Appendix A

Review of Partial Fraction Expansions

Generally, one meets the method of partial fraction expansions in the integral calculus, where the method is used to express a difficult integral as a linear combination of simpler ones. For example,

$$\begin{aligned}\int \frac{dx}{x^2+3x+2} &= \int \left(\frac{1}{2}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x+3}\right) dx \\ &= \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x+3} \\ &= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + \text{constant}.\end{aligned}$$

In this text we use the method primarily to help us to invert Laplace and Fourier transforms such as the Laplace transform $F(s) = 1/(s^2+3s+2)$. For convenient reference, this appendix contains a review of the method.

Let $p(x)$ and $q(x)$ be finite-degree polynomials in x, of degree P and Q, respectively. Then

$$f(x) = \frac{p(x)}{q(x)} \tag{A1}$$

is called a **rational function** of x. Let P be less than Q. [If $P \geq Q$, then we can, by the long division of q into p, express f as a polynomial of degree $P-Q$ plus a rational function $r(x)/q(x)$, where the degree R of r *is* less than Q. For instance, long division gives

$$\frac{x^5+6x^2-5x+6}{x^3-x^2+x+1} = x^2 - x + 2 + \frac{x^2-4x+1}{x^3+x^2-x+3}.$$

Whereas the method of partial fractions cannot be applied to the rational function on the left (because $P=5$ is not less than $Q=3$), it can be applied to the one on the right (because $P=2$ *is* less than $Q=3$).]

Distinct roots. Let $q(x)$ in (A1) have the distinct roots $x_1, \ldots, x_Q$. Then f admits the **partial fraction expansion**

$$\boxed{f(x) = \frac{p(x)}{q(x)} = \frac{a_1}{x - x_1} + \frac{a_2}{x - x_2} + \cdots + \frac{a_Q}{x - x_Q},} \tag{A2}$$

where the a_j's are constants. One way to determine the a_j's is to recombine the terms on the right-hand side over a common denominator [namely, $q(x)$] and require its numerator to be identical to $p(x)$.

EXAMPLE 1. Expand $f(x) = (x - 1)/(x^2 + 5x + 6)$ in partial fractions. Since $x^2 + 5x + 6 = (x + 2)(x + 3)$, we can expand f as

$$f(x) = \frac{x - 1}{x^2 + 5x + 6} = \frac{a_1}{x + 2} + \frac{a_2}{x + 3}.$$

To determine a_1 and a_2, write

$$\frac{x - 1}{x^2 + 5x + 6} = \frac{(a_1 + a_2)x + (3a_1 + 2a_2)}{x^2 + 5x + 6}.$$

For the numerators to be identical we need

$$\begin{aligned} x^0 &: \quad -1 = 3a_1 + 2a_2, \\ x^1 &: \quad 1 = a_1 + a_2. \end{aligned}$$

Solving these equations gives $a_1 = -3$ and $a_2 = 4$ so

$$\frac{x - 1}{x^2 + 5x + 6} = -\frac{3}{x + 2} + \frac{4}{x + 3}. \quad \blacksquare$$

However, it is simpler to proceed as follows. To calculate a_1, multiply (A2) by $x - x_1$, then let $x \to x_1$ in the result. That step gives

$$\lim_{x \to x_1} \left[(x - x_1) \frac{p(x)}{q(x)} \right] = a_1 + 0 + \cdots + 0.$$

To find a_2 multiply by $x - x_2$ and let $x \to x_2$, and so on. Thus,

$$a_j = \lim_{x \to x_j} \left[(x - x_j) \frac{p(x)}{q(x)} \right] \tag{A3}$$

or, applying l'Hôpital's rule to the indeterminate part, $(x - x_j)/q(x)$,

$$\boxed{a_j = \frac{p(x_j)}{q'(x_j)}.} \tag{A4}$$

In Example 1, $x_1 = -2$ and $x_2 = -3$ so (A4) gives $a_1 = [(x-1)/(2x+5)]|_{x=-2} = -3$ and $a_2 = [(x-1)/(2x+5)]|_{x=-3} = 4$ as obtained above. In summary, if $q(x)$ has distinct roots, then $f(x) = p(x)/q(x)$ can be expanded in partial fractions according to (A2) and the a_j's can be found readily according to (A4).

Repeated roots. If any root x_j of $q(x)$ is of multiplicity k [i.e., $(x-x_j)^k$ is a factor of $q(x)$], then the jth term on the right-hand side of (A2) must be modified to the form

$$\boxed{\frac{a_{j1}}{x-x_j} + \frac{a_{j2}}{(x-x_j)^2} + \cdots + \frac{a_{jk}}{(x-x_j)^k}} \tag{A5}$$

or, equivalently,

$$\frac{b_{j0} + b_{j1}x + \cdots + b_{j,k-1}x^{k-1}}{(x-x_j)^{k-1}}.$$

To solve for $a_{j1}, \ldots, a_{jk}$ in (A5), we can recombine terms over a common denominator [namely, $q(x)$] and equate coefficients of powers of x in the numerator (since powers of x are linearly independent functions of x) as we did in Example 1.

EXAMPLE 2. To expand $(4x^2+5)/[(x-2)^3(x+3)]$ in partial fractions, write

$$\begin{aligned}\frac{4x^2+5}{(x-2)^3(x+3)} &= \left[\frac{a}{x-2} + \frac{b}{(x-2)^2} + \frac{c}{(x-2)^3}\right] + \frac{d}{x+3}\\ &= \frac{\left[a(x-2)^2 + b(x-2) + c\right](x+3) + d(x-2)^3}{(x-2)^3(x+3)}\\ &= [(a+d)x^3 + (-a+b-6d)x^2 + (-8a+b+c+12d)x\\ &\quad + (12a-6b+3c-8d)]/[(x-2)^3(x+3)],\end{aligned}$$

where the notation a, b, c, d will be simpler than using subscripted a_{jk}'s. Thus,

$$\begin{aligned} x^0 &: \quad 5 = 12a - 6b + 3c - 8d,\\ x^1 &: \quad 0 = -8a + b + c + 12d,\\ x^2 &: \quad 4 = -a + b - 6d,\\ x^3 &: \quad 0 = a + d,\end{aligned}$$

with solution $a = 41/125$, $b = 59/25$, $c = 21/5$, $d = -41/125$ so

$$\frac{4x^2+5}{(x-2)^3(x+3)} = \frac{41}{125}\frac{1}{x-2} + \frac{59}{25}\frac{1}{(x-2)^2} + \frac{21}{5}\frac{1}{(x-2)^3} - \frac{41}{125}\frac{1}{x+3}. \quad \blacksquare$$

In Example 2 we could have used (A4) to compute d but we could not have used it to compute a, b, c because $x = 2$ is a repeated root. To compute $a_{j1}, \ldots, a_{jk}$ in (A5), we need a modified version of (A4), namely,

$$\boxed{a_{jm} = \frac{1}{(k-m)!}\frac{d^{(k-m)}}{dx^{(k-m)}}\left[(x-x_j)^k\frac{p(x)}{q(x)}\right]\Bigg|_{x\to x_j}} \tag{A6}$$

for $m = 1, \ldots, k$. For instance, if we apply (A6) to the calculation of a, b, c in Example 2 we obtain

$$a = a_{11} = \frac{1}{2!}\frac{d^2}{dx^2}\left[(x-2)^3 \frac{4x^2+5}{(x-2)^3(x+3)}\right]\Bigg|_{x\to 2} = \frac{41}{125},$$

$$b = a_{12} = \frac{1}{1!}\frac{d}{dx}\left[(x-2)^3 \frac{4x^2+5}{(x-2)^3(x+3)}\right]\Bigg|_{x\to 2} = \frac{59}{25},$$

and

$$c = a_{13} = \frac{1}{0!}\left[(x-2)^3 \frac{4x^2+5}{(x-2)^3(x+3)}\right]\Bigg|_{x\to 2} = \frac{21}{5},$$

which results agree with those obtained above.

Partial fraction expansions can also be carried out using computer software. For example, the relevant *Maple* command is **convert**, and the commands

$$\text{convert}((x-1)/(x\hat{}2+5*x+6), \text{parfrac}, x);$$

and

$$\text{convert}((4*x\hat{}2+5)/((x-2)\hat{}3*(x+3)), \text{parfrac}, x);$$

give the results that we obtained in Examples 1 and 2, respectively.

We leave the derivation of (A6) for you as an exercise.

Appendix B

Existence and Uniqueness of Solutions of Systems of Linear Algebraic Equations

This appendix is intended as a minimal prerequisite for Chapter 3. Alternatively, to integrate linear algebra more heavily with the ODE chapters, we suggest following Chapter 2 with Sections 8.1–10.6 before beginning Chapter 3.

