Circuit Techniques for Thermodynamic Analysis

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This paper demonstrates that the two-port transmission-matrix analysis method may be used for thermodynamics calculations, and compares it to classic energy methods of thermodynamic analysis. A simple thermodynamics problem is proposed and solved using the two different methods, and their advantages and disadvantages are compared. We conclude that using using impedance methods linearizes thermodynamic energy relations, making linear algebra methods an applicable solution method.

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I. INTRODUCTION

The purpose of this correspondence is to investigate the possibility and utility of using the 2-port transmission matrix method to analyze thermodynamic problems, commonly used in modeling electrical and mechanical systems, such as LRC electrical circuits and spring-mass-damper systems (?).

Specifically, this report will investigate the use of the Laplace frequency domain to model thermodynamic systems, and to draw connections with components of electrical, mechanical and thermodynamic analysis, using the two-port transmission line methods.

Traditionally, thermodynamics is analyzed in the time domain using energy relationships (?). Energy relationships are nonlinear in the *conjugate variables*, the *product* of which define the power (energy-rate). For example, voltage times current (coulomb/sec) or temperature times entropy-rate are each a power, having units of watts (?, Appendix I). Unlike thermodynamics, which is formulated in terms of energy, electrical and mechanical circuits use impedance, defined as the *ratio of conjugate variables* (e.g., $Z(s) = \texttt{voltage/current}$) when modeling electrical circuits, or $Z(s) = \texttt{force/velocity}$ for mechanical systems.

The definition of an impedance $Z(s)$ utilizes the Laplace frequency s. The Laplace transform replaces calculus with algebra in the Laplace frequency variable ($s = \sigma + \jmath\omega$). This is primarily because electrical and mechanical circuits are second order (or higher) systems, that benefit greatly from this type of analysis. Presently thermodynamics is modeled using only RC circuits (first order system).

A. Problem Statement

To show how the two-port transmission line analysis works with Thermodynamics, a simple and classic thermodynamic problem is proposed and solved, using the classic method (**?**), followed by a two-port analysis.

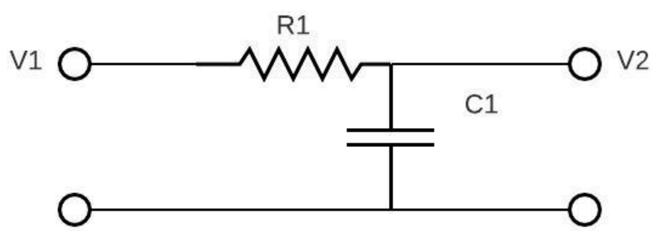

FIG. 1. *The two-port matrix representation of an RC Circuit*

The example electrical (i.e., initial) problem is shown in Fig. 1. The thermodynamic version will include the heat generated in the resistor R_1, due to electrical current, causing it to produce heat energy.

Assume this RC circuit, with resistance (R_1), placed in an incompressible fluid (e.g., water) having mass (m_w) and specific heat capacity under constant pressure (c_p), which is otherwise isolated from the environment. Let the source voltage and current be $[V_1, I_1]$. After the RC circuit has been turned on and has reached equilibrium, we study the change in temperature $T(t)$ of the fluid as a function of time t. Stated another way, what is the time response of temperature of the fluid, as the capacitor is charging? Finally, what is the impact of the power lost to heating the water around the resistor, on the charging of the capacitor? The final voltage on C_1 will be different due to the power lost to the water.

B. Classic Solution

To determine the energy dissipated by the resistor into the fluid, the current passing through the resistor must be determined. To find this current the RC circuit may be analyzed as a two port transmission line. Figure 1 can then be analyzed using a 2x2 representation matrix relation

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix},$$

where the currents are defined into the ports and the voltages across the ports (?). This may be found by collapsing the matrix product,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 + sRC & R_1 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

which then provids the relations between the input and outputs

$$V_1 = 1 + sR_1C_1V_2 - RI_2$$

$$I_1 = sC_1V_2 - I_2.$$

Setting $I_2 = 0$ and combining the two equations, we find

$$V_2 = \frac{V_1}{1 + sR_1C_1}$$

$$I_1 = sC_1V_2,$$

or

$$\begin{aligned} I_1 &= \frac{C_s V_1}{1 + sR_1C_1} \\ &= C_1 V_1 \frac{s}{1 + sR_1C_1} \\ &= \frac{V_1}{R_1} \frac{s}{s + 1/R_1C_1}. \end{aligned} \tag{1}$$

From the initial condition ($t = 0$)

$$V_1(t) = V_0 u(t) \leftrightarrow V_0/s. \tag{2}$$

Substituting $V_1(0)$ in into Eq. 1 gives

$$I_1 = \frac{V_0}{R_1} \frac{1}{s + 1/R_1C_1}.$$

This equation is in the Laplace frequency domain.

