The Impedance Concept and Its Application to Problems of Reflection, Refraction, Shielding and Power Absorption

By S. A. SCHELKUNOFF

This paper calls attention to the practical value of a more extended use of the impedance concept. It brings out a certain underlying unity in what otherwise appear diverse physical phenomena. Although an attempt has been made to trace the history of the concept of "impedance" and many interesting early suggestions have been found, reference to these lies beyond the scope of this paper. Apparently, Sir Oliver Lodge was the first to use the word "impedance," but the concept has been developed gradually as circumstances demanded through the efforts of countless workers.

The main body of the paper is divided into three parts: Part I, dealing with the exposition of the impedance idea as applied to different types of physical phenomena; Part II, in which the general formulæ are deduced for reflection and transmission coefficients; Part III, presenting some special applications illustrating the practical utility of the foregoing manner of thought.

THE term "impedance" has had an interesting history, in which one generalization has suggested another with remarkable rapidity. Introduced by Oliver Lodge,[1] it meant the ratio V/I in the special circuit comprised of a resistance and an inductance, I and V being the *amplitudes* of an alternating current and the driving force which produced it. This was soon extended to the somewhat more general circuit consisting of a resistance, an inductance coil and a condenser.[2] The usage did not develop much further until the use of

[1] Dr. Oliver Lodge, F.R.S., "On Lightning, Lightning Conductors, and Lightning Protectors," *Electrical Review*, May 3, 1889, p. 518

[2] It is interesting to note that the first impulse was to introduce a new word rather than to extend the meaning of the old term. Thus in 1892, F. Bedell and A. Crehore write as follows: "From the analogy of this equation to Ohm's law, we see that the expression $\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}$ is of the nature of a resistance, and is the apparent resistance of a circuit containing resistance, self-inductance and capacity. This expression would quite properly be called 'impedance' but the term impedance has for several years been used as a name for the expression $\sqrt{R^2 + L^2\omega^2}$, which is the apparent resistance of a circuit containing resistance and self-inductance only. We would suggest, therefore, that the word 'impediment' be adopted as a name for the expression $\sqrt{R^2 + \left(\frac{1}{C\omega} - L\omega\right)^2}$ which is the apparent resistance of a circuit containing resistance, self-induction and capacity, and the term impedance be retained in the more limited meaning it has come to have, that is $\sqrt{R^2 + L^2\omega^2}$, the

complex quantities, which had begun early in the nineteenth century among mathematicians, was popularized among engineers by Kennelly and Steinmetz. Then the proportionality relation $V = ZI$, which had previously been true only if V and I were interpreted as amplitudes, acquired a more general significance, for it was found that this relation could express the phase relationship as well, provided Z was given a suitable complex value.

An important generalization came when the close similarity of the laws connecting V and I in an electric circuit to those governing force and velocity in mechanical systems suggested that the ratio "force/velocity" be called a "mechanical impedance." This usage is now well nigh universal.

The next step was a short one: it amounted to extending the term to include also the ratio "force per unit area/flow per unit area"; that is, "pressure/flux." This usage is well known in such fields as acoustics, but it has not penetrated as far into the electrical field as convenience seems to warrant.

If we read these remarks with a view to appraising the direction in which future growth might be expected, we are immediately impressed by the strong trend toward interpreting the ratio "force/velocity" in an ever widening sense. It is my purpose in the present paper to indicate some further extensions which I have found to be useful. They are founded upon five basic ideas. The first is to recognize and use whenever possible analogies between dynamical fields in which the impedance concept is common and others (heat, for instance) in which it is not. The second is the idea of extending the V/I relation from circuits to radiation fields, in much the same way that the "force/velocity" concept has been made to embrace "pressure/flux" in hydrodynamics. The third is, to regard the impedance as an attribute of the *field* as well as of the body or the medium which supports the field, so that the impedance to a plane wave is not the same as the impedance to a cylindrical wave, even when both are propagated in infinite "free space." The fourth basic idea is that of assigning *direction* to the impedances of fields. This does not mean, however, that the impedances are vectors; in fact, they are not, since they fail to obey the laws of addition and the laws of transformation peculiar to vectors. And finally the fifth is a generalization of the idea of a *one-dimensional transmission line* or simply a *transmission line.* While

apparent resistance of a circuit containing resistance and self-induction only." Frederick Bedell and Albert C. Crehore, "Derivation and Discussion of the General Solution of the Current Flowing in a Circuit Containing Resistance, Self-Induction and Capacity, With Any Impressed Electromotive Force," *Journal A. I. E. E.*, Vol. IX, 1892, pp. 303–374, see p. 340.

all physical phenomena are essentially three-dimensional, frequently all but one are irrelevant and can be ignored or are relatively unimportant and can be neglected. In the mathematical language, this means that only one coordinate (distance, angle, etc.) is retained explicitly in the *equations of transmission.*

The paper is divided into three parts. Part I discusses broadly the ratios to which the term "impedance" can appropriately be applied in a wide variety of physical fields, ranging from electric circuits and heat conduction to electromagnetic radiation. In this part the concept is gradually broadened until at the end it has acquired the property of direction mentioned above. Parts II and III consider the general laws governing reflection, refraction, shielding and power absorption, and rephrase them as theorems regarding the generalized impedances. To make the illustrations more effective, familiar examples are chosen.

PART I

THE IMPEDANCE CONCEPT

Electric Circuits

In an electric circuit comprised of a resistance R and an inductance L, the instantaneous voltage-current relation is described by the following differential equation

$$L\frac{dI_0}{dt} + RI_0 = V_0, \tag{1}$$

where V_0 is the applied electromotive force. If V_0 varies harmonically with frequency f, ultimately I_0 will also vary harmonically with frequency f. What happens is that the solution of (1) consists of two parts, the *transient* part and the *steady state* part, the former decreasing exponentially with time and the latter being periodic.

The steady state solution of (1), or indeed of the most general linear differential equation with constant coefficients, can be found by means of a simple mathematical device based upon the use of complex numbers. Thus if V_0 and I_0 vary harmonically, they may be regarded as real parts of the corresponding complex expressions $Ve^{i\omega t}$ and $Ie^{i\omega t}$, where $f = \omega/2\pi$ is the frequency. The quantities V and I are complex numbers whose moduli represent the amplitudes and whose phases are the initial phases (at the instant $t = 0$) of the electromotive force and the electric current. The time rate of change of I_0 is then the real part of the derivative of $Ie^{i\omega t}$, that is, the real part of $i\omega Ie^{i\omega t}$.

If we form another equation after the pattern of (1), replacing I_0 and V_0 by the *imaginary* part of $Ie^{i\omega t}$ and $Ve^{i\omega t}$, and add the new

equation to (1), we shall have

$$L\frac{d(Ie^{i\omega t})}{dt} + R(Ie^{i\omega t}) = V_0 e^{i\omega t}.$$

Differentiating and cancelling the time factor $e^{i\omega t}$, we obtain

$$(R + i\omega L)I = V.$$

The ratio $Z = V/I = Ve^{i\omega t}/Ie^{i\omega t}$ is called the *impedance* of the electric circuit. In the present instance

$$Z = R + i\omega L.$$

In general, the impedance $Z = R + iX$ has a real and an imaginary part, the former being the *resistive* component of the impedance and the latter the *reactive.*

Mechanical Circuits

Linear oscillations of a mass in a resisting medium are described by equations identical with (1) and (2) except for the customary difference in lettering

$$m\frac{d(ve^{i\omega t})}{dt} + r(ve^{i\omega t}) = Fe^{i\omega t}.$$

In this equation, v represents the velocity and F the applied force, m the mass and r the resistance coefficient. The mechanical impedance is then

$$Z = r + i\omega m.$$

Similarly, for torsional vibrations the impedance is defined as the ratio "torque/angular velocity."

Electric Waves in Transmission Lines

Let x be the distance coordinate specifying a typical section of an electric transmission line. Let the complex quantities V and I be the voltage across and the electric current in the transmission line.[3] Then the space rate of change of the voltage is proportional to the current and the space rate of change of the current is proportional to the voltage

$$\frac{dV}{dx} = -ZI, \qquad \frac{dI}{dx} = -YV. \tag{2}$$

[3] The time factor $e^{i\omega t}$ is usually implicit.

The coefficients of proportionality Z and Y are known as the distributed *series impedance* and the distributed *shunt admittance* of the line; they depend upon the distributed series resistance R, shunt conductance G, series inductance L and shunt capacity C in the following manner:

$$Z = R + i\omega L, \qquad Y = G + i\omega C. \tag{3}$$

In a generalized transmission line Z and Y may be functions of x and may depend upon ω in a more complicated manner than that suggested in (3).

