Chapter 1

Differentiable Manifolds

§1–1 Definition of Differentiable Manifolds

Differentiable manifolds are generalizations of Euclidean spaces. Roughly speaking, any given point in a manifold has a neighborhood which is homeomorphic to an open set of a Euclidean space. Hence we can establish local coordinates in a neighborhood of every point. A manifold is then the result of pasting together pieces of a Euclidean space.

We will use $\mathbb{R}$ to represent the field of real numbers. Let

$$\mathbb{R}^m = \left\{ x = (x^1, \dots, x^m) \,\middle|\, x^i \in \mathbb{R}, \qquad 1 \le i \le m \right\}, \tag{1.1}$$

that is, the set of all ordered m-tuples of real numbers. The number x^i is called the i-th **coordinate** of the point $x \in \mathbb{R}^m$. For any $x, y \in \mathbb{R}^m, a \in \mathbb{R}$, let

$$\begin{cases} (x+y)^i = x^i + y^i, \\ \quad (ax)^i = ax^i. \end{cases} \tag{1.2}$$

This defines addition and scalar multiplication in $\mathbb{R}^m$, making $\mathbb{R}^m$ an m-dimensional vector space over $\mathbb{R}$.

Besides this linear structure, $\mathbb{R}^m$ also has a standard topological structure. For $x, y \in \mathbb{R}^m$, define

$$d(x, y) = \sqrt{\sum_{i=1}^{m} (x^i - y^i)^2}. \tag{1.3}$$

It is easy to verify that the function $d(x, y)$ satisfies the following three conditions:

1) $d(x, y) \ge 0$, the equality holds if and only if $x = y$;

2) $d(x,y) = d(y,x)$;
3) for any $x, y, z \in \mathbb{R}^m$, we have the inequality $d(x,y) + d(y,z) \geq d(x,z)$.

Hence $d(x,y)$ is a metric on $\mathbb{R}^m$, which makes $\mathbb{R}^m$ a metric space. As such, $\mathbb{R}^m$ has the natural topological structure[a]: the unions of open balls $B_{x,r} = \{y \in \mathbb{R}^m \,|d(x,y) < r\}$ $(x \in \mathbb{R}^m, r > 0)$ are the open sets. The m-dimensional vector space $\mathbb{R}^m$ with the metric (1.3) is called the m-dimensional **Euclidean space**.

Suppose f is a real-valued function defined on an open set $U \subset \mathbb{R}^m$. If all the k-th order partial derivatives of f exist and are continuous for $k \leq r$, then we say $f \in C^r(U)$. Here r is some positive integer. If $f \in C^r(U)$ for every positive integer r, then we say $f \in C^\infty(U)$. If f is analytic, i.e., if f can be expressed as a convergent series in a neighborhood of any point of U, then we say $f \in C^\omega(U)$.

Definition 1.1. Suppose M is a Hausdorff space. If for any $x \in M$, there exists a neighborhood U of x such that U is homeomorphic to an open set in $\mathbb{R}^m$, then M is called an m-dimensional **manifold** (or m-dimensional **topological manifold**).

If the homeomorphism in Definition 1.1 is $\varphi_U : U \longrightarrow \varphi_U(U)$, where $\varphi_U(U)$ is an open set in $\mathbb{R}^m$, we call (U, φ_U) a **coordinate chart** of M. Since φ_U is a homeomorphism, for any $y \in U$, we can define the coordinates of y to be the coordinates of $u = \varphi_U(y) \in \mathbb{R}^m$, i.e.

$$u^i = (\varphi_U(y))^i, \qquad i = 1, \ldots, m. \tag{1.4}$$

The $u^i, i = 1, \ldots, m$, are called the **local coordinates** of the point $y \in U$.

Suppose (U, φ_U) and (V, φ_V) are two coordinate charts of M. If $U \cap V \neq \varnothing$, then $\varphi_U(U \cap V)$ and $\varphi_V(U \cap V)$ are two nonempty open sets in $\mathbb{R}^m$, and the map

$$\varphi_V \circ \varphi_U^{-1}\Big|_{\varphi_U(U\cap V)} : \varphi_U(U \cap V) \longrightarrow \varphi_V(U \cap V)$$

defines a homeomorphism between these two open sets, with inverse given by

$$\varphi_U \circ \varphi_V^{-1}\Big|_{\varphi_V(U\cap V)}.$$

These are both maps between open sets in a Euclidean space. Expressed in coordinates, $\varphi_V \circ \varphi_U^{-1}$ and $\varphi_U \circ \varphi_V^{-1}$ each represents m real-valued functions

[a]For fundamental topological concepts, see for instance Munkres 1975.

