INTRODUCTION

Basic Network Postulates

An *electric network* is defined as a structure with a set of *ports* or *accessible terminal pairs* at which voltages and currents may be measured. Such ports are available for the transport of electromagnetic energy into or out of the structure. The ports may be connected across branches of a low-frequency circuit, or they may be connected to a complicated branch mesh configuration, or under suitable conditions they may correspond to terminal planes of a wave-guide or other distributed system across which power may flow. In any case, whether the system operates at low or high frequencies, the voltages and currents must be suitably defined so that electromagnetic field quantities may be directly computed from these voltages and currents. The subject of network theory is generally only concerned with the voltages and currents (rather than the fields) at specified ports of an electrical system and with the constraints imposed by the system on these variables. The treatment presented here will therefore concentrate on the port or terminal pair quantities. Other sources† have dealt in considerable detail with the relations between voltages, currents, and electromagnetic fields.

For the most part, the discussions in this text will deal with properties of circuits (e.g., network geometric structure) which are independent of frequency, network analysis techniques in which frequency is only incidental to the main presentation, or they will involve the study of methods which are valid at a single frequency. It should be emphasized, however, that many of the techniques described here may be carried over directly into the domain of variable frequency network theory. In view of this particular scope, the major emphasis will be algebraic rather than function-theoretic and, particularly in regard to networks with many ports (e.g., branches or node

†S. Ramo and J. R. Whinnery, *Fields and Waves in Modern Radio* (New York: John Wiley & Sons, Inc., 1953).

pairs), the treatment will utilize the classic (and generally elementary) properties of matrices.

It should be clear from the above discussion that the properties of networks considered as processors of signals in the time domain will not be directly invoked. Rather, the presentation is mostly limited to steady-state responses. Nevertheless, it is important at the outset to consider briefly the physical time domain constraints of the networks whose structural properties will be dealt with in detail in the remainder of this book.

The physical nature of the networks considered here is defined in terms of a set of postulates which describe the system in terms of its response to real time signals at the network ports†. The simplest presentation of these postulates is given in terms of a 1-port with a single voltage and a single current. We presume that the voltage $v(t)$ is the excitation‡ and is limited to some broad class of time functions which correspond to signals in the physical world. The current $i(t)$ at the available port of the system is the response function. We will generally be concerned with networks which satisfy the following constraints on $v(t)$, $i(t)$.

P-1 Linearity postulate: Generally, we mean by linearity that the response is proportional to the excitation. Then, if $v(t)$ is the excitation and $i(t)$ the response, written

$$v(t) \longrightarrow i(t)$$

[this is read as, "$v(t)$ implies $i(t)$"] linearity means

$$\alpha v^{(1)}(t) + \beta v^{(2)}(t) \longrightarrow \alpha i^{(1)}(t) + \beta i^{(2)}(t)$$

where

$$v^{(1)}(t) \longrightarrow i^{(1)}(t)$$

$$v^{(2)}(t) \longrightarrow i^{(2)}(t)$$

and α and β are arbitrary constants, and $v^{(1)}(t)$, $v^{(2)}(t)$ are any two of the permissible excitation signals.

P-2 Time invariance postulate: Time invariance means that the system parameters do not change as a function of time. Thus, a given excitation produces the same response no matter when it is applied.

†D. C. Youla, L. J. Castriota, and H. J. Carlin, "Bounded Real Scattering Matrices and the Foundation of Linear Passive Network Theory" (*IRE Transactions of the Professional Group on Circuit Theory*, Vol. CT-6, No. 1, (Mar. 1959).

‡Note that either voltage or current may be the excitation variable, and in fact $v(t)$ can stand for either one. For simplicity of presentation we assume voltage excitation and current response.

Thus with

$$v(t) \longrightarrow i(t)$$

then

$$v(t + t_o) \longrightarrow i(t + t_o)$$

where t_o is an arbitrary time interval.

P-3 Passivity postulate: Many of the networks discussed in this book satisfy the requirement that they only absorb or store energy; that is, they do not amplify it, nor do they return more energy to the source than is supplied. This property may be described in terms of an energy integral in which the total energy delivered to the system (plus if absorbed, minus if outward and back to the source) is measured from some very early time when the circuit is completely quiescent with no stored energy; i.e., starting from $t = -\infty$. Then, the passivity restriction requires that for the 1-port in question, and for real signals in real time

$$\text{total delivered energy} = \int_{-\infty}^{t} v(\tau) i(\tau)\, d\tau \geq 0, \qquad t > -\infty$$

Thus, this restriction requires that the energy delivered be nonnegative for any time t after the excitation $v(t)$ is applied, a condition that must be satisfied for all permissible $v(t)$ signals.