If Z and Y are independent of x, (2) possesses two exponential solutions:

$$\begin{aligned} I^+ &= Ae^{-\Gamma x + i\omega t}, & V^+ &= Z_0 I^+; \\ I^- &= Be^{\Gamma x + i\omega t}, & V^- &= -Z_0 I^-; \end{aligned} \tag{4}$$

where

$$\Gamma = \alpha + i\beta = \sqrt{ZY}, \qquad Z_0 = \sqrt{\frac{Z}{Y}} = \frac{\Gamma}{Y} = \frac{Z}{\Gamma}.$$

It is customary to designate by Γ that value of the square root which is in the first quadrant of the complex plane or on its boundaries; the other value of the square root is $-\Gamma$.

The two "secondary" constants Γ and Z_0 are called, respectively, the *propagation constant* and the *characteristic impedance.* The real part α of the propagation constant is the *attenuation constant* and β is the *phase constant.*

Equations (4) represent *progressive* waves because an observer moving along the line with a certain finite velocity beholds an unchanging phase of V and I. This velocity c is called the *phase velocity* of the wave. Setting $x = ct$ in the upper pair of (4), we obtain the condition for the stationary phase

$$-\beta c + \omega = 0, \qquad c = \frac{\omega}{\beta}.$$

Hence, V^+ and I^+ represent a wave traveling in the positive x-direction. Similarly we find that V^- and I^- represent a wave traveling in the opposite direction.

Consider two points in which the phases of V and I differ by 2π when observed at the same instant; the distance λ between these points is called the wave-length. By definition

$$\beta\lambda = 2\pi, \qquad \lambda = \frac{2\pi}{\beta}.$$

If the transmission line is non-uniform, that is, if Z and Y are functions of x, then the solutions of (2) are usually more complicated. In any case, however, there are two linearly independent solutions $I^+(x)$ and $I^-(x)$ in terms of which the most general solution can always be expressed

$$I(x) = AI^+(x) + BI^-(x).$$

These independent solutions may represent either progressive waves in two opposite directions or certain convenient combinations of such waves.

The corresponding V-functions are found by differentiation from (2); thus

$$V^+(x) = -\frac{1}{Y}\frac{dI^+}{dx}, \qquad V^-(x) = -\frac{1}{Y}\frac{dI^-}{dx}.$$

The *impedance* of the V^+, I^+-wave is then

$$Z_0^+(x) = \frac{V^+(x)}{I^+(x)} = -\frac{1}{YI^+}\frac{dI^+}{dx} = -\frac{1}{Y}\frac{d}{dx}(\log I^+).$$

Similarly the impedance of the V^-, I^--wave is

$$Z_0^-(x) = -\frac{V^-(x)}{I^-(x)} = \frac{1}{YI^-}\frac{dI^-}{dx} = \frac{1}{Y}\frac{d}{dx}(\log I^-). \tag{5}$$

The negative sign in (5) is merely a matter of convention: the "positive" and the "negative" directions of the transmission line are so defined that the real parts of Z_0^+ and Z_0^- are positive.

In general, Z_0^+ and Z_0^- are not equal to each other. Moreover, there is a considerable amount of arbitrariness in our choice of the basic solutions I^+ and I^-. Thus, we are brought face to face with the fact that we must regard the impedance as an attribute of the wave as well as of the transmission line. This point of view will become even more prominent when we come to deal with the wave transmission in three-dimensional media. There even progressive waves may have different characters (they may be plane, cylindrical, spherical, etc.) and the impedances of the *same* medium to these waves will be different. And naturally, it goes without saying that the impedances to like waves in *different* media may also be different. One could, perhaps, take the position that geometrically similar waves in different media are not really alike if the corresponding "force/velocity" ratios are not equal and that under all circumstances the "impedance" is the property of a wave. However, "intrinsic im-

pedance" will be used to designate a constant of the medium without reference to any particular wave.

Vibrating Strings

In strings under constant tension τ, simply periodic waves may be described by the following two equations:

$$\frac{dF}{dx} = -(r + i\omega m)v, \qquad \frac{dv}{dx} = -\frac{i\omega}{\tau}F,$$

where m is the mass and r the resistance per unit length of the string. The variable F represents the force on a typical point of the string at right angles to the string and v is the velocity at that point.

Hence the characteristic impedance and the propagation constant are given by

$$Z_0 = \sqrt{\frac{(r + i\omega m)\tau}{i\omega}}, \qquad \Gamma = \sqrt{(r + i\omega m)\frac{i\omega}{\tau}}.$$

In the non-dissipative case we have simply

$$Z_0 = \sqrt{m\tau}, \qquad \Gamma = i\omega\sqrt{\frac{m}{\tau}}.$$

Heat Waves

Transmission of heat waves is also a special case of the generalized transmission line theory. In the one-dimensional case we have

$$\frac{\partial T}{\partial x} = -\frac{v}{K}, \qquad \frac{\partial v}{\partial x} = -c\delta\frac{\partial T}{\partial t},$$

where: T is the temperature, v the rate of heat flow, K the thermal conductivity, δ the density and c the specific heat. For simply periodic waves, we obtain

$$\frac{dT}{dx} = -\frac{1}{K}v, \qquad \frac{dv}{dx} = -i\omega c\delta T.$$

Thus the characteristic impedance and the propagation constant of heat waves are

$$Z_0 = \frac{1}{\sqrt{i\omega c\delta K}}, \qquad \Gamma = \sqrt{\frac{i\omega c\delta}{K}}.$$

The ratio "the temperature of the source/the rate of heat flow from the source" is the impedance "seen" by the heat source.

Electromagnetic Waves

The transmission equations of *uniform linearly polarized*[4] plane waves are:

$$\frac{dE}{dx} = -i\omega\mu H, \qquad \frac{dH}{dx} = -(g + i\omega\epsilon)E,$$

where: E is the electric intensity, H the magnetic intensity, and g, ϵ, μ are, respectively, the conductivity, the dielectric constant and permeability of the medium. These equations are of the same form as (2). Even the physical meanings of E and H are closely related to those of V and I; thus E is V per unit length and H is I per unit length.

The propagation constant and the characteristic impedance of an unbounded medium to linearly polarized plane waves are:

$$\sigma = \sqrt{i\omega\mu(g + i\omega\epsilon)}, \qquad \eta = \sqrt{\frac{i\omega\mu}{g + i\omega\epsilon}} = \frac{\sigma}{g + i\omega\epsilon} = \frac{i\omega\mu}{\sigma}.$$

These constants are so directly related to the fundamental electromagnetic constants of the medium that they themselves may be regarded as fundamental constants. On this account, we call σ and η, respectively, the *intrinsic propagation constant* and the *intrinsic impedance of the medium.* The intrinsic impedance will frequently occur as a multiplier in the expressions for the impedances of various types of waves.

The intrinsic impedance of a non-dissipative medium is simply $\eta = \sqrt{\mu/\epsilon}$; in air, this is equal to 120π or approximately 377 ohms.* Thus in the uniform linearly polarized plane wave traveling in free space, the relation between E and H is

$$E = 120\pi H \quad \text{or} \quad E \doteq 377H,$$

provided the positive directions of E and H are properly chosen.

An electromagnetic field of general character can be described by means of three electric components E_x, E_y, E_z, and three magnetic components H_x, H_y, H_z. We can form the following matrix whose components can be regarded as impedances:

[4] In this connection the word "uniform" is used to mean that equiphase planes are also equi-amplitude planes.

* See the letter from G. A. Campbell to Dean Harold Pender reproduced at the end of this paper.

$$\left\| \begin{matrix} \frac{E_x}{H_x}, & \frac{E_x}{H_y}, & -\frac{E_x}{H_z} \\ -\frac{E_y}{H_x}, & \frac{E_y}{H_y}, & \frac{E_y}{H_z} \\ \frac{E_z}{H_x}, & -\frac{E_z}{H_y}, & \frac{E_z}{H_z} \end{matrix} \right\|$$

The algebraic signs preceding the ratios of components with different subscripts are assigned as follows. If a right-hand screw is rotated through 90° from the positive axis indicated by the subscript in the numerator toward the positive axis indicated by the subscript in the denominator, it will advance either in the positive or in the negative direction of the remaining axis. In the former case the ratio is given the positive sign and in the latter the negative sign. This convention happens to be particularly convenient in expressions for the Poynting vector.

Thus two impedances are associated with any pair of perpendicular directions, the x-axis and the y-axis, let us say; these impedances are:

$$Z_{xy} = \frac{E_x}{H_y}, \qquad Z_{yx} = -\frac{E_y}{H_x}.$$

If these two impedances are equal, then we define the *impedance in the direction of the positive z-axis* as follows:

$$Z_z = \frac{E_x}{H_y} = -\frac{E_y}{H_x}.$$

Similar definitions hold for the impedances in other directions.

While the impedances as now defined possess an attribute of direction, they are neither vectors nor tensors because they do not add in the proper fashion. However, in practical applications this lack of vectorial properties does not seem to be a drawback.

