1 Problems NS1

Topic of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton's root finding method, Riemann zeta function.

Deliverable: Answers to problems

Note: The term 'analytic' is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Problem *# 1: A two-port network application for the Laplace transform*

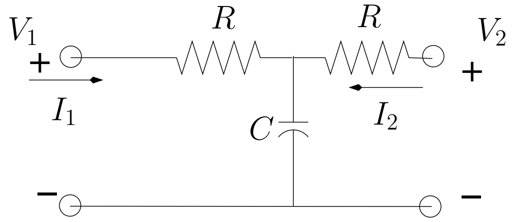

Figure 1: *This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$. It is convienent to define the dimentionless ratio $s/s_c = RCs$ in terms of a time constant $\tau = RC$ and cutoff frequency $s_c = 1/\tau$.*

– 1.1: Find the 2×2 ABCD matrix representation of Fig. 1. Express the results in terms of the dimentionless ratio s/s_c where $s_c = 1/\tau$ is the cutoff frequency and $\tau = RC$ is the time constant.

– 1.2: Find the eigenvalues of the 2×2 ABCD matrix.

– 1.3: Assuming that $I_2 = 0$, find the transfer function $H(s) \equiv V_2/V_1$. From the results of the ABCD matrix you determined above, show that

$$H(s) = \frac{s_c}{s + s_c}. \tag{NS-1.1}$$

– 1.4: The transfer function $H(s)$ has one pole. Where is the pole and residue*?*

– 1.5: Find $h(t)$, the inverse Laplace transform of $H(s)$.

– 1.6: Assuming that $V_2 = 0$ find $Y_{12}(s) \equiv I_2/V_1$.

– 1.7: Find the input impedance to the right-hand side *of the system, $Z_{22}(s) \equiv V_2/I_2$ for two cases:*

1. $I_1 = 0$

2. $V_1 = 0$

– *1.8: Compute the determinant of the ABCD matrix.* Hint: It is always ± 1.

– *1.9: Given the result of the previous problem Eq. NS-1.1, compute the derivative of* $H(s) = \frac{V_2}{V_1}\Big|_{I_2=0}$.

Problem # 2: Train-mission-line *We wish to model the dynamics of a freight-train having N such cars, and study the* velocity transfer function *under various load conditions.*

As shown in Fig. 2, the train model consists of masses connected by springs.

Physical description:

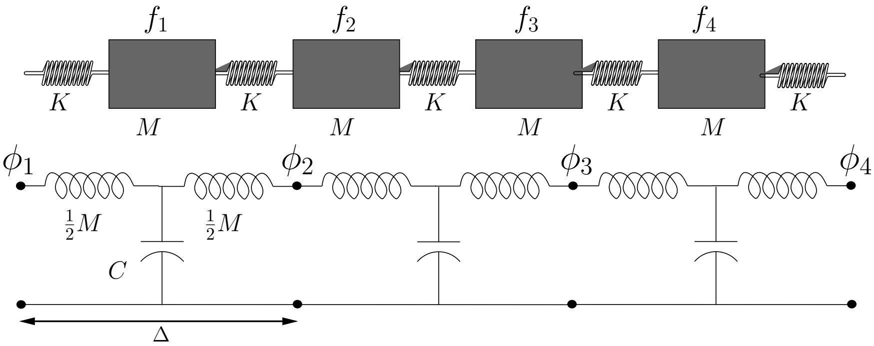

Figure 2: *Depiction of a train consisting of cars, treated as a mass M and linkages, treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is Δ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL.*

Use the ABCD method to find the matrix representation of the system of Fig. 2. Define the force on the nth train car $f_n(t) \leftrightarrow F_n(\omega)$, and velocity $v_n(t) \leftrightarrow V_n(\omega)$.

Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($\mathcal{A} = \mathcal{D}$) identical cell matrix $\mathcal{T}(s)$.