Definitions. We call the m equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m, \end{aligned} \tag{B1}$$

where the a_{jk} coefficients and the c_j's are known, a **system** of m linear algebraic equations on the n unknowns $x_1, \ldots, x_n$. Any set of numbers $x_1, \ldots, x_n$ that renders each of the m equations a numerical equality is called a **solution** of the system. A system is said to be **consistent** if it admits at least one solution, and **inconsistent** if it admits none. It is shown in Chapter 8 that if (B1) is consistent, then it admits either a **unique** solution (one solution) or an infinity of them, never three solutions or 27 solutions, for instance.

We call the array of coefficients

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{B2}$$

the **coefficient matrix**, and enclose the a_{jk} elements between brackets simply to show that the entire array is being regarded as a single entity. We say that $\mathbf{A}$ has

m rows and n columns. For instance, the second row consists of the elements $a_{21}, a_{22}, \ldots, a_{2n}$.

The case where m=n. In applications, the number of equations (m) is usually, but not always, equal to the number of unknowns (n); in general, m can be less than, equal to, or greater than n. Consider first the case where $m = n$. As the simplest case, let $m = n = 1$ so (B1) becomes $a_{11}x_1 = c_1$, or simply

$$ax = c. \tag{B3}$$

If $a \neq 0$, then (B3) admits a unique solution, namely, $x = c/a$. If $a = 0$, however, then there are two possibilities, as can be seen from (B3): if $c \neq 0$, then there is no solution and (B3) is inconsistent, and if $c = 0$, then (B3) is consistent and there is the infinity of solutions $x = \alpha$, where α is arbitrary.

The upshot, for this simple case, is that whether or not the coefficient a is zero is crucial: if $a \neq 0$ (the generic case), then there is a unique solution, and if $a = 0$ (the nongeneric or *singular* case) then there is either no solution or an infinity of them, depending upon the value of c. This idea generalizes to the case where $m = n > 1$ as indicated in (i) and (ii) below. First, we define a so-called **determinant** of the **A** matrix, and denote it as

$$\det\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \tag{B4}$$

that is, with straight line braces instead of square brackets. The determinant of **A** is a number (which can be positive, negative, or zero), defined as

$$\begin{vmatrix} a_{11} \end{vmatrix} \equiv a_{11}, \tag{B5a}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \equiv a_{11}a_{22} - a_{21}a_{12}, \tag{B5b}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \equiv \begin{aligned} & a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) \\ & + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \end{aligned} \tag{B5c}$$

for $n = 1, 2$, and 3, respectively. The general definition for any n is given in Section 10.4, but the cases (B5a,b,c) should suffice for Chapter 3. Do not confuse determinant with absolute value, especially in (B5a) where we cannot tell, from the left-hand side, whether we are signifying the absolute value of a_{11} or the determinant of the tiny matrix $\mathbf{A} = [a_{11}]$, without being told or from the context. For instance, $|-6| = -6$, $\begin{vmatrix} 2 & 5 \\ -3 & 4 \end{vmatrix} = 8 - (-15) = 23$, and

$$\begin{vmatrix} 3 & -2 & 5 \\ 0 & 4 & 6 \\ 1 & 1 & 8 \end{vmatrix} = (3)(32-6) - (-2)(0-6) + (5)(0-4) = 46.$$

Now, whether or not $\det \mathbf{A}$ is zero is crucial insofar as the existence and uniqueness of solutions of the system (B1):

(i) If $\det \mathbf{A} \neq 0$, then (B1) is consistent and has a unique solution.

(ii) If $\det \mathbf{A} = 0$, then (B1) has either no solution (is inconsistent) or an infinity of solutions, depending upon the c_j's.

For instance, consider the three systems

$$\begin{aligned} 2x_1 - x_2 &= 4 \\ 3x_1 + x_2 &= 11, \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= 5 \\ 4x_1 - 2x_2 &= 3, \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x_1 - x_2 &= 5 \\ 4x_1 - 2x_2 &= 10. \end{aligned}$$

In the first case, $\det \mathbf{A} = 5 \neq 0$ so there is a unique solution (namely, $x_1 = 3$ and $x_2 = 2$); in the second case, $\det \mathbf{A} = 0$ and there is no solution (as is not surprising since the second left-hand side is twice the first left-hand side whereas 3 is not twice 5); and in the third case, $\det \mathbf{A} = 0$ and there is an infinity of solutions [namely, $x_2 = \alpha$ and $x_1 = (5 + \alpha)/2$, where α is arbitrary].

