Chapter 4

Vector differential equations

4.1 Problems VC-1

4.1.1 Topics of this homework:
Vector algebra and fields in \( \mathbb{R}^3 \), gradient and scalar Laplacian operators, definitions of divergence and curl, Gauss’s (divergence) and Stokes’s (curl) laws, system classification (postulates).

4.1.2 Scalar fields and the \( \nabla \) operator

Problem # 1: Let \( T(x, y) = x^2 + y \) be an analytic scalar temperature field in two dimensions (single-valued \( \in \mathbb{R}^2 \)).

– 1.1: Find the gradient of \( T(x) \) and make a sketch of \( T \) and the gradient.

Ans:

– 1.2: Compute \( \nabla^2 T(x) \) to determine whether \( T(x) \) satisfies Laplace’s equation.

Ans:

– 1.3: Sketch the iso-temperature contours at \( T = -10, 0, 10 \) degrees.
Ans:

– 1.4: The heat flux\(^1\) is defined as \( J(x, y) = -\kappa(x, y)\nabla T \), where \( \kappa(x, y) \) is a constant that denotes thermal conductivity at the point \((x, y)\). Given that \( \kappa = 1 \) everywhere (the medium is homogeneous), plot the vector \( J(x, y) = -\nabla T \) at \( x = 2, y = 1 \). Be clear about the origin, direction, and length of your result.

Ans:

– 1.5: Find the vector perpendicular to \( \nabla T(x, y) \)—that is, tangent to the iso-temperature contours. Hint: Sketch it for one \((x, y)\) point (e.g., 2, 1) and then generalize.

Ans:

– 1.6: The thermal resistance \( R_T \) is defined as the potential drop \( \Delta T \) over the magnitude of the heat flux \( |J| \). At a single point the thermal resistance is

\[
R_T(x, y) = -\nabla T / |J|.
\]

How is \( R_T(x, y) \) related to the thermal conductivity \( \kappa(x, y) \)?

Ans:

Problem # 2: Acoustic wave equation
Note: In this problem, we will work in the frequency domain.

\(^1\)The heat flux is proportional to the change in temperature times the thermal conductivity \( \kappa \) of the medium.
– 2.1: The basic equations of acoustics in one dimension are

\[-\frac{\partial}{\partial x} P = \rho_o s \psi \quad \text{and} \quad -\frac{\partial}{\partial x} \psi = \frac{s}{\eta_o P_o} P.\]

Here \( P(x, \omega) \) is the pressure (in the frequency domain), \( \psi'(x, \omega) \) is the volume velocity (the integral of the velocity over the wavefront with area \( A \)), \( s = \sigma + \omega j \), \( \rho_o = 1.2 \) is the specific density of air, \( \eta_o = 1.4 \), and \( P_o \) is the atmospheric pressure (i.e., \( 10^5 \) Pa). Note that the pressure field \( P \) is a scalar (pressure does not have direction), while the volume velocity field \( \psi' \) is a vector (velocity has direction).

We can generalize these equations to three dimensions using the \( \nabla \) operator

\[-\nabla P = \rho_o s \psi \quad \text{and} \quad -\nabla \cdot \psi' = \frac{s}{\eta_o P_o} P.\]

– 2.2: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure \( P \),

\[\nabla^2 P = \frac{s^2}{c_0^2} P,\]

where \( c_0 \) is a constant representing the speed of sound.

**Ans:**

– 2.3: What is \( c_0 \) in terms of \( \eta_0 \), \( \rho_0 \), and \( P_0 \)?

**Ans:**

– 2.4: Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform [e.g., \( \frac{dx}{dt} \leftrightarrow sX(s) \)]. For your notation, define the time–domain signal using a lowercase letter, \( p(x, y, z, t) \leftrightarrow P \).

**Ans:**
4.1.3 Vector fields and the $\nabla$ operator

4.1.4 Vector algebra

**Problem #3:** Let $\mathbf{R}(x, y, z) \equiv x(t)\mathbf{\hat{x}} + y(t)\mathbf{\hat{y}} + z(t)\mathbf{\hat{z}}$.

- 3.1: If $a$, $b$, and $c$ are constants, what is $\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c)$?
  
  **Ans:**

- 3.2: If $a$, $b$, and $c$ are constants, what is $\frac{d}{dt}(\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c))$?
  
  **Ans:**

**Problem #4:** Find the divergence and curl of the following vector fields:

- 4.1: $\mathbf{v} = \mathbf{\hat{x}} + \mathbf{\hat{y}} + 2\mathbf{\hat{z}}$
  
  **Ans:**

- 4.2: $\mathbf{v}(x, y, z) = x\mathbf{\hat{x}} + xy\mathbf{\hat{y}} + z^2\mathbf{\hat{z}}$
  
  **Ans:**

- 4.3: $\mathbf{v}(x, y, z) = x\mathbf{\hat{x}} + xy\mathbf{\hat{y}} + \log(z)\mathbf{\hat{z}}$
  
  **Ans:**
4.1.5 Vector and scalar field identities

Problem # 5: Find the divergence and curl of the following vector fields:

- 5.1: \( \mathbf{v} = \nabla \phi \), where \( \phi(x, y) = xe^{y} \)
   Ans:

- 5.2: \( \mathbf{v} = \nabla \times \mathbf{A} \), where \( \mathbf{A} = x\mathbf{\hat{x}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}} \)
   Ans:

- 5.3: \( \mathbf{v} = \nabla \times \mathbf{A} \), where \( \mathbf{A} = y\mathbf{\hat{x}} + x^{2}\mathbf{\hat{y}} + z\mathbf{\hat{z}} \)
   Ans:

- 5.4: For any differentiable vector field \( \mathbf{V} \), write two vector calculus identities that are equal to zero.
   Ans:
5.5: What is the most general form a vector field may be expressed in, in terms of scalar \( \Phi \) and vector \( \mathbf{A} \) potentials?

\( \text{Ans:} \)

Problem # 6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.

- 6.1: Let \( \mathbf{v} = \sin(x)\mathbf{\hat{x}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}} \). Find \( \nabla \cdot (\nabla \times \mathbf{v}) \).

\( \text{Ans:} \)

- 6.2: Let \( \mathbf{v} = \sin(x)\mathbf{\hat{x}} + y\mathbf{\hat{y}} + z\mathbf{\hat{z}} \). Find \( \nabla \times (\nabla \sqrt{\mathbf{v} \cdot \mathbf{v}}) \)

\( \text{Ans:} \)

- 6.3: Let \( \mathbf{v}(x, y, z) = \nabla (x + y^2 + \sin(\log(z))) \). Find \( \nabla \times \mathbf{v}(x, y, z) \).

\( \text{Ans:} \)

4.1.6 Integral theorems

Problem # 7: For each of the following problems, in a few words, identify either Gauss’s or Stokes’s law, define what it means, and explain the formula that follows the question.

- 7.1: What is the name of this formula?

\[ \int_S \mathbf{n} \cdot \mathbf{v} \, dA = \int_V \nabla \cdot \mathbf{v} \, dV. \]
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Ans:

– 7.2: What is the name of this formula?

$$\int_S (\nabla \times \mathbf{V}) \cdot dS = \oint_C \mathbf{V} \cdot d\mathbf{R}$$

Give one important application. Ans:

– 7.3: Describe a key application of the vector identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}.$$ 

Ans: