1.5.5 Lec 35 (II): Scalar Wave Equations (Acoustics)

In this section we discuss the general solution to the wave equation. The wave equation has two forms: scalar waves (acoustics) and vector waves (electromagnetics). These have an important mathematical distinction, but have a similar solution space, one scalar and the other vector. To understand the differences we start with the scalar wave equation.

The scalar wave equation: A good starting point for understanding PDEs is to explore the scalar wave equation (Eq. 1.27, p. 68). Thus, we shall limit our analysis to acoustics, the classic case of scalar waves. Acoustic wave propagation was first analyzed mathematically by Isaac Newton (electricity had yet to be discovered) in his famous book *Principia* (1687), in which he first calculated the speed of sound based on the conservation of mass and momentum.

Early history: The study of wave propagation begins at least as early as Huygens (ca. 1678), followed soon after (ca. 1687) by Sir Isaac Newton's calculation of the speed of sound (Pierce, 1981, p. 15). To obtain a wave, one must include two basic components: the stiffness of air, and its mass. These two equations shall be denoted (1) *Newton's 2nd law* ($F = ma$) and (2) *Hooke's law* ($F = kx$), respectively. In vector form these equations are (1) *Euler's equation* (i.e., conservation of momentum density)

$$-\nabla\varrho(\boldsymbol{x},t) = \rho_o \frac{\partial}{\partial t}\boldsymbol{u}(\boldsymbol{x},t) \leftrightarrow \rho_o s\mathcal{V}(\boldsymbol{x},s), \tag{1.129}$$

which assumes the density ρ_o is independent of time and position $\boldsymbol{x}$, and (2) the *continuity equation* (i.e., conservation of mass density)

$$-\boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x},s) = \frac{1}{\eta_o P_o}\frac{\partial}{\partial t}\varrho(\boldsymbol{x},t) \leftrightarrow \frac{s}{\eta_o P_o}\mathcal{P}(\boldsymbol{x},s) \tag{1.130}$$

(Pierce, 1981; Morse, 1948, p. 295). Here $P_o = 10^5$ [Pa], is the barometric pressure, $\eta_o = 1.4$ and $\eta_o P_o$ is the dynamic (adiabatic) stiffness. Combining Eqs. 1.129 and 1.130 (removing $\boldsymbol{u}(\boldsymbol{x},t)$) results in the 3-dimensional (3D) scalar pressure wave equation

$$\nabla^2\varrho(\boldsymbol{x},t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(\boldsymbol{x},t) \leftrightarrow \frac{s^2}{c_o^2}\mathcal{P}(\boldsymbol{x},s) \tag{1.131}$$

with $c_o = \sqrt{\eta_o P_o/\rho_o}$ is the sound velocity. Because the merged equations describe the pressure, which is a scalar field, this is an example of the *scalar wave equation*

Exercise: Show that Eqs. 1.129 and 1.130 can be reduced to Eq. 1.131. **Solution:** Taking the divergence of Eq. 1.129 gives

$$-\boldsymbol{\nabla}\cdot\nabla\varrho(\boldsymbol{x},t) = \rho_o\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x},t). \tag{1.132}$$

Note that $\boldsymbol{\nabla}\cdot\nabla = \nabla^2$. Next, substituting Eq. 1.130 into the above relation results in the scalar wave equation Eq. 1.131, since $c_o = \sqrt{\eta_o P_o/\rho_o}$.

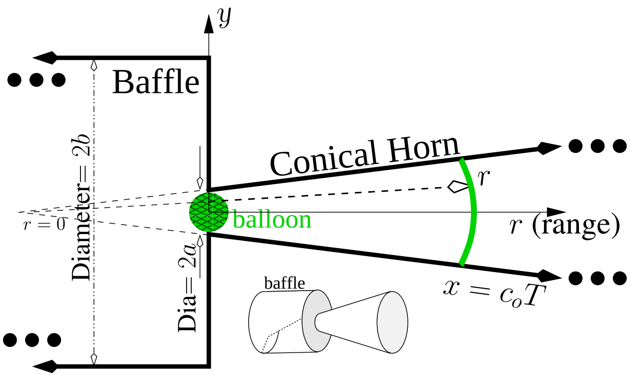

Figure 1.22: *Experimental setup showing a large pipe on the left terminating the wall containing a small hole with a balloon, shown in green. At time $t = 0$ the balloon is pricked and a pressure pulse is created. The baffle on the left is meant to represent a semi-∞ long tube having a large radius compared to the horn input diameter $2a$, such that the acoustic admittance looking to the left ($A/\rho_o c_o$ with $A \to \infty$) is very large compared to the horn's throat admittance (Eq. 1.149). At time T the outbound pressure pulse $p(r,T) = \delta(t - x/c_o)/r$ has reached a radius $x = r - r_o = c_oT$ where $r = x$ is the location of the source at the throat of the horn and r is measured from the vertex. At the throat of the horn $\mathcal{V}_+/\boldsymbol{A}_+ = \mathcal{V}_-/\boldsymbol{A}_-$.*

1.5.6 Lec 36a: The Webster horn equation (I)

There is an important generalization of the problem of lossless plane-wave propagation in 1-dimensional (1D) uniform tubes (e.g., transmission line theory). By allowing the area $A(r)$ of the horn to vary along the *range* axis r (the direction of wave propagation), as depicted in Fig. 1.22 for a *conical horn* (i.e., $A(r) = A_o(r/r_o)^2$), general solutions to the wave equation may be explored. Classic applications of horns include vocal tract acoustics, loudspeaker design, cochlear mechanics, the hydrogen atom, and cases having wave propagation in periodic media (Brillouin, 1953).

For the 1D scalar wave equation (guided waves, aka, acoustic horns), the *Webster Laplacian* is

$$\nabla_r^2\, \varrho(r,t) = \frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\,\varrho(r,t). \tag{1.133}$$

The Webster Laplacian is based on the *quasi-static approximation* (P10: p. 129) which requires that the frequency lies below the critical value $f_c = c_o/2d$, namely that a half wavelength is greater than the horn diameter d (i.e., $d < \lambda/2$).[118] For the case of the adult human ear canal, $d = 7.5$ [mm] and $f_c = (343/2 \cdot 7.5) \times 10^{-3} \approx 22.87$ [kHz].

