1-D Wave Equations

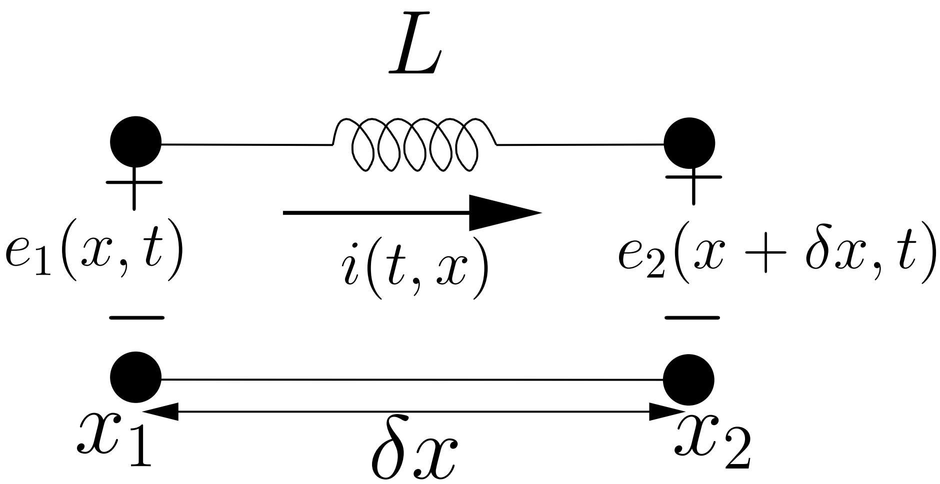
Transmission lines

Jont Allen

ECE-537

The electrical case

- Series elements:
  e is the *electromotive force*



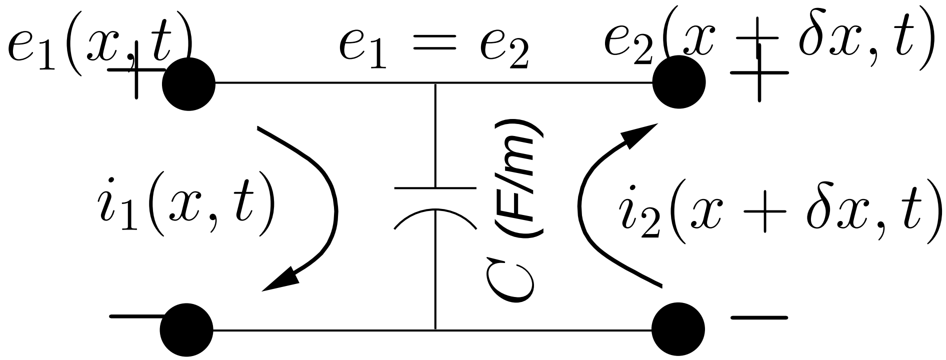
From Ohm's Law

$$\lim_{\delta x \to 0} \frac{e_1 - e_2}{x_1 - x_2} = -\frac{\partial e}{\partial x} = L\frac{\partial i}{\partial t}$$

The electrical case, cont.

- Shunt elements:
  i is the flow of charge

$$\frac{i_1 - i_2}{\delta x} = C\frac{\partial e}{\partial t}$$



thus from Ohm's Law

(1) $$-\frac{\partial i}{\partial x} = C\frac{\partial e}{\partial t}.$$

The electrical case, cont.

- Remove i to obtain an expression for e
- From Eq. 1, operating with $\partial/\partial x$

$$-\frac{\partial^2 i}{\partial x \partial t} = \frac{1}{L}\frac{\partial^2 e}{\partial x^2}.$$

- From Eq. 2, operating with $\partial/\partial t$

$$-\frac{\partial^2 i}{\partial t \partial x} = C\frac{\partial^2}{\partial t^2}.$$

- Equate the RHSs to obtain the wave equation in e:

(2) $$\frac{1}{c^2}\frac{\partial^2 e}{\partial t^2} = \frac{\partial^2 e}{\partial x^2}; \quad c = 1/\sqrt{LC},\ z_0 = \sqrt{L/C}$$

The acoustic case

- Pressure $p(x,t)$ is similar to $e(x,t)$ (scaler force, Ohm's Law)

$$-\frac{\partial p}{\partial x} = \rho_0 \frac{\partial u}{\partial t}$$

- Volume velocity u similar to i (vector flow, Ohm's Law)

$$-\gamma P_0 \frac{\partial u}{\partial x} = \frac{\partial p}{\partial t}.$$

- thus

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2}$$

- Defs: $P_0 = 10^5$ [Pa], $\gamma = 1.4$, $\rho_0 = 1.18$ [kgm/m^3]; $c = \sqrt{\frac{\gamma P_0}{\rho_0}}$ [m/s], $z_0 = \sqrt{\rho_0 \gamma P_0}$ [acoustic ohms]

The mechanical case

- Force f is similar to e and p: From Newton's Law

$$-\frac{\partial f}{\partial x} = m\frac{\partial u}{\partial t}$$

- Particle velocity u similar to i: From Hooke's Law

$$-k\frac{\partial u}{\partial x} = \frac{\partial f}{\partial t}.$$

- thus, as before:

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

- where $c = \sqrt{\frac{k}{m}}$ m/s; $z_0 = \sqrt{km}$, with k [Nt/m²]; m [kgm/m].

Solution to the wave equation

- Assume two arbitrary functions $f(\zeta)$ and $g(\zeta)$, traveling with velocity c and $-c$, respectively:

$$\Phi(x,t) = f(ct - x) + g(ct + x)$$

- The proof of this is nice. Let $f' \equiv \partial f(\zeta)/\partial\zeta$.

$$\frac{\partial\Phi(x,t)}{\partial x} = -f' + g'; \qquad \frac{\partial^2\Phi(x,t)}{\partial x^2} = (-1)^2 f'' + g''$$

$$\frac{\partial\Phi(x,t)}{\partial t} = c(f' + g'); \qquad \frac{\partial^2\Phi(x,t)}{\partial t^2} = c^2(f'' + g'')$$

- Thus:

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\partial^2\Phi}{\partial x^2}$$

Summary

- The wave equation describes any *linear system* that allows waves (arbitrary functions) to travel at the wave speed, in different directions, at the same time
- It obeys linearity (superposition)
- Note that this proof applies to the 1-dimension case
- All transmission line theory can be understood with these simple ideas (i.e., two waves traveling in opposite directions)

Pressure $p(t)$ *[voltage* $v(t)$*, force* $f(t)$*]*
Volume velocity $u(t)$ *[current* $i(t)$*, particle velocity* $v(t)$*]*