\[ \int_{-\infty}^{\infty} P(\xi - x, y) d\xi = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\xi}{(\xi - x)^2 + y^2} = 1, \tag{14} \]
and that its graph becomes more and more sharply focused at \( \xi = x \) as \( y \to 0 \), as seen in Fig. 2. Thus, \( P(\xi - x, y) \) looks like a delta sequence at \( \xi = x \), as indeed must be true since the boundary condition (1b) really means that \( \lim_{y \to 0} u(x, y) = f(x) \) or, since \( u(x, y) \) is given by (12),
\[ \lim_{y \to 0} \int_{-\infty}^{\infty} P(\xi - x, y) f(\xi) d\xi = f(x), \tag{15} \]
which, by definition, means that \( P \) becomes a delta function at \( x \) as \( y \to 0 \).

**COMMENT 3.** As a check case, let us use (12) for the case where \( u(x, 0) = f(x) = \) constant = \( f_0 \), since then the solution should, by inspection, be \( u(x, y) = f_0 \) everywhere. In fact, (12) does give that correct result — even though the assumed conditions at infinity (\( u \to 0 \) as \( x \to \pm \infty \) and as \( y \to \infty \)) are not satisfied. That is, (12) is even more robust than anticipated.  


Closure. Besides illustrating the use of the Fourier transform in solving the Laplace equation on an infinite domain, we also obtain an important specific solution, the Poisson integral formula (12) for the half plane. As usual (see Exercise 12 of Section 20.3), the solution due to Dirichlet boundary data \( f \) can be expressed as an integration, over the boundary, of a kernel times the boundary values. As a boundary point is approached from within the domain, the kernel becomes a delta function and picks out the value of \( f \) at that point, thereby satisfying the boundary condition.

EXERCISES 20.4

1. (a) Use (12) to evaluate \( u(x,y) \) if \( f(x) = 100H(x) \) (where \( H \) is the Heaviside function).
   (b) Draw the isotherms \( u = 25, 50, 75 \).
   (c) Plot \( u(x,y) \) versus \( x \) at \( y = 0, 1, 3 \).

2. (a) Use (12) to evaluate \( u(x,y) \) if \( f(x) = 100[H(x + 1) - H(x - 1)] \).
   (b) Plot \( u(x,y) \) versus \( x \) at \( y = 0 \) and at \( y = 2 \).
   (c) Show that \( u(x,y) \sim 200/(\pi y) \) as \( y \to \infty \). HINT: Note that
   \[
   \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots
   \]
   for \( |x| < 1 \), and
   \[
   \tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots
   \]
   for \( |x| < 1 \) where, in each case, \( \tan^{-1} \) denotes the choice (of the multivalued \( \tan^{-1} \) function) lying between \(-\pi/2 \) and \(+\pi/2 \).

3. (a) Show, from (12), that if \( f(x) \) is an even function of \( x \) then so is \( u(x,y) \).
   (b) Show, from (12), that if \( f(x) \) is an odd function of \( x \) then so is \( u(x,y) \).

4. Use (12) and the method of images (explained in the optional Section 18.5) to solve \( \nabla^2 u = u_{xx} + u_{yy} = 0 \) in the first quadrant \( x > 0, y > 0 \), with \( u(x,0) = f(x) \), with suitable conditions at \( x = \infty \) and at \( y = \infty \), and with
   
   (a) \( u(0,y) = 0 \)
   (b) \( u_x(0,y) = 0 \)

5. Consider the infinite strip problem
   \[
   \nabla^2 u = u_{xx} + u_{yy} = 0, \quad (|x| < \infty, \ 0 < y < a)
   \]
   \[
   u(x,0) = f(x), \quad u(x,a) = g(x).
   \]
   (a) Show that
   \[
   u(x,y) = F^{-1} \left\{ f(\omega) \frac{\sinh \omega(a - y)}{\sinh \omega a} + \hat{g}(\omega) \frac{\sinh \omega y}{\sinh \omega a} \right\},
   \]
   but do not try to evaluate that Fourier inverse.
   (b) With \( f(x) = g(x) = 100H(x) \), use intuition to sketch the isotherms, say \( u = 10, 25, 50, 75, 90 \).

6. In Comment 2 we discuss the delta function behavior of the Poisson kernel \( P \) as \( y \to 0 \). In fact, letting \( f(x) = \delta(x - x_0) \), show that \( P(x_0 - x, y) = P(x - x_0, y) \) is itself the solution or “response” due to a boundary temperature that is a delta function at \( x_0 \). NOTE: With this result in mind, we can interpret (12) as a superposition principle. For let us break \( f \) into narrow vertical rectangles. What is the response \( du(x,y) \) due to
the single shaded rectangular pulse (shown in the figure). The pulse is a delta function at \( \xi \) (as \( d\xi \to 0 \)), so (as noted above) its response at \( x,y \) is \( P(\xi - x,y) \); actually, the pulse is a delta function scaled by \( f(\xi) \) \( d\xi \) because its area is \( f(\xi) \) \( d\xi \) rather than unity, so its response is likewise scaled,

\[
d u(x,y) = P(x - \xi, y) f(\xi) \, d\xi
\]

or, since \( P \) is an even function of its first argument,

\[
d u(x,y) = P(\xi - x, y) f(\xi) \, d\xi.
\]

And superimposing these infinitesimal responses (by integration) gives (12). Thus, we can understand (12) as a superposition principle.

![Figure 1](image_url)  
**Figure 1.** The problem (11).

![Figure 2](image_url)  
**Figure 2.** Finite-difference mesh for \( M = N = 3 \).