The term on the right of Eq. 1.133, which is identical to Eq. 1.116 (p. 176), is also the Laplacian for thin tubes (e.g., rectangular, spherical, and cylindrical coordinates). Thus the Webster horn "wave" equation is

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\varrho(r,t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(r,t) \leftrightarrow \frac{s^2}{c_o^2}\mathcal{P}(r,s) \tag{1.134}$$

where $\varrho(r,t) \leftrightarrow \mathcal{P}(r,s)$ is the average pressure (Hanna and Slepian, 1924; Mawardi, 1949; Morse, 1948), Olson (1947, p. 101), Pierce (1981, p. 360). Extensive experimental analysis for various types of horns (conical, exponential, parabolic) along with a review of horn theory may be found in Goldsmith and Minton (1924).

The limits of the Webster horn equation: It is frequently (i.e., always) stated that the Webster horn equation (WHEN) is fundamentally limited, thus is an approximation that only applies to

[118]This condition may be written several ways, the most common being $ka < 1$, where $k = 2\pi/\lambda$ and a is the horn radius. This may be expressed in terms of the diameter as $\frac{2\pi}{\lambda}\frac{d}{2} < 1$, or $d < \lambda/\pi < \lambda/2$. Thus $d < \lambda/2$ may be a more precise metric by the factor $\pi/2 \approx 1.6$. This is called the *half-wavelength assumption* a synonym for the quasi-static approximation.

frequencies much lower than f_c. However in all these discussions it is assumed that the area function $A(r)$ is the horn's cross-sectional area, not the area of the iso-pressure wave-front (Morse, 1948; Shaw, 1970; Pierce, 1981).

In the next section it is shown that this "limitation" may be avoided (subject to the $f < f_c$ quasi-static limit (P10, p. 130)), *making the Webster horn theory an "exact" solution for the lowest order "plane-wave" eigenfunction*. The nature of the quasi-static approximation is that it "ignores" higher order evanescent modes, which are naturally small since they are in cutoff (evanescent modes do not propagate) (Hahn, 1941; Karal, 1953). This is the same approximation that is required to define an impedance, since every eigenmode defines an impedance (Miles, 1944).

To apply this theory, the acoustic variables (eigenfunctions) are redefined for the *average pressure* and its corresponding *volume velocity*, each defined on the iso-pressure wave-front boundary (Webster, 1919; Hanna and Slepian, 1924). The resulting impedance is then the ratio of the average pressure over the volume velocity. This approximation is valid up to the frequency where the next mode begins to propagate ($f > f_c$), which may be estimated from the roots of the Bessel eigenfunctions (Morse, 1948). Perhaps it should be noted that these ideas, that come from acoustics, apply equally well to electromagnetics, or any other wave phenomena described by eigenfunctions.

The best known examples of wave propagation are electrical and acoustic *transmission lines*. Such systems are loosely referred to as the *telegraph* or *telephone equations*, referring back to the early days of their discovery (Brillouin, 1953; Heaviside, 1892; Campbell, 1903b; Feynman, 1970a). In acoustics, guided waves are called horns, such as the horn connected to the first phonographs from around the turn of the century (Webster, 1919). Thus the names reflect the historical development, to a time when the mathematics and the applications were running in close parallel.

1.5.7 Lec 36b: Webster horn equation (II): Derivation

Here we transform the acoustic equations Eq. 1.129 and 1.130 (p. 192) into their equivalent integral form Eq. 1.134 (p. 193). This derivation is similar (but not identical) to that of Hanna and Slepian (1924) and Pierce (1981, p. 360).

Conservation of momentum: The first step is an integration of the normal component of Eq. 1.129 (p. 192) over the iso-pressure surface $\mathcal{S}$, defined by $\nabla p = 0$

$$-\int_{\mathcal{S}} \nabla p(\mathbf{x}, t) \cdot d\mathbf{A} = \rho_o \frac{\partial}{\partial t} \int_{\mathcal{S}} \mathbf{u}(\mathbf{x}, t) \cdot d\mathbf{A}.$$

The *average pressure* $\varrho(x, t)$ is defined by dividing by the total area

$$\varrho(x, t) \equiv \frac{1}{A(x)} \int_{\mathcal{S}} p(x, t)\, \hat{\boldsymbol{n}} \cdot d\boldsymbol{A}. \tag{1.135}$$

From the definition of the gradient operator

$$\nabla p = \frac{\partial p}{\partial x} \hat{\boldsymbol{n}}, \tag{1.136}$$

where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the iso-pressure surface $\mathcal{S}$. Thus the left side of Eq. 1.129 reduces to $\partial\varrho(x, t)/\partial x$.

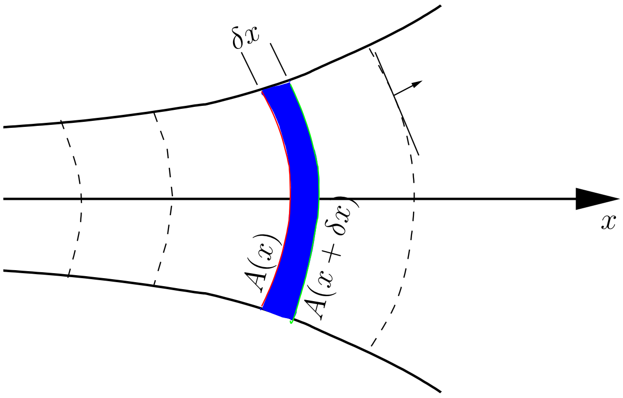

Figure 1.23: *Derivation of horn equation using Gauss's law: The divergence of the velocity $\nabla \cdot \mathbf{u}$, within δx, shown as the filled* shaded region*, is integrated over the enclosed volume. Next the divergence theorem is applied, transforming the integral to a surface integral normal to the surface of propagation. This results in the difference of the two volume velocities $\delta\nu = \nu(x+\delta x) - \nu(x) = [\mathbf{u}(x+\delta x)\cdot \mathbf{A}(x+\delta x) - \mathbf{u}(x)\cdot \mathbf{A}(x)]$. The flow is always perpendicular to the iso-pressure contours.*