The above definitions can be extended to other systems of coordinates. Let r be the distance of a point P (r, θ, φ) from the origin of the spherical coordinate system, θ the polar angle or colatitude and φ the meridian angle or the longitude (Fig. 1). Then the "radial" impedance in the outward direction is defined as

$$Z_r = \frac{E_\theta}{H_\varphi} = -\frac{E_\varphi}{H_\theta}, \tag{6}$$

provided the two ratios of the field components are equal. The radial impedance looking *toward* the origin is defined as the negative of (6). Similarly the "meridian" impedance in the direction of increasing θ

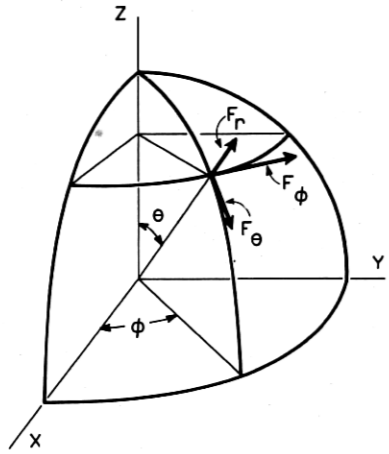

Fig. 1—Spherical coordinates. The positive directions of r-, θ-, and φ-components of a vector are, respectively, the directions of increasing r, θ, and φ.

and the impedance in the direction of increasing φ are:

$$Z_\theta = -\frac{E_r}{H_\varphi} = \frac{E_\varphi}{H_r}, \qquad Z_\varphi = \frac{E_r}{H_\theta} = -\frac{E_\theta}{H_r}.$$

In cylindrical coordinates we have (Fig. 2):

$$Z_\rho = -\frac{E_z}{H_\varphi} = \frac{E_\varphi}{H_z}, \qquad Z_\varphi = \frac{E_z}{H_\rho} = -\frac{E_\rho}{H_z}, \qquad Z_z = \frac{E_\rho}{H_\varphi} = -\frac{E_\varphi}{H_\rho}.$$

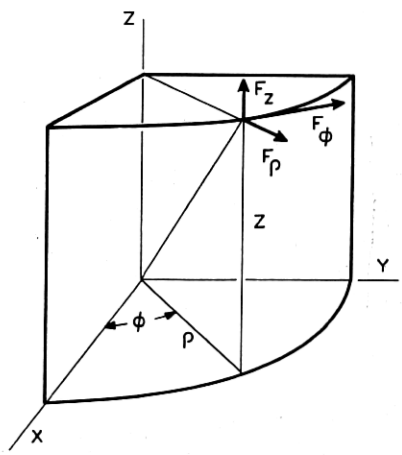

Fig. 2—Cylindrical coordinates. The positive directions of ρ-, φ-, and z-components of a vector are, respectively, the directions of increasing ρ, φ, and z.

Usually it is only one of the entire set of three-dimensional impedances that is of particular importance, the preferred direction being frequently the direction of the wave under consideration. When the ratios involved in the above definitions are unequal, it is expedient to resolve the field into component fields for which the ratios are equal. We shall now consider some special examples.

The field of the spherical electromagnetic wave emitted by a Hertzian doublet is known to be

$$\begin{aligned} E_\theta^+ &= \frac{i\omega\mu Ile^{-\sigma r}}{4\pi r}\left(1 + \frac{1}{\sigma r} + \frac{1}{\sigma^2 r^2}\right)\sin\theta, \\ E_r^+ &= \frac{\eta Ile^{-\sigma r}}{2\pi r^2}\left(1 + \frac{1}{\sigma r}\right)\cos\theta, \\ H_\varphi^+ &= \frac{\sigma Ile^{-\sigma r}}{4\pi r}\left(1 + \frac{1}{\sigma r}\right)\sin\theta, \end{aligned} \tag{7}$$

where: Il is the moment of the doublet in ampere-meters, r is the distance from the doublet, θ the angle made by a typical direction in space with the axis of the doublet, and φ is the angle between two planes containing the doublet, one of which is kept fixed for reference. The radial impedance of this wave is

$$Z_r^+ = \frac{E_\theta}{H_\varphi} = \eta\,\frac{1 + \dfrac{1}{\sigma r} + \dfrac{1}{\sigma^2 r^2}}{1 + \dfrac{1}{\sigma r}}.$$

In a non-dissipative medium this becomes

$$Z_r^+ = \eta\,\frac{1 + \dfrac{1}{i\beta r} - \dfrac{1}{\beta^2 r^2}}{1 + \dfrac{1}{i\beta r}}.$$

At a distance large compared with the wave-length, the radial impedance to the spherical wave emitted by a doublet is substantially equal to the intrinsic impedance of the medium. Very close to the doublet (compared with the wave-length) the radial impedance is substantially a capacitive reactance; in fact, we have approximately $Z_0^+ = 1/i\omega\epsilon r$.

Reversing the sign of σ in (7), we obtain a spherical wave traveling toward the origin. At first sight, this *inward bound* wave appears to be the natural mate to the outward bound wave. Two such waves move in opposite directions in the same sense in which two plane

waves spread out from a plane source in an infinite homogeneous space. However, the analogy is not complete. The inward bound spherical wave cannot exist without an appropriate receiver of energy at the origin. In absence of such a receiver, the energy condenses at the origin and spreads outward again. The result of interference between two such progressive waves will be called the "internal" spherical wave.[5] It is natural to regard a thin spherical source in an infinite *homogeneous* medium as an analogue of a thin plane source and to consider the waves *on the two sides* of such a spherical source as the mates. In accordance with this idea the (+) and the (−) signs are used to distinguish between the waves produced by a source on its two sides rather than to indicate "progressive" waves moving in opposite directions. This attitude is not only a possible and a natural attitude but almost a necessary one in view of the fact that no generally applicable criterion is known by which "progressive" waves could be identified in any particular case. As often happens, in simple situations there is no need for arguing as to which attitude is the more proper one; thus the waves on the two sides of a plane source in an infinite homogeneous medium are two progressive waves moving in opposite directions.

The field of the internal spherical wave is

$$E_\theta^- = \frac{i\omega\mu A}{2\pi r}\left(\sinh \sigma r - \frac{\cosh \sigma r}{\sigma r} + \frac{\sinh \sigma r}{\sigma^2 r^2}\right)\sin \theta,$$

$$E_r^- = \frac{\eta A}{\pi r^2}\left(\frac{\sinh \sigma r}{\sigma r} - \cosh \sigma r\right)\cos \theta,$$

$$H_\varphi^- = \frac{\sigma A}{2\pi r}\left(\frac{\sinh \sigma r}{\sigma r} - \cosh \sigma r\right)\sin \theta.$$

The corresponding impedance is then

$$Z_r^- = -\frac{E_\theta^-}{H_\varphi^-} = \eta\,\frac{\sinh \sigma r - \dfrac{\cosh \sigma r}{\sigma r} + \dfrac{\sinh \sigma r}{\sigma^2 r^2}}{\cosh \sigma r - \dfrac{\sinh \sigma r}{\sigma r}}.$$

Close to the origin we have approximately

$$Z_r^- = \frac{2}{(g + i\omega\epsilon)r}.$$

[5] If the medium is non-dissipative, this wave is a *standing* wave; but, in general, it is simply a combination of two progressive waves in such proportions that the field is finite at the origin.

If the source of electromagnetic waves is a small coil rather than a small doublet, the field is

$$
\begin{aligned}
E_\varphi^+ &= -\frac{\eta\sigma^2 SIe^{-\sigma r}}{4\pi r}\left(1+\frac{1}{\sigma r}\right)\sin\theta,\\
H_\theta^+ &= \frac{\sigma^2 SIe^{-\sigma r}}{4\pi r}\left(1+\frac{1}{\sigma r}+\frac{1}{\sigma^2 r^2}\right)\sin\theta,\\
H_r^+ &= \frac{\sigma SIe^{-\sigma r}}{2\pi r^2}\left(1+\frac{1}{\sigma r}\right)\cos\theta.
\end{aligned}
\tag{8}
$$

In this equation I is the current in the loop and S is the area. The corresponding radial impedance is then:

$$
Z_r^+ = -\frac{E_\varphi^+}{H_\theta^+} = \eta\,\frac{1+\dfrac{1}{\sigma r}}{1+\dfrac{1}{\sigma r}+\dfrac{1}{\sigma^2 r^2}}.
$$

This impedance approaches η as r increases indefinitely. Close to the loop we have approximately

$$
Z_r^+ = i\omega\mu r. \tag{9}
$$

The field of the internal wave having the same type of amplitude distribution over equiphase surfaces as the diverging wave (8) is

$$
\begin{aligned}
E_\varphi^- &= \frac{\eta\sigma^2 A}{2\pi r}\left(\cosh\sigma r-\frac{\sinh\sigma r}{\sigma r}\right)\sin\theta,\\
H_\theta^- &= \frac{\sigma^2 A}{2\pi r}\left(\sinh\sigma r-\frac{\cosh\sigma r}{\sigma r}+\frac{\sinh\sigma r}{\sigma^2 r^2}\right)\sin\theta,\\
H_r^- &= \frac{\sigma A}{\pi r^2}\left(\frac{\sinh\sigma r}{\sigma r}-\cosh\sigma r\right)\cos\theta.
\end{aligned}
$$