### 20.5 Numerical Solution

#### 20.5.1 Rectangular domains.

Following the same general lines as in Section 18.6, where we develop the finite-difference solution technique for the diffusion equation, here we do the same for the Laplace equation or, more generally and with no additional difficulty, for the Poisson equation \( \nabla^2 u = f \).

Limiting our attention to the two-dimensional case, we begin with the problem

\[
\begin{align*}
\nabla^2 u &= u_{xx} + u_{yy} = f(x,y), & (0 < x < a, \ 0 < y < b) \\
\nu(0, y) &= p(y), \ u(x, 0) = q(x), \ u(a, y) = r(y), \ u(x, b) = s(x)
\end{align*}
\]

depicted in Fig. 1, and generalize to nonrectangular domains in Section 20.5.2. Seeking an approximate numerical solution, we discretize the problem by dividing \( a \) into \( M \) equal parts of dimension \( \Delta x = a/M \), dividing \( b \) into \( N \) equal parts of dimension \( \Delta y = b/N \), and defining nodal points \( P_{jk} = (x_j, y_k) = (j\Delta x, k\Delta y) \) for \( j = 0, 1, 2, \ldots, M \) and \( k = 0, 1, 2, \ldots, N \). Accordingly, we seek \( u \) not everywhere in the domain but only at the nodal points – more specifically, at the interior nodal points since \( u \) is prescribed at the boundary nodal points by the Dirichlet boundary conditions (1b). The grid is shown in Fig. 2 for the choice \( M = N = 3 \).

Next, we replace the PDE (1a) by a finite-difference approximation that will lead to a set of linear algebraic equations in the unknown nodal values of \( u \). As in Section 18.6, we adopt the approximations

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &\approx \frac{u(x - \Delta x, y) - 2u(x, y) + u(x + \Delta x, y)}{(\Delta x)^2}, \\
\frac{\partial^2 u}{\partial y^2} &\approx \frac{u(x, y - \Delta y) - 2u(x, y) + u(x, y + \Delta y)}{(\Delta y)^2}.
\end{align*}
\]

Putting (2a, b) into the PDE, with \( x = x_j, x - \Delta x = x_{j-1}, x + \Delta x = x_{j+1} \) (and similarly for \( y \)) gives
Thus, we adopt the algebraic equation

\[
\frac{U_{j-1,k} - 2U_{j,k} + U_{j+1,k}}{(\Delta x)^2} + \frac{U_{j,k-1} - 2U_{j,k} + U_{j,k+1}}{(\Delta y)^2} = f_{j,k}.
\]

as our finite-difference approximation of (1a), where \(f_{j,k}\) is shorthand for \(f(x_j, y_k)\).

As in Section 18.6, we use different letters (lowercase and uppercase) to distinguish between the exact solution \(u(x, y)\) of (1) and the approximating solution \(U_{j,k}\) generated by the finite-difference equation (4). We call \(u(x_j, y_k) - U_{j,k}\) the truncation error at \(P_{j,k}\), namely, the error incurred by replacing \(u_{xx}\) and \(u_{yy}\) in (1a) by the finite-difference approximations (2). Observe that whereas in Section 18.6 we distinguish between the local truncation error (incurred in carrying out a single time step) and the accumulated truncation error (incurred in carrying out all the time steps up until the time in question) – here we do not – because there are no time steps. Thus, there is simply “the truncation error.”

Suppose that we compute \(U_{j,k}\) at a particular point \(P\) in the domain, then again using a finer mesh, again using a finer mesh, and so on. If, as the mesh becomes infinitely fine (i.e., as \(\Delta x\) and \(\Delta y\) both tend to zero) the computed values converge to the exact solution at \(P\), then the finite-difference scheme is convergent. Recall that in our study of the diffusion equation (Section 18.6) we pay comparable attention to the companion questions of convergence and stability; the difference scheme is said to be stable if the accumulated roundoff error remained small. For the Poisson and Laplace equations, however, we do not “march out” a solution in time, so the issue of stability is not relevant. In fact, roundoff error should be quite negligible compared with the truncation error for the methods considered in this section. If we choose \(\Delta x = \Delta y = h\), say, then (4) becomes

\[
\begin{bmatrix}
U_{j-1,k} + U_{j,k-1} + U_{j+1,k} + U_{j,k+1} - 4U_{j,k} = h^2f_{j,k}.
\end{bmatrix}
\]

which is often expressed, schematically, in the form

\[
\begin{bmatrix}
1 & -4 & 1 \\
1 & -4 & 1 \\
1 & -4 & 1
\end{bmatrix} U = h^2f.
\]

If, in addition, \(f(x, y) = 0\), so that (1a) reduces to the Laplace equation, then (5) gives

\[
U_{j,k} = \frac{1}{4} (U_{j-1,k} + U_{j,k-1} + U_{j+1,k} + U_{j,k+1}).
\]
We call these **five-point formulas** because they involve five grid points, which we denote as $P$, $W$(east), $S$(outh), $E$(ast), and $N$(orth), respectively, in Fig. 3. It is striking that (6) is a discrete and approximate version of the average value property of the two-dimensional Laplace equation, namely, that $u$ at any given point $P$ in the solution domain is the average value of $u$ over any circle centered at $P$ and lying entirely within the domain.

**EXAMPLE 1.** To illustrate the use of (5), let $a = b = 1$ and $f(x, y) = p(y) = q(x) = s(x) = 0$, and $r(y) = 100 \sin \pi y$ so that, by separation of variables, we have the simple exact solution

$$u(x, y) = 100 \frac{\sinh \pi x}{\sinh \pi} \sin \pi y$$

available for comparison with our computed approximate solution. As the simplest (and crudest) case, let $M = N = 2$. Then $\Delta x = \Delta y = 0.5$ and there is only one internal node, (Fig. 4). Writing out (5) for that point (i.e., with $j = k = 1$) gives

$$P_{11} : \quad U_{01} + U_{10} + U_{21} + U_{12} - 4U_{11} = 0$$

or, recalling the given boundary conditions, $0 + 0 + 100 \sin (\pi/2) - 4U_{11} = 0$. Solving, $U_{11} = 25$ and we have the following comparison.

**Computed:** \quad $U_{11} = 25$ \quad (9a)

**Exact:** \quad $u_{11} = 100 \frac{\sinh (\pi/2)}{\sinh \pi} \sin \frac{\pi}{2} = 19.9$, \quad (9b)

where $u_{jk}$ means the exact solution $u(x, y)$ evaluated at $P_{jk}$; note that we will generally omit the comma between the two subscripted indices, for brevity. It is not surprising that the error is so great because the grid is so coarse; that is, $h = 0.5$ is not small compared to $a = b = 1$.

Next, let $M = N = 4$ (right-hand member of Fig. 4). Write out (5) for the nine internal grid points $P_{11}, P_{21}, \ldots, P_{33}$:

![Figure 4. Example 1.](image)
\[ P_{11} = 0 + 0 + U_{21} + U_{12} - 4U_{11} = 0, \]
\[ P_{21} = U_{11} + 0 + U_{31} + U_{22} - 4U_{21} = 0, \]
\[ P_{22} = U_{13} + U_{22} + U_{33} + 0 - 4U_{23} = 0, \]
\[ P_{33} = U_{23} + U_{32} + 100\sin \frac{3\pi}{4} + 0 - 4U_{33} = 0. \]

or, in matrix form,

\[
\begin{bmatrix}
-4 & 1 & 0 \\
1 & -4 & 1 \\
0 & 1 & -4 \\
\end{bmatrix}
\begin{bmatrix}
U_{11} \\
U_{21} \\
U_{31} \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
-100\sin \frac{3\pi}{4} \\
\end{bmatrix},
\]

where matrix elements not shown are zeros and the partitioning lines are to be ignored for the moment. Roughly speaking, the first equation in (10) ensures the satisfaction of the Laplace equation in the neighborhood of \( P_{11} \), the second equation in (10) ensures the satisfaction of the Laplace equation in the neighborhood of \( P_{21} \), and so on, so that the satisfaction of (10) is equivalent to the approximate satisfaction of the Laplace equation in the entire domain (as well as the Dirichlet boundary conditions).

Solving (11) by a computer algebra system (e.g., using the Maple `linsolve` command described at the end of Section 8.3), we obtain these values.

**Computed:**
\[ U_{11} = 5.8, \quad U_{21} = 15.1, \quad U_{31} = 33.2, \]
\[ U_{12} = 8.3, \quad U_{22} = 21.3, \quad U_{32} = 46.9, \]
\[ U_{13} = 5.8, \quad U_{23} = 15.1, \quad U_{33} = 33.2. \]

**Exact:**
\[ u_{11} = 5.3, \quad u_{21} = 14.1, \quad u_{31} = 32.0, \]
\[ u_{12} = 7.5, \quad u_{22} = 19.9, \quad u_{32} = 45.3, \]
\[ u_{13} = 5.3, \quad u_{23} = 14.1, \quad u_{33} = 32.0. \]

These results are seen to be in better agreement than (9a) and (9b) but are still quite crude. If this were a realistic application, rather than only an illustration of the method, we might choose \( h \) to be 0.05 or smaller.

**COMMENT 1.** Observe from Fig. 5 that both the domain and the boundary conditions are symmetric about the mid-line \( y = 0.5 \), so it is evident that the solution \( u(x,y) \) should,
likewise, be symmetric about that line.* Thus, our numerical solution, for the case where \( M = N = 4 \), is wasteful because we know in advance that

\[
U_{13} = U_{11}, \quad U_{23} = U_{21}, \quad U_{33} = U_{31},
\]

(13)
so there are really only six unknowns rather than nine. It suffices to apply (5) to the six points \( P_{11}, P_{21}, P_{31}, P_{12}, P_{22}, P_{32} \), and to use (13). Thus, the reduced system is as follows:

\[
\begin{align*}
P_{11} : & \quad 0 + 0 + U_{12} + U_{11} - 4U_{11} = 0, \\
P_{21} : & \quad U_{11} + 0 + U_{22} + U_{21} - 4U_{21} = 0, \\
P_{31} : & \quad U_{21} + 0 + 100\sin \frac{\pi}{2} + U_{32} - 4U_{31} = 0, \\
P_{12} : & \quad 0 + U_{11} + U_{22} + U_{11} - 4U_{12} = 0, \\
P_{22} : & \quad U_{12} + U_{21} + U_{22} + U_{21} - 4U_{22} = 0, \\
P_{32} : & \quad U_{22} + U_{31} + 100\sin \frac{\pi}{2} + U_{32} - 4U_{32} = 0,
\end{align*}
\]

(14)
where the underlined terms are those that result from the symmetry relations (13).

