Topic of this homework: Linear Algebra: Solutions of non-symmetric matrices (Tall and Fat), Singular value decomposition (SVD)

Deliverables: Show your work. Numerical results may not be sufficient, unless specifically requested.

1 Least-square solution of a non-square matrix

1.1 Tall (over-specified) systems

You are given the system

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
\]

1. Find the least squares (LS) solution of the tall (over-specified) system of equations \( Ax = b \). Mentally note that leaving out the third equation (third row of \( A \)), which is very different than the first two, would result in a trivial solution of \( x_1 = 1 \) and \( x_2 = 1 \).

2. Justify that the inverse of the “tall” over-specified system of equations as being \((A^T A)^{-1} A^T\).

3. Find the LS solution, but unlike Eq. 1, the third (row) is close to the first row

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

where \(|\epsilon| \ll 1\).

4. Starting from Eq. 1, find the eigen-values and eigen-vectors of the LS solution \( A^T A \).

5. Summarize your conclusion about the impact of Eq. 2 on the LS solution, as a function of \(|\epsilon| \ll 1\).

1.2 Fat (under-specified) systems of equations

You are given the under-specified system

\[
\begin{bmatrix}
1 & 2 & \epsilon \\
2 & -1 & -\epsilon
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-1
\end{bmatrix}.
\]

with \(|\epsilon| \ll 1\).

1. Find the inverse of the fat under-specified system

2. Identify \(\text{span}(A)\) and \(\text{null}(A)\) for

\[
A = \begin{bmatrix}
1 & 2 \\
2 & -1
\end{bmatrix}
\]

Define the operational definitions of these concepts.

3. Ignore this question. It was incomplete.
2 Singular Value Decomposition (SVD)

You are given

\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}_{2 \times 3}. \]

1. Find \( U, \Sigma \) and \( V \) such that \( A = U \Sigma V^T \).

Hint: If \( \text{eig}(S) \) computes the matrix of eigenvectors of matrix \( S \), then \( U = \text{eig}(AA^T) \in \mathbb{R}^{2 \times 2} \) and \( V = \text{eig}(A^T A) \in \mathbb{R}^{3 \times 3} \).

Hint 2: \( U^T U = I \) and \( V^T V = I \).

Hint 3: This problem was intended to be done using Matlab/Octave. For those of you that don’t have access, for what ever reason, here are the eigenvectors of \( V \):

\[
V = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ \sqrt{2} \\ \sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} = \text{eig}(A^T A)
\]

Note that the three are orthogonal.

2. Repeat calculation for \( A' \)

\[ A' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \]

Hint: Take the transpose of the formula for \( A \) in terms of \( U, \Sigma, V \).

3. Define the property of a unitary matrix.

4. Given that \( A = U \Sigma V^T \), where \( U \) and \( V \) are orthogonal (real-unitary) and \( \Sigma \) is diagonal, show that \( U \) and \( V \) are the eigen-matrices of \( AA^T \) and \( A^T A \), respectively.

5. What are the ranks of \( A^T A \) and \( AA^T \). Give a full explanation.

3 Operator symmetry

Each matrix (operator) \( A \) below has dimensions \( m \times n \) (m rows and n columns). Define \( r \) as the smaller, and \( l \) as the larger, of \( m, n \).

For each given matrix symmetry, provide the following information:

1. The name of the symmetry (e.g., Hermitian, unitary, self-adjoint, analytic, causal, etc.)
2. Definiteness: i.e., positive, semi-positive, negative, semi-negative definite
3. Eigen-vector properties (e.g., real, imaginary, complex, conjugate, zero, N, ON)
4. Eigen-value spectrum: i.e., discrete fixed set, infinite set, continuous infinite, etc.

3.1 Matrix Symmetry

1. \( A = A^T \)
2. \( A = \overline{A} \)
3. \( A = -A \)
4. \( A^T A \)
5. \( A A^T \)
6. \( A = A^\dagger \)
7. \( A^\dagger A \)
8. \( AA^\dagger \)
9. Prove that the eigenvalues of \( A^\dagger = A \), a Hermitian matrix, are real.
10. Prove that the eigenvectors of a symmetric matrix are orthogonal.

### 3.2 Complex matrix symmetry

1. Impedance matrix \( Z(s) = R(s) + jX(s) \), for a network having two nodes is given as
   \[
   \begin{bmatrix}
   V_1(\omega) \\
   V_2(\omega)
   \end{bmatrix} = \begin{bmatrix} 1 + s & 1/s \\ 1/s & 1 + 1/s \end{bmatrix} \begin{bmatrix} I_1(\omega) \\
   I_2(\omega)
   \end{bmatrix}
   \]  
   (4)
   Is this matrix Hermitian?

2. Admittance matrix for the above impedance matrix \( Y(s) = Z^{-1}(s) \) is given as
   \[
   \begin{bmatrix} I_1(\omega) \\
   I_2(\omega)
   \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 1 + 1/s & -1/s \\ -1/s & 1 + s \end{bmatrix} \begin{bmatrix} V_1(\omega) \\
   V_2(\omega)
   \end{bmatrix}
   \]  
   (5)
   with \( \Delta = (1 + s)(1 + 1/s) - 1/s^2 = 2 + s + 1/s - 1/s^2 \)
   Is this matrix Hermitian?

### 3.3 Continuous Symmetry

1. Given the following differential operator
   \[
   A = \frac{d^2}{dt^2} + 2\frac{d}{dt} + 1.
   \]
   Find the eigen values and vectors. Hint: Take the Laplace Transform.