– *2.1: Find the elements of the ABCD matrix $\mathcal{T}$ for the single cell that relate the input node 1 to output node 2*

$$\begin{bmatrix} F \\ V \end{bmatrix}_1 = \mathcal{T} \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2 . \qquad \text{(NS-1.2)}$$

– *2.2: Express each element of $\mathcal{T}(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi j f$, $s_c = 2\pi j f_c$). The Nyquist sampling cutoff frequency f_c is defined in terms the minimum number of cells (i.e., 2) of length Δ per wavelength:*

The Nyquest sampling theorm says that there are at least two cars per wavelength (more than two time samples at the highest frequency). From the figure, the distance between cars $\Delta = c_o T_o$ [m] where

$$c_o = \frac{1}{\sqrt{MC}}. \quad \text{[m/s]}$$

The cutoff frequency obeys $f_c\lambda_c = c_o$ where the Nyquist wavelength is $\lambda_c = 2\Delta$. Therefore the Nyquist sampling condition is

$$\omega < \omega_c = 2\pi f_c \equiv \frac{2\pi c_o}{\lambda_c} = \frac{2\pi c_o}{2\Delta} = \frac{\pi}{\Delta\sqrt{MC}}. \quad \text{[Hz]} \tag{NS-1.3}$$

– 2.3: Use the property of the Nyquist sampling frequency $f < f_c$ *(Eq. NS-1.3) to remove higher order powers of frequency*

$$1 + \cancelto{0}{\left(\frac{s}{s_c}\right)^2} \approx 1 \tag{NS-1.4}$$

to determine a band-limited approximation of $\mathcal{T}(s)$.

Problem *# 3*
Now consider the cascade of N such $\mathcal{T}(s)$ matrices, and perform an eigen–analysis.

– 3.1: Find the eigenvalues and eigenvectors of $\mathcal{T}(s)$, as functions of s/s_c.

Problem *# 4*
Finally, find the velocity transfer function:

– 4.1: Assuming that $N = 2$ and that $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the $\mathcal{T}$ matrix, find

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

Express H_{12} in terms of a residue expansion.

– 4.2: Find $h_{21}(t) \leftrightarrow H_{21}(s)$.

– 4.3: What is the input impedance $Z_1 = F_1/V_1$, assuming $F_2 = -r_0V_2$?

– 4.4: Simplify the expression for Z_2 assuming:

1. Assuming the *characteristic impedance* $r_0 = \sqrt{M/C}$
2. terminate the system in r_o: $F_2 = -r_0V_2$ (i.e., $-V_2$ cancels)
3. Assume higher order frequency terms are less than 1 ($|s/s_c| < 1$)
4. Let the number of cells $N \to \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteric impedance r_o, the input impedance $Z_1(s) = r_o$. Thus when we simplify the expression for $\mathcal{T}(s)$ it should be equal to r_o. Show that this is true for this setup.

– 4.5: State the ABCD matrix relationship between the first and Nth node in terms of the cell matrix. Write out the transfer function for one cell: H_{21}?

– 4.6: What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?

System Classification

Problem *# 5: Answer the following system classification questions about physical systems, in terms of the system postulates.*

– *5.1: Provide a brief definition of the following properties:* L/NL : linear(L)/nonlinear(NL):

TI/TV : time-invariant(TI)/time varying(TV):

P/A : passive(P)/active(A):

C/NC : causal(C)/non-causal(NC):

Re/Clx : real(Re)/complex(Clx):

– *5.2: Along the rows of the table, classify the following* systems*: In terms of a table having 5 columns, labeled with the abbreviations: L/NL, TI/TV, P/A, C/NC, Re/Clx:*

					Category		
#	Case:	Definition	L/NL	TI/TV	P/A	C/NC	Re/Clx
1	Resistor	$v(t) = r_0\, i(t)$					
2	Inductor	$v(t) = L\frac{di}{dt}$					
3	Switch	$v(t) \equiv \begin{cases} 0 & t \leq 0 \\ V_0 & t > 0. \end{cases}$					
5	Transistor	$I_{out} = g_m(V_{in})$					
7	Resistor	$v(t) = r_0\, i(t+3)$					
8	modulator	$f(t) = e^{i2\pi t} g(t)$					

– *5.3: Using the same classification scheme, characterize the following equations:*

#	Case:	L/NL	TI/TV	P/A	C/NC	Re/Clx
1	$A(x)\frac{d^2y(t)}{dt^2} + D(t)y(x,t) = 0$					
2	$\frac{dy(t)}{dt} + \sqrt{t}\, y(t) = \sin(t)$					
3	$y^2(t) + y(t) = \sin(t)$					
4	$\frac{\partial^2 y}{\partial t^2} + xy(t+1) + x^2 y = 0$					
5	$\frac{dy(t)}{dt} + (t-1)\, y^2(t) = ie^t$					