**COMMENT 2.** Observe that (11) may be partitioned, according to the thin lines in (11), as

\[
\begin{bmatrix}
\mathbf{B} & 
\mathbf{I} & 
\cdots & 
0 \\
\mathbf{I} & 
\mathbf{B} & 
\mathbf{I} & 
\vdots \\
\vdots & 
\mathbf{I} & 
\mathbf{B} & 
\mathbf{I} \\
0 & 
\cdots & 
\mathbf{I} & 
\mathbf{B}
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
\vdots \\
U_{N-1}
\end{bmatrix}
= 
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_{N-1}
\end{bmatrix},
\]

(15)
where \( \mathbf{B} \) is the \((N - 1) \times (N - 1)\) matrix

\[
\mathbf{B} =
\begin{bmatrix}
-4 & 1 & \cdots & 0 \\
1 & -4 & 1 & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & -4
\end{bmatrix},
\]

(16)
and \( \mathbf{I} \) is an identity matrix of order \( N - 1 \). In (11), for instance, \( N = 4 \) so \( \mathbf{B} \) and \( \mathbf{I} \) are \( 3 \times 3 \). Each vector is comprised of the unknown nodal values across the \( j \)th row of the mesh. Whereas \( \mathbf{B} \) is tridiagonal, \( \mathbf{A} \) is not tridiagonal; it is **block** tridiagonal.

The method is powerful because it enables us to obtain solutions even if the inputs \( p(y), q(x), r(y), s(x), \) and \( f(x, y) \) are nonconstant functions, in which case

*It surely seems clear, if only intuitively, that the solution is symmetric about the line \( y = 0.5 \) as claimed, but to put that claim on solid ground we can put forward arguments similar to those given in Section 18.5 on the method of images. We will leave that point for the exercises.
analytical solution becomes extremely laborious. However, it is to be appreciated that the computer calculation is not trivial if we seek good accuracy because (in the absence of helpful symmetries) we need to solve a system of \((M - 1) \times (N - 1)\) linear algebraic equations. If, for instance, we choose \(M = N = 50\), for the sake of accuracy, then we have a system of 2,401 equations! In such cases one concern is how to solve that large system of equations efficiently, to which topic we return in Section 20.5.3. First, in Section 20.5.2, we indicate how to extend the method to handle domains of essentially arbitrary shape.

### 20.5.2. Nonrectangular domains

Thus far we have dealt with cases where the boundary curve is rectangular so that grid lines can coincide with the edges of the domain as in Fig. 4, that is, where the mesh “fits” the domain. What happens if the mesh does not fit, as illustrated in Fig. 6? We cannot apply the finite-difference scheme (5) at grid points such as \(P\) because the points \(N\) and \(E\) do not fall on the boundary curve \(C\); they fall outside the domain. To handle this case we slide \(N\) and \(E\) so that they do fall on \(C\), as shown in Fig. 7, and revise the difference quotient approximations (2a,b) accordingly. [We need to modify (2a), for instance, because it gives \(u_{xx}\) at \(P\) as a linear combination of the values of \(u\) at \(W\), \(P\), and \(E\). If we move \(E\), then the weighting of those values will change; we can expect \(U_E\) to be weighted more heavily than \(U_W\) because \(E\) is closer to \(P\) than \(W\). Similarly for (2b).] For the geometry shown in Fig. 7 we need to slide \(N\) and \(E\), but in other cases we may need to slide \(W\) and/or \(S\) as well, so let us consider the most general case shown in Fig. 8, where \(0 < \alpha \leq 1, 0 < \beta \leq 1, 0 < \gamma \leq 1,\) and \(0 < \delta \leq 1\).

We begin with Taylor expansions about \(P\) (namely, the point \(x_j, y_k\)) in the eastern and western directions, respectively,

\[
\begin{align*}
    u(x_j + \alpha h, y_k) &= u(x_j, y_k) + u_x(x_j, y_k)\alpha h + \frac{1}{2!} u_{xx}(x_j, y_k)(\alpha h)^2 + \cdots, \\
    u(x_j - \gamma h, y_k) &= u(x_j, y_k) + u_x(x_j, y_k)(-\gamma h) + \frac{1}{2!} u_{xx}(x_j, y_k)(-\delta h)^2 + \cdots.
\end{align*}
\]

or, using \(N, E, S, W, P\) subscript notation instead,

\[
\begin{align*}
    u_E &= u_P + u_x|_P \alpha h + \frac{1}{2} u_{xx}|_P \alpha^2 h^2 + \cdots, \\
    u_W &= u_P - u_x|_P \gamma h + \frac{1}{2} u_{xx}|_P \gamma^2 h^2 - \cdots. \\
\end{align*}
\]  (17a, 17b)

Multiplying (17a) by \(\gamma\) and (17b) by \(\alpha\) and adding, to cancel the \(u_x\) terms, gives

\[
\gamma u_E + \alpha u_W = (\gamma + \alpha) u_P + \frac{1}{2} (\alpha^2 \gamma + \alpha \gamma^2) h^2 u_{xx}|_P + \cdots. \tag{18}
\]

so that if we neglect terms of order \(h^3\) and higher in (18) then we obtain

\[
u_{xx}|_P \approx \frac{2}{\alpha \gamma (\alpha + \gamma)} h^2 [\gamma u_E + \alpha u_W - (\gamma + \alpha) u_P] \tag{19a}
\]
in place of (2a). Similarly, Taylor expansions about $P$ in the northern and southern directions give the result

$$ u_{yy}|_P \approx \frac{2}{\beta \delta (\beta + \delta) h^2} \left[ \delta u_N + \beta u_S - (\delta + \beta) u_P \right] $$

(19b)

in place of (2b). Using (19a,b), our finite-difference approximation to the Poisson equation $u_{xx} + u_{yy} = f(x,y)$, at $P$, becomes

$$ \frac{2}{\gamma(\gamma + \alpha)} U_W + \frac{2}{\delta(\delta + \beta)} U_S + \frac{2}{\alpha(\alpha + \gamma)} U_E + \frac{2}{\beta(\beta + \delta)} U_N - 2 \frac{\alpha \gamma + \beta \delta}{\alpha \beta \gamma \delta} U_P = h^2 f_P. $$

(20)

Naturally, if $\alpha = \beta = \gamma = \delta = 1$, then (20) reduces to (5).