The radial impedance to this wave is then

$$
Z_r^- = \frac{E_\varphi^-}{H_\theta^-} = \eta\,\frac{\cosh\sigma r-\dfrac{\sinh\sigma r}{\sigma r}}{\sinh\sigma r-\dfrac{\cosh\sigma r}{\sigma r}+\dfrac{\sinh\sigma r}{\sigma^2 r^2}}.
$$

Close to the origin we have approximately

$$
Z_r^- = \tfrac{1}{2}i\omega\mu r. \tag{10}
$$

A line doublet formed by two parallel electric current filaments produces a cylindrical wave. Close to the doublet (compared with

the wave-length) we have

$$H_\varphi^+ = \frac{Il}{2\pi\rho^2}\cos\varphi, \qquad H_\rho^+ = -\frac{Il}{2\pi\rho^2}\sin\varphi. \tag{11}$$

In this equation Il is the moment of the doublet per unit length, I being the current and l the distance between the filaments. These equations are well known in the elementary theory of electromagnetism. The electric field is obtainable from (11) with the aid of Faraday's law of electromagnetic induction. This field and the corresponding radial impedance are

$$E_z^+ = -\frac{i\omega\mu Il}{2\pi\rho}\cos\varphi, \qquad Z_\rho^+ = i\omega\mu\rho. \tag{12}$$

The exact field of the line doublet and the corresponding radial impedance are:

$$E_z^+ = -\frac{\eta\sigma^2 Il}{2\pi}K_1(\sigma\rho)\cos\varphi, \qquad H_\varphi^+ = -\frac{\sigma^2 Il}{2\pi}K_1'(\sigma\rho)\cos\varphi,$$

$$H_\rho^+ = -\frac{\sigma Il}{2\pi\rho}K_1(\sigma\rho)\sin\varphi, \qquad Z_\rho^+ = -\eta\frac{K_1(\sigma\rho)}{K_1'(\sigma\rho)}.$$

The internal cylindrical wave with the same relative amplitude distribution over equiphase surfaces as in the wave originated by the line doublet is[6]

$$E_z^- = i\omega\mu A\, I_1(\sigma\rho)\cos\varphi, \qquad H_\varphi^- = \sigma A\, I_1'(\sigma\rho)\cos\varphi,$$

$$H_\rho^- = \frac{A}{\rho}I_1(\sigma\rho)\sin\varphi, \qquad Z_\rho^- = \eta\frac{I_1(\sigma\rho)}{I_1'(\sigma\rho)}.$$

Close to the doublet we have approximately

$$E_z^- = i\omega\mu P\rho\cos\varphi, \qquad H_\varphi^- = P\cos\varphi,$$
$$H_\rho^- = P\sin\varphi, \qquad Z_\rho^- = i\omega\mu\rho.$$

Another familiar field is that produced by two parallel line charges in a perfect dielectric. Close to the doublet this field is

$$E_\varphi^+ = \frac{ql\sin\varphi}{2\pi\epsilon\rho^2}, \qquad E_\rho^+ = \frac{ql\cos\varphi}{2\pi\epsilon\rho^2},$$
$$H_z^+ = \frac{i\omega ql\sin\varphi}{2\pi\rho}, \qquad Z_\rho^+ = \frac{1}{i\omega\epsilon\rho}, \tag{13}$$

[6] The symbols $I_n(x)$ and $K_n(x)$ designate the modified Bessel functions as defined in G. N. Watson's "Bessel Functions."

ql being the moment of the doublet. The last equation is obtained from the first two with the aid of Ampére's law. The exact expressions for any medium are

$$E_\varphi^+ = -\frac{i\omega ql\sigma\eta}{2\pi} K_1'(\sigma\rho) \sin \varphi, \qquad E_\rho^+ = \frac{i\omega ql\eta}{2\pi\rho} K_1(\sigma\rho) \cos \varphi,$$

$$H_z^+ = \frac{i\omega ql\sigma}{2\pi} K_1(\sigma\rho) \sin \varphi, \qquad Z_\rho^+ = -\eta \frac{K_1'(\sigma\rho)}{K_1(\sigma\rho)}.$$

For an internal cylindrical wave, we have

$$E_\varphi^- = A\sigma I_1'(\sigma\rho) \sin \varphi, \qquad E_\rho^- = -\frac{A}{\rho} I_1(\sigma\rho) \cos \varphi,$$

$$H_z^- = -A(g + i\omega\epsilon) I_1(\sigma\rho) \sin \varphi, \qquad Z_\rho^- = \eta \frac{I_1'(\sigma\rho)}{I_1(\sigma\rho)}.$$

Close to the doublet this becomes substantially

$$E_\varphi^- = P \sin \varphi, \qquad E_\rho^- = -P \cos \varphi,$$

$$H_z^- = -P(g + i\omega\epsilon)\rho \sin \varphi, \qquad Z_\rho^- = \frac{1}{(g + i\omega\epsilon)\rho}.$$

In concluding this set of examples we shall emphasize the fact that the impedance to a wave depends upon the particular manner in which the applied electromotive force is distributed in space, in very much the same way as it depends upon the manner of distribution of this force in time, that is, upon the frequency of the wave. Just as the impedance has a meaning only if the applied electromotive force varies harmonically with a certain well defined frequency,[7] there are definite types of applied force distribution in space for which the impedance has a meaning and other types for which it has not. Arbitrary spatial distributions of force may be decomposed into "space harmonics" in a manner analogous to Fourier's frequency analysis of arbitrary time distributions of force. This is just another way of interpreting the well-known method of solving Maxwell's equations with the aid of characteristic wave functions.

Here is a simple example of the dependence of the impedance to a wave upon the manner of applied force distribution. Consider the wave generated by an infinite electric current filament of radius a

[7] Strictly speaking, the impedance concept is applicable to any impressed force which varies exponentially with time, the exponent being in general a complex number. The only exceptions are the exponents which are either zeros or infinities of the impedance function. Undamped impressed forces constitute merely an important subclass of exponential forces.

when the electromotive force driving the current is distributed uniformly along the filament. In this case we have

$$E_z = -\frac{\eta I K_0(\sigma\rho)}{2\pi a K_1(\sigma a)}, \qquad H_\varphi = \frac{I K_1(\sigma\rho)}{2\pi a K_1(\sigma a)}, \qquad Z_\rho = \eta \frac{K_0(\sigma\rho)}{K_1(\sigma\rho)},$$

where I is the current in the filament. On the other hand if the electromotive force is applied to the filament with a uniform progressive phase delay so that it varies along the filament as e^{-ikz} for instance, then the field and the impedance are

$$E_z = -\frac{\eta I K_0(\Gamma\rho)}{2\pi a K_1(\Gamma a)} e^{-ikz}, \qquad H_\varphi = \frac{I K_1(\Gamma\rho)}{2\pi a K_1(\Gamma a)} e^{-ikz},$$

$$Z_\rho = \eta \frac{K_0(\Gamma\rho)}{K_1(\Gamma\rho)}, \qquad \Gamma = \sqrt{\sigma^2 + k^2}.$$

PART II

REFLECTION, REFRACTION, SHIELDING AND POWER ABSORPTION—GENERAL FORMULÆ

Uniform Transmission Lines

While the following discussion refers specifically to an electric transmission line, the results apply to all generalized transmission lines of which the former may be considered typical. These results depend upon certain boundary conditions and are not influenced by the names of the variables.

Consider a semi-infinite transmission line terminated by a prescribed impedance Z_t. Suppose that an "impressed" wave is coming from infinity. If V_i and I_i are the voltage and the current, their ratio must equal the characteristic impedance Z_0 of the wave. On the other hand, the ratio of the voltage across the impedance Z_t to the current through it is Z_t by definition. Thus, unless Z_t is equal to Z_0, a "reflected" wave must originate at the terminal and travel backwards. Let V_r and I_r be the voltage and the current of the reflected wave at the terminal. The total values of the voltage and the current will be designated by V_t and I_t. Then at the terminal

$$I_i + I_r = I_t, \qquad V_i + V_r = V_t. \tag{14}$$

By (9) and by the definition of Z_t, we have

$$V_i = Z_0 I_i, \qquad V_r = -Z_0 I_r, \qquad V_t = Z_t I_t. \tag{15}$$

Designating the ratio of the characteristic impedance of the line to

the terminal impedance by k, we use (15) to rewrite (14) in the following form:

$$I_i + I_r = I_t, \qquad I_i - I_r = \frac{I_t}{k}, \qquad k = \frac{Z_0}{Z_t}.$$

Solving, we obtain the reflection and transmission coefficients:

$$\begin{aligned} R_I &= \frac{I_r}{I_i} = \frac{k-1}{k+1}, \qquad & R_V &= \frac{V_r}{V_i} = \frac{1-k}{1+k}, \\ T_I &= \frac{I_t}{I_i} = \frac{2k}{1+k}, \qquad & T_V &= \frac{V_t}{V_i} = \frac{2}{1+k}. \end{aligned} \tag{16}$$

Thus *when $k = 1$, that is when the terminal impedance equals the characteristic impedance, there is no reflection. When the ratio of the impedances is zero or infinity the reflection is complete: in the first case the current vanishes and the voltage is doubled, and in the second the current is doubled and the voltage vanishes. The amount of reflection is completely determined by the ratio of the impedances.*

The terminal impedance may be another semi-infinite transmission line and its characteristic impedance will play the part of Z_t. It is important to note that *neither the propagation constants nor the velocities of the wave in the lines have anything to do with reflection.* No reflection will take place if the lines have equal impedances and there will be reflection in the case of unequal impedances even if the velocities are the same.

