Impedance and Admittance

Waves in 1-ports

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Summary from the last lecture

- The wave equation

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = \frac{\partial^2 \Phi}{\partial x^2}$$

  describes any system that allows waves to travel at the wave speed in different directions
- It obeys linearity (superposition)
- All transmission line theory can be understood with these simple ideas that waves traveling in opposite directions:

$$\Phi(x, t) = f(ct - x) + g(ct + x)$$

- Note that this solution only works in 1-dimension

1-Ports networks

- The 1-port is a pair of terminals:

Pressure $p(t)$ *[voltage* $v(t)$*, force* $f(t)$*]*
Volume velocity $u(t)$ *[current* $i(t)$*, particle velocity* $v(t)$*]*

- The Thevenin/Norton model is almost a 1-port

  http://auditorymodels.org/GEAR/PAPERS/Stuff/Johnson1-03.djvu

- A 2-port has an input and an output (i.e., two 1-ports)

Definition of the *characteristic impedance* z

- Pressure as the sum of two waves:

$$P(x,s) \equiv P^+(x,s) + P^-(x,s)$$

- Velocity as the sum of two waves:

$$U(x,s) \equiv U^+(x,s) - U^-(x,s)$$

- The *characteristic impedance for each* $\pm$ *wave is:*

$$z_0 \equiv \frac{P^+}{U^+} = \frac{P^-}{U^-}$$

  - Parity: *symmetry with respect to reflection*

Definition of the *reflectance* $R(x, s)$

- Next rearrange the terms in z_0:

$$R(x, s) \equiv \frac{P^-}{P^+} = \frac{U^-}{U^+}$$

  - This shows that $R(s)$ relates the two waves, independent of the wave type, P or U
- thus:

$$P = P^+ + RP^+ = (1 + R)P^+$$

- and:

$$U = U^+ - RU^+ = (1 - R)U^+$$

Impedance to Reflectance II

- The impedance $Z(s)$ may be defined in terms of the reflectance as follows:
  - Starting from Ohm's Law:

$$Z \equiv \frac{P(s)}{U(s)} = \frac{P^+ + P^-}{U^+ - U^-} = \frac{P^+}{U^+}\left(\frac{1 + P^-/P^+}{1 - U^-/U^+}\right) = z_o \frac{1+R}{1-R}$$

  - Solving for R in terms of Z and z_o gives:

$$R = \frac{Z - z_o}{Z + z_o}$$

How to determine $R(s)$?

- The reflectance $R(s)$ may be computed from z_o and $Z(s)$ as

$$R(s) \equiv \frac{Z(s) - z_o}{Z(s) + z_o}$$

where $s = \sigma + i\omega$

  - Example: Z(s) = sL

$$R(s) \equiv \frac{s - z_o/L}{s + z_o/L}.$$

Thus $R(s)$ is an all-pass filter with a pole and zero on either side of the s-plane origin. Since $|R(s)| = 1$, the system is lossless.

1-Port Absorbed Power

- Let $\mathcal{P}(t)$ be the absorbed power, defined as

$$\mathcal{P}(t) \equiv \int_{-\infty}^{t} p(\tau)u(\tau)d\tau$$

- In terms of waves

$$\mathcal{P}(t) \equiv \int_{-\infty}^{t} [p^{+} + p^{-}][u^{+} - u^{-}]d\tau$$

$$= \int_{-\infty}^{t} (p^{+})^{2}(\tau)d\tau/z_o - \int_{-\infty}^{t} p_{-}^{2}(\tau)d\tau/z_o$$

$$= \mathcal{P}^{+}(t) - \mathcal{P}^{-}(t) \geq 0.$$

- The total power is incident – retrograde.
  For a passive system absorbed power ≥ 0.

The lossless transmission line

Pressure $p(t)$ [voltage $v(t)$, force $f(t)$]
Volume velocity $u(t)$ [current $i(t)$, particle velocity $v(t)$]

- Example: If Z_{load} is and open circuit, $R(x=0)=1$,
  - Then $R(\omega, x=L) = e^{-i\frac{\omega}{c}2L} \leftrightarrow r(t) = \delta(t - 2L/c)$
  - Thus

$$Z(\omega) = z_o \frac{1 + e^{-i\frac{\omega}{c}2L}}{1 - e^{-i\frac{\omega}{c}2L}}$$

  - or

$$Z(\omega) = z_o \frac{e^{-i\frac{\omega}{c}L} + e^{-i\frac{\omega}{c}L}}{e^{-i\frac{\omega}{c}L} - e^{-i\frac{\omega}{c}L}} = i z_o \cot\left(\frac{\omega}{c}L\right)$$

- $Z_{load} = 0 \rightarrow R(x=0) = -1 \rightarrow Z(\omega) = i z_o \tan\left(\frac{\omega}{c}L\right)$