The integral on the right side defines the *volume velocity*

$$\nu(x,t) \equiv \int_S \mathbf{u}(x,t)\cdot d\boldsymbol{A}. \tag{1.137}$$

Thus the integral form of Eq. 1.129 becomes

$$-\frac{\partial}{\partial x}\varrho(x,t) = \frac{\rho_o}{A(x)}\frac{\partial}{\partial t}\nu(x,t) \leftrightarrow \mathscr{Z}(x,s)\mathcal{V} \tag{1.138}$$

where

$$\mathscr{Z}(s,x) = s\rho_o/A(x) = sM(x) \tag{1.139}$$

and $M(x) = \rho_o/A(x)$ [kgm/m^5] is the per-unit-length mass density of air.

Conservation of mass: Integrating Eq. 1.130 (p. 192) over the volume $\mathcal{V}$ gives

$$-\int_{\mathcal{V}} \nabla\cdot\boldsymbol{u}\,dV = \frac{1}{\eta_o P_o}\frac{\partial}{\partial t}\int_{\mathcal{V}} p(\boldsymbol{x},t)dV.$$

Volume $\mathcal{V}$ is defined by two iso-pressure surfaces between x and $x+\delta x$ (Fig. 1.23). On the right-hand side we use our definition for the *average pressure* (i.e., Eq. 1.135), integrated over the thickness δx.

Applying Gauss's law to the left-hand side,[119] and using the definition of ϱ (on the right), in the limit $\delta x \to 0$, gives

$$\frac{\partial\nu}{\partial x} = -\frac{A(x)}{\eta_o P_o}\frac{\partial\varrho}{\partial t} \leftrightarrow -\mathscr{Y}(x,s)\mathcal{P} \tag{1.140}$$

where

$$\mathscr{Y}(s,x) = sA(x)/\eta_o P_o = sC(x).$$

$C(x) = A(x)/\eta_o P_o$ [m^4/N] is the per-unit-length compliance of the air. These two equations Eq. 1.138 and 1.140 accurately characterize the Webster plane-wave mode up to the frequency where the higher order eigenmodes begin to propagate (i.e, $f > f_c$).

Speed of sound c_o: In terms of $M(x)$ and $C(x)$, the speed of sound and the acoustic admittance are

$$c_o = \sqrt{\frac{\textit{stiffness}}{\textit{mass}}} = \frac{1}{\sqrt{C(x)M(x)}} = \sqrt{\frac{\eta_o P_o}{\rho_o}} \tag{1.141}$$

[119] As shown in Fig. 1.23, we convert the divergence into the difference between two volume velocities, namely $\nu(x+\delta x) - \nu(x)$, and $\partial\nu/\partial x$ as the limit of this difference over δx, as $\delta x \to 0$.

Characteristic admittance $\mathcal{Y}(x)$: Since the horn equation (Eq. 1.134) is 2d order, it has pressure eigenfunction solutions $\mathcal{P}^+$ and $\mathcal{P}^-$ and their corresponding velocity eigenfunctions $\mathcal{V}^+$ and $\mathcal{V}^-$, related through Eq. 1.138, which defines the *characteristic admittance* $\mathcal{Y}(x)$

$$\mathcal{Y}(x) = \frac{1}{\sqrt{\textit{stiffness} \cdot \textit{mass}}}. = \sqrt{\frac{C(x)}{M(x)}} = \frac{A(x)}{\rho_o c_o} = \frac{\mathcal{V}^+}{\mathcal{P}^+} = \frac{\mathcal{V}^-}{\mathcal{P}^-} \tag{1.142}$$

(Campbell, 1903a, 1910, 1922). The *characteristic impedance* $\mathcal{Z}(x) = 1/\mathcal{Y}(x)$. Based on physical requirements that the admittance must be positive, thus only the positive square root is allowed.

Since the horn (Eq. 1.134) is loss less, $\mathcal{Y}(x)$ must be real and positive. If losses are introduced, the *propagation function* $\kappa(s)$ (p. 142) and the characteristic impedance $\mathcal{Y}(x,s)$ would become complex analytic functions of the Laplace frequency s (Kirchhoff, 1974; Mason, 1928; Ramo *et al.*, 1965; Pierce, 1981, p. 532-4).

One must be carefully in the definition the area $A(x)$: The area is *not* the cross-sectional area of the horn, rather it is the wave-front area, as discussed next. Since $A(x)$ is independent of frequency, it is independent the wave direction.

1.5.8 Matrix formulation of WHEN (III)

Newton's conservation of momentum law (Eq. 1.129), along with conservation of mass (Eq. 1.130), are modern versions of Newton's starting point for accurately calculating the horn lowest order plane-wave eigenmode. Following the simplification of averaging the normal component of the *particle velocity* over the iso-pressure wave front, Eqs. 1.138, 1.140 may be rewritten as a 2x2 matrix in the acoustic variables, average pressure $\mathcal{P}(r,\omega)$ and volume velocity $\mathcal{V}(r,\omega)$ (here we replace the range-variable x by r)

$$-\frac{d}{dr}\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix} = \begin{bmatrix}0 & sM(r)\\ sC(r) & 0\end{bmatrix}\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix}, \tag{1.143}$$

where $M(r) = \rho_o/A(r)$ and $C(r) = A(r)/\eta_o P_o$ are the unit-length mass and compliance of the horn (Ramo *et al.*, 1965, p. ???). The acoustic variables $\mathcal{P}_c(r,\omega)$ and $\mathcal{V}(r,\omega)$ are sometimes referred to as *conjugate variables*.[120]

To obtain the Webster horn pressure equation Eq. 1.134 (p. 193) from Eq. 1.143 take the partial derivative of the top equation

$$-\frac{\partial^2\mathcal{P}}{\partial r^2} = s\frac{\partial M(r)}{\partial r}\mathcal{V} + sM(r)\frac{\partial\mathcal{V}}{\partial r}.$$

Use the lower equation to remove $\partial\mathcal{V}/\partial r$

$$\frac{\partial^2\mathcal{P}}{\partial r^2} + s\frac{\partial M(r)}{\partial r}\mathcal{V} = s^2 M(r)C(r)\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P},$$

and the upper equation a second time to remove $\mathcal{V}$. Thus Eq. 1.143 reduces to

$$\frac{\partial^2}{\partial r^2}\mathcal{P}(r,s) + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\mathcal{P}_r = \frac{s^2}{c_o^2}\mathcal{P}(r,s). \tag{1.144}$$

[120] `https://en.wikipedia.org/wiki/Conjugate_variables_(thermodynamics)` The product of conjugate variables defines an *intensity* while their ratio defines an *impedance* (Pierce, 1981, p. 37-41).

