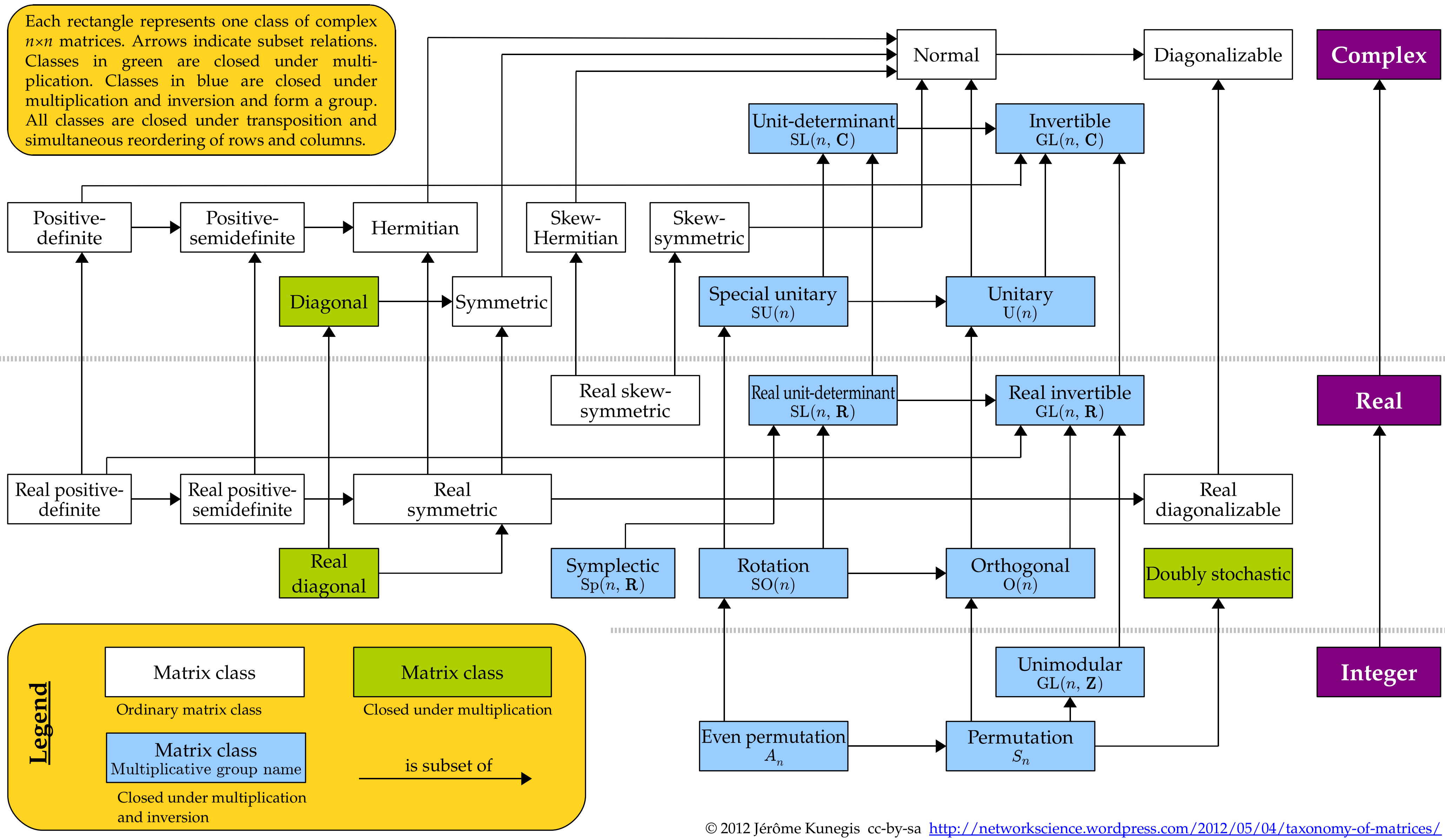
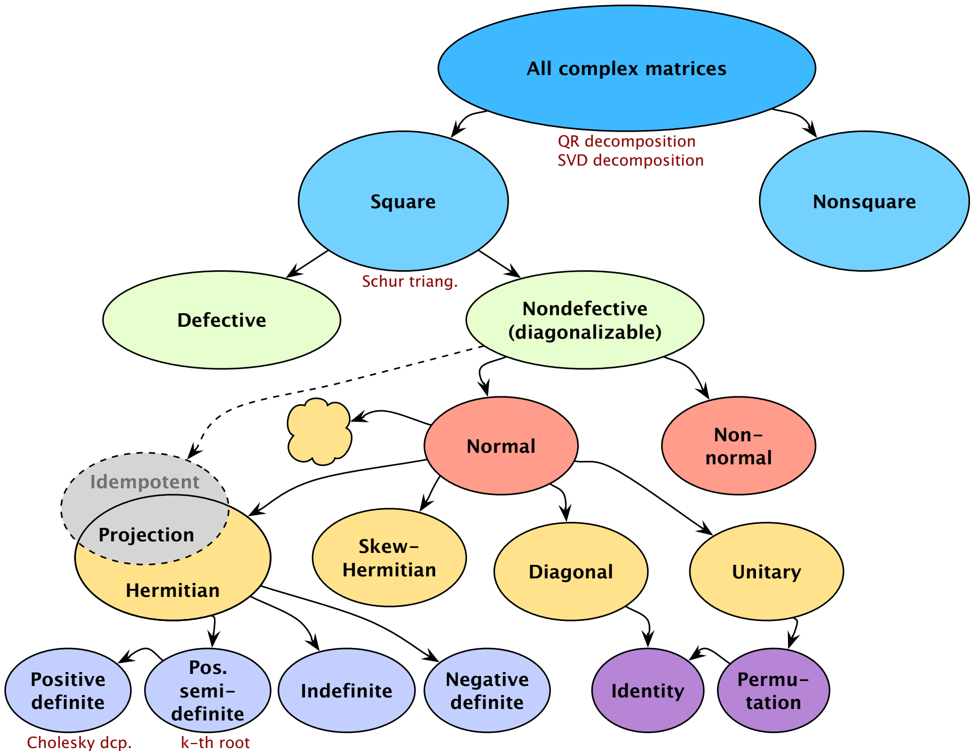
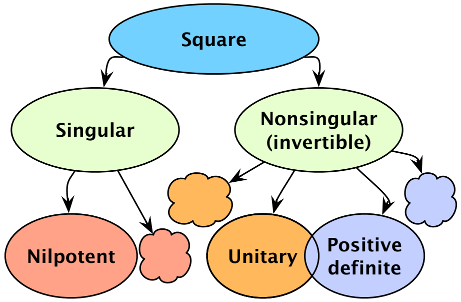
Taxonomy of $n{\times}n$ Matrices