P-4 Causality postulate: A system is causal if it yields no response until after the excitation is applied. More precisely, the system response at any time t is only a function of the past and present excitation and cannot be influenced by the future form of the signal. Thus, suppose

$$v^{(1)}(t) \longrightarrow i^{(1)}(t), \qquad -\infty \leq t \leq \infty$$

and

$$v^{(2)}(t) = v^{(1)}(t), \qquad t \leq t_o$$

i.e., the presumption is that $v^{(2)}(t)$ may differ from $v^{(1)}(t)$ for $t > t_o$, then causality implies that with

$$v^{(2)}(t) \longrightarrow i^{(2)}(t)$$

we must have

$$i^{(2)}(t) = i^{(1)}(t), \qquad t \leq t_o$$

Thus, the response up to time $_o t$ can only be affected by the signal properties up to time t_o.

The usual definition of causality, which we may term *P-4a*, states that if

$$v(t) \longrightarrow i(t)$$

then

$$v(t) = 0, \qquad t \le t_o$$

implies

$$i(t) = 0, \qquad t \le t_o$$

In general, P-4 is less restrictive than P-4a since it does not require the excitation to be zero for $t < 0$. However, in *a linear system* we can deduce the causality postulate P-4 from P-4a, or vice versa (see Prob. 1.1). Therefore, the two postulates are equivalent in such a system. Thus, suppose we hypothesize P-4a. Now consider two excitations $v_a(t)$, $v_b(t)$ defined for $-\infty \le t \le \infty$, with

$$v_a(t) = v_b(t), \qquad t \le t_o$$

Then define

$$v(t) = v_a(t) - v_b(t) = 0, \qquad t \le t_o$$

$$v(t) \longrightarrow i(t)$$

Now, by P-4a

$$i(t) = 0, \qquad t \le t_0$$

By the linearity postulate P-1, however, since $v_a \to i_a$, $v_b \to i_b$

$$0 = i(t) = i_a(t) - i_b(t), \qquad t \le t_o$$

or

$$i_a(t) = i_b(t), \qquad t \le t_0$$

which is P-4.

P-5 Real time function postulate: When a signal $v(t)$, which is a real function of real time, is impressed on the network, it must give rise to a response $i(t)$, which is also a real function of real time.

It is important to bear this postulate in mind despite the fact that we often work with signals which are complex functions of time. Thus, it is customary to employ $v(t) = \varepsilon^{j\omega t}$ ($\omega = 2\pi f$, $f =$ frequency) as an excitation. Of course, such signals generally produce a complex response, but the postulate states that if the signal has a form such as $v(t) = \text{Re}\,\varepsilon^{j\omega t} = \cos \omega t$ (Re = *real part of*), then the response will likewise be a real function of time. Time signals discussed below are all real.

Since the presentations which follow are primarily concerned with steady-state responses at a real frequency ω, *complex phasor notation* is used throughout. Thus, if $z(j\omega)$ is the impedance of a network branch expressed as a complex number, and

$$z(j\omega) = r(j\omega) + jx(j\omega) = |z|\varepsilon^{j\phi}$$

$$i(j\omega) = \frac{v(j\omega)}{z(j\omega)}$$

with $v(j\omega) = |v|\varepsilon^{j\theta}$, the phasor corresponding to the signal excitation $v(t) = \text{Re}\,|v|\varepsilon^{j[\omega t + \theta]}$, then in accordance with the usual conventions the real steady-state response is given by

$$i(t) = \text{Re}\,\frac{|v|}{|z|}\,\varepsilon^{j[\omega t + \theta - \phi]}$$

$$= \frac{|v|}{|z|}\cos(\omega t + \theta - \phi)$$

Because spatially independent network elements are presumed in most of the text (thus, in the above discussion z is a complex number without a spatial variable), the treatment centers about lumped systems, or distributed systems at reference planes fixed in space.