In fact, if $\det \mathbf{A} \neq 0$, then we can give a formula for the solution of (B1). Namely, for each j from 1 to n, x_j is given as the ratio of two determinants: the one in the denominator is $\det \mathbf{A}$, and the one in the numerator is the same except that its jth column is replaced by the column of c's on the right-hand side of (B1). This result is known as **Cramer's rule**. For instance, if

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 2x_1 - 2x_2 + x_3 &= 0 \\ 4x_1 + 3x_2 - x_3 &= -5, \end{aligned} \tag{B6}$$

then Cramer's rule gives

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -5 & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & 3 & -1 \end{vmatrix}} = \frac{-6}{5} = -\frac{6}{5}, \qquad x_2 = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 4 & -5 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & 3 & -1 \end{vmatrix}} = \frac{11}{5}, \tag{B7}$$

and, in similar fashion, $x_3 = 34/5$. Notice from Cramer's rule that since each x_j is given as the ratio of two determinants, with the determinant in each denominator being the determinant of $\mathbf{A}$, it follows that if $\det \mathbf{A} \neq 0$, then there exists a unique solution for each x_j. That result is identical to (i) stated above.

Remember that (i) and (ii), and Cramer's rule, hold only in the case where $m = n$. If $m \neq n$, then the "determinant of $\mathbf{A}$" is not even defined. More about this in Section 10.4.

The homogeneous case. If all of the c_j's in (B1) are zero, then we say that the system is homogeneous. It should be evident that *homogeneous systems are always consistent* since they necessarily admit the solution $x_1 = x_2 = \cdots = x_n = 0$, which is known as the **trivial solution**. (When we give it that name, we do not mean to imply that it is in some way beneath our dignity; it is a perfectly legitimate solution.) Further, *if (B1) is a homogeneous system with $m < n$, then it necessarily admits not only the trivial solution, but also an infinity of nontrivial solutions.* For instance, it can be verified by substitution that the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 0 \end{aligned} \tag{B8}$$

admits the solution $x_1 = x_3 = \alpha$ and $x_2 = -\alpha$ for any α (which solutions include the trivial solution for the choice $\alpha = 0$).

Appendix C

Table of Laplace Transforms

$f(t)$	$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$
NOTE: s is regarded as real here.	
1. 1	$\frac{1}{s}$ $(s > 0)$
2. e^{at}	$\frac{1}{s-a}$ $(s > a)$
3. $\sin at$	$\frac{a}{s^2+a^2}$ $(s > 0)$
4. $\cos at$	$\frac{s}{s^2+a^2}$ $(s > 0)$
5. $\sinh at$	$\frac{a}{s^2-a^2}$ $(s > \lvert a \rvert)$
6. $\cosh at$	$\frac{s}{s^2-a^2}$ $(s > \lvert a \rvert)$
7. t^n $(n = \text{positive integer})$	$\frac{n!}{s^{n+1}}$ $(s > 0)$
8. t^p $(p > -1)$	$\frac{\Gamma(p+1)}{s^{p+1}}$ $(s > 0)$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$ $(s > a)$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$ $(s > a)$
11. $t\sin at$	$\frac{2as}{(s^2+a^2)^2}$ $(s > 0)$
12. $t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$ $(s > 0)$
13. $t\sinh at$	$\frac{2as}{(s^2-a^2)^2}$ $(s > a)$

$f(t)$	$\overline{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$
14. $t \cosh at$	$\dfrac{s^2+a^2}{(s^2-a^2)^2}$ $(s > a)$
15. $t^n e^{at}$ $(n = \text{positive integer})$	$\dfrac{n!}{(s-a)^{n+1}}$ $(s > a)$
16. $t^p e^{at}$ $(p > -1)$	$\dfrac{\Gamma(p+1)}{(s-a)^{p+1}}$ $(s > a)$
17. $\ln t$	$-\dfrac{\gamma + \ln s}{s}$ $(s > 0)$
	(γ = Euler's constant ≈ 0.577215665)
18. $H(t-a)$ $(a \geq 0)$	$\dfrac{e^{-as}}{s}$ $(s > 0)$
19. $\delta(t-a)$ $(a > 0)$	e^{-as}
20. $\dfrac{e^{-a^2/t}}{\sqrt{t}}$ $(a \geq 0)$	$\sqrt{\dfrac{\pi}{s}}\, e^{-2a\sqrt{s}}$ $(s > 0)$
21. $\dfrac{e^{-a^2/t}}{t^{3/2}}$ $(a > 0)$	$\dfrac{\sqrt{\pi}}{a} e^{-2a\sqrt{s}}$ $(s > 0)$
22. $J_0(t)$	$\dfrac{1}{\sqrt{s^2+1}}$ $(s > 0)$