In order to convert this equation back to the time domain, the inverse Laplace transform must be taken, giving

$$I_1(s) = \frac{V_0}{R_1}\ \frac{1}{s + 1/R_1C_1} \ \leftrightarrow \ \ i_1(t) = \frac{V_0}{R_1} e^{-t/R_1C_1} \tag{3}$$

$$= I_0 e^{-t/\tau}, \tag{4}$$

where $\tau = R_1C_1$ and $I_0 = \frac{V_0}{R_1}$.

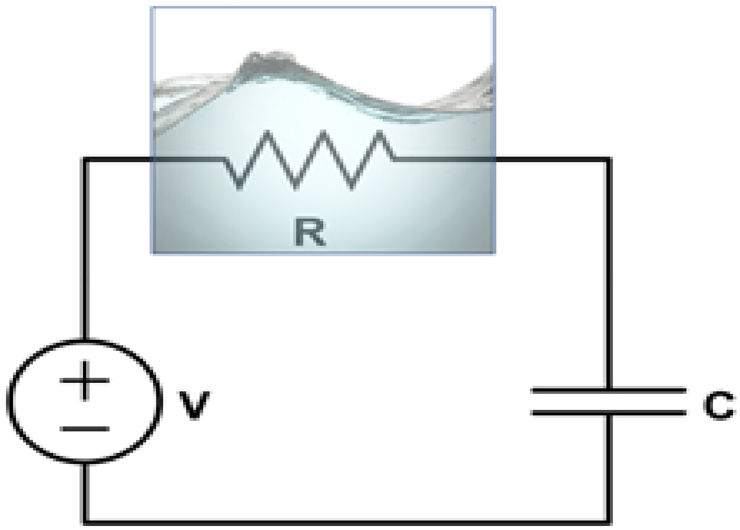

FIG. 2. *Equivalent RC circuit of Fig. 1, including the thermal losses in the resistor immersed in a water bath.*

C. Thermodynamic relations

The power dissipated by the resistor at time t is

$$P(t) = V_1(t)I_1(t) = I_0(t)^2 R_1 = I_0^2 e^{-2t/\tau} R_1.$$

The total energy dissipated in the resistor is the time integral of $P(t)$

$$Q(t) = \int_0^t P(t)dt = I_0^2 R_1 \int_0^t e^{-2t/RC} dt = I_0^2 R_1 \tfrac{\tau}{2} \left(1 - e^{-2t/\tau}\right).$$

Assuming all the energy dissipated by the resistor is absorbed by the fluid, the relationship between the energy absorbed by the fluid and the change in temperature (?)

$$Q = m_f c_p \Delta T.$$

TABLE I. *Table of parameters for the circuit of Fig. 2.*

Parameters	symbol	Value	Units
Voltage	V_o	10	[V]
Resistance	R_1	10	[Ω]
Capacitance	C_1	1	[F]
Mass of water	m_f	1	[g]
Specific Heat Capacity	c_f	4186	[$\frac{J}{kg\cdot K}$]

Rearranging and substituting

$$\Delta T(t) = \frac{Q(t)}{m_f c_p} = \frac{I_0^2 R_1 \frac{\tau}{2}(1 - e^{-2t/\tau})}{m_f c_p} \tag{5}$$

The values for the constants are given in Table 2.

Thus we find

$$\tau = RC = 10 \cdot 1 = 10 \ \ \texttt{[sec]}$$

$$I_0 = \frac{V_0}{R_1} = \frac{10}{10} = 1 \ \ \texttt{[Amp]}$$

$$\Delta T(t) = \frac{I_0^2 R_1 \frac{\tau}{2}(1 - e^{-2t/\tau})}{m_f c_f} = \frac{1^2 \cdot 10 \cdot \frac{10}{2} \cdot (1 - e^{-2t/10})}{0.001 \cdot 4186} \ \ \texttt{[°C]}, \tag{6}$$

as shown in Fig. 3, where we visualize $\Delta T(t)$.