Equations of this form may be directly integrated by parts by use of the chain rule

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\mathcal{P}(r,s) = \frac{\partial^2}{\partial r^2}\mathcal{P}(r,s) + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\mathcal{P}_r(r,s), \tag{1.145}$$

where the integration factor is the horn area function $A(r)$.

Merging Eqs. 1.144 and 1.145 results in the Webster horn equation (Eq. 1.134, p. 193):

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\mathcal{P}(r,s) = \frac{s^2}{c_o^2}\mathcal{P}(r,s).$$

Equations having this integrated form are known as *Sturm-Liouville equations*. This important class of differential equations follow from the use of separation of variables on the Laplacian, in any (i.e., every) separable coordinate system (Morse and Feshbach, 1953, Ch. 5.1, p. 494-523).

Summary: Applying Gauss's law to the 3D wave equation (Eq. 1.131, p. 192) results in a 1D Webster horn equation (WHEN, Eq. 1.134, p. 193), which is a non-singular Sturm-Liouville equation, where the area function is the integration factor $A(r)$.[121]

Thus Eqs. 1.131 and 1.143 are equivalent to the WHEN (Eq. 1.134).

1.5.9 Lec 37a: d'Alembert's eigenvector superposition principle

Since the Webster horn equation (Eq. 1.134) is second order in time, it has two unique pressure eigenfunctions $\mathcal{P}^+(r,s)$ and $\mathcal{P}^-(r,s)$. The general solution may always be written as the superposition of pressure eigenfunctions, with amplitudes determined by the boundary conditions.

Based on d'Alembert's superposition principle, the pressure $\mathcal{P}$ and velocity $\mathcal{V}$ may be decomposed in terms of the pressure eigenfunctions $\mathcal{P}^+$ and $\mathcal{P}^-$

$$\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix} = \begin{bmatrix}1 & 1\\ \mathcal{Y}(r) & -\mathcal{Y}(r)\end{bmatrix}\begin{bmatrix}\mathcal{P}^+(r,\omega)\\ \mathcal{P}^-(r,\omega)\end{bmatrix}. \tag{1.146}$$

This equation has several applications.

Generalized admittance/impedance: The *generalize admittance*[122] $Y_{in}(r,s)$ looking into the horn is

$$Y_{in}(r,s) \equiv \frac{\mathcal{V}(r,\omega)}{\mathcal{P}(r,\omega)} = \frac{\mathcal{V}^+ - \mathcal{V}^-}{\mathcal{P}^+ + \mathcal{P}^-} = \frac{\mathcal{V}_+}{\mathcal{P}_+}\left(\frac{1-\mathcal{V}^-/\mathcal{V}^+}{1+\mathcal{P}^-/\mathcal{P}^+}\right) = \mathcal{Y}(r)\frac{1-\Gamma(r,s)}{1+\Gamma(r,s)}. \tag{1.147}$$

Here we have factored out the forward traveling eigenfunction $\mathcal{V}^+$ and $\mathcal{P}^+$, and re-expressed Y_{in} in terms of two ratios, the *characteristic admittance* $\mathcal{Y}(r)$ (Eq. 1.142) and the reflectance $\Gamma(r,s)$. $Y_{in}(s)$ depends on the entire horn. In the case of a finite length horn, it depends on the terminating admittance. When the horn is terminated, reflections occur, resulting in the horn having poles and zeros at frequencies $s_k \in \boldsymbol{C}$, where $\Gamma(r,s_k) = \pm 1$.

[121] The Webster Horn equation is also related to Schrödenger's Eq., the corner stone of quantum mechanics 1.5.6, p. 331.

[122] It is "generalized" in the sense that it is not a Brune, rational function, impedance.

Table 1.5: *Table of horns and their properties for N = 1, 2 or 3 dimensions, along with the exponential horn (EXP). In this table the horn's range variable is x, having* area $A(x)$, *diameter* $r_o(x) = \sqrt{A(x)/4\pi}$. *$F(x)$ is the coefficient on $\mathcal{P}_x$, $\kappa(s) \equiv s/c_o$, where c_o is the speed of sound and $s = \sigma + \omega\jmath$ is the Laplace frequency. The range variable x may be rendered dimensionless (see Fig. 1.24) by normalizing it as $x \equiv (\xi - \xi_o)/(L - \xi_o)$, with ξ the linear distance along the horn axis, from $x = \xi_o$ to L corresponding to $x = 0$ to 1. The horn's eigenfunctions are $\mathcal{P}^{\pm}(x, \omega) \leftrightarrow \varrho^{\pm}(x, t)$. When $\pm$ is indicated, the outbound solution corresponds to the negative sign. Eigenfunctions $H_o^{\pm}(x, s)$ are outbound and inbound Hankel functions. The last column is the* radiation admittance *normalized by the* characteristic admittance $\mathcal{Y}(x) = A(x)/\rho_o c_o$.