Linearity of transform and inverse (Theorems 5.3.1, 5.3.2):

23. $\alpha u(t) + \beta v(t)$	$\alpha\overline{u}(s) + \beta\overline{v}(s)$

Transform of derivative (Theorem 5.3.3):

24. $f'(t)$	$s\overline{f}(s) - f(0)$
25. $f''(t)$	$s^2\overline{f}(s) - sf(0) - f'(0)$
26. $f^{(n)}(t)$	$s^n\overline{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

Transform of integral (Theorem 5.7.3):

27. $\int_0^t f(\tau)\,d\tau$	$\dfrac{\overline{f}(s)}{s}$

Laplace convolution theorem (Theorem 5.3.4):

28. $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$\overline{f}(s)\overline{g}(s)$

$f(t)$	$\overline{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$

s-Shift (Theorem 5.7.1):

29. $e^{-at}f(t)$	$\overline{f}(s+a)$

t-Shift (Theorem 5.7.2):

30. $H(t-a)f(t-a)$	$e^{-as}\overline{f}(s)$

Multiplication by t and $1/t$ (Theorems 5.7.4 and 5.7.5):

31. $tf(t)$	$-\dfrac{d\overline{f}(s)}{ds}$
32. $\dfrac{f(t)}{t}$	$\int_s^\infty \overline{f}(s')\,ds'$

Transform of periodic function (Theorem 5.7.8):

33. $f(t)$, of period T	$\dfrac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt$

Appendix D

Table of Fourier Transforms

	$f(x)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$
1.	$\dfrac{1}{x^2 + a^2} \quad (a > 0)$	$\dfrac{\pi}{a} e^{-a\|\omega\|}$
2.	$H(x)e^{-ax} \quad (\mathrm{Re}\, a > 0)$	$\dfrac{1}{a + i\omega}$
3.	$H(-x)e^{ax} \quad (\mathrm{Re}\, a > 0)$	$\dfrac{1}{a - i\omega}$
4.	$e^{-a\|x\|} \quad (a > 0)$	$\dfrac{2a}{\omega^2 + a^2}$
5.	e^{-x^2}	$\sqrt{\pi} e^{-\omega^2/4}$
6.	$\dfrac{1}{2a\sqrt{\pi}} e^{-x^2/(2a)^2} \quad (a > 0)$	$e^{-a^2\omega^2}$
7.	$\dfrac{1}{\sqrt{\|x\|}}$	$\sqrt{\dfrac{2\pi}{\|\omega\|}}$
8.	$e^{-a\|x\|/\sqrt{2}} \sin\left(\dfrac{a}{\sqrt{2}}\|x\| + \dfrac{\pi}{4}\right) \quad (a > 0)$	$\dfrac{2a^3}{\omega^4 + a^4}$
9.	$H(x + a) - H(x - a)$	$\dfrac{2 \sin \omega a}{\omega}$
10.	$\delta(x - a)$	$e^{-i\omega a}$
11.	$f(ax + b) \quad (a > 0)$	$\dfrac{1}{a} e^{ib\omega/a} \hat{f}\left(\dfrac{\omega}{a}\right)$
12.	$\dfrac{1}{a} e^{-ibx/a} f\left(\dfrac{x}{a}\right) \quad (a > 0,\ b \text{ real})$	$\hat{f}(a\omega + b)$
13.	$f(ax) \cos cx \quad (a > 0,\ c \text{ real})$	$\dfrac{1}{2a}\left[\hat{f}\left(\dfrac{\omega - c}{a}\right) + \hat{f}\left(\dfrac{\omega + c}{a}\right)\right]$
14.	$f(ax) \sin cx \quad (a > 0,\ c \text{ real})$	$\dfrac{1}{2ai}\left[\hat{f}\left(\dfrac{\omega - c}{a}\right) - \hat{f}\left(\dfrac{\omega + c}{a}\right)\right]$
15.	$f(x + c) + f(x - c) \quad (c \text{ real})$	$2\hat{f}(\omega) \cos \omega c$