Properties of: $r(t) \leftrightarrow R(s)$

- $|R(s)| \leq 1 \quad \Rightarrow \quad ||r(t)|| \leq 1.$ passive
- $r(t) = 0$ for all $t < 0$ causal
  - $r(t) = 0$ for all $t \leq 0$ strictly causal
- In the time domain $R(s)$ defines convolution relations

$$p^-(t) = r(t) \star p^+(t)$$

$$u^-(t) = r(t) \star u^+(t)$$

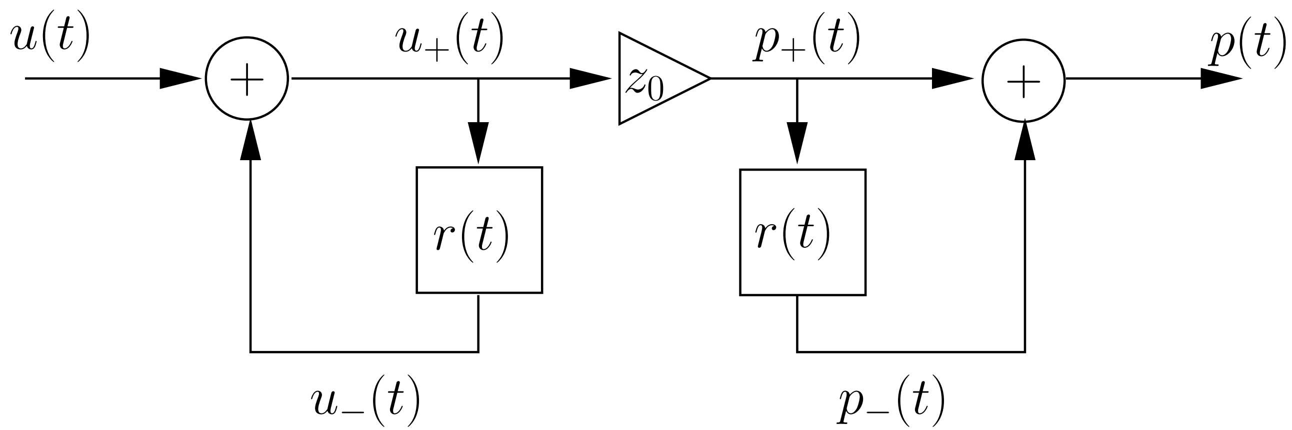
  - If we let $p^+(t) = \delta(t)$ then $p^-(t) = r(t)$
  - Likewise, if we let $u^+(t) = \delta(t)$, then $u^-(t) = r(t)$
- Thus $r(t)$ relates the two traveling waves, independent of the wave type, p or u.

1-port reflectance

- Ohm's law is great for simple functions of $s = \sigma + i\omega$
- However for systems with delay, such as the vocal tract, the middle ear, or with nonlinear systems such as the cochlear, it is simply inadequate.
- Formulating Ohm's law in the time domain, with waves, can be much more physical, and informative.

1-Port impedance via waves

- The 1-port impedance, described by wave variables:



- $p^+ = z_o u^+$, , $z_o = \frac{\rho c}{A}$ is the *surge impedance*, *reflectance* $r(t)$ is given by:

$$u^-(t) = r(t) \star u^+(t), \quad p^-(t) = r(t) \star p^+(t),$$

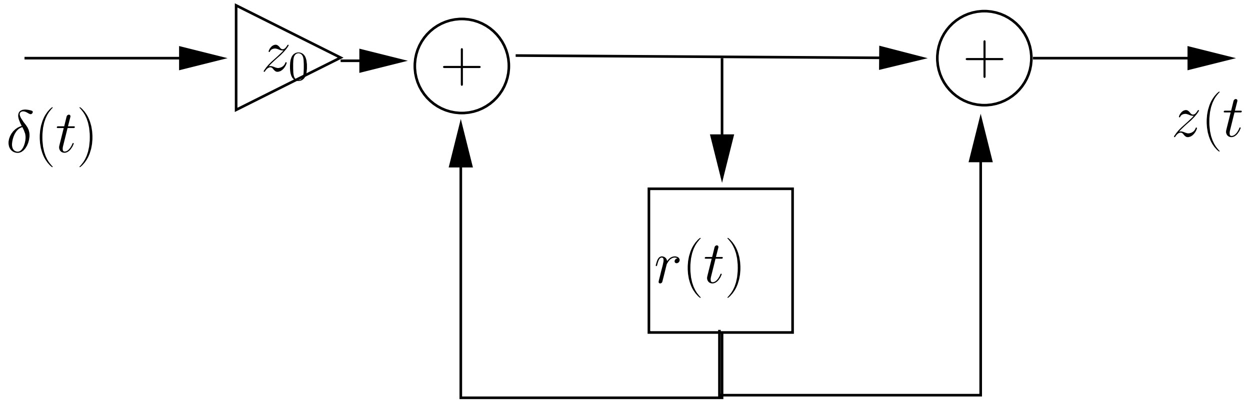
and

$$u^+(t) = u(t) + u^-(t), \quad p(t) = p^+(t) + p^-(t).$$

Explicit formula for $z(t)$

- If we let $u(t) = \delta(t)$ then we recover $z(t)$

$$z(t) = r(t) \star z(t) + z_o[\delta(t) + r(t)]$$



- This justifies calling z_o the *surge impedance* since

$$z_o(t = 0^+) \equiv \frac{1}{2\pi} \int_{\omega=\infty}^{\infty} \Re\, z_0(x, s)\big|_{\sigma=0}\, d\omega$$