Matrix Taxonomy



Nonsingular Matrices *(0.5)*.



A matrix $\mathbf{A} \in \mathcal{M}_n$ is nonsingular iff

- $\mathbf{A}^{-1}$ exists
- $\det \mathbf{A} \neq 0$
- $\operatorname{rnk} \mathbf{A} = n$
- $\dim \mathcal{R}(\mathbf{A}) = n$
- rows (cols) are linearly independent
- $0 \notin \sigma(\mathbf{A})$

Nilpotent Matrices *(p39)*.

A matrix $\mathbf{A} \in \mathcal{M}_n$ is nilpotent to index k if $\mathbf{A}^k = \mathbf{0}$ but

Hermitian Matrices *(4.1.1)*.

A matrix $\mathbf{A} \in \mathcal{M}_n$ is Hermitian if $\mathbf{A} = \mathbf{A}^H$.

- A Hermitian matrix is parametrized by n^2 free real variables.
- *(4.1.1.1)* For any $\mathbf{B} \in \mathcal{M}_n$, $\mathbf{B} + \mathbf{B}^H$, $\mathbf{B}^H\mathbf{B}$, and $\mathbf{B}\mathbf{B}^H$ are Hermitian.
- *(4.1.1.2,3)* Hermitian matrices are closed under addition, multiplication by a scalar, raising to an integer power, and (if nonsingular) inversion.
- *(4.1.2)* Any matrix $\mathbf{A}$ has a unique decomposition $\mathbf{A} = \mathbf{B} + j\mathbf{C}$ where $\mathbf{B}$ and $\mathbf{C}$ are Hermitian: $\mathbf{B} = (\mathbf{A} + \mathbf{A}^H)/2$ and $\mathbf{C} = (\mathbf{A} - \mathbf{A}^H)/(2j)$
- *(4.1.3c)* The eigenvalues of a Hermitian matrix are all real.
- *(4.1.4a)* $\mathbf{A}$ is Hermitian iff $\mathbf{x}^H\mathbf{A}\mathbf{x}$ is real for all $\mathbf{x} \in \mathbb{C}^n$.
- *(4.1.4c)* $\mathbf{A}$ is Hermitian iff $\mathbf{S}^H\mathbf{A}\mathbf{S}$ is Hermitian for all $\mathbf{S} \in \mathcal{M}_n$ (or: iff $\mathbf{x}^H\mathbf{A}\mathbf{y} =$

Matrix Properties

Eigenvalues and -vectors *(1.1.2)*. If $\lambda \in \mathbb{C}$ and $\mathbf{x} \neq \mathbf{0}$ satifsy the equation $\mathbf{Ax} = \lambda\mathbf{x}$, they are considered eigenvalue and eigenvector, respectively, of the matrix $\mathbf{A} \in \mathcal{M}_n$. The *eigenvalue spectrum* $\sigma(\mathbf{A})$ of $\mathbf{A}$ is the set of all eigenvalues.

- *(1.2.4)* The eigenvalue spectrum $\sigma(\mathbf{A})$ coincides with the roots of the *characteristic polynomial* $p_{\mathbf{A}}(t) = \det(t\mathbf{I} - \mathbf{A})$.
- *(1.2.6)* Every $\mathbf{A} \in \mathcal{M}_n$ has exactly n eigenvalues, counting multiplicities (c.m.).
- *(2.4.2, Cayley-Hamilton)* $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$. If $\mathbf{A}$ is nonsingular, then there is a polynomial $q(t)$ of degree at most $n-1$, such that $\mathbf{A}^{-1} = q(\mathbf{A})$.
- *(1.4.1)* The eigenvalues of $\mathbf{A}^T$ are the same as those of $\mathbf{A}$, c.m. The eigenvalues of $\mathbf{A}^H$ are the complex conjugates of the eigenvalues of $\mathbf{A}$, c.m.
- *(1.4.2)* The set of eigenvectors associated with a particular $\lambda \in \sigma(\mathbf{A})$ is a subspace of $\mathbb{C}^n$, called *eigenspace* of $\mathbf{A}$ corresponding to λ.
- *(1.4.3)* The dimension of the eigenspace of $\mathbf{A} \in \mathcal{M}_n$ corresponding to the eigenvalue λ is its *geometric multiplicity*.
- *(1.4.3)* The multiplicity of λ as a zero of $p_{\mathbf{A}}(t)$ is the *algebraic multiplicity* of λ. The algebraic multiplicity is greater or equal than the geometric multiplicity.
- *(1.3.8)* Eigenvectors corresponding to different eigenvalues are linearly independent.
- *(1.3.20)* Be $\mathbf{A} \in \mathcal{M}_{m,n}$ and $\mathbf{B} \in \mathcal{M}_{n,m}$. Then $\mathbf{BA}$ has the same eigenvalues as $\mathbf{AB}$, c.m., together with an additional $n-m$ eigenvalues equal to 0.
- *(p37)* For nonsingular $\mathbf{A} \in \mathcal{M}_n$, if $\lambda \in \sigma(\mathbf{A})$, then $\lambda^{-1} \in \sigma(\mathbf{A}^{-1})$, corresponding to the same eigenvector.
- *(p43)* If $\lambda \in \sigma(\mathbf{A})$, then $\lambda^k \in \sigma(\mathbf{A}^k)$.
- If $\lambda \in \sigma(\mathbf{A})$, then $(1+\lambda) \in \sigma(\mathbf{I}+\mathbf{A})$.
- *(4.3.1, Weyl)* For Hermitian $\mathbf{A}, \mathbf{B} \in \mathcal{M}_n$, with their eigenvalues arranged in increasing order, for $k = 1, \ldots, n$:
$$\lambda_k(\mathbf{A}) + \lambda_1(\mathbf{B}) \leq \lambda_k(\mathbf{A}+\mathbf{B}) \leq \lambda_k(\mathbf{A}) + \lambda_n(\mathbf{B}).$$
- *(4.5.9, Ostrowski)* For $\mathbf{A} \in \mathcal{M}_n$ Hermi-