The variables V and I can stand for any two physical quantities satisfying equations (2). It will be observed that if we disregard the physical significance of the variables V and I, the characteristic impedance can be defined either as the ratio V/I or as I/V. We are perfectly free to make our choice. It is evident from (16) that if we interchange V and I and replace k by its reciprocal, the expressions for the reflection and transmission coefficients remain unaltered.

Non-Uniform Transmission Lines

The foregoing analysis has to do only with *uniform* lines. In the case of non-uniform lines the impedances looking in the opposite directions may be different. These two impedances will be defined by the following equations:

$$Z_0^+ = \frac{V^+}{I^+}, \qquad Z_0^- = -\frac{V^-}{I^-},$$

where V^+, I^+ and V^-, I^- refer to the two waves.

At the terminal we have as before:

$$I_i + I_r = I_t, \qquad Z_0^+ I_i - Z_0^- I_r = Z_t I_t.$$

Hence the more general expressions for the reflection and transmission coefficients are:

$$R_I = \frac{Z_0^+ - Z_t}{Z_0^- + Z_t}, \qquad R_V = -\frac{Z_0^-}{Z_0^+} R_I,$$

$$T_I = \frac{Z_0^- + Z_0^+}{Z_0^- + Z_t}, \qquad T_V = \frac{Z_t}{Z_0^+} T_I.$$

These reflection and transmission coefficients can be expressed in terms of the ratios of the line impedances to the terminal impedance.

Shielding

When a source of electromagnetic waves is enclosed in a metallic box, the field outside the box is substantially weaker than it would have been in the absence of the box. The box is said to act as a "shield." Under some conditions, transmission of electromagnetic waves in free space and in the metallic shield is governed by equations of the form (1). In those cases the shielding effect can evidently be regarded as due to a reflection loss at the boundaries of the shield and to an attenuation loss in the shield itself. A schematic representation of a single layer shield is shown in Fig. 3. The source of disturbance

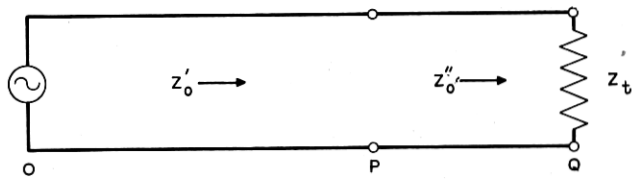

Fig. 3—Transmission line representation of a shield. The generator represents the source of the electromagnetic disturbance, the section OP the space surrounding the source, the section PQ the shield, and the impedance Z_t the space outside the shield.

is shown as a generator, the space around this source is represented by a piece of a transmission line OP, the shield by a piece PQ and the space outside the shield by the impedance Z_t.

The simplest case to consider is that of an *electrically thick* shield, in which the attenuation between P and Q is so great that waves reflected at Q do not affect appreciably the situation at P. In such a case the impedance at P looking toward Q equals the characteristic impedance Z_0'' and the same is true of the impedance at Q looking

toward P. The effect of the inserted piece is comprised of two independent reflections at P and Q and of attenuation with concomitant phase change between P and Q. Thus the transmission coefficients across PQ, that is, the ratios of the quantities at Q to the impressed quantities at P, are

$$T_I = T_{I,P}\, T_{I,Q}\, e^{-\Gamma'' l}, \qquad T_V = T_{V,P}\, T_{V,Q}\, e^{-\Gamma'' l},$$

where $T_{I,P}$ is the transmission coefficient for I at P and the remaining T's have similar meanings.

If PQ is a piece of a uniform transmission line inserted into a uniform semi-infinite line, $Z_t = Z_0'$. In this case, we have

$$T_I = T_V = \frac{4k}{(k+1)^2}\, e^{-\Gamma'' l},$$

where k is the ratio of the characteristic impedances. The factor $4k/(k+1)^2$ represents the reflection loss and $e^{-\alpha'' l}$ the attenuation loss.

Let us now assume that PQ is electrically short and that all the transmission lines in question are uniform. By the transmission line theory, the ratios of the total currents and voltages at P and Q are:

$$\frac{I_Q}{I_P} = \frac{Z_0''}{Z_0'' \cosh \Gamma'' l + Z_t \sinh \Gamma'' l},$$

$$\frac{V_Q}{V_P} = \frac{Z_t}{Z_t \cosh \Gamma'' l + Z_0'' \sinh \Gamma'' l}.$$

On the other hand, we have

$$\frac{I_P}{I_i} = \frac{2Z_0'}{Z_0' + Z_P}, \qquad \frac{V_P}{V_i} = \frac{2Z_P}{Z_0' + Z_P},$$

where Z_P is the impedance at P looking toward Q

$$Z_P = Z_0'' \frac{Z_t \cosh \Gamma'' l + Z_0'' \sinh \Gamma'' l}{Z_0'' \cosh \Gamma'' l + Z_t \sinh \Gamma'' l}.$$

The transmission coefficients across PQ can be represented as

$$T_I = \frac{I_Q}{I_i} = \frac{I_Q}{I_P} \cdot \frac{I_P}{I_i}, \qquad T_V = \frac{V_Q}{V_i} = \frac{V_Q}{V_P} \cdot \frac{V_P}{V_i}.$$

Making appropriate substitutions into this equation, we obtain

$$T_I = p(1 - qe^{-2\Gamma'' l})^{-1} e^{-\Gamma'' l}, \qquad T_V = \frac{Z_t}{Z_0'}\, T_I, \tag{17}$$

where

$$p = \frac{4Z_0'Z_0''}{(Z_0'' + Z_0')(Z_0'' + Z_t)},$$

$$q = \frac{(Z_0'' - Z_0')(Z_0'' - Z_t)}{(Z_0'' + Z_0')(Z_0'' + Z_t)}.$$

In the special case when $Z_t = Z_0'$, we have

$$p = \frac{4k}{(k+1)^2}, \qquad q = \left(\frac{k-1}{k+1}\right)^2, \qquad \text{and} \qquad T_V = T_I.$$

If PQ is electrically long, (17) becomes simply

$$T_I = pe^{-\Gamma''l}, \qquad T_V = \frac{Z_t}{Z_0'} T_I.$$

An interesting physical interpretation of (17) will follow if we expand the factor in parentheses into a series

$$T_I = pe^{-\Gamma''l} + pqe^{-3\Gamma''l} + pq^2e^{-5\Gamma''l} + \cdots.$$

The first term represents what remains of the original wave on the first passage through PQ. A part of the original wave is reflected back at Q and then partially re-reflected from P; the second term represents that fraction of the re-reflected wave which is transmitted beyond Q. The following terms represent succeeding reflections. In making this analysis, we must remember that $p = p_1p_2$ where p_1 and p_2 are respectively the transmission coefficients across the first and the second boundaries on the supposition that the inserted piece is infinitely long. Similarly, $q = q_1q_2$, the product of the two reflection coefficients.

Let us now consider a non-uniform transmission line. The propagation of a disturbance is no longer exponential and we introduce the ratios $\kappa^+ = V^+(x_2)/V^+(x_1)$ and $\kappa^- = V^-(x_1)/V^-(x_2)$ for the voltage ratios in the waves moving in opposite directions. In what follows x_1 and x_2 are the coordinates of the beginning and the end of the inserted piece. The transmission coefficient T across the insertion, that is, the ratio of the total quantity at $x = x_2$ to the impressed quantity at $x = x_1$, is then

$$T = p_1\kappa^+p_2 + (p_1\kappa^+)(q_2\kappa^-q_1\kappa^+)p_2 + (p_1\kappa^+)(q_2\kappa^-q_1\kappa^+)(q_2\kappa^-q_1\kappa^+)p_2 + \cdots.$$

This can be rewritten as follows:

$$T = p[1 + q\kappa + (q\kappa)^2 + (q\kappa)^3 + \cdots]\kappa^+ = \frac{p}{1 - q\kappa}\kappa^+,$$

where

$$p = p_1 p_2, \qquad q = q_1 q_2, \qquad \kappa = \kappa^+ \kappa^-.$$

The same formula applies of course to I provided we interpret the p's, q's and κ's as referring to the variable I rather than the variable V. If the inserted piece is electrically long, we have approximately

$$T = p\kappa^+.$$

In many practical applications the inserted piece is a uniform transmission line, so that

$$\kappa^+ = \kappa^- = e^{-\Gamma l}, \qquad \kappa = e^{-2\Gamma l},$$

where Γ is the propagation constant and l is the length of the piece. In this case

$$T = \frac{p}{1 - qe^{-2\Gamma l}} e^{-\Gamma l}.$$

Power Absorption and Radiation

The power transferred from left to right across P is the real part of the following function Ψ:

$$\Psi_P = \tfrac{1}{2} V_P I_P^* = \tfrac{1}{2} Z_P I_P I_P^*, \tag{18}$$

where the asterisk denotes the complex number conjugate to the one represented by the letter itself. The power absorbed by the impedance Z_t is

$$\Psi_T = \tfrac{1}{2} Z_t I_Q I_Q^*. \tag{19}$$

The difference between (18) and (19) represents the power absorbed by the section PQ.