N	Name	radius	Area/A_o	$F(x)$	$\mathcal{P}^{\pm}(x,s)$	$\varrho^{\pm}(x,t)$	$Y^{\pm}_{rad}/\mathcal{Y}$
1D	uniform	1	1	0	$e^{\pm\kappa(s)x}$	$\delta(t \mp x/c_o)$	1
2D	parabolic	$\sqrt{x/x_o}$	x/x_o	$1/x$	$H_o^{\pm}(-\jmath\kappa(s)x)$	—	$\frac{-\jmath x H_1^{\pm}}{H_o^{\pm}}$
3D	conical	x	x^2	$2/x$	$e^{\pm\kappa(s)x}/x$	$\delta(t \mp x/c_o)/x$	$1 \pm c_o/sx$
EXP	exponential	e^{mx}	e^{2mx}	$2m$	$e^{-\left(m \pm \sqrt{m^2+\kappa^2}\right)x}$	$e^{-mx}E(t)$	Eq. 1.150

The *reflectance* is defined as

$$\Gamma(r,s) \equiv \frac{\mathcal{V}^-(r,\omega)}{\mathcal{V}^+(r,\omega)} = \frac{\mathcal{P}^-(r,\omega)}{\mathcal{P}^+(r,\omega)}, \tag{1.148}$$

which follows by a rearrangement of terms in Eq. 1.142. The magnitude of the reflections depends $|\Gamma|$, which must be between 0 and 1. Alternatively this equation may be obtained by solving Eq. 1.147 for $\Gamma(r,s)$.

Horn radiation admittance: A horn's acoustic *radiation admittance* $Y^{\pm}_{rad}(r,s)$ is the input admittance (Eq. 1.147) when there is no terminating load[123]

$$Y^{\pm}_{rad}(r,s) = \lim_{r\rightarrow\infty} Y^{\pm}_{in}(r,s) = -\lim_{r=\rightarrow\infty} \frac{A(r)}{s\rho_o}\frac{d}{dr}\ln \mathcal{P}^{\pm}(r,s). \tag{1.149}$$

The input admittance becomes the radiation admittance when the horn is infinite in length, namely it is the input admittance for an eigenfunction. A table of properties is given in Table 1.5 for four different simple horns.

Expressions for $Y_{rad}(x,s)$ are given in the last column of Table 1.5. For the case of the exponential horn with $\kappa = s/c_o$

re-derive

$$Y^{\pm}_{rad}(x,s) = -\frac{A(x)}{s\rho_o}\left(m \pm \sqrt{m^2+\kappa^2}\right)x. \tag{1.150}$$

Kleiner (2013) gives the expression for $Y_{rad}(x,\omega)$ for the exponential horn as

$$Y_{rad}(x,\omega) = \frac{S(x)}{\jmath\omega\rho}\left[\frac{m}{2} + \jmath\frac{\sqrt{4\omega^2 - (mc)^2}}{2c}\right]$$

and impedance

$$Z_{in}(r,s) = \frac{\rho c}{S_T}\left[\jmath\frac{\omega_c}{\omega} + \sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}\right],$$

where $\omega_c(r)$ is the cutoff frequency.

[123] To compute the radiation impedance $Y^{\pm}_{rad}$ one must know the eigenfunctions $\mathcal{P}^{\pm}(r,s)$.

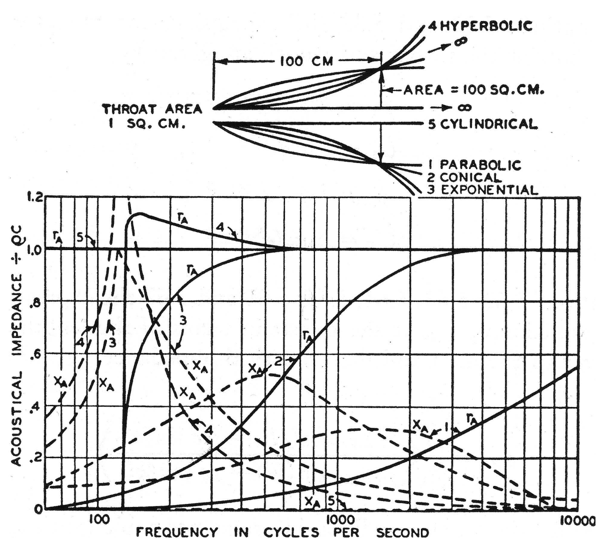

Figure 1.24: *Throat acoustical resistance r_A and acoustical reactance x_A, frequency characteristics of infinite eigenfunctions of the parabolic, conical, exponential, hyperbolic and cylindrical horns,having a throat area of 1 square centimeter. Note how the "critical" frequency (defined here as the frequency where the reactive and real parts of the radiation impedance are equal) of the horn reduces dramatically with the type of horn. For the uniform horn, the reactive component is zero, so there is no cutoff frequency. For the parabolic horn (1) the cutoff is around 3 kHz. For the conical horn (2) the cutoff is at 0.6 [kHz]. For the exponential horn (3) the critical frequency is around 0.18 [kHz], which is one 16th that of the parabolic horn. For each horn the cross-sectional area is defined as 100 [cm^2] at a distance of 1 [m] from the throat (Olson, 1947, p. 101), (Morse, 1948, p. 283).*

1.5.10 Lec37b: Complex-analytic $\Gamma(s)$ and $Y_{in}(s)$

When defining the complex reflectance $\Gamma(s)$ as a function of the complex frequency $s = \sigma + j\omega$, a *very* important assumption has been made: even though $\Gamma(s)$ is defined by the ratio of two functions of real (radian) frequency ω, like the impedance, the reflectance must be *causal* (postulate P1, p. 128). Namely $\Gamma(s) \leftrightarrow \gamma(t)$ is zero for $t < 0$. The same holds for the time-domain admittance and impedance $\zeta(t) \leftrightarrow Z_{in}(s) = 1/Y_{in}(s)$. That $\gamma(t)$ and $\zeta(t)$ are causal is required by the physics.

The forward and retrograde waves are functions of frequency ω, as they depend on the source pressure (or velocity) and the point of horn excitation. The reflectance is a transfer function (thus the source term cancels) that depends only on the Thévenin impedance (or reflectance) looking into the system (at any position r).