Submatrices *(0.7)*. With index sets $\alpha \subseteq \{1, \ldots, m\}$, $\beta \subseteq \{1, \ldots, n\}$ and $\mathbf{A} \in \mathcal{M}_{m,n}$, $\mathbf{A}(\alpha, \beta)$ denotes the submatrix that is indexed by rows α and columns β.

- For $\mathbf{A} \in \mathcal{M}_n$, if $\alpha = \beta$, $\mathbf{A}(\alpha, \beta) = \mathbf{A}(\alpha)$ is a *principal submatrix* of $\mathbf{A}$.
- The determinant of a square submatrix of $\mathbf{A} \in \mathcal{M}_{m,n}$ is called a *minor*.
- The determinant of a principal submatrix of $\mathbf{A} \in \mathcal{M}_n$ is a *principal minor*.

Adjoint Matrix *(0.8.2)*. The adjoint of $\mathbf{A} \in \mathcal{M}_n$ is the matrix $\mathbf{B} \in \mathcal{M}_n$ of *cofactors* defined by
$$b_{ij} = (-1)^{i+j} \det \mathbf{A}_{ji}.$$

- $(\operatorname{adj}\mathbf{A})\mathbf{A} = \mathbf{A}(\operatorname{adj}\mathbf{A}) = (\det\mathbf{A})\mathbf{I}$.
- If $\mathbf{A}$ invertible, $\mathbf{A}^{-1} = (\det^{-1}\mathbf{A}) \operatorname{adj}\mathbf{A}$.

Range and Null Space *(0.2.3)*. The range of a matrix $\mathbf{A} \in \mathcal{M}_{m,n}$ is $\mathcal{R}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{C}^m : \mathbf{y} = \mathbf{Ax},\ \mathbf{x} \in \mathbb{C}^n\}$. The null space of a matrix $\mathbf{A} \in \mathcal{M}_{m,n}$ is $\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{C}^n : \mathbf{Ax} = \mathbf{0}\}$.

- *(0.2.3)* $\dim\mathcal{R}(\mathbf{A}) + \dim\mathcal{N}(\mathbf{A}) = n$.
- *(0.4.4g)* $\dim\mathcal{R}(\mathbf{A}) = \operatorname{rnk}\mathbf{A}$.
- *(0.6.6)* $\mathcal{R}(\mathbf{A}) = \mathcal{N}^{\perp}(\mathbf{A}^H)$.

Rank *(0.4)*. The rank of a matrix $\mathbf{A} \in \mathcal{M}_{m,n}$ is the largest number of rows (columns) of $\mathbf{A}$ that constitute a linearly independent set.

- *(0.4.4)* $\operatorname{rnk}\mathbf{A} = \dim\mathcal{R}(\mathbf{A}) = n - \dim\mathcal{N}(\mathbf{A})$.
- *(0.4.6a)* $\mathbf{A}, \mathbf{A}^H, \mathbf{A}^T, \mathbf{A}^*$ have the same rank.
- *(0.4.6b)* rank is unchanged by left/right multiplication with a nonsingular matrix.
- *(0.4.6c)* If $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{m,n}$, then $\operatorname{rnk}\mathbf{A} = \operatorname{rnk}\mathbf{B}$ iff there are nonsingular $\mathbf{X} \in \mathcal{M}_n$ and $\mathbf{Y} \in \mathcal{M}_m$ s.t. $\mathbf{B} = \mathbf{XAY}$.
- *(0.4.6d)* $\operatorname{rnk}\mathbf{A}^H\mathbf{A} = \operatorname{rnk}\mathbf{A}$.
- *(0.4.5a)* $\operatorname{rnk}\mathbf{A} \leq \min(m, n)$.
- *(0.4.5c)* If $\mathbf{A} \in \mathcal{M}_{m,k}$ and $\mathbf{B} \in \mathcal{M}_{k,n}$
$$\operatorname{rnk}\mathbf{A} + \operatorname{rnk}\mathbf{B} - k \leq \operatorname{rnk}\mathbf{AB}$$
$$\operatorname{rnk}\mathbf{AB} \leq \min(\operatorname{rnk}\mathbf{A}, \operatorname{rnk}\mathbf{B}).$$
- *(0.4.5d)* If $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{m,n}$, $\operatorname{rnk}(\mathbf{A}+\mathbf{B}) \leq \operatorname{rnk}\mathbf{A} + \operatorname{rnk}\mathbf{B}$.
- *(p95)* For $\mathbf{A} \in \mathcal{M}_n$,

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$\mathbf{A}^{k-1} \neq \mathbf{0}$.