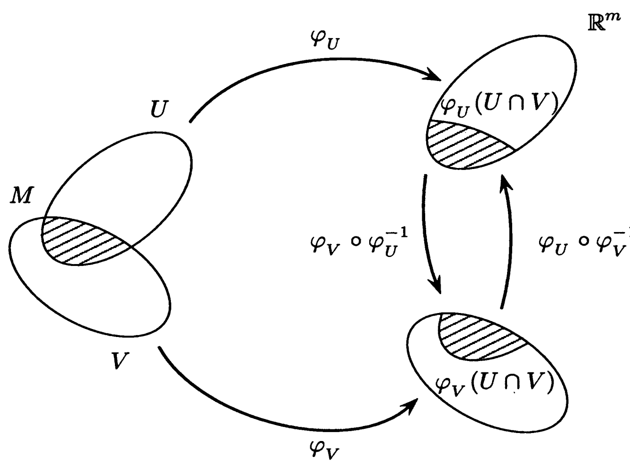

FIGURE 1.

on an open set of a Euclidean space (see Figure 1). We may write

$$y^i = f^i\left(x^1, \dots, x^m\right) = \left(\varphi_V \circ \varphi_U^{-1}\left(x^1, \dots, x^m\right)\right)^i, \quad \left(x^1, \dots, x^m\right) \in \varphi_U(U \cap V); \tag{1.5}$$

$$x^i = g^i\left(y^1, \dots, y^m\right) = \left(\varphi_U \circ \varphi_V^{-1}\left(y^1, \dots, y^m\right)\right)^i, \quad \left(y^1, \dots, y^m\right) \in \varphi_V(U \cap V). \tag{1.6}$$

Since $\varphi_V \circ \varphi_U^{-1}$ and $\varphi_U \circ \varphi_V^{-1}$ are homeomorphisms inverse to each other, f^i and g^i are continuous functions, and

$$\begin{cases} f^i\left(g^1\left(y^1, \dots, y^m\right), \cdots, g^m\left(y^1, \dots, y^m\right)\right) = y^i, \\ g^i\left(f^1\left(x^1, \dots, x^m\right), \cdots, f^m\left(x^1, \dots, x^m\right)\right) = x^i. \end{cases} \tag{1.7}$$

We say that the coordinate charts (U, φ_U) and (V, φ_V) are **C^r-compatible** if $U \cap V = \varnothing$, and if $f^i\left(x^1, \dots, x^m\right)$ and $g^i\left(y^1, \dots, y^m\right)$ are C^r when $U \cap V \neq \varnothing$.

Definition 1.2. Suppose M is an m-dimensional manifold. If a given set of coordinate charts $\mathcal{A} = \left\{(U, \varphi_U), (V, \varphi_V), (W, \varphi_W), \cdots\right\}$ on M satisfies the following conditions, then we call $\mathcal{A}$ a **C^r-differentiable structure** on M:

1) $\{U, V, W, \dots\}$ is an open covering of M;

2) any two coordinate charts in $\mathcal{A}$ are C^r-compatible;
3) $\mathcal{A}$ is **maximal**, i.e., if a coordinate chart $(\tilde{U}, \varphi_{\tilde{U}})$ is C^r-compatible with all coordinate charts in $\mathcal{A}$, then $(\tilde{U}, \varphi_{\tilde{U}}) \in \mathcal{A}$.

If a C^r-differentiable structure is given on M, then M is called a **C^r-differentiable manifold**. A coordinate chart in a given differentiable structure is called a **compatible (admissible) coordinate chart** of M. From now on, a **local coordinate system** of a point p on a differentiable manifold M refers to a coordinate system obtained from an admissible coordinate chart containing p.

Remark 1. Conditions 1) and 2) in Definition 1.2 are primary. It is not hard to show that if a set $\mathcal{A}'$ of coordinate charts satisfies 1) and 2), then for any positive integer s, $0 < s \leq r$, there exists a unique C^s-differentiable structure $\mathcal{A}$ such that $\mathcal{A}' \subset \mathcal{A}$. In fact, suppose $\mathcal{A}$ represents the set of all coordinate charts which are C^s-compatible with every coordinate chart in $\mathcal{A}'$, then $\mathcal{A}$ is a C^s-differentiable structure uniquely determined by $\mathcal{A}'$. Hence, to construct a differentiable manifold, we need only choose a covering by compatible charts.

Remark 2. In this book, we also assume that any manifold M is a second countable topological space, i.e., M has a countable topological basis (see footnote on page 2).