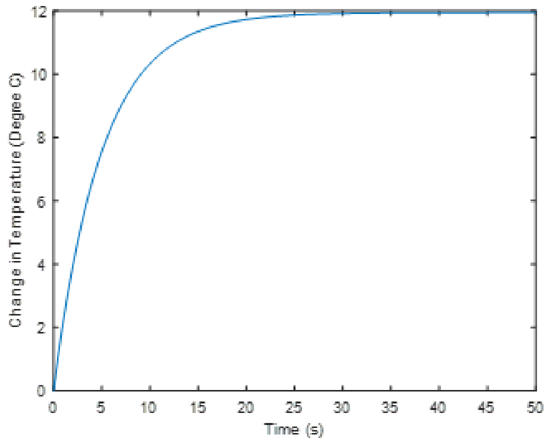

FIG. 3. *Time response of the temperature $\Delta T(t)$ of the water.*

D. Discussion

While the classic analysis provides the current in the resistor, allowing us to calculate the power and total energy dissipated in the resistor as a function of time, it is not actually correct, since the energy absorbed by the water will change the energy balance relations. Thus the classic $I_1(t)$ is not the true current. To obtain the correct answer, we must include the energy dissipated in the water. This requires a thermodynamic calculation, which we shall provide in §II.

II. TWO-PORT ANALYSIS METHOD

The system including the heat lost can also be modeled as a two-port transmission line, with a resistor, an ideal transformer and two capacitors, as shown in Fig. 4.

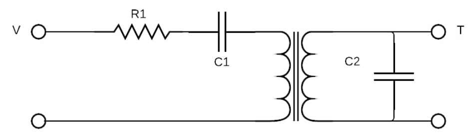

FIG. 4. *Two-port model including the iso-baric heat lost to resistor R_1 in the water bath. The turns ratio of the transformer (a) relates the voltage and current to the temperature and entropy–rate. For example $T = V/a$ and $\dot{\mathcal{S}} = aI$. Thus the units on a are either [V/°C] or [entropy-rate/A].*

Evaluating the transmission matrix of Fig.4 gives

$$\mathcal{T}(s) = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_2 & 1 \end{bmatrix},$$

where

$$\begin{bmatrix} V(s) \\ I(s) \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} T(s) \\ -\dot{\mathcal{S}} \end{bmatrix} = \mathcal{T}(s) \begin{bmatrix} T(s) \\ -\dot{\mathcal{S}}(s) \end{bmatrix}.$$

Thus

$$\begin{bmatrix} V \\ I \end{bmatrix}(s) = \frac{1}{a} \begin{bmatrix} \frac{s+a^2C_1+C_2}{C_1} & \frac{sR_1C_1+1}{sC_1} \\ sC_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ -\dot{\mathcal{S}} \end{bmatrix}(s).$$

Since the system is isolated from the environment is adiabatic, the entropy flux out of the water (i.e., heat flow $\dot{\mathcal{S}}$) is zero. This allows us to find the relationship between the input voltage and the temperature:

$$V(s) = \left(\frac{C_1C_2R_1s + a^2C_1 + C_2}{aC_1} \right) T(s).$$

Assuming that $V(s)$ is a unit step function (see Eq. 2),

$$T(s) = \frac{V_o}{s}\frac{aC_1}{C_1C_2R_1s + a^2C_1 + C_2} = \frac{V_o}{s}\frac{a}{C_2R_1s + a^2 + C_2/C_1}.$$

Expressing this in pole-residue form (**?**)

$$T(s) = \frac{V_oa}{s}\frac{1}{C_2R_1s + a^2 + C_2/C_1} = \frac{V_oa}{C_2R_1}\frac{1}{s}\frac{1}{s + \frac{a^2+C_2/C_1}{C_2R_1}},$$

the inverse Laplace is

$$T(s) \leftrightarrow T(t) = \frac{V_oa}{C_2R_1}\int_0^t e^{\frac{-t(a^2+C_2/C_1)}{C_2R_1}}dt.$$

In this case we can define $\tau_2 = C_2R_1/(a^2 + C_2/C_1))$. Evaluating the integral gives

$$T(t) = \frac{V_oa}{C_2R_1}\tau_2\left(e^{-t/\tau_2}\right) + T_o.$$

Since the temperature rise ΔT is of interest, the boundary condition is $T(t=0) \equiv T_o = 0$.
This can be seen in the following equation.

$$\Delta T(t) = \frac{V_oa}{C_2R_1}\tau_2 e^{-t/\tau_2} + \frac{V_oa}{C_2R_1}\tau_2,$$

or

$$\Delta T(t) = \frac{V_oa}{C_2R_1}\,\tau_2\left(1 - e^{-t/\tau_2}\right). \tag{7}$$

A. Discussion

Note that this is now of the same form as Eq. 5 (τ_2 is quite different), and is identical if we set $C_2 = 0$.