- *(p37)* $\lambda_i = 0$, $i = 1, \ldots, n$..

Idempotent Matrices *(p37)*. A matrix $\mathbf{P} \in \mathcal{M}_n$ is idempotent if $\mathbf{P}^2 = \mathbf{P}$.

- $\operatorname{rnk} \mathbf{P} = \operatorname{tr} \mathbf{P}$.
- *(p37)* $\lambda_i \in \{0, 1\}$, $i = 1, \ldots, n$. The geometric multiplicity of the eigenvalue 1 is $\operatorname{rnk} \mathbf{P}$.
- $\mathbf{P}^H$, $\mathbf{I}-\mathbf{P}$, and $\mathbf{I}-\mathbf{P}^H$ are all idempotent.
- $\mathbf{P}(\mathbf{I}-\mathbf{P}) = (\mathbf{I}-\mathbf{P})\mathbf{P} = \mathbf{0}$.
- $\mathbf{Px} = \mathbf{x}$ iff $\mathbf{x}$ lies in the range of $\mathbf{P}$.
- $\mathcal{N}(\mathbf{P}) = \mathcal{R}(\mathbf{I}-\mathbf{P})$.
- *(TODO)* $\mathbf{P}$ is its own generalized inverse, $\mathbf{P}^\dagger$.

Nondefective (Diagonalizable) Matrices *(1.3.6)*. A matrix $\mathbf{A} \in \mathcal{M}_n$ is diagonalizable if it is similar to a diagonal matrix, i.e., $\mathbf{A} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1}$.

- *(1.3.7)* The columns of $\mathbf{S}$ are eigenvectors of $\mathbf{A}$, the diagonal entries of $\boldsymbol{\Lambda}$ corresponding eigenvalues.
- *(1.4.4)* $\mathbf{A}$ is diagonalizable iff it is nondefective, i.e., the geometric multiplicity is equal to the algebraic multiplicity for each eigenvalue.
- *(1.3.9)* If $\mathbf{A}$ has n distinct eigenvalues, it is diagonalizable.
- *(p46)* $\operatorname{rnk} \mathbf{A} = \#$ nonzero eigenvalues.

Normal Matrices *(2.5)*. A matrix $\mathbf{A} \in \mathcal{M}_n$ is normal if $\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H$.

- *(2.5.4)* $\mathbf{A}$ is normal iff it is unitarily similar to a diagonal matrix.
- *(2.5.4)* $\mathbf{A}$ is normal iff it has an orthonormal set of n eigenvectors.
- Normal matrices are closed under unitary equivalence, raising to an integer power, and (if nonsingular) inversion.
- The singular values of a normal matrix are the absolute values of the eigenvalues.
- The eigenvalues of $\mathbf{A}^H$ are the conjugates of the eigenvalues of $\mathbf{A}$ and have the same eigenvectors.
- A normal matrix is Hermitian iff its eigenvalues are all real.
- A normal matrix is skew-Hermitian iff its eigenvalues all have zero real parts.
- A normal matrix is unitary iff its eigenvalues all have an absolute value of 1.
- If $\mathbf{A}$ and $\mathbf{B}$ are normal and $\mathbf{AB} = \mathbf{BA}$ then $\mathbf{AB}$ is normal.
- *(2.5.4)* $\|A\|_\mathrm{F}^2 = \sum_{i=1}^{n} |\lambda_i|^2$.

$\mathbf{x}^H\mathbf{A}^H\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$).

- *(4.1.5, Spectral Theorem)* $\mathbf{A}$ is Hermitian iff there is a unitary matrix $\mathbf{U} \in \mathcal{M}_n$ and a real diagonal matrix $\boldsymbol{\Lambda} \in \mathcal{M}_n$ s.t. $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H$.
- If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian then so are $\mathbf{AB} + \mathbf{BA}$ and $j(\mathbf{AB} - \mathbf{BA})$.
- *(4.2.2, Rayleigh-Ritz)* $\min(\lambda) \leq \mathbf{x}^H\mathbf{Ax} \leq \max(\lambda)$, $\mathbf{x}^H\mathbf{x} = 1$.

Positive Definite Matrices *(4.1.1)*. A Hermitian matrix $\mathbf{A} \in \mathcal{M}_n$ is *positive definite* (pd) if $\mathbf{x}^H\mathbf{Ax} > 0$ for all nonzero $\mathbf{x} \in \mathbb{C}^n$. If $\mathbf{x}^H\mathbf{Ax} \geq 0$, $\mathbf{A}$ is said to be *positive semidefinite* (psd).

- *(7.1.2)* Any principal submatrix of a pd matrix is pd.
- *(7.1.3)* Any nonnegative linear combination of psd matrices is psd.
- *(7.1.4,5)* For p(s)d $\mathbf{A} \in \mathcal{M}_n$, λ_i (for $i = 1, \ldots, n$), $\det \mathbf{A}$, $\operatorname{tr} \mathbf{A}$, and all principal minors are positive (nonnegative).
- *(7.2.1)* $\mathbf{A} \in \mathcal{M}_n$ is p(s)d iff all eigenvalues are positive (nonnegative).
- *(7.2.8)* $\mathbf{A} \in \mathcal{M}_n$ is pd iff there is a nonsingular $\mathbf{C} \in \mathcal{M}_n$ s.t. $\mathbf{A} = \mathbf{C}^H\mathbf{C}$. Any solution $\mathbf{C}$ can be written as $\mathbf{C} = \mathbf{V}\mathbf{A}^{1/2}$ with $\mathbf{V} \in \mathcal{M}_n$ unitary.
- *(p409)* Any psd rank-m matrix $\mathbf{A} \in \mathcal{M}_n$ may be written as $\mathbf{A} = \mathbf{C}^H\mathbf{C}$ with some $\mathbf{C} \in \mathcal{M}_{m,n}$.
- *(7.1.6)* For pd $\mathbf{A} \in \mathcal{M}_n$ and $\mathbf{C} \in \mathcal{M}_{n,m}$, $\mathbf{C}^H\mathbf{AC}$ is *psd*, and $\operatorname{rnk} \mathbf{C}^H\mathbf{AC} = \operatorname{rnk} \mathbf{C}$.