	$f(x)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$
16.	$f(x+c) - f(x-c) \quad (c \text{ real})$	$2i\hat{f}(\omega)\sin\omega c$
17.	$x^n f(x) \quad (n = 1, 2, \ldots)$	$i^n \dfrac{d^n}{d\omega^n}\hat{f}(\omega)$
Linearity of transform and inverse:		
18.	$\alpha f(x) + \beta g(x)$	$\alpha\hat{f}(\omega) + \beta\hat{g}(\omega)$
Transform of derivative:		
19.	$f^{(n)}(x)$	$(i\omega)^n\hat{f}(\omega)$
Transform of integral:		
20.	$f(x) = \int_{-\infty}^{x} g(\xi)\,d\xi$, where $f(x) \to 0$ as $x \to \infty$	$\hat{f}(\omega) = \dfrac{1}{i\omega}\hat{g}(\omega)$
Fourier convolution theorem:		
21.	$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$	$\hat{f}(\omega)\hat{g}(\omega)$

Appendix E

Table of Fourier Cosine and Sine Transforms

	$f(x)$	$\hat{f}_C(\omega) = \int_0^\infty f(x) \cos \omega x \, dx$
1C.	$e^{-ax} \quad (a > 0)$	$\dfrac{a}{\omega^2 + a^2}$
2C.	$x^n e^{-ax} \quad (a > 0)$	$\dfrac{n! \, \mathrm{Re}\, (a + i\omega)^{n+1}}{(\omega^2 + a^2)^{n+1}}$ (Re = real part)
3C.	$\dfrac{1}{x^2 + a^2} \quad (a > 0)$	$\dfrac{\pi}{2a} e^{-a\omega}$
Linearity of transform and inverse:		
4C.	$\alpha f(x) + \beta g(x)$	$\alpha \hat{f}_C(\omega) + \beta \hat{g}_C(\omega)$
Transform of derivative:		
5C.	$f'(x)$	$\omega \hat{f}_S(\omega) - f(0)$
6C.	$f''(x)$	$-\omega^2 \hat{f}_C(\omega) - f'(0)$
Convolution theorem:		
7C.	$\dfrac{1}{2} \int_0^\infty [f(\lvert x - \xi \rvert) + f(x + \xi)] g(\xi) \, d\xi$	$\hat{f}_C(\omega) \hat{g}_C(\omega)$

	$f(x)$	$\hat{f}_S(\omega) = \int_0^\infty f(x) \sin \omega x \, dx$
1S.	$e^{-ax} \quad (a > 0)$	$\dfrac{\omega}{\omega^2 + a^2}$
2S.	$x^n e^{-ax} \quad (a > 0)$	$\dfrac{n! \operatorname{Im} (a + i\omega)^{n+1}}{(\omega^2 + a^2)^{n+1}}$ (Im = imaginary part)
3S.	$\dfrac{x}{x^2 + a^2} \quad (a > 0)$	$\dfrac{\pi}{2} e^{-a\omega}$

Linearity of transform and inverse:

4S.	$\alpha f(x) + \beta g(x)$	$\alpha \hat{f}_S(\omega) + \beta \hat{g}_S(\omega)$

Transform of derivative:

5S.	$f'(x)$	$-\omega \hat{f}_C(\omega)$
6S.	$f''(x)$	$-\omega^2 \hat{f}_S(\omega) + \omega f(0)$

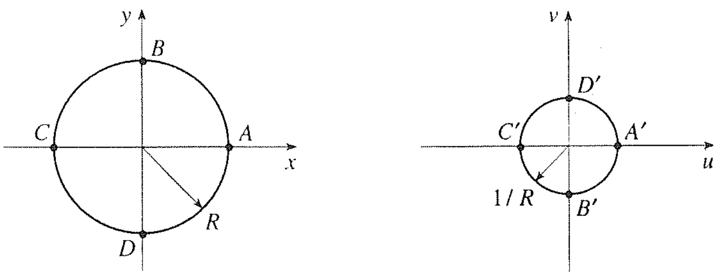
Convolution theorem:

7S.	$\dfrac{1}{2} \int_0^\infty [f(\lvert x - \xi \rvert) - f(x + \xi)] g(\xi) \, d\xi$	$\hat{f}_C(\omega) \hat{g}_S(\omega)$