Remark 3. If a C^∞-differentiable structure is given on M, then M is called a **smooth manifold**. If M has a C^ω-differentiable structure, then M is called an **analytic manifold**. In this book, we are mostly interested in smooth manifolds. When there is no confusion, the term manifold will mean smooth manifold.

Example 1. For $M = \mathbb{R}^m$, let $U = M$ and φ_U be the identity map. Then $\{(U, \varphi_U)\}$ is a coordinate covering of $\mathbb{R}^m$. This provides a smooth differentiable structure on $\mathbb{R}^m$, called the **standard differentiable structure** of $\mathbb{R}^m$.

Example 2. Consider the m-dimensional unit sphere

$$S^m = \left\{ x \in \mathbb{R}^{m+1} \,\middle|\, \left(x^1\right)^2 + \cdots + \left(x^{m+1}\right)^2 = 1 \right\}.$$

For $m = 1$, take the following four coordinate charts:

$$\begin{cases} U_1 \left\{x \in S^1 \left| x^2 > 0 \right.\right\}, \varphi_{U_1}(x) = x^1, \\ U_2 \left\{x \in S^1 \left| x^2 < 0 \right.\right\}, \varphi_{U_2}(x) = x^1, \\ V_1 \left\{x \in S^1 \left| x^1 > 0 \right.\right\}, \varphi_{V_1}(x) = x^2, \\ V_2 \left\{x \in S^1 \left| x^1 < 0 \right.\right\}, \varphi_{V_2}(x) = x^2. \end{cases} \tag{1.8}$$

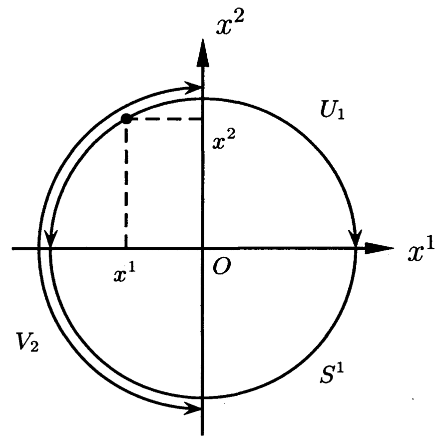

FIGURE 2.

Obviously, $\{U_1, U_2, V_1, V_2\}$ is an open covering of S^1. In the intersection $U_1 \cap V_2$, we have (see Figure 2)

$$\begin{cases} x^2 = \sqrt{1 - (x^1)^2} > 0, \\ x^1 = -\sqrt{1 - (x^2)^2} < 0. \end{cases} \tag{1.9}$$

These are both C^∞ functions, thus $\left(U_1, \varphi_{U_1}\right)$ and $\left(V_2, \varphi_{V_2}\right)$ are C^∞-compatible. Similarly, any other pair of the given coordinate charts are C^∞-compatible. Hence these coordinate charts suffice to make S^1 a 1-dimensional smooth manifold. For $m > 1$, the smooth structure on S^m can be defined similarly.

Example 3. The m-dimensional projective space P^m. Define a relation $\sim$ in $\mathbb{R}^{m+1} - \{0\}$ as follows: for $x, y \in \mathbb{R}^{m+1} - \{0\}$, $x \sim y$ if and only if there exists a real number a such that $x = ay$. Obviously, $\sim$ is an equivalence relation. For $x \in \mathbb{R}^{m+1} - \{0\}$, denote the equivalence class of x by

$$[x] = \left[x^1, \dots, x^{m+1}\right].$$

The m-dimensional projective space is the quotient space

$$\begin{aligned} P^m &= \left(\mathbb{R}^{m+1} - \{0\}\right) / \sim \\ &= \left\{[x] \,|\, x \in \mathbb{R}^{m+1} - \{0\}\right\}. \end{aligned} \tag{1.10}$$

The numbers of the $(m+1)$-tuple $(x^1, \dots, x^{m+1})$ are called the **homogeneous coordinates** of $[x]$. They are determined by $[x]$ up to a a nonzero factor. P^m is thus the space of all straight lines in $\mathbb{R}^{m+1}$ which pass through the origin.