Skew-Hermitian Matrices. A matrix $\mathbf{K} \in \mathcal{M}_n$ is skew-Hermitian if $\mathbf{K} = -\mathbf{K}^H$.

- *(p175)* $\mathbf{S}$ is Hermitian iff $j\mathbf{S}$ is skew-Hermitian.
- $\mathbf{K}$ is skew-Hermitian iff $\mathbf{x}^H\mathbf{Ky} = -\mathbf{x}^H\mathbf{K}^H\mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$.
- Skew-Hermitian matrices are closed under addition, multiplication by a scalar, raising to an odd power, and (if nonsingular) inversion.
- *(p175)* $\mathbf{K}^2$ is Hermitian.
- *(p175)* The eigenvalues of a skew-Hermitian matrix are either 0 or pure imaginary.

Triangular Matrices *(0.9.3)*. A matrix $\mathbf{T} \in \mathcal{M}_n$ is upper (lower) triangular if $t_{ij} = 0$ for $j < i$ ($j > i$). Strictness if true for $j \leq i$ ($j \geq i$).

- Upper/lower triangular matrices are closed under addition, multiplication,

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tian and $\mathbf{S} \in \mathcal{M}_n$ nonsingular, with all eigenvalues arranged in increasing order, for all $k = 1, \ldots, n$ there is a positive real number θ_k in the range

$$\lambda_1(\mathbf{S}\mathbf{S}^H) \leq \theta_k \leq \lambda_n(\mathbf{S}\mathbf{S}^H) \quad \text{s.t.}$$
$$\lambda_k(\mathbf{S}\mathbf{A}\mathbf{S}^H) = \theta_k \lambda_k(\mathbf{A}).$$

Inverse Matrix. For any $\mathbf{A} \in \mathcal{M}_{m,n}$, having a SVD $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H$, there is a *Moore-Penrose generalized inverse* or *pseudoinverse* $\mathbf{A}^\dagger = \mathbf{V}\boldsymbol{\Sigma}^\dagger\mathbf{U}^H$, where $\boldsymbol{\Sigma}^\dagger$ is the transpose of $\boldsymbol{\Sigma}$ in which the positive singular values are replaced by their reciprocals.

- *(p421)* $\mathbf{A}^\dagger\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\dagger$ are Hermitian.
- *(p421)* $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$ and $\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$.
- *(0.5)* For any nonsingular $\mathbf{A} \in \mathcal{M}_n$, there is a unique $\mathbf{A}^\dagger = \mathbf{A}^{-1}$ s.t. $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.
- Any full-rank skinny $\mathbf{A} \in \mathcal{M}_{m,n}$, $m \geq n$, has a left inverse $\mathbf{B} \in \mathcal{M}_{n,m}$ s.t. $\mathbf{B}\mathbf{A} = \mathbf{I}_n$. The left inverse with the smallest norm is the pseudoinverse $\mathbf{B} = \mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$.
- Any full-rank fat $\mathbf{A} \in \mathcal{M}_{m,n}$, $m \leq n$, has a right inverse $\mathbf{B} \in \mathcal{M}_{n,m}$ s.t. $\mathbf{A}\mathbf{B} = \mathbf{I}_m$. The right inverse with the smallest norm is the pseudoinverse $\mathbf{B} = \mathbf{A}^\dagger = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^{-1}$
- $(\mathbf{A}^H)^\dagger = (\mathbf{A}^\dagger)^H$, and $(\mathbf{A}^H)^{-1} = (\mathbf{A}^{-1})^H$ (if $\mathbf{A}$ nonsingular).
- *(0.7.4, Matrix Inversion Lemma)* Be $\mathbf{A} \in \mathcal{M}_m$ and $\mathbf{R} \in \mathcal{M}_n$ nonsingular, and $\mathbf{X} \in \mathcal{M}_{m,n}$ and $\mathbf{Y} \in \mathcal{M}_{n,m}$. Then
$$(\mathbf{A} + \mathbf{X}\mathbf{R}\mathbf{Y})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{X}\left(\mathbf{R}^{-1} + \mathbf{Y}\mathbf{A}^{-1}\mathbf{X}\right)^{-1}\mathbf{Y}\mathbf{A}^{-1}$$
$$(c\mathbf{I}_m + \mathbf{X}\mathbf{Y})^{-1} = \frac{1}{c}\left(\mathbf{I}_m - \mathbf{X}\left(c\mathbf{I}_n + \mathbf{Y}\mathbf{X}\right)^{-1}\mathbf{Y}\right)$$
$$(c\mathbf{I}_m + \mathbf{X}\mathbf{X}^H)^{-1} = \frac{1}{c}\left(\mathbf{I}_m - \mathbf{X}\left(c\mathbf{I}_n + \mathbf{X}^H\mathbf{X}\right)^{-1}\mathbf{X}^H\right).$$
- For $\mathbf{X} \in \mathcal{M}_{m,n}$ and $\mathbf{Y} \in \mathcal{M}_{n,m}$,
$$\mathbf{Y}(c\mathbf{I} + \mathbf{X}\mathbf{Y})^{-1} = (c\mathbf{I} + \mathbf{Y}\mathbf{X})^{-1}\mathbf{Y}.$$
- For nonsingular $\mathbf{A}, \mathbf{B} \in \mathcal{M}_m$
$$\mathbf{A}^{-1} = \mathbf{B}^{-1} + \mathbf{B}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{A}^{-1}.$$
- If $\mathbf{A} \in \mathcal{M}_m$ and the inverses exist, any pair of $\mathbf{A}$, $(c\mathbf{I} - \mathbf{A})^{-1}$, and $(c\mathbf{I} + \mathbf{A})^{-1}$ commutes. Further,
$$\mathbf{A}(\mathbf{I} - \mathbf{A})^{-1} = (\mathbf{I} - \mathbf{A})^{-1} - \mathbf{I}$$
$$\mathbf{A}(\mathbf{I} + \mathbf{A})^{-1} = \mathbf{I} - (\mathbf{I} + \mathbf{A})^{-1}$$