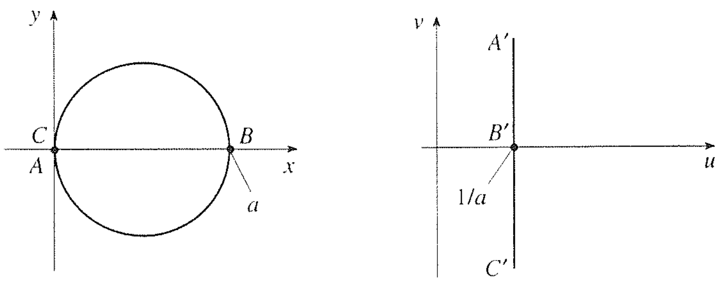
Appendix F

Table of Conformal Maps

1. $w = 1/z$

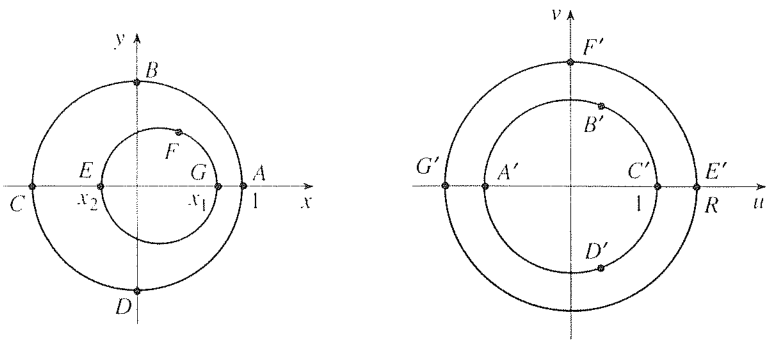


2. $w = 1/z$



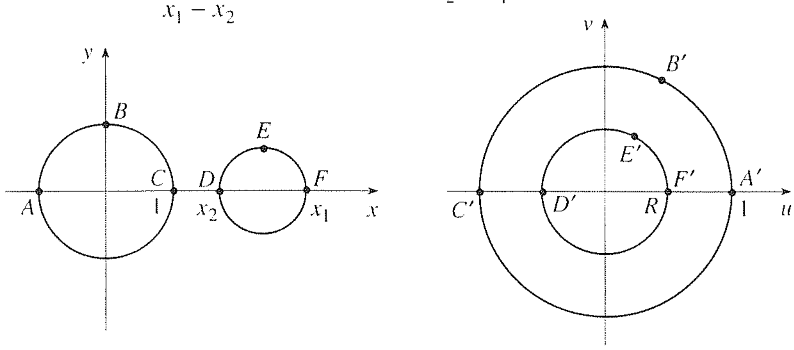
3. $w = \dfrac{z-a}{az-1}$; $\quad a = \dfrac{1 + x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 + x_2}$,

$$R = \frac{1 - x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 - x_2}, \qquad -1 < x_2 < x_1 < 1$$

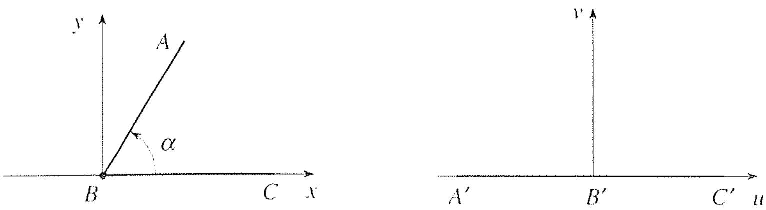


4. $w = \dfrac{z-a}{az-1}$; $\quad a = \dfrac{x_1 x_2 + 1 + \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1 + x_2}$,

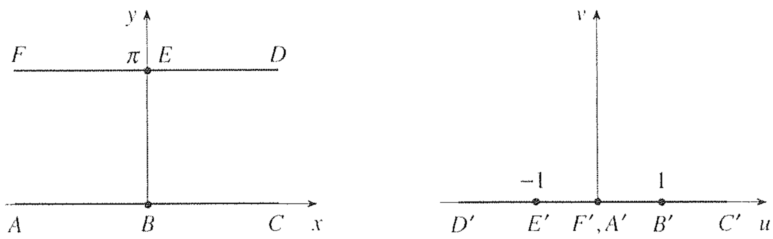
$$R = \frac{x_1 x_2 - 1 - \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1 - x_2}, \qquad 1 < x_2 < x_1$$