The power absorbed by a shield is calculated in a similar manner. The energy flow per unit area of the shield is given by an expression closely analogous to (18); the tangential component of H appears in the place of I and Z_P is to be interpreted as the impedance in the direction normal to the shield. The formula is derivable from the Poynting expression for energy flow. Thus the power flow per unit area is $\frac{1}{2} Z_n H_t H_t^*$ where H_t is the tangential component of H and Z_n is the impedance in the direction normal to the shield.

PART III

REFLECTION, REFRACTION, SHIELDING AND POWER ABSORPTION

Special Applications

The general formulæ derived in the preceding part are directly applicable to a variety of special cases such as reflection of plane waves at a plane boundary, shielding action of cylindrical shields upon electric waves produced by an infinite parallel pair of electric current filaments, shielding action of spherical shields upon electric waves produced by a coil or a condenser, etc. Of course, most of these results have already been obtained and published, each special problem having been treated on its own merits rather than as a particular case of a general formula. For this reason, we shall confine ourselves largely to a discussion of those aspects of reflection which are particularly illuminated by the general point of view.

Cylindrical Waves

Consider two parallel wires carrying equal and opposite alternating currents. At a distance from the wires two or three times as large as their interaxial separation, the wave is substantially that of a line doublet and the radial impedance in free space is approximately [8] $i\omega\mu\rho$ so long as ρ is much less than the wave-length. This restriction on ρ is permissible in the present communication art. In metallic media this expression for the radial impedance is good only at very low frequencies. At high frequencies the radial impedance in metallic media is substantially [9] $\sqrt{i\omega\mu/g}$.

If the pair of wires is surrounded by a metal cylinder, the latter will act as a shield by virtue of reflections taking place at the boundary and attenuation through the shield.

The attenuation constant is substantially $\sqrt{\pi\mu g f}$ nepers per meter.[10] Thus the attenuation in logarithmic units through the shield is proportional to the first power of its thickness and to the square roots of the conductivity, the permeability and the frequency.

The reflection loss depends upon the impedance ratio. In the neighborhood of $f = 0$, the impedance ratio is seen to be equal to the ratio of the permeabilities. Consequently, at very low frequencies non-magnetic shields are relatively inefficient since there is no reflec-

[8] Equation (12).

[9] The approximate error is $1/2\sigma\rho$; at 10 kc. the error is about 2.5 per cent at a distance 1 cm. from the line source.

[10] This is true even at low frequencies if the shield is thin compared to its diameter. Otherwise, the cylindrical divergence of the wave must be taken into account.

tion loss. Inasmuch as the radial impedance in air is proportional to the first power of the frequency and in metal it is proportional only to the square root of the frequency, a point is reached beyond which the radial impedance in air always exceeds the radial impedance in metals. Thus the air-to-magnetic metal impedance ratio is less than unity near $f = 0$ and greater than unity for sufficiently high frequencies. Consequently, the absolute value of this impedance ratio is equal to unity at some intermediate frequency at which the reflection

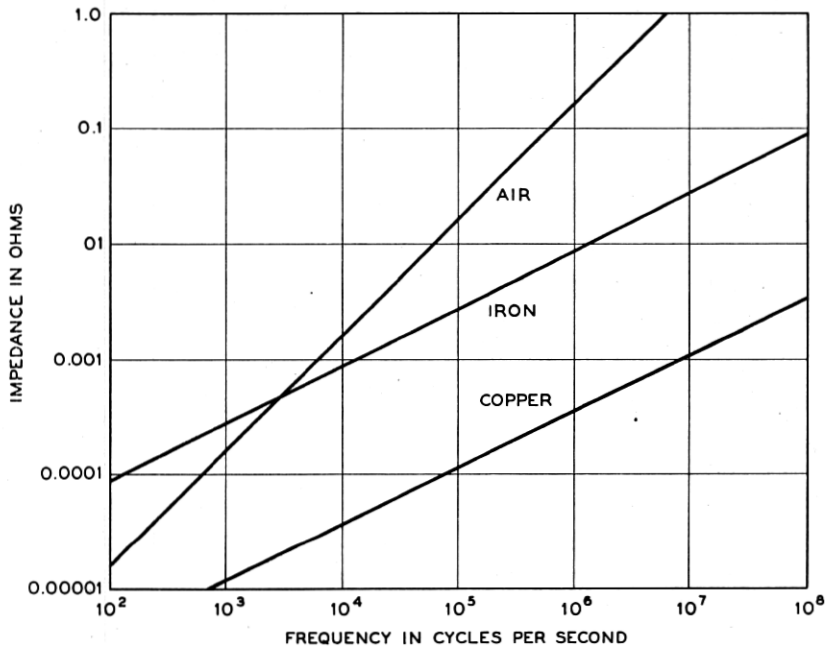

Fig. 4—The radial impedances in air, copper and iron at a distance of 2 centimeters from the axis for cylindrical waves generated by line doublets comprised of infinitely long electric current filaments. The conductivity of copper = 5.8005×10^7 mhos per meter, the conductivity of iron = 10^7 mhos per meter, the permeability of air and copper = 1.257×10^{-6} henries per meter, the permeability of iron = 1.257×10^{-4} henries per meter.

loss will be quite small.[11] Some typical curves of radial impedances are shown in Fig. 4. The radial impedances in non-magnetic metals are always less than the impedance in air.

At high frequencies the reflection loss between metals is substantially independent of the frequency. At copper-iron boundaries this loss is always high and at copper-air boundaries it increases steadily with the frequency and becomes quite substantial at frequencies as

[11] A small reflection loss exists because the impedances have different phases.

high as 100,000 cycles. On the other hand, in a certain frequency range the reflection loss at iron-air boundaries may be very low. Since the attenuation loss of a complete shield made of coaxial layers of copper and iron is independent of the sequence of the layers, considerable gain in shielding may be secured by placing an iron layer between two copper layers rather than a copper layer between two iron layers so as to take advantage of the added reflection loss, assuming of course that the amounts of copper and iron are the same in both cases.

Since the high-frequency impedance ratio is proportional to the diameter of the shield, the size has a substantial influence upon the effectiveness of the shield. Each time the diameter of a *non-magnetic* shield is doubled, the shielding is increased by 6 decibels. In the case of *magnetic* shields, this is true only at frequencies considerably higher than the critical frequency at which the reflection loss is minimum. Considerably below this frequency, the effectiveness of a magnetic shield is *decreased* by 6 decibels with each doubling of the diameter of the shield. For the transition region we can say that with increasing size the effectiveness of the magnetic shield decreases below the critical frequency and increases above it.

"Electrostatic Shielding"

If the cylindrical wave is originated by two parallel oppositely charged wires, alternating with a given frequency f, the radial impedance in free space is $1/i\omega\epsilon\rho$ provided ρ is small compared with the wave-length.[12] As in the preceding case, in metallic media the radial impedance is $\sqrt{i\omega\mu/g}$ provided the frequency is not too low; for very low frequencies the radial impedance becomes $1/g\rho$.

It is clear at once that for these waves the reflection loss is tremendous. Thus in air $\epsilon = (1/36\pi)10^{-9}$ farads per meter; if $f = 10^6$ and $\rho = 0.01$ m., then the radial impedance is 1,800,000 ohms. The corresponding impedance in copper is only 0.000369 ohms. At lower frequencies the disparity between the radial impedances becomes even greater. The impedance ratio tends to infinity as the frequency approaches zero.

In the elementary theory a metal shield is regarded as a perfect shield against this "electrostatic field." An "electrostatic" field alternating 1,000,000 cycles per second is probably a misnomer. And the shielding is excellent but not perfect. Nevertheless the distinction between two possible types of waves is a valid one, at least in the frequency range usually employed in the communication art. In one wave the electric field is normal to the direction of propagation and

[12] Equation (13).

in the other the magnetic field is so disposed. The former wave may be called *transverse electric* and the latter *transverse magnetic.* The product of the corresponding radial impedances of these waves is equal to the square of the intrinsic impedance. Hence if one wave is a low impedance wave (as compared to the intrinsic impedance), the other is a high impedance wave. Under the usual engineering conditions these waves are unmistakably different in air, although this distinction disappears in metallic media. It must be pointed out, however, that for micro-waves the dimensions of the shield may be comparable to the wave-length, in which case the radial impedances may be of the same order of magnitude.