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rnk $\mathbf{A} \geq$ # nonzero eigenvalues.
(p46) Equality for diagonalizable $\mathbf{A}$.

Trace *(0.4)*. For $\mathbf{A} \in \mathcal{M}_n$, $\operatorname{tr}\mathbf{A} = \sum_{i=1}^n a_{ii}$.

- *(1.2.12)* $\operatorname{tr}\mathbf{A} = \sum_{i=1}^n \lambda_i(\mathbf{A})$.
- $\operatorname{tr}(\alpha\mathbf{A}) = \alpha\operatorname{tr}(\mathbf{A})$.
- $\operatorname{tr}(\mathbf{A}^T) = \operatorname{tr}(\mathbf{A})$.
- $\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B})$.
- $\operatorname{tr}(\mathbf{A}\mathbf{B}) = \operatorname{tr}(\mathbf{B}\mathbf{A})$.
- $\operatorname{tr}(\mathbf{a}\mathbf{b}^T) = \mathbf{a}^T\mathbf{b}$.
- $\operatorname{tr}(\mathbf{a}\mathbf{b}^H) = \overline{\mathbf{a}^H\mathbf{b}}$.
- $\operatorname{tr}(\mathbf{A}\mathbf{B}\mathbf{C}\mathbf{D}) = \operatorname{tr}(\mathbf{B}\mathbf{C}\mathbf{D}\mathbf{A}) = \operatorname{tr}(\mathbf{C}\mathbf{D}\mathbf{A}\mathbf{B}) = \operatorname{tr}(\mathbf{D}\mathbf{A}\mathbf{B}\mathbf{C})$.
- $\operatorname{tr}(\mathbf{A} \otimes \mathbf{B}) = \operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{B})$.

Determinant *(0.3)*. For $\mathbf{A} \in \mathcal{M}_n$,
$$\det\mathbf{A} = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det\mathbf{A}_{ij} = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det\mathbf{A}_{ij},$$
where $\mathbf{A}_{ij} \in \mathcal{M}_{n-1}$ is the submatrix obtained by deleting row i and column j.

- *(1.2.12)* $\det\mathbf{A} = \Pi_{i=1}^n \lambda_i(\mathbf{A})$.
- *(0.3.1)* $\det\mathbf{A}^T = \det\mathbf{A}$.
- *(0.3.1)* $\det\mathbf{A}^H = \overline{\det\mathbf{A}}$.
- $\det c\mathbf{A} = c^n \det\mathbf{A}$.
- Interchanging any pair of columns of $\mathbf{A}$ multiplies $\det\mathbf{A}$ by -1 (likewise rows).
- Multiplying any column of $\mathbf{A}$ by c multiplies $\det\mathbf{A}$ by c (likewise rows).
- Adding any multiple of one column onto another column leaves $\det\mathbf{A}$ unaltered (likewise rows).
- For $\mathbf{A}, \mathbf{B} \in \mathcal{M}_n$, $\det\mathbf{A}\mathbf{B} = \det\mathbf{A}\,\det\mathbf{B}$.
- For nonsingular $\mathbf{A} \in \mathcal{M}_n$, $\det(\mathbf{A} + \mathbf{x}\mathbf{y}^H) = \det(\mathbf{A})\left(1 + \mathbf{y}^H\mathbf{A}^{-1}\mathbf{x}\right)$.
- If $\mathbf{A} \in \mathcal{M}_n$ and $\mathbf{D} \in \mathcal{M}_k$,
$$\det\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \det\mathbf{Q} = \det(\mathbf{A})\det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}) = \det(\mathbf{D})\det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}).$$
The quantity $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is called the *Schur complement* of $\mathbf{A}$ in $\mathbf{Q}$.
For $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{M}_n$
$$\mathbf{A}\mathbf{C} = \mathbf{C}\mathbf{A}: \ \det\mathbf{Q} = \det(\mathbf{A}\mathbf{D} - \mathbf{C}\mathbf{B})$$
$$\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}: \ \det\mathbf{Q} = \det(\mathbf{D}\mathbf{A} - \mathbf{C}\mathbf{B})$$
$$\mathbf{D}\mathbf{B} = \mathbf{B}\mathbf{D}: \ \det\mathbf{Q} = \det(\mathbf{D}\mathbf{A} - \mathbf{B}\mathbf{C})$$
$$\mathbf{D}\mathbf{C} = \mathbf{C}\mathbf{D}: \ \det\mathbf{Q} = \det(\mathbf{A}\mathbf{D} - \mathbf{B}\mathbf{C})$$
- *(7.8.1, Hadamard's inequality)* If $\mathbf{A}$ is positive semidefinite, then $\det\mathbf{A} \leq \prod_{i=1}^n a_{ii}$.

- *(p111)* The direct sum $\mathbf{A}_1 \oplus \ldots \oplus \mathbf{A}_k$ is normal iff each $\mathbf{A}_j$ is normal.