Thus it is a matter of determining the value of a, C_1, and $C2$. Assuming that C_1 stays the same for the two solutions, a and C_2 can be determined by renormalizing the two solutions to have the same functional form. Given

$$T(t) = \frac{V_o a}{C_2 R_1}\tau_2\left(1 - e^{\frac{-t}{\tau_2}}\right) = \frac{I_0^2 R_1 \frac{\tau_1}{2}}{m_f c_f}(1 - e^{-2t/\tau_1}), \tag{8}$$

where

$$\tau_1 = \tau = RC_1 \tag{9}$$

then if we reapply the definition of $\tau_2 = C_2 R_1/(a^2 + C_2/C_1))$

$$\frac{V_o a}{C_2 R_1}\tau_2 = \frac{V_o a}{C_2 R_1}(\frac{C_2 R_1}{a^2 + C_2/C_1}) = \frac{V_o a}{a^2 + C_2/C_1} \tag{10}$$

and if we equate the linear constants on both sides

$$\frac{V_o a}{a^2 + C_2/C_1} = \frac{I_0^2 R_1 \frac{\tau}{2}}{m_f c_f} \tag{11}$$

and if we equate the exponent

$$\frac{-1}{\tau_2} = \frac{-2}{\tau_1} \Rightarrow \frac{-(a^2 + C_2/C_1)}{C_2 R_1} = \frac{-2}{R_1 C_1}. \tag{12}$$

we have a linear set of two equations and two unknowns.

$$a^2 R_1 C_1 + R_1 C_2 = 2 R_1 C_2 \Rightarrow C_2 = a^2 C_1$$

Substituting Eq. 12 back into Eq. 11 gives

$$\frac{V_o a}{a^2 + (a^2 C_1)/C_1} = \frac{I_0^2 R_1 \frac{R_1 C_1}{2}}{m_f c_f}.$$

Simplifying

$$\frac{V_o a}{2a^2} = \frac{V_o^2 C_1}{2 m_f c_f}$$

$$a = \frac{m_f c_f}{V_o C_1}$$

Substituting back into Eq. 12 and solving for C_2

$$C_2 = \frac{m_f^2 c_f^2}{C_1} \tag{13}$$

Plotting Eq. 7 and comparing to Eq. 6, we see that the solution has the same functional form, but is numerically distinct, due to the added heat loss into the water, thus accounting for this important missing term in the classic solution. They are identical when $C_2 = 0$, thus decoupling the entropy-rate (heat loss) and the electrical current and voltage.

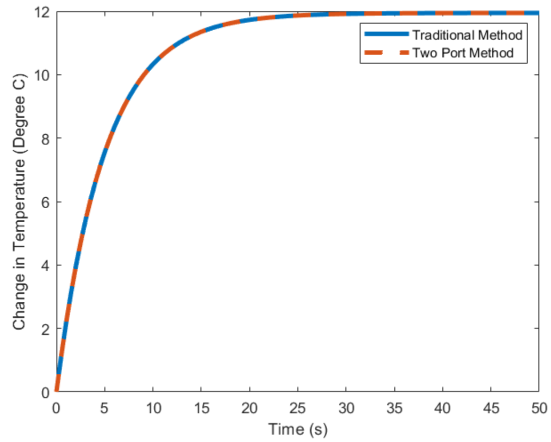

FIG. 5. *Time response of the water for both methods.*

III. CONCLUSION

From the above demonstration of both methods, the advantages and disadvantages of both the classical method to thermodynamics become transparent. For trivial thermodynamics problems, such as the one demonstrated above, it is often easier to use the classic method of power and energy conversions. However this ignores the heat lost to the resistor during the charging of the capacitor.

The classical method lends itself to a more instinctual understanding of the problem, as most of the problem is solved in the time domain. However, the two-port representation naturally includes the heat lost to the water, and is an algorithmic approach to solving such problems. As an interesting example, consider the case where C_1 is replaced by an inductor. In this case the circuit's resonant frequency is dramatically reduced (becomes finite) by adding the heat capacity of the water.

The transmission matrix method lends itself to much more complex versions of the thermodynamic problem, where, for example, the voltage applied is not be a simple unit step function. This method would also be more useful in creating simulated environment algorithms that are more accurate and efficient compared to methods that are based around time integration such as modeling more complex thermodynamic phenomenons such as triple point and super cooling. By understanding this analysis and being able to apply this method to thermodynamics, it may open up new insights into the discipline of thermodynamics.