P-6 Reciprocity postulate: A substantial portion of this book (Chapter 5) is devoted to properties of *nonreciprocal networks*, but the first four chapters deal mainly with *reciprocal networks*. The reciprocity postulate is primarily associated with network structure rather than with time domain behavior, and a detailed discussion of this property is given later. However, in its simplest form, P-6 states that if $v_1 = v$ is applied to *port* 1 of a network with two ports (a 2-port) with resultant current $i_2 = i$ in the lead short-circuiting *port* 2, then if $v_2 = v$ is applied to *port* 2, the short-circuit current at *port* 1 is $i_1 = i$. In both cases the current polarity i_1, i_2 is *into* the network from the terminal designated as positive polarity for the applied voltages v_1, v_2. It should be emphasized that reciprocity and passivity (the latter extended to an n-port) are independent postulates. Passive networks may be reciprocal or nonreciprocal, as also may be active networks (i.e., those that do not satisfy P-3).

An important general question may be raised concerning the independence of the five time-domain postulates. In fact, P-1 (*linearity*), P-2 (*time-invariance*), P-3 (*passivity*), P-5 (*reality*) are independent, but a surprising result first stated by Youla *et al.*† is that except for trivial exceptions, *P-4 (causality) is a consequence of linearity and passivity.* An elementary proof of this assertion for a 1-port is given below. The reader is referred to the literature† for a more extended discussion.

Suppose therefore that a 1-port admits a set of signal excitations $v(t) \not\equiv 0$. Thus, at this point we immediately exclude a short circuit, which only admits the excitation $v(t) = 0$. In the strict sense we may indeed regard the short-circuited 1-port as noncausal though it is linear and passive, since for $v(t) = 0$ any current whatever may flow. (Regarding $i(t)$ as excitation, $v(t)$ as response, the open circuit is similarly noncausal.)

Now, choose a real time excitation $v(t)$

$$v(t) \longrightarrow i(t), \qquad -\infty \le t \le \infty$$

with

$$v(t) = 0, \qquad t \le t_o$$

We will show that under the assumptions of linearity (P-1) and passivity (P-3)

$$i(t) = 0, \qquad t \le t_o$$

from which P-4 follows.

Consider a signal

$$\bar{v}(t) = v_o(t) + \alpha v(t), \qquad -\infty \le t \le \infty$$

where α is an arbitrary constant and

$$v_o(t) \longrightarrow i_o(t), \qquad -\infty \le t \le \infty$$

$$\bar{v}(t) \longrightarrow \bar{i}(t), \qquad -\infty \le t \le \infty$$

$$v_0(t) \not\equiv 0$$

Furthermore, linearity (P-1) yields

$$\bar{i}(t) = i_o(t) + \alpha i(t)$$

Hence, the passivity relation (P-3) which applies to *all* admissible signals

$$\int_{-\infty}^{t} \bar{v}(\tau)\bar{i}(\tau)\, d\tau \ge 0, \qquad t \ge -\infty$$

becomes

$$\int_{-\infty}^{t} [v_o(\tau)i_o(\tau) + \alpha v_o(\tau)i(\tau) + \alpha v(\tau)i_o(\tau) + \alpha^2 v(\tau)i(\tau)]\, d\tau \ge 0, \qquad t \ge -\infty$$

†Youla, Castriota, and Carlin, *Op. cit.*, p. 2.

For $t \le t_o$, $v(t) = 0$, hence the above integral reduces to

$$\int_{-\infty}^{t} v_o(\tau)i_o(\tau)\, d\tau + \alpha \int_{-\infty}^{t} v_o(\tau)i(\tau)\, d\tau \ge 0, \qquad t \le t_o$$

But for any $t \le t_o$ the first integral is nonnegative (by passivity), and if the second integral is nonzero we may always choose the arbitrary constant α so as to violate the inequality, e.g., we may choose α large in magnitude and negative if the second integral is positive. But the passivity hypothesis requires that the integral inequality (i.e., on the sum of the two integrals) be satisfied, and this can only be the case when

$$\int_{-\infty}^{t} v_o(\tau)i(\tau)\, d\tau = 0, \qquad t \le t_o$$

and since this is true for all $t \le t_o$, the integrand must be zero and

$$v_o(t)i(t) = 0, \qquad t \le t_o$$

By hypothesis $v_o(t) \ne 0$, hence

$$i(t) = 0, \qquad t \le t_o$$

This proves that, given P-1 and P-3, P-4a follows. But we have already shown that P-4a implies P-4. Hence [except for the trivial restriction on $v(t)$] a *passive linear system must be causal.* This is true whether the network is time invariant or not. This causality theorem has a good deal of importance even beyond the confines of pure network theory (as indeed do many other network theorems) since it applies to any *linear passive system.* For example, it has been cited by workers in the field of thermodynamics and elasticity.†

Notation

The philosophy governing notation has aimed at maximum simplicity. The following rules have been generally adhered to, though from time to time where the context ruled out any possibility of confusion, the rules may have been relaxed in the interest of naturalness of presentation.