To calculate $\Gamma(r, s)$ we invert d'Alembert's superposition equation (Eq. 1.146)

$$\begin{bmatrix} \mathcal{P}^+(r,s) \\ \mathcal{P}^-(r,s) \end{bmatrix} = \frac{1}{2\mathcal{Y}(r)} \begin{bmatrix} \mathcal{Y}(r) & 1 \\ \mathcal{Y}(r) & -1 \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ \mathcal{V} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & \mathcal{Z}(r) \\ 1 & -\mathcal{Z}(r) \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ \mathcal{V} \end{bmatrix}. \tag{1.151}$$

The reflectance is defined as the ratio of the two pressure eigenfunctions

$$\Gamma(r, s) \equiv \frac{\mathcal{P}^-}{\mathcal{P}^+} = \frac{\mathcal{P} - \mathcal{Z}\mathcal{V}}{\mathcal{P} + \mathcal{Z}\mathcal{V}} = \frac{Z_{in} - \mathcal{Z}}{Z_{in} + \mathcal{Z}} = -\frac{Y_{in} - \mathcal{Y}}{Y_{in} + \mathcal{Y}}, \tag{1.152}$$

which is related to Eq. 1.147.

Given some experience with $Y_{in}(r, s)$ and $\Gamma(r, s)$, one soon appreciates the advantage of working with the reflectance over the radiation impedance/admittance $Z_{rad}(s)$ (aka immittance). The impedance has complicated properties, all of which are difficult to verify, whereas the reflectance is easily understood (it is closer to the physics). For example, we know that for a physical passive impedance $\Re Z \geq 0$. The corresponding property for the reflectance is $|\Gamma(\omega)| \leq 1$, with equality when the input resistance is zero.

It is important to note that because the area $A(x)$ is varying along the direction of propagation, energy is continuously being scattered back to the input, as captured by the area-dependent eigenfunctions. It is because of this scattering that the input admittance $Y_{in}(s)$ (Eq. 1.147) and the reflectance $\Gamma(r, s)$ (Eq. 1.152) depends on frequency, as explicitly shown in Fig. 1.24 (Morse, 1948, p. 283).

Exercise:

1. Show that $\Re Y_{in}(s) \ge 0$ if and only if $|\Gamma \le 1|$. Hint: Use Eq 1.152 (or 1.147).

2. Showing that the unit circle in the $\Gamma(s)$ plane maps onto the $\omega\jmath$ axis in the impedance plane.

Solution: To prove this take the real part of $Y_{in}(s)$ (Eq. 1.147) and show that it is greater than zero if $|\Gamma(s)| \le 1$

$$\begin{aligned}\frac{2}{\mathcal{Y}(r)}\Re Y_{in}(s) &= \frac{1-\Gamma}{1+\Gamma} + \frac{1-\Gamma^*}{1+\Gamma^*}\\ &= \frac{(1-\Gamma)(1+\Gamma^*) + (1+\Gamma)(1-\Gamma^*)}{|1+\Gamma|^2}\\ &= \frac{2(1-|\Gamma|^2)}{|1+\Gamma|^2} \ge 0.\end{aligned}$$

In conclusion:

1. if $|\Gamma| < 1$, then $\Re Z_{in} > 0$.

2. if $|\Gamma| = 1$, then $\Re Z_{in} = 0$.

1.5.11 Lec 37c Finite length horns

For a horn of finite length L the acoustic variables $\mathcal{P}(x,s), \mathcal{V}(x,s)$ may be expressed in terms of pressure eigenfunctions. If we define the forward wave $\mathcal{P}^+(x,\omega)$ as launched from $x=0$ and the retrograde wave $\mathcal{P}^-(x,\omega)$ as launched from $x=L$, we may write the pressure and velocity as

$$\begin{bmatrix}\mathcal{P}(x)\\ \mathcal{V}(x)\end{bmatrix} = \begin{bmatrix}\mathcal{P}^+(x) & \mathcal{P}^-(x-L)\\ \mathcal{Y}(x)\mathcal{P}^+(x) & -\mathcal{Y}(x)\mathcal{P}^-(x-L)\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}. \tag{1.153}$$

Here $\alpha(x,\omega)$ scales the forward wave and $\beta(x,\omega)$ scales the retrograde wave. Thus the reflectance $\Gamma(L,\omega) = \beta/\alpha$ is defined at the site of reflection ($x=L$). Typically the *characteristic admittance* $\mathcal{Y}(x) = A(x)/\rho_o c_o$ only depends on the location x but not on the Laplace frequency s. This formula may not be correct if the horn has losses ($\mathcal{Y} \in \boldsymbol{C}$), as discussed in Kirchhoff (1868); Mason (1927, 1928); Robinson (2017).

To evaluate the coefficients $\alpha(\omega)$ and $\beta(\omega)$ we must invert Eq. 1.153. The eigenfunction scale factors α, β are determined by the load admittance Y_{load} at the cite of reflection $x=L$.

Notation: Adopting subscript notation: $\mathcal{P}_x^\pm \equiv \mathcal{P}^\pm(x)$, $\mathcal{P}_0^+ \equiv \mathcal{P}^+(0) = 1$, $\mathcal{P}^-(L) \equiv \mathcal{P}_L^- = 1$, $\mathcal{V}_x^\pm \equiv \mathcal{V}^\pm(x)$. $\mathcal{Y}_x = \mathcal{Y}(x)$, and inverting Eq. 1.153 at $x=L$ gives

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_L = \frac{-1}{\Delta_L}\begin{bmatrix}-\mathcal{Y}_L\cancelto{1}{\mathcal{P}^-(L-L)} & -\cancelto{1}{\mathcal{P}^-(L-L)}\\ -\mathcal{Y}_L\mathcal{P}_L^+ & \mathcal{P}_L^+\end{bmatrix}_L \begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_L \tag{1.154}$$

where the determinant is

$$\Delta_L = -2\mathcal{Y}_L\mathcal{P}_L^+\cancelto{1}{\mathcal{P}^-(L-L)}.$$

Since $\mathcal{Z}_L = 1/\mathcal{Y}_L$.