**EXAMPLE 2.** To illustrate the use of (20), let us solve the Poisson problem shown in Fig. 9a, using the grid shown in Fig. 9b. Obviously, the grid is quite coarse, but it should suffice for the purpose of illustration. In this case the source term is $f(x,y) = 5x - y$ and we have chosen $h = 2$. Since there are not many grid points it will be more convenient to denote the points as $a, b, \ldots, r, s$ rather than with the double subscript notation. We need to write (20) for each of the internal grid points $o, p, q, r, s$. Thus, we need to determine $\alpha, \beta, \gamma, \delta$ for each of these points. At $r$, for instance, we compute $\alpha$ from $\overline{rg} \equiv \alpha h$ (where $\overline{rg}$ denotes the length of the line $r g$). Thus, $\alpha = \overline{rg}/h = (x_r - x_r)/h = (5.5 - 4)/2 = 0.75$. Next, $\overline{rg} \equiv \beta h$ gives $\beta = \overline{rg}/h = 0.75/1 = 1$, $\overline{rk} \equiv \gamma h$ gives $\gamma = (x_r - x_k)/h = (4 - \sqrt{4^2 - 2^2})/2 \approx 0.27$, and $\overline{rg} \equiv \delta h$ gives $\delta = 1$. Similarly at $o, p, \text{ and } s$, so we have these values:

(a)

(b)

Figure 9. Example 2.
Thus, writing (20) at these points gives the equations

\begin{align}
o & : \quad U_o + U_e + U_p + U_q - 4U_o = 2^2[5(2) - 6], \\
p & : \quad \frac{2}{1.25}U_o + U_q + \frac{2}{0.25(1.25)}U_d + U_e - \frac{2}{0.25}U_p \\
 & = 2^2[5(4) - 6], \\
q & : \quad \frac{2}{1.5}U_s + U_r + \frac{2}{0.5(1.5)}U_e + U_p - \frac{2}{0.5}U_q \\
 & = 2^2[5(4) - 4], \\
r & : \quad \frac{2}{0.27(1.02)}U_k + U_j + \frac{2}{0.75(1.02)}U_g + U_q - \frac{2}{0.20}U_r \\
 & = 2^2[5(4) - 2], \\
s & : \quad U_m + \frac{2}{0.27(1.27)}U_l + U_q + \frac{2}{1.27}U_o - \frac{2}{0.27}U_s \\
 & = 2^2[5(2) - 4].
\end{align}

With \( U_o = 0 \), \( U_b = 20 \), \( U_e = U_d = U_e = U_q = 40 \), \( U_f = (60 + 30)/2 = 45 \), \( U_k = U_i = 30 \), and \( U_k = U_i = 30 \), and \( U_m = (30 + 0)/2 = 15 \) from the boundary conditions (where we've used average values at the corners \( j \) and \( m \), as suggested in Section 18.6), (21) becomes

\begin{align}
-4U_o + U_p + U_s & = -4, \\
1.6U_o - 10U_p + U_q & = -240, \\
U_p - 6U_q + U_r + 1.33U_s & = -42.7, \\
U_q - 12U_r & = -295.5, \\
1.57U_o + U_q - 9.41U_s & = -166.
\end{align}

with the solution

\( U_o = 13.6, \quad U_p = 28.3, \quad U_q = 21.1, \quad U_r = 26.4, \quad U_s = 22.2 \). (23)

COMMENT. If these values don’t look correct, relative to the given boundary values, don’t forget that besides the boundary condition inputs there is also the internal source distribution \( f(x, y) = 5x - y \). Mathematically, we call the forcing term \( f \) in \( \nabla^2 u = f \) a “source” term. However, if we think of this problem in terms of steady-state heat conduction, then we need to recall, from (39) in Section 16.8, that the heat source term there has a minus sign in front of it. Therefore, the \( f(x, y) = 5x - y \) term in our PDE is, in physical terms,
a heat sink, and it is the presence of that heat sink distribution that causes the interior temperatures to be lower than we would expect if we considered only the boundary conditions. For instance, if there were no sink distribution \((f = 0)\) then we would expect \(U_p\) to be around 35, rather than the value 28.3 given in (23).

### 20.5.3. Iterative algorithms. (Optional)
The resulting systems of linear algebraic equations on the \(U_{jk}\)'s are of the form \(AU = c\), where \(A\), typically, is quite large. To illustrate, suppose that the domain is square (as in Example 1) and that \(j = 0, 1, 2, \ldots, N\) and \(k = 0, 1, 2, \ldots, N\). Then there are \((N - 1)^2\) unknown \(U_{jk}\)'s and \(A\) is \((N - 1)^2 \times (N - 1)^2\). For instance, in Example 1, \(N = 4\) so \(A\) is \(9 \times 9\). If, for greater accuracy, we choose \(N = 25\), say, then \(A\) is \(576 \times 576\). Thus, it is important to develop efficient methods of solution of the typically large systems of linear algebraic equations that arise.

Recall that a similar difficulty arose in Section 18.6, where the choice of an implicit (rather than explicit) scheme led to coupled (tridiagonal) systems of linear algebraic equations. However, that situation was not nearly as difficult since the problem was of initial-value type and we merely needed to solve for one time-line of unknowns at a time. Thus, with \(N = 25\), say, the \(A\) matrix was only \(24 \times 24\) rather than \(576 \times 576\). It's true that we need to solve these 24th-order systems for each time step, but (supposing that there are 24 time steps, say) it is much easier to solve twenty four 24th-order systems than to solve one 576th-order system.