Projection Matrices. A matrix $\mathbf{P} \in \mathcal{M}_n$ is a projection matrix if it is Hermitian and idempotent, i.e., $\mathbf{P}^H = \mathbf{P}^2 = \mathbf{P}$.

- $\mathbf{P}$ is positive semidefinite.
- $\mathbf{I} - \mathbf{P}$ is a projection matrix.
- With any subspace $\mathcal{S} \subseteq \mathbb{C}^n$, any vector $\mathbf{x} \in \mathbb{C}^n$ can be decomposed as $\mathbf{x} = \mathbf{P}_{\mathcal{S}}\,\mathbf{x} + \mathbf{P}_{\mathcal{S}}^{\perp}\mathbf{x}$ with $\mathbf{P}_{\mathcal{S}}^{\perp} = \mathbf{I}_n - \mathbf{P}_{\mathcal{S}}$ where $\mathbf{P}_{\mathcal{S}}$ is the unique projection matrix on $\mathcal{S}$. Also, $\mathcal{R}(\mathbf{P}_{\mathcal{S}}) = \mathcal{S}$.
- Be $\mathcal{S} = \mathcal{R}(\mathbf{A})$. Then the projection matrix on $\mathcal{S}$ is $\mathbf{P}_{\mathcal{S}} = \mathbf{A}\mathbf{A}^{\dagger}$.
  If $\mathbf{B} \in \mathcal{M}_{n,k}$ is a basis for the subspace $\mathcal{S}$, then $\mathbf{P}_{\mathcal{S}} = \mathbf{B}(\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H$.
  If $\mathbf{U}$ is a unitary basis for the subspace $\mathcal{S}$, then $\mathbf{P}_{\mathcal{S}} = \mathbf{U}\mathbf{U}^H$.
- If $\mathbf{P}$ and $\mathbf{Q}$ are projection matrices, then the following are equivalent:
  - $\mathbf{P} - \mathbf{Q}$ is a projection matrix
  - $\mathbf{P} - \mathbf{Q}$ is positive semidefinite
  - $\mathbf{PQ} = \mathbf{Q}$
  - $\mathbf{QP} = \mathbf{Q}$

Nonderogatory Matrices *(1.4.4)*. A matrix $\mathbf{A} \in \mathcal{M}_n$ is nonderogatory if every eigenvalue has geometric multiplicity 1.

raising to an integer power, and (if nonsingular) inversion.

- $\lambda_i = t_{ii}$, $i = 1, \ldots, n$.
- $\det \mathbf{T} = \Pi_{i=1}^{n} t_{ii}$.
- $\operatorname{rnk} \mathbf{T} \geq \#$ of nonzero t_{ii}.
- *(0.9.4, p62)* For a block-triangular
$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$
the eigenvalues are those of $\mathbf{A} \in \mathcal{M}_n$ together with those of $\mathbf{C} \in \mathcal{M}_m$, c.m. Thus, $\det \mathbf{T} = \det \mathbf{A} \det \mathbf{C}$. Further,
$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{C}^{-1} \\ \mathbf{0} & \mathbf{C}^{-1} \end{bmatrix}.$$

Unitary Matrices *(2.1)*. A complex matrix $\mathbf{U} \in \mathcal{M}_n$ is unitary if $\mathbf{U}^H\mathbf{U} = \mathbf{I}$.

- Unitary matrices are closed under multiplication, raising to an integer power, and inversion.
- $\mathbf{U}$ is unitary iff $\mathbf{U}^H$ is unitary.
- $\mathbf{U}$ is unitary iff $\|\mathbf{Ux}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$.
- $|\lambda_i| = 1$, $i = 1, \ldots, n$.
- $|\det \mathbf{U}| = 1$.
- A square matrix is unitary iff its columns form an orthonormal basis.
- The space of unitary matrices in $\mathcal{M}_n$ is parametrized by n^2 free real variables and called *unitary group*.
- $\mathbf{U}$ is unitary iff $\mathbf{U} = e^{j\mathbf{K}}$ or $j\mathbf{K} = \ln(\mathbf{U})$ for some Hermitian $\mathbf{K}$ (no 1:1 mapping).
- *(Cayley transform)* $\mathbf{U}$ is unitary iff $\mathbf{U} = (\mathbf{I} + j\mathbf{K})^{-1}(\mathbf{I} - j\mathbf{K})$ for some Hermitian $\mathbf{K}$.

Permutation Matrices *(0.9.5)*. $\mathbf{A} \in \mathcal{M}_n$ is a permutation matrix if its columns are a permutation of the columns of $\mathbf{I}$.

Matrix Relations

Similarity *(1.3)*. A matrix $\mathbf{A} \in \mathcal{M}_n$ is similar to a matrix $\mathbf{B} \in \mathcal{M}_n$ if there exists a nonsingular matrix $\mathbf{S} \in \mathcal{M}_n$ s.t. $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

- *(p43)* Similarity invariants are: trace, determinant, rank, characteristic polynomial, eigenvalues.

Unitary Equivalence *(2.2)*. A matrix $\mathbf{A} \in \mathcal{M}_n$ is unitarily equivalent (or unitarily similar) to a matrix $\mathbf{B} \in \mathcal{M}_n$ if there exists a unitary matrix $\mathbf{U} \in \mathcal{M}_n$ s.t. $\mathbf{B} = \mathbf{U}^H\mathbf{A}\mathbf{U}$.

- *(2.2.2)* Additional *unitary* similarity invariants are: Frobenius norm (and thus $\operatorname{tr} \mathbf{A}^H\mathbf{A}$).

$\mathbf{A}(\mathbf{I}+\mathbf{A})^{-1} = \mathbf{I} - (\mathbf{I}+\mathbf{A})^{-1}$.