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_L = \frac{1}{2\mathcal{P}_L^+}\begin{bmatrix}1 & \mathcal{Z}_L\\ \mathcal{P}_L^+ & -\mathcal{Z}_L\mathcal{P}_L^+\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.155}$$

This quantifies the general expression for the eigenfunction amplitudes α, β at the reflection site $x = L$, where the two mix for the first time. Note that the sign of $\mathcal{V}_L$ must be negative to satisfy the definition of every ABCD matrix (i.e., the output velocity must equal the input velocity of the next cell). The reflection coefficient is given by the ratio of β/α at $x = L$, which depends on the load impedance

$$Z_{load}(x = L, s) = -\mathcal{P}_L/\mathcal{V}_L.$$

When the load impedance equals the local characteristic impedance $\mathcal{Z}_L$, $\beta = 0$.

Substituting Eq. 1.155 into Eq. 1.153 results in an expression for the input acoustic variables at $x = 0$ in terms of those at $x = L$, with $\mathcal{P}_L^+ = 1$ and $\mathcal{P}_L^- = 1$:

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_0 = \frac{1}{2\mathcal{P}_L^+}\begin{bmatrix}1 & \mathcal{P}_0^-\\ \mathcal{Y}_0 & -\mathcal{Y}_0\mathcal{P}_0^-\end{bmatrix}\begin{bmatrix}1 & -\mathcal{Z}_L\\ \mathcal{P}_L^+ & \mathcal{Z}_L\mathcal{P}_L^+\end{bmatrix}_L\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.156}$$

Thus

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_0 = \frac{1}{2\mathcal{P}_L^+}\begin{bmatrix}1 + \mathcal{P}_L^+\mathcal{P}_0^- & -\mathcal{Z}_L(1 + \mathcal{P}_L^+\mathcal{P}_0^-)\\ \mathcal{Y}_0^+(\mathcal{P}_0^+ - \mathcal{P}_L^+\mathcal{P}_0^-) & -\mathcal{Y}_0\mathcal{Z}_L(\mathcal{P}_0^+ - \mathcal{P}_L^+\mathcal{P}_0^-)\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.157}$$

It may be more useful to leave this expression in terms of $\Gamma_L(s)$ than to substitute Eq. 1.154 into Eq. 1.153. These expressions will be verified by comparing the special cases with this general case.

Consider changing the sig[n] ... $-\mathcal{V}$, for simplicity.

Three examples of horns

Summary of four classic horns: Figure 1.24 is taken from the classic book of Olson (1947, p. 101), showing the radiation impedance $Z_{rad}(r, \omega)$ for five horns: 1-parabolic, 2-conical, 3-exponential, 4-hyperbolic, and 5-cylindrical. A summary of the properties of four of these horns: 1) the uniform (cylindrical) ($A(x) = A_o$), 2) parabolic ($A(r) = A_o r$), 3) conical (spherical) ($A(r) = A_o r^2$) and 4) exponential ($A(r) = A_o e^{2mr}$), as summarized in Table 1.5 (p. 198).

1) The uniform horn

The 1D wave equation [$A(r) = A_o$]

$$\frac{d^2}{dr^2}\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P}.$$

Solution: The two eigenfunctions of this equation are the two d'Alembert waves (Eq. 1.85, p. 141)

$$\varrho(x,t) = \alpha(\omega)\varrho^+(t - x/c) + \beta(\omega)\varrho^-(t + x/c) \;\leftrightarrow\; \alpha(\omega)e^{-\kappa(s)x} + \beta(\omega)e^{\kappa(s)(x-L)},$$

where $\kappa(s) = s/c_o = \jmath\omega/c$ is denoted the wave-evolution function, propagation constant, or wave number and α, β are defined in Eq. 1.153.

Note that for the uniform horn $\omega/c_o = 2\pi/\lambda$. It is convenient to normalize $\mathcal{P}_0^+ = 1$ and $\mathcal{P}_L^- = 1$, as was done for the general case, above. When the area is not constant, λ is a complex function of frequency, resulting in a complex input impedance (admittance), internal standing waves and wave propagation loss.

The characteristic admittance (Eq. 1.142) is independent of direction. The signs must be "physically chosen," with the velocity $\mathcal{V}^{\pm}$ *into* the port, to assure that $\mathcal{Y} > 0$, for both waves, where $\mathcal{Y}$ is independent of direction and x.

Applying the boundary conditions: The general solution in terms of the eigen vector matrix (Eq. 1.153), evaluated at $x = L$, is

$$\begin{bmatrix}\mathcal{P}(x)\\ \mathcal{V}(x)\end{bmatrix}_L = \begin{bmatrix}e^{-\kappa x} & e^{\kappa(x-L)}\\ \mathcal{Y}e^{-\kappa x} & -\mathcal{Y}e^{\kappa(x-L)}\end{bmatrix}_L \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_x = \begin{bmatrix}e^{-\kappa L} & 1\\ \mathcal{Y}e^{-\kappa L} & -\mathcal{Y}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_L, \tag{1.158}$$

where α, β are the relative weights on the two unknown eigenfunctions, to be determined by the boundary conditions at $x = 0, L$, $\kappa = s/c$ and $\mathcal{Y} = 1/\mathcal{Z} = A_o/\rho_o c$.

Solving Eq. 1.158 for α and β with determinant $\Delta = -2\mathcal{Y}e^{-\kappa L}$,

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_L = \frac{-1}{2\mathcal{Y}e^{-\kappa L}}\begin{bmatrix}-\mathcal{Y} & -1\\ -\mathcal{Y}e^{-\kappa L} & e^{-\kappa L}\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_L = \frac{1}{2}\begin{bmatrix}e^{\kappa L} & -\mathcal{Z}e^{\kappa L}\\ 1 & \mathcal{Z}\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.159}$$

In the final step we swapped all the signs, including on $\mathcal{V}$, and moved $\mathcal{Z} = 1/\mathcal{Y}$ inside the matrix.

We may uniquely determine these two weights given the pressure and velocity at the boundary $x = L$, which is typically determined by the load impedance ($\mathcal{P}_L/\mathcal{V}_L$).

The weights may now be substituted back into Eq. 1.158, to determine the pressure and velocity amplitudes at any point $0 \le x \le L$.