1. (a) Capital letters usually indicate matrices; thus

$$Z,\ Y,\ S,\ A_1,\ B_1$$

In certain cases (the text presentation clearly defines the symbol) upper case is used for scalar elements

P (power), G (gain), Z_A (an impedance element)

†S. R. Groot and P. Mazur, *Non-Equilibrium Thermodynamics*, (Interscience, 1962), Chap. 8; and J. Meixner and H. Konig, Rheologica Acta, Vol. 1 2/3, 1958.

(b) Determinants are indicated by det ( ), e.g., det Z, and determinant arrays by vertical rules, e.g., $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Matrix arrays are indicated by brackets, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2. Lower case letters generally indicate scalar elements such as branch impedances, driving point impedances, voltages, currents, and scattering variables

$$z, y, v, i, a, b$$

3. Lower case letters with double subscripts indicate matrix elements. Single subscripts are usually used for elements of a column vector.

$$z_{ij}, y_{ij}, s_{ij}$$

$$a_k, v_k, b_k$$

Occasionally, capital letters with double subscripts are used for matrix elements.

R_{kk} (resistance), L_{kk} (inductance), C_{kk} (capacitance)

4. In Chapter 4, *special notation* is used to distinguish normalized from nonnormalized quantities, but in other chapters where the distinction is generally unnecessary, the special notation is omitted.

(a) Nonnormalized matrices in Chapter 4 are indicated as

$$\not{Z}, \not{Y}, \not{S}$$

$$V, \underline{I}$$

(b) Nonnormalized scalars are underscored

$$\underline{v}_k, \underline{i}_k, \underline{z}_{ij}$$

But special symbols are used for nonnormalized (rarely used) scattering variables, i.e., α_k, β_k.

(c) Normalized quantities are unbarred

$$Z, Y, S$$

$$V, I$$

5. (a) Single *column matrices* (*column vectors*) are indicated by a lower case letter above a tilde. Thus for normalized incident and reflected scattering variables

$$\underset{\sim}{a}, \underset{\sim}{b}$$

Lower case letters above a tilde are also occasionally used to indicate row vectors.

(b) In the specific cases of voltage and current single column matrices, capital letters are used to indicate column vectors.

$$V, I$$

6. Transformed variables and matrices are indicated with an upper bar, circumflex, or other distinguishing mark. Thus

$$\bar{V}, \bar{I} \text{ or } \hat{V}, \hat{I}$$

$$\bar{\underset{\sim}{a}}, \bar{\underset{\sim}{b}} \text{ or } \hat{\underset{\sim}{a}}, \hat{\underset{\sim}{b}}$$

$$\bar{Z}, \bar{Y} \text{ or } \hat{Z}, \hat{Y}$$

7. Various operations on matrices and scalars are denoted as follows:

(a) Complex conjugate $(\ )^*$

$$s^*, z^*, v_k^*$$

$$S^* = (s_{ij}^*), \qquad V^* = \begin{bmatrix} v_1^* \\ v_2^* \\ \cdot \\ \cdot \\ \cdot \\ v_n^* \end{bmatrix}$$

(b) Transpose of a matrix or column vector

$Z' = (z_{ji})$, with $Z = (z_{ij})$

Also

$$V = (v_1, v_2, \ldots, v_n)'$$

and

$$\underset{\sim}{a}' = (a_1, a_2, \ldots, a_n)$$

where $\underset{\sim}{a}$, V are column vectors

(c) Conjugate transpose or adjoint of a matrix

$$Z^{*\prime} = \tilde{Z} = (z_{ji}^*), \qquad \text{with } Z = (z_{ij})$$

$$\underset{\sim}{a}^{*\prime} = \tilde{\underset{\sim}{a}} = (a_1^*, a_2^*, \ldots, a_n^*)$$

where $\underset{\sim}{a}$ is a column vector.