In the above discussion we have supposed that the line source was on the axis of the shield. If it is not, it is possible to represent the actual source by means of an equivalent system of sources along the axis and calculate the shielding effect. The latter is different for cylindrical waves of different orders. This will result in somewhat different shielding for different positions outside the shield. Ordinarily, however, the difference is not large enough to be considered in practical problems.

Spherical Waves

A small coil carrying an alternating current will give rise to a transverse electric spherical wave and a small condenser to a transverse magnetic wave. Consider a shield concentric with the coil or the condenser. In the shield the radial impedance is $\sqrt{i\omega\mu/g}$, again excepting very low frequencies. In air the radial impedance of the outward bound electric wave is [13] $i\omega\mu r$ and that of the internal wave $\frac{1}{2}i\omega\mu r$. The corresponding impedances of transverse magnetic waves are $1/i\omega\epsilon r$ and $2/i\omega\epsilon r$. The conditions for reflection and shielding are substantially the same as in the case of cylindrical waves. Some quantitative difference results from the inequality of the radial impedances in opposite directions.

Plane Waves

The next example of uniform linearly polarized plane waves is particularly well known.[14] When the boundary between two media coincides with an equiphase surface of the impinging wave, the formulæ

[13] Equations (9), (10).

[14] The general formulæ for the reflection and transmission coefficients have been obtained by T. C. Fry on the basis of the Maxwell theory in his paper "Plane Waves of Light II," published in the *Journal of the Optical Society of America* and *Review of Scientific Instruments*, Vol. 16, pp. 1–25 (1928). The earliest formulæ are probably due to A. Cauchy, who obtained them from the "elastic solid" theory of light waves.

of Part II are directly applicable and the reflection coefficient depends upon the ratio of the intrinsic impedances of the media.

A more interesting situation arises when the incidence is oblique. Let the xy-plane be the boundary between two homogeneous media and let the *electric vector be parallel* to this boundary. We may assume it to be parallel to the x-axis. In this case the electric field strength is given by

$$E_x = E_0 e^{-\sigma s + i\omega t}, \qquad E_y = E_z = 0, \tag{20}$$

where E_0 is the amplitude and s is the distance from the equiphase surface passing through the origin. If the angle of incidence is ϑ (Fig. 5), this distance may be expressed as: $s = y \sin \vartheta + z \cos \vartheta$.

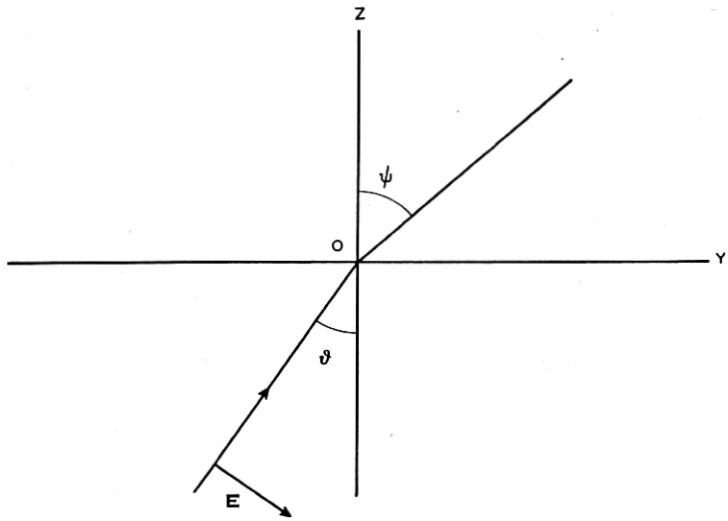

Fig. 5—Reflection of plane waves. The x-axis is toward the reader, the xy-plane is the boundary between the media, the E-vector is toward the reader, the angle of incidence $= \vartheta$, and the angle of reflection $= \psi$.

The magnetic vector is perpendicular to E and to the ray and its value is

$$H = H_0 e^{-\sigma s + i\omega t}, \qquad E_0 = \eta H_0. \tag{21}$$

The cartesian components are then

$$H_y = H_0 \cos \vartheta \, e^{-\sigma s + i\omega t}, \qquad H_z = -H_0 \sin \vartheta \, e^{-\sigma s + i\omega t}, \qquad H_x = 0.$$

Equations (20) and (21) represent the motion of equiphase planes in the direction specified by the angle ϑ. It is equally possible to regard them as representing the motion of *phase-amplitude patterns*

in the direction *normal* to the xy-plane. We need only to rewrite these equations as follows:

$$\begin{aligned} E_x &= (E_0 e^{-\sigma y \sin \vartheta}) e^{-\sigma z \cos \vartheta + i\omega t}, \\ H_y &= (H_0 \cos \vartheta \, e^{-\sigma y \sin \vartheta}) e^{-\sigma z \cos \vartheta + i\omega t}. \end{aligned} \tag{22}$$

The relative distribution of the amplitude and the phase of the wave are governed by the factor $e^{-\sigma y \sin \vartheta}$ and this phase-amplitude pattern is propagated in the direction of the z-axis, the propagation constant being $\sigma \cos \vartheta$.

The advantages of this point of view are clear. In attempting to find the reaction of the second medium upon the incident wave, it is necessary to satisfy certain boundary conditions at *every point* of the interface. This can be insured by requiring the reflected and the refracted waves to have the same phase-amplitude patterns at the interface and by adjusting their relative amplitude and phases to secure the fulfilment of the boundary conditions at some one point. In other words, the problem is reduced to that for which the general solution was given in Part II.

The impedance to the incident wave in the z-direction is found from (22):

$$Z_z = \frac{E_x}{H_y} = \frac{E_0}{H_0 \cos \vartheta} = \eta \sec \vartheta.$$

This impedance is seen to be a function of the intrinsic impedance of the medium and of the angle of incidence.

For the refracted wave in the second medium the transmission equations are similar to (22):

$$\begin{aligned} E_x' &= (E_0' e^{-\sigma' y \sin \psi}) e^{-\sigma' z \cos \psi + i\omega t}, \\ H_y' &= (H_0' \cos \psi \, e^{-\sigma' y \sin \psi}) e^{-\sigma' z \cos \psi + i\omega t}, \qquad E_0' = \eta' H_0. \end{aligned} \tag{23}$$

The "angle of refraction" ψ is, in general, different from ϑ. In our equations we may regard ψ merely as a parameter. Its value is obtained from the condition that at the xy-plane the phase-amplitude pattern of the incident and the refracted waves must be the same, and consequently

$$\sigma \sin \vartheta = \sigma' \sin \psi. \tag{24}$$

In dielectrics this relation is known as Snell's law of refraction.

By (23), the impedance to the refracted wave in the z-direction is

$$Z_z' = \frac{E_x'}{H_y'} = \eta' \sec \psi.$$

The reflection and the transmission coefficients are then obtained from (22) in terms of the impedance ratio

$$k = \frac{\eta \sec \vartheta}{\eta' \sec \psi} = \frac{\eta \cos \psi}{\eta' \cos \vartheta}. \tag{25}$$

Thus, we have

$$R_H = \frac{k-1}{k+1}, \qquad R_E = \frac{1-k}{1+k},$$

$$T_H = \frac{2k}{k+1}, \qquad T_E = \frac{2}{1+k}.$$

These coefficients refer to the *tangential* components of the field.

In a similar way we can deal with the case in which the magnetic vector of the incident wave is parallel to the boundary. The parts played by E and H are interchanged and the impedance ratio becomes

$$k = \frac{\eta \cos \vartheta}{\eta' \cos \psi}. \tag{26}$$

The cosine factors have changed their places.

The general case, in which neither E nor H is parallel to the boundary, cannot be treated in the above manner. In this case the components of E and H which are parallel to the boundary are not perpendicular to each other, the impedances Z_{xy} and Z_{yx} are not equal to each other and the unique impedance $Z_z = Z_{xy} = Z_{yx}$, upon which the results of Part II are based, does not exist. In accordance with a suggestion made in Part I, the incident wave must be resolved into components possessing unique impedances in the direction normal to the boundary. It is well known that such a decomposition is possible for ordinary plane waves; the latter can always be decomposed into two components, in one of which E is parallel to the boundary and in the other H is so disposed.

It is not surprising that reflection of arbitrarily oriented waves cannot be treated directly. The impedance ratios (25) and (26) for two basic orientations are in general different and the polarization of the reflected wave will be changed. An exceptional case arises *when the intrinsic propagation constants of the media are equal.* In this case $\psi = \vartheta$, as seen from (24), and *the impedance ratio is independent of the angle of incidence and of the particular orientation of the wave.* Consequently, *the reflection and the transmission coefficients depend solely upon the ratio of the intrinsic impedances of the media.*

Frequently the permeabilities of the media are assumed to be the same, in which case the ratio of the intrinsic impedances is equal to

the inverse ratio of the "indices of refraction" of the media. Much could be said, however, in favor of not making such an assumption when formulating the general results since in many applications the permeabilities may be unequal.