Fortunately, the \(A\) matrix is strongly diagonal, so that we can use the same iterative techniques that were described in Section 18.6. For instance, suppose that the rectangular grid fits the domain so that we can use the scheme (5) rather than its generalization (20). Because the \(-4\) coefficient of \(U_{jk}\) dominates the other coefficients on the left-hand side of (5) we can, to a first approximation, write \(-4U_{jk} \approx h^2 f_{jk}\). Solving the latter for the \(U_{jk}\)'s and calling them \(U_{jk}^{(0)}\) gives

\[
U_{jk}^{(0)} = -\frac{h^2}{4} f_{jk}. \tag{24}
\]

Next, we put those values into the thus-far-neglected first four terms on the left-hand side of (5), transpose them to the right, and obtain the improved values

\[
U_{jk}^{(1)} = \frac{1}{4} \left( U_{j-1,k}^{(0)} + U_{j+1,k}^{(0)} + U_{j,k-1}^{(0)} + U_{j,k+1}^{(0)} - h^2 f_{jk} \right).
\]

Repeating this process gives the Jacobi iterative scheme

\[
U_{jk}^{(n+1)} = \frac{1}{4} \left( U_{j-1,k}^{(n)} + U_{j,k-1}^{(n)} + U_{j+1,k}^{(n)} + U_{j,k+1}^{(n)} - h^2 f_{jk} \right) \tag{25}
\]

*If the truth behind this claim is not obvious to you, try changing the numbers: wouldn't you rather solve twenty 2nd-order systems than one 40th-order system?*
for $n = 0, 1, 2, \ldots$, with the “starting values” given by (24). As discussed in Section 18.6 for the diffusion version of (25), we can improve on (25) by using the latest iterates as soon as they become available and moving systematically across the first row (left to right), then the second, and so on. Known as the Gauss–Seidel or Liebmann method, it is expressed as

$$U_{j,k}^{(n+1)} = \frac{1}{4} \left( U_{j-1,k}^{(n+1)} + U_{j,k-1}^{(n+1)} + U_{j+1,k}^{(n)} + U_{j,k+1}^{(n)} - h^2 f_{j,k} \right).$$

(26)

The latter converges more rapidly than the Jacobi method and is more readily programmed.

Re-expressing (26) as

$$U_{j,k}^{(n+1)} = \frac{1}{4} \left( U_{j-1,k}^{(n+1)} + U_{j,k-1}^{(n+1)} + U_{j+1,k}^{(n)} + U_{j,k+1}^{(n)} + U_{j,k}^{(n)} - h^2 f_{j,k} \right)
\equiv U_{j,k}^{(n)} + \Delta U_{j,k}^{(n)},$$

(27)

we can insert a numerical control parameter $\omega$ as follows,

$$U_{j,k}^{(n+1)} = U_{j,k}^{(n)} + \omega \Delta U_{j,k}^{(n)},$$

(28)

and choose $\omega$ so as to speed the convergence. It has been shown that the optimal value of $\omega$ is

$$\omega_{\text{opt}} = 2 \frac{1 - \sin \left( \frac{\pi}{N} \right)}{\cos^2 \left( \frac{\pi}{N} \right)}.$$  

(29)

For large $N$, $\omega_{\text{opt}} \approx 2$. Since $\omega > 1$ amounts to an overcorrection, in (28), the method (28) is called successive overrelaxation, or SOR, for brevity. We will omit a numerical illustration of these methods because the ideas are the same as for the diffusion equation; see Example 3 of Section 18.6.

One final point. We stated that the $A$ matrix is strongly diagonal, but that situation will be obtained only if we write the scalar equations on the $U_{j,k}$’s in the correct sequence, which we clarify by means of the following example.

**EXAMPLE 3.** Applying (5) to the Laplace problem shown in Fig. 10 gives the scalar equations

\begin{align*}
a : & \quad -4U_a + U_b = -100, \quad (30a) \\
b : & \quad U_a - 4U_b + U_c = -50, \quad (30b) \\
c : & \quad U_b - 4U_c + U_d = -30, \quad (30c) \\
d : & \quad U_c - 4U_d = -90 \quad (30d)
\end{align*}

\[1\] Or, equivalently and more easily programmed, we can take $U_{j,k}^{(0)} = 0$. Then (25) gives $U_{j,k}^{(1)} = -((h^2)/4)f_{j,k}$, which is identical to the right-hand side of (24), so the subsequent iterates are the same as before.

or, in matrix form,

\[
\begin{bmatrix}
-4 & 1 & 0 & 0 \\
1 & -4 & 1 & 0 \\
0 & 1 & -4 & 1 \\
0 & 0 & 1 & -4
\end{bmatrix}
\begin{bmatrix}
U_a \\
U_b \\
U_c \\
U_d
\end{bmatrix}
= 
\begin{bmatrix}
-100 \\
-50 \\
-30 \\
-90
\end{bmatrix}
\] (31)

However, if we interchange the scalar equations (30a) and (30c), say, then we obtain, in place of (31), the equivalent system

\[
\begin{bmatrix}
0 & 1 & -4 & 1 \\
1 & -4 & 1 & 0 \\
-4 & 1 & 0 & 0 \\
0 & 0 & 1 & -4
\end{bmatrix}
\begin{bmatrix}
U_a \\
U_b \\
U_c \\
U_d
\end{bmatrix}
= 
\begin{bmatrix}
-30 \\
-50 \\
-100 \\
-90
\end{bmatrix}
\] (32)

Thus, whereas the matrix in (31) is strongly diagonal, the one in (32) is not.