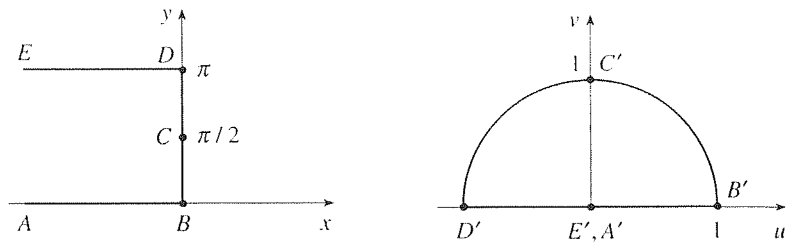
5. $w = z^{\pi/\alpha}$



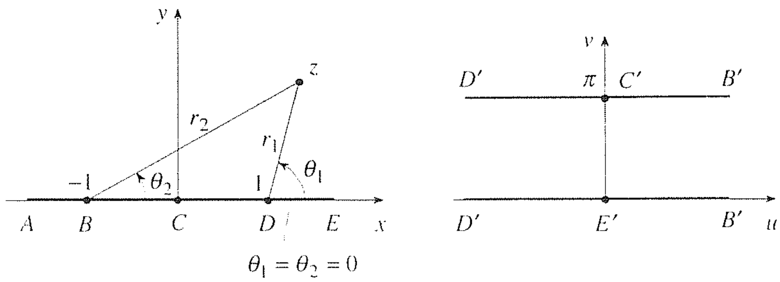
6. $w = e^z$



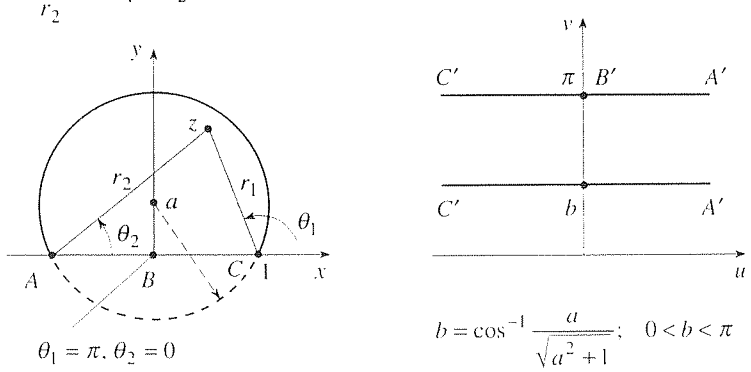
7. $w = e^z$



8. $w = \log \dfrac{z-1}{z+1} = \ln \dfrac{r_1}{r_2} + i(\theta_1 - \theta_2)$

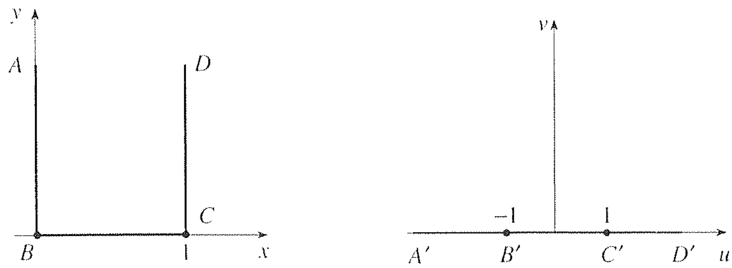


9. $w = \log \dfrac{z-1}{z+1} = \ln \dfrac{r_1}{r_2} + i(\theta_1 - \theta_2)$

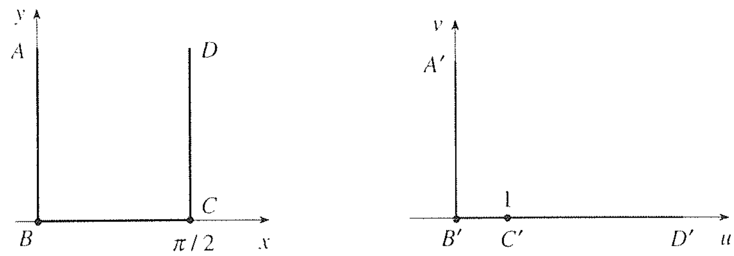


$b = \cos^{-1} \dfrac{a}{\sqrt{a^2+1}}; \quad 0 < b < \pi$

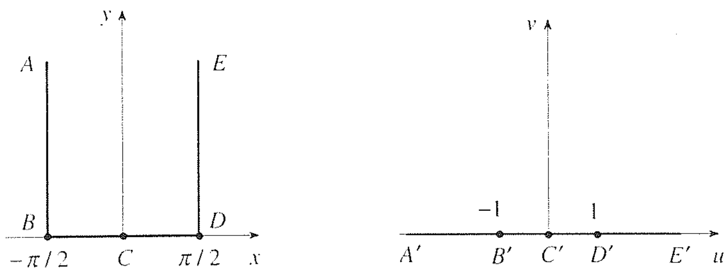
10. $w = -\cos \pi z$



11. $w = \sin z$



12. $w = \sin z$



13. $w = z + \dfrac{a^2}{z}$