Images

A few additional interesting results can be obtained for the special case of two semi-infinite homogeneous media having equal propagation constants. If the media are separated by a plane boundary, problems of reflection and refraction can be solved by the *method of images*. This method is frequently used in electrostatics and one or

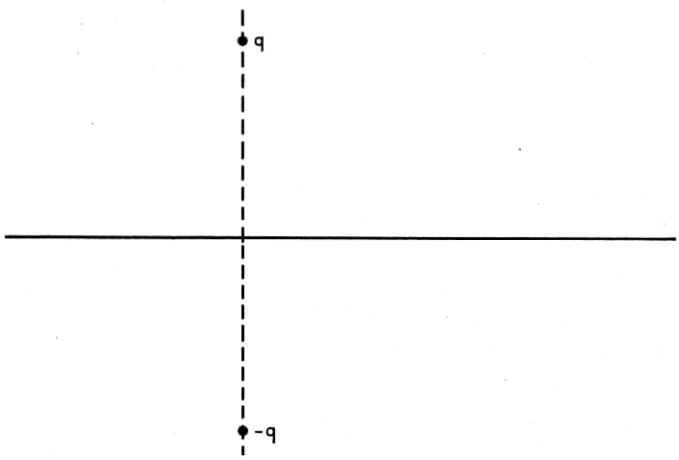

Fig. 6

two simple examples from that science will serve as an introduction to the later generalizations.

The field of a point charge q above a conducting plane can be found by assuming another point charge $(-q)$. This "image" charge (Fig. 6) is the same distance below the plane as the actual charge is above the plane, both charges lying on the same perpendicular. The field due to the original charge and to the image charge satisfies the boundary conditions at the conducting plane since it makes the latter an equipotential. This combined field gives the correct resultant field on the *same* side of the plane as the original charge; on the opposite side the field is zero.

If the boundary is the interface between two perfect dielectrics (Fig. 7) with dielectric constants respectively equal to ϵ_1 and ϵ_2, the results are almost equally simple. Above the boundary we have

a reflected field in addition to the original field. This reflected field is produced by an image charge $q' = (\epsilon_1 - \epsilon_2)q/(\epsilon_1 + \epsilon_2)$ on the supposition that the dielectric constant is everywhere equal to ϵ_1. Below the plane the field is such as would be produced by a charge $q'' = 2\epsilon_1 q/(\epsilon_1 + \epsilon_2)$ if placed where the original charge is, also on the assumption that the dielectric constant is everywhere ϵ_1. The charge producing the correct field below the boundary would be $q''' = 2\epsilon_2 q/(\epsilon_1 + \epsilon_2)$ if we were to assume ϵ_2 as the dielectric constant of the whole space.

Inspecting equations (7) for an electric current element, which we assume to be perpendicular to the plane interface of two homogeneous

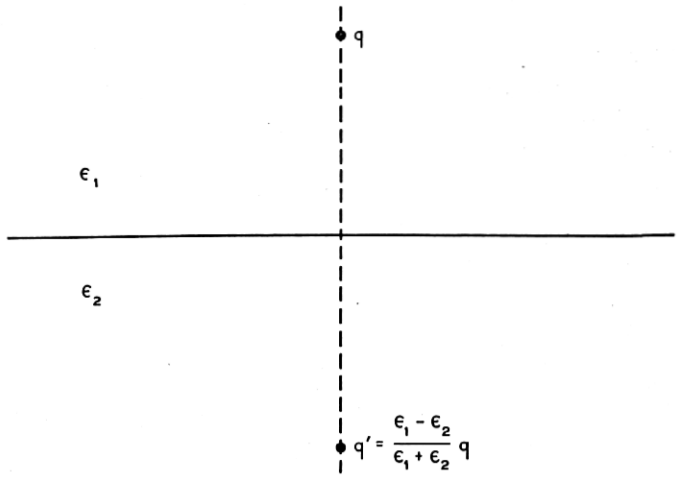

Fig. 7

media, we see that the method of images can readily be extended to *dynamic* fields *provided* the intrinsic propagation constants of the media are equal. In order to make this conclusion more evident, we replace $i\omega\mu$ in the first equation by the equivalent product $\eta\sigma$ and then calculate the component of E *tangential* to the interface

$$E_p^+ = E_\theta^+ \cos\theta + E_r^+ \sin\theta = \eta I l \frac{\sigma e^{-\sigma r}}{4\pi r}\left(1 + \frac{3}{\sigma r} + \frac{3}{\sigma^2 r^2}\right) \sin\theta \cos\theta.$$

It is easy to see that the continuity of the tangential field components will be preserved if we assume a reflected field on the same side of the boundary and a refracted field on the opposite side in accordance with the following specifications. *The reflected field is such as would be produced by an image current element of moment* $(\eta_1 - \eta_2)Il/(\eta_1 + \eta_2)$

and the refracted field is such as could be produced by a current element of moment $2\eta_1 Il/(\eta_1 + \eta_2)$, *occupying the same position as the source. In calculating these fields a uniform intrinsic impedance* η_1 *is assumed throughout the whole space.*

Since the current I in the element implies two point charges, $-I/i\omega$ and $I/i\omega$, at its terminals, we can interpret the above rule of images in terms of the charges. *The image of a point charge* q *for calculating the reflected field is* $(\eta_2 - \eta_1)q/(\eta_2 + \eta_1)$. *For calculating the refracted field a charge* $2\eta_1 q/(\eta_1 + \eta_2)$ *must be assumed in the same position as the original charge.* For perfect dielectrics the expressions of the image charges reduce to those given by electrostatics.

ACKNOWLEDGMENT

I wish to express my appreciation to Dr. T. C. Fry for his valuable criticism in the preparation of this paper.

AN HISTORICAL NOTE

The following memorandum written in 1932 by Dr. G. A. Campbell, formerly of the American Telephone and Telegraph Company, represents an interesting historical comment and it is reprinted with Dr. Campbell's permission.

A letter discussing the characteristic impedance of free space, written to me seven years ago by Dr. H. W. Nichols, is of possible interest in connection with both this impedance and the question of superfluous units. He derives the impedance from the Poynting vector by simple substitutions. Specific use is made, however, of five systems of units. The letter also supplies an illustration of confusion arising from the multiplicity of units in use. Apparently, Heaviside's 30 ohms ("Electrical Papers," ii, p. 377, 1888) was in ordinary ohms and not in Heaviside's own units, as Nichols quite naturally assumed. The correct explanation of the 30 ohms seems to be that Heaviside's "resistance-operator of an infinitely long tube of unit area" was not intended to be the characteristic impedance, as I define it.

In definitive units the characteristic impedance of free space equals the square of the effective volts per meter, in a plane electromagnetic wave, divided by the transmitted watts per square meter. For a numerical example, take the figures for strong sunlight (Maxwell, ii, footnote p. 441) which correspond to 666.1 effective volts per meter and 1176 watts per square meter. The characteristic impedance of free space implicitly assumed was thus 377.3 ohms, which checks well with my 376.54 international ohms.

If free space could be bounded in one direction by a thin, plane film having surface resistivity equal to the characteristic impedance of space, a normally incident plane wave would be completely absorbed by the film; there would be neither reflected wave nor transmitted wave beyond the

film. This picture is suggested by the analogy of a transmission line terminated, at the receiving end, in its characteristic impedance, so that there is no reflected wave. The difficulty with the analogy is that free space exists beyond the film and cannot be cut off. This idealized picture may serve, however, to indicate the simplification made possible by the introduction of characteristic impedances in practical problems involving reflection, refraction and absorption.

The characteristic impedance of free space may be usefully introduced into formulas for the characteristic impedances of transmission lines. Thus, assuming perfect conductors, we have:

For flat strips, width w, separation d, if w/d is large or the guard-ring method is employed in measurements,

$$K_f = 376.54 \frac{d}{w};$$

For concentric cylinders, with radii b and a,

$$K_c = \frac{376.54}{2\pi} \log \frac{b}{a}.$$

These characteristic impedances will each agree with the characteristic impedance of free space if $w = d$ and $b = 535.49\, a$. Since these strips are not wide compared with the separation, it would be necessary to employ the guard-ring method to maintain the plane wave assumed in the square shaft between the two strips. These two characteristic impedances would each become one ohm if $w = 376.54\, d$, and $b = 1.0168\, a$.

Practically, the finite conductivity of copper would add a reactance component and change the resistance component. It would be interesting to investigate simple cases numerically and include mutual characteristic impedances between two metallic circuits.

My own interest in the applications of the impedance concept to the electromagnetic field theory dates back to the last quarter of 1931.