To obtain the strongly diagonal form, in any given example, proceed as follows. Order the elements of the U vector (in \(A U = c\)) any way you like, but then be sure to write the scalar equations in the same order, as we did in (30) and (31). \(\blacksquare\)

**Closure.** We derive the finite-difference schemes (4) and (if \(\Delta x = \Delta y = h\)) (5), and the generalization (20) for curvilinear boundaries. We illustrate their implementation in Examples 1 and 2, respectively, using coarse grids for simplicity. Efficient iterative solution techniques, that are needed for fine grids (i.e., for large A matrices), are described in the optional Section 20.5.3. Dirichlet boundary conditions are the simplest and are used throughout, although other boundary conditions are considered in the exercises.
EXERCISES 20.5

1. Consider the Poisson problems $u_{xx} + u_{yy} = f(x, y)$ labeled A, B, C, D in the accompanying figures, each with Dirichlet boundary conditions. Write the linear algebraic equations governing the unknown nodal values and solve for those values, either by hand or by computer.

(a) Problem A with $u_1(y) = 10y$, $u_2 = u_3 = 20$, $f(x, y) = -20xy$.
(b) Problem A with $u_1 = u_2 = 5$, $u_3 = 50$, $f(x, y) = 0$.
(c) Problem A with $u_1 = u_2 = u_3 = 0$, $f(x, y) = -100$.
(d) Problem B with $u_1 = u_2 = u_3 = u_4 = 0$, $f(x, y) = -50$.
(e) Problem B with $u_2 = 100$, $u_1 = u_3 = u_4 = f(x, y) = 0$.
(f) Problem B with $u_1 = u_2 = u_3 = 0$, $u_4 = 200$, $f(x, y) = x^2 + y^2$.
(g) Problem C with $u_1 = u_2 = u_3 = u_4 = 0$, $u_5 = 50$, $f(x, y) = -20$.
(h) Problem C with $u_1 = u_2 = u_3 = u_4 = u_5 = 100$, $f(x, y) = 30$.
(i) Problem C with $u_1 = u_3 = u_4 = u_5 = 0$, $u_2(x) = 10x$, $f(x, y) = 50(x^2 + y^2)$.
(j) Problem D with $u_1 = u_3 = f(x, y) = 0$, $u_2 = u_4 = u_5 = 100$. HINT: Note the symmetry about $y = 4$.
(k) Problem D with $u_1 = u_3 = u_4 = f(x, y) = 0$, $u_2 = u_5 = 50$, $u_2 = -50$. HINT: Note the antisymmetry about $y = 4$.

2. Consider the Poisson equation $u_{xx} + u_{yy} = f(x, y)$ on the domain between two nested squares, one with corners at $(1, 1), (-1, -1), (1, -1)$, and the other with corners at $(0.6, 0.6), (-0.6, 0.6), (-0.6, -0.6), (0.6, -0.6)$. Let $\Delta x = \Delta y = h = 0.2$. Given $f(x, y)$ and the boundary conditions, use (5) to solve for $u$ at each of the 32 nodal points using any symmetries or antisymmetries that are present to re-
duce the number of unknowns.

(a) \( f(x, y) = -100, \ u = 0 \) on inner and outer boundaries
(b) \( f(x, y) = 0, \ u = 0 \) on inner boundary, \( u = 100 \) on outer boundary
(c) \( f(x, y) = 0, \ u(0.6, y) = u(1, y) = u(x, 0.6) = u(x, 1) = 50, \ u(-0.6, y) = u(-1, y) = u(x, -0.6) = u(x, -1) = -50 \)
(d) \( f(x, y) = 0, u(-1, y) = u(-0.6, y) = u(0.6, y) = u(1, y) = 0, u(x, -1) = u(x, -0.6) = -20, u(x, 0.6) = u(x, 1) = 20 \)

3. We developed equation (20), based on the pattern in Fig. 8, to enable us to handle the domains with irregular boundaries. However, we can also use (20) [or, equivalently, (4) with \( \Delta \xi \neq \Delta \eta \)] to increase the nodal point density in regions where greater resolution is needed, as indicated in the figure below. Solve for \( U_8, U_9, \ldots, U_i \) using any symmetries or antisymmetries that are present to reduce the number of unknowns. HINT: At \( a \) use \( h = 0.5 \), at \( b \) use \( h = 1 \) and so on.

![Diagram](image)

(a) \( f(x, y) = -10, \ u_1 = \cdots = u_8 = 0 \)
(b) \( f(x, y) = 0, \ u_1 = u_2 = u_4 = u_5 = u_6 = u_8 = 0, \ u_3 = u_7 = 50 \)
(c) Same as (b), but with \( u_7 \) changed to \(-50\).

4. First, read Exercise 3. For the Laplace equation on the domain shown, with the Dirichlet boundary conditions \( u(0, y) = 100, u(x, 0) = u(x, 1) = u(4, y) = 0 \), \( u(x, y) \) will vary rapidly only near the left end. Thus, let us bunch the nodal points as shown.

(a) Solve for \( u \) at the 21 nodal points using the finite-difference method. HINT: Use symmetry to reduce the number of unknowns to 12.
(b) Compare your computed values at the six nodal points on the horizontal center line \( y = 0.5 \) with the exact values, obtained by separation of variables.

5. Show that (20) agrees with (4), as it should, if we set \( \alpha h = \gamma \Delta \xi \) and \( \beta h = \delta \Delta \eta \).