Decompositions

Minimum-Rank Factorization *(0.4.6e)*. Every rank-k $\mathbf{A} \in \mathcal{M}_{m,n}$ may be written as $\mathbf{A} = \mathbf{XBY}$ with $\mathbf{X} \in \mathcal{M}_{m,k}$, $\mathbf{Y} \in \mathcal{M}_{k,n}$, and $\mathbf{B} \in \mathcal{M}_k$. In particular, a rank-1 matrix may be written as $\mathbf{A} = \mathbf{x}\mathbf{y}^H$.

Singular Value Decomposition *(7.3.5)*. Every rank-k matrix $\mathbf{A} \in \mathcal{M}_{m,n}$ may be written as

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H,$$

where $\mathbf{U} \in \mathcal{M}_m$ and $\mathbf{V} \in \mathcal{M}_n$ are unitary. The matrix $\mathbf{\Sigma} \in \mathcal{M}_{m,n}$ contains the k nonzero entries $\sigma_{11}, \ldots, \sigma_{kk}$ in nonincreasing order and zeros elsewhere.

- The real *singular values* $\sigma_i = \sigma_{ii}$ are the nonnegative square roots of the eigenvalues of $\mathbf{A}\mathbf{A}^H$ and therefore unique.
- The columns of $\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^H$, the columns of $\mathbf{V}$ are eigenvectors of $\mathbf{A}^H\mathbf{A}$.
- *(p422)* The first k columns of $\mathbf{U}$ form an orthonormal basis for the range of $\mathbf{A}$, the last $n-k$ columns of $\mathbf{V}$ form an orthonormal basis for the null space of $\mathbf{A}$.
- *(p418)* The singular values are invariant under conjugation, transposition, and left or right multiplication with a unitary matrix.
- $\operatorname{rnk} \mathbf{A} = \#$ nonzero singular values.

Schur Triangularization *(2.3)*. Every $\mathbf{A} \in \mathcal{M}_n$ is unitarily similar to an upper triangular matrix, i.e., $\mathbf{U}^H\mathbf{A}\mathbf{U} = \mathbf{T}$, with $t_{ii} = \lambda_i$, $i = 1, \ldots, n$. Neither $\mathbf{U}$ nor $\mathbf{T}$ are unique.

QR Factorization *(2.6)*. Every $\mathbf{A} \in \mathcal{M}_{n,m}$ with $n \geq m$ can be written as $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q} \in \mathcal{M}_{n,m}$ has orthonormal columns and $\mathbf{R} \in \mathcal{M}_m$ is upper triangular.

- If $\mathbf{A}$ is nonsingular, then $\mathbf{R}$ may be chosen s.t. all r_{ii} are positive, in which event both $\mathbf{Q}$ and $\mathbf{R}$ are unique.

Cholesky Decomposition *(7.2.9)*. A matrix $\mathbf{A} \in \mathcal{M}_n$ is positive definite iff there is a nonsingular lower triangular $\mathbf{L} \in \mathcal{M}_n$ with positive diagonal entries s.t. $\mathbf{A} = \mathbf{L}\mathbf{L}^H$ (or an upper triangular $\mathbf{R}$ with $\mathbf{A} = \mathbf{R}^H\mathbf{R}$).

kth Root *(7.2.6)*. For any positive (semi)definite matrix $\mathbf{A} \in \mathcal{M}_n$, there is a unique positive (semi)definite Hermitian kth ($k > 0$) root $\mathbf{B} \in \mathcal{M}_n$ s.t. $\mathbf{B}^k = \mathbf{A}$. This kth root is denoted by $\mathbf{B} = \mathbf{A}^{1/k}$.

- $\mathbf{AB} = \mathbf{BA}$.
- $\operatorname{rnk} \mathbf{B} = \operatorname{rnk} \mathbf{A}$.
- $\mathbf{B}$ is positive definite iff $\mathbf{A}$ is.
- $\mathbf{B} = p(\mathbf{A})$ for some polynomial $p(\cdot)$.
- *(p54)* Every diagonalizable matrix has a square root.

Matrix Operators

Kronecker Product *(II 4)*. For $\mathbf{A} \in \mathcal{M}_{m,n}$ and $\mathbf{B} \in \mathcal{M}_{p,q}$,

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \cdots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

- *(II 4.2)* $(\mathbf{A} \otimes \mathbf{B})^H = \mathbf{A}^H \otimes \mathbf{B}^H$
  $$(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$$
  $$(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}.$$
- *(II 4.2.10)* Mixed products:
  $$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}.$$
- Inversion: If $\mathbf{A}$ and $\mathbf{B}$ nonsingular, then
  $$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}.$$
- *(II 4.3)* $\operatorname{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A}) \operatorname{vec} \mathbf{B}$
  $$\operatorname{vec}(\mathbf{AB}) = (\mathbf{B}^T \otimes \mathbf{I}) \operatorname{vec} \mathbf{A} = (\mathbf{I} \otimes \mathbf{A}) \operatorname{vec} \mathbf{B}.$$
- *(II p. 252)*
  $\operatorname{vec}^T(\mathbf{Y})(\mathbf{A}\otimes\mathbf{B}) \operatorname{vec}(\mathbf{X}) = \operatorname{tr}\big(\mathbf{A}^T\mathbf{Y}^T\mathbf{B}\mathbf{X}\big)$

Hadamard Product *(II 5)*. For $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{m,n}$

$$\mathbf{A} \circ \mathbf{B} = [a_{ij}b_{ij}] \in \mathcal{M}_{m,n}.$$

- *(II 5.1.7)*
  $\operatorname{rnk} \mathbf{A} \circ \mathbf{B} \leq \operatorname{rnk} \mathbf{A} \operatorname{rnk} \mathbf{B}$.
- *(II p311)* $\det \mathbf{A} \circ \mathbf{B} \geq \det \mathbf{A} \det \mathbf{B}$.

lg.eps (0, 2)