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_x = \frac{1}{2}\begin{bmatrix}e^{-\kappa x} & e^{\kappa(x-L)}\\ \mathcal{Y}e^{-\kappa x} & -\mathcal{Y}e^{\kappa(x-L)}\end{bmatrix}_x \begin{bmatrix}e^{\kappa L} & -\mathcal{Z}e^{\kappa L}\\ 1 & \mathcal{Z}\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.160}$$

Setting $x = 0$ and multiplying these out gives the final transmission matrix

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_0 = \frac{1}{2}\begin{bmatrix}e^{\kappa L} + e^{-\kappa L} & \mathcal{Z}(e^{\kappa L} - e^{-\kappa L})\\ \mathcal{Y}(e^{\kappa L} - e^{-\kappa L}) & e^{\kappa L} + e^{-\kappa L}\end{bmatrix}_x \begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.161}$$

Note the diagonal terms are $\cosh \kappa L$ and off-diagonal terms are $\sinh \kappa L$.

Applying the last boundary condition, we evaluate Eq. 1.159 to obtain the ABCD matrix at the input ($x = 0$) (Pipes, 1958)

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_0 = \begin{bmatrix}\cosh \kappa L & \mathcal{Z}\sinh \kappa L\\ \mathcal{Y}\sinh \kappa L & \cosh \kappa L\end{bmatrix}\begin{bmatrix}\mathcal{P}\\ -\mathcal{V}\end{bmatrix}_L. \tag{1.162}$$

Exercise: Evaluate this expression in terms of the load impedance. **Solution:** Since $Z_{load} = P_L/V_L$,

$$\begin{bmatrix}\mathcal{P}\\ \mathcal{V}\end{bmatrix}_0 = \begin{bmatrix}\mathcal{Z}_{load}\cosh \kappa L - \mathcal{Z}\sinh \kappa L\\ \mathcal{Z}_{load}\mathcal{Y}\sinh \kappa L - \cosh \kappa L\end{bmatrix}. \tag{1.163}$$

Impedance matrix: Expressing Eq. 1.163 as an impedance matrix gives (algebra required)

$$\begin{bmatrix}\mathcal{P}_o\\ \mathcal{P}_L\end{bmatrix} = \frac{\mathcal{Z}}{\sinh(\kappa L)}\begin{bmatrix}\cosh(\kappa L) & 1\\ 1 & \cosh(\kappa L)\end{bmatrix}\begin{bmatrix}\mathcal{V}_o\\ \mathcal{V}_L\end{bmatrix}.$$

Input admittance Y_{in}: Given the input admittance of the horn, it is possible to determine if it is uniform, without further analysis. Namely, if the horn is uniform and infinite in length, the input impedance at $x = 0$ is

$$Y_{in}(0, s) \equiv \frac{\mathcal{V}(0, \omega)}{\mathcal{P}(0, \omega)} = \mathcal{Y},$$

since $\alpha = 1$ and $\beta = 0$. That is, for an infinite uniform horn, there are no reflections.

When the horn is terminated with a fixed impedance Z_L at $x = L$, one may substitute pressure and velocity measurements into Eq. 1.159 to find α and β, and given these, one may calculate the *reflectance* at $x = L$ (Eq. 1.148, p. 198)

$$\Gamma_L(s) \equiv \left.\frac{\mathcal{P}^-}{\mathcal{P}^+}\right|_{x=L} = \frac{\beta}{\alpha} = \frac{\mathcal{P}(L, \omega) - \mathcal{Z}\mathcal{V}(L, \omega)}{\mathcal{P}(L, \omega) + \mathcal{Z}\mathcal{V}(L, \omega)} = \frac{Z_L - \mathcal{Z}}{Z_L + \mathcal{Z}}$$

given sufficiently accurate measurements of the throat pressure $\mathcal{P}(0, \omega)$, velocity $\mathcal{V}(0, \omega)$, and the characteristic impedance of the input $\mathcal{Z} = \rho_o c/A(0)$.

2) Conical horn

For each horn we must find the natural normalization from the range variable to the normalized range variable x. For the conical horn the radius is proportional to the range variable r, thus

$$A(r) = 4\pi \sin^2 (\Theta/2)\ r^2. \quad [\text{m}^2]$$

The angle Θ is a measure of the solid (cone) angle. When $\Theta = \pi$ we have the case of the entire sphere, so the solid angle is 4π [steradian] and the area is $4\pi r^2$. The formula for the area may be simplified by defining $A_\theta \equiv 4\pi \sin^2(\Theta/2)\, r_o^2\ [m^2]$, resulting in the more convenient relation

$$A(r) = A_\theta\, (r/r_o)^2. \quad [\text{m}^2].$$

This is a bit tricky because A_θ is not a constant since it depends on the place where the area was normalized, in this case r_o.

Using the conical horn area $A(r) \propto r^2$ in Eq. 1.134, p. 193 [or Eq. 1.143 (p. 196)] results in the spherical wave equation (Section 1.5.2, p. 176)

$$\mathcal{P}_{rr}(r, \omega) + \frac{2}{r}\mathcal{P}_r(r, \omega) = \kappa^2 \mathcal{P}(r, \omega). \tag{1.164}$$

Here $F(r) = \partial_r \ln A(r) = 2/r$, the eigenfunctions are $\delta(t \mp r/c_o) \leftrightarrow e^{\mp\kappa r}/r$, and the input admittance is $Y_{in}^{\pm} = \mathcal{Y}\,(1 \pm c_o/sr)$ (Table 1.5, p. 198).

3) Exponential horn:

The case of the *exponential horn*

$$\mathcal{P}^{\pm}(r, \omega) = e^{-mr} e^{\mp \jmath \sqrt{\omega^2 - \omega_c^2}\ r/c} \tag{1.165}$$

is of special interest because the radiation impedance is purely reactive below the horn's cutoff frequency ($\omega < \omega_c = mc_o$), as may be seen from curves 3 and 4 of Fig. 1.24, since no energy can radiate from an open horn below ω_c, because

$$\kappa(s) = -m \pm \frac{\jmath}{c_o}\sqrt{\omega^2 - \omega_c^2}$$

becomes purely real for $\omega < \omega_c$ (non-propagating evanescent waves).