6. (Empirical estimate of the order of the method) (a) Show that the truncation error in (3) is \( O(h^2) \) so the method is of second order.
(b) To test the assertion in part (a), note that if the method is of order \( p \) that means that

\[
u(x_j, y_k) - U_{j,k} \sim Ch^p\]

as \( h \rightarrow 0 \), where \( x_j, y_k \) is any fixed field point within the domain, \( u(x_j, y_k) \) is the exact solution there, \( U_{j,k} \) is the computed solution there, and \( C \) is some constant. It suffices to use a concrete example. For that purpose, use the point \( x = y = 0.5 \) in Example 1. We found that \( (0.5, 0.5) = 19.9, \) with \( h = 0.5 \) we obtained \( U_{1.1} = 25 \) there, and with \( h = 0.25 \) we obtained \( U_{2.2} = 21.3 \) there. Writing (6.1) for each of these two cases gives two equations in the unknowns \( C \) and \( p \). Solving for \( p \), show that \( p \approx 1.87 \). NOTE: In (6.1) \( U_{j,k} \) denotes the value obtained using a perfect computer, one with no roundoff error, and that is virtually the case since the roundoff error should be extremely small compared to the truncation error.

(c) In obtaining \( p \approx 1.87 \), in part (b), we treated (6.1) as an equation (i.e., with an equal sign), whereas it is only true as \( h \rightarrow 0 \). Thus, we can expect a better empirical estimate of \( p \) if we use two successive small values of \( h \) such as 1/4 and 1/6, rather than 1/2 and 1/4. Here, we ask you to compute \( U \) at \( x = 0.5, \ y = 0.5 \) using \( h = 1/6 \) (in which case there will be 25 equations in 25 unknowns, which can be reduced to 15 equations in 15 unknowns by using symmetry) and to use the \( h = 1/4 \) and \( h = 1/6 \) results to obtain a more accurate estimate of \( p \). NOTE: Remember that when we say \( h \) is small we mean small relative to the size of the domain. In this case the domain is a unit square so the values \( h = 1/4 \) and \( h = 1/6 \) are not especially small. If, instead, the domain were of dimension 50 \times 50, then these values would be quite small.

7. (Neumann and Robin boundary conditions) First, recall the forward, backward and central difference quotients.

\[
f'(x) \approx \frac{f(x+h) - f(x)}{h}, \quad \text{(forward)}
\]

\[
f'(x) \approx \frac{f(x) - f(x-h)}{h}, \quad \text{(backward)}
\]
Thus figure (7.1, 2, 3) shows the difference quotients (backward)

\[ f'(x) \approx \frac{f(x) - f(x - h)}{h} \]

and (central)

\[ f'(x) \approx \frac{f(x + h) - f(x - h)}{2h} \]

with truncation errors that are \( O(h) \), \( O(h) \), and \( O(h^2) \), respectively. Thus far in this section we have considered boundary conditions only of Dirichlet type. To see how to handle a Neumann condition consider the representative problem shown in the figure.

(a) Writing out equation (5) at \( a, b, c, g \) gives four equations in six unknowns \( U_a, U_b, U_c, U_e, U_g \), and \( U_i \). To apply the Neumann condition \( \frac{\partial u}{\partial n} = \frac{\partial u}{\partial x} \) at \( e \) and \( i \), use the backward difference quotient (7.2), show that the resulting system is

\[
\begin{bmatrix}
-4 & 1 & 0 & 1 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & -1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
U_a \\
U_b \\
U_c \\
U_e \\
U_g \\
U_i \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
2 \\
0 \\
0 \\
1 \\
\end{bmatrix},
\] (7.4)

and solve for \( U_a, \ldots, U_i \).

(b) However, whereas the first, second, fourth, and fifth scalar equations in (7.4) [namely, those resulting from the application of (5)] have a truncation error that is \( O(h^2) \) the third and sixth [those resulting from the application of (7.2) to the Neumann boundary condition] have a truncation error that is \( O(h) \). Just as "the chain is as weak as the weakest link," these \( O(h) \) errors contaminate the whole system (7.4) and cause an overall truncation error that is \( O(h) \). To prevent this reduction in accuracy we can use the central difference quotient (7.3) instead of the backward difference quotient (7.2). The idea is to extend the domain as indicated by the dashed lines. Applying (5) at \( a, b, c, e, g, i \) gives six equations in the eight unknowns. To apply the Neumann condition at \( c \) and \( i \) use (7.3), thereby obtaining two more equations. The result is eight linear algebraic equations on \( U_a, \ldots, U_j \). Obtain that system and solve it for \( U_a, \ldots, U_j \). NOTE: Of course, in the end you can discard the auxiliary values \( U_d \) and \( U_y \).

(c) Solve the problem exactly, by separation of variables, and compare your values of \( U_c \), say, from parts (a) and (b), with the exact values at \( a, b, c, \) and \( g \).

(d) With the Neumann boundary condition \( u_x(1, y) = 9y \) changed to the Robin boundary condition

\[ u_y(1, y) + 3u(1, y) = 9y, \]

modify (7.4) accordingly.

8. (Other elliptic PDE's) The Laplace and Poisson equations are elliptic, and the methods developed in this section are applicable to other elliptic PDE's as well. In each case, first verify that the PDE is elliptic. Then derive a finite-difference scheme (with \( \Delta x = \Delta y = h \)) analogous to (5), using (2a), (2b), and central difference quotients (see Exercise 7) for first-order derivatives. Then, apply the finite-difference scheme at each interior node for the case where the domain is \( 0 < x < 1, 0 < y < 1 \) with \( u(x, 0) = 0, u(0, y) = 0, u(x, 1) = 50, u(1, y) = 10y, \) and \( h = 1/3 \). (You need not solve the resulting system of equations.)

(a) \((1 + x^2)u_{xx} + u_{yy} = 0\)

(b) \(u_{xx} + 2u_{yy} - u_x = 0\)

(c) \(u_{xx} + u_{yy} - u = 20x - 5y\)

(d) \(u_{xx} + (1 + x^2 + y^2)u_{yy} + 2u_y = 15(x^2 + y^2)\)