Let

$$\begin{cases} U_i = \{ [x^1, \dots, x^{m+1}] \mid x^i \neq 0 \}, \\ \varphi_i([x]) = ({}_i\xi_1, \dots, {}_i\xi_{i-1}, {}_i\xi_{i+1}, \dots, {}_i\xi_{m+1}), \end{cases} \tag{1.11}$$

where $1 \le i \le m+1$, ${}_i\xi_h = x^h/x^i \, (h \neq i)$. Obviously, $\{U_i, 1 \le i \le m+1\}$ forms an open covering of P^m. On $U_i \cap U_j$, $i \neq j$, the change of coordinates is given by

$$\begin{cases} {}_j\xi_h = \dfrac{{}_i\xi_h}{{}_i\xi_j}, & h \neq i, j; \\ {}_j\xi_i = \dfrac{1}{{}_i\xi_j}. \end{cases} \tag{1.12}$$

Hence $\{(U_i, \varphi_i)\}_{1 \le i \le m+1}$ suffices to generate a smooth structure on P^m.

Remark. In each of the above examples, the respective coordinate charts given are in fact C^ω-compatible also, and so provide the structures for $\mathbb{R}^m$, S^m, and P^m as analytic manifolds.

Example 4 (Milnor's Exotic Sphere). There may exist distinct differentiable structures on a single topological manifold. J. Milnor gave a famous example (Milnor 1956), which shows that there exist nonisomorphic smooth structures on homeomorphic topological manifolds (see the discussion following the remark to definition 1.3 below). Hence a differentiable structure is more than a topological structure. A complete understanding of the Milnor sphere is outside the scope of this text. Here we will give only a brief description of the main ideas. [A more recent example is the existence of distinct smooth structures on $\mathbb{R}^4$ discovered by S. K. Donaldson (see Donaldson and Kronheimer 1991)].

Choose two antipodal points A and B in S^4. Let

$$U_1 = S^4 - \{A\}, \qquad U_2 = S^4 - \{B\}. \tag{1.13}$$

Then U_1 and U_2 form an open covering of S^4. We wish to paste the trivial sphere bundles $U_1 \times S^3$ and $U_2 \times S^3$ together to get the 3-sphere bundle Σ^7 over S^4.

Under the stereographic projection, U_1 and U_2 are both homeomorphic to $\mathbb{R}^4$, and $U_1 \cap U_2$ is homeomorphic to $\mathbb{R}^4 - \{0\}$. Identify the elements of $\mathbb{R}^4 - \{0\}$ as quaternions, and choose an odd number κ, where $\kappa^2 - 1 \not\equiv 0 \bmod 7$.

Consider the map $\tau : (\mathbb{R}^4 - \{0\}) \times S^3 \longrightarrow (\mathbb{R}^4 - \{0\}) \times S^3$, such that for every $(u, v) \in (\mathbb{R}^4 - \{0\}) \times S^3$, we have

$$\tau(u, v) = \left(\frac{u}{\|u\|^2}, \frac{u^h v u^j}{\|u\|} \right), \tag{1.14}$$

where

$$h = \frac{\kappa + 1}{2}, \qquad j = \frac{1 - \kappa}{2}, \tag{1.15}$$

and in (1.14) the multiplication and the norm $\|\|$ are in the sense of quaternions. Obviously τ is a smooth map. We can thus paste $U_1 \times S^3$ and $U_2 \times S^3$ together using τ. It can be proved that the Σ^7 constructed in this way is homeomorphic to the 7-dimensional unit sphere S^7, but its differentiable structure is different from the standard differentiable structure of S^7 (Example 2).

On a smooth manifold, the concept of a smooth function is well-defined. Let f be a real-valued function defined on an m-dimensional smooth manifold M. If $p \in M$, and (U, φ_U) is a compatible coordinate chart containing p, then $f \circ \varphi_U^{-1}$ is a real-valued function defined on the open subset $\varphi_U(U)$ of the Euclidean space $\mathbb{R}^m$. If $f \circ \varphi_U^{-1}$ is C^∞ at the point $\varphi_U(p) \in \mathbb{R}^m$, we say that the function f is C^∞ at $p \in M$.

The differentiability of the function f at the point p is independent of the choice of the compatible coordinate chart containing p. In fact, for another compatible coordinate chart (V, φ_V) containing p such that $U \cap V \neq \varnothing$, we have

$$f \circ \varphi_V^{-1} = \left(f \circ \varphi_U^{-1} \right) \circ \left(\varphi_U \circ \varphi_V^{-1} \right).$$

Since $\varphi_U \circ \varphi_V^{-1}$ is smooth, we see that $f \circ \varphi_V^{-1}$ and $f \circ \varphi_U^{-1}$ are differentiable at the same point p.

If the real-valued function f is C^∞ at every point in M, then we call f a C^∞, or **smooth**, function on M. We shall denote the set of all smooth functions on M by $C^\infty(M)$.

Smooth real-valued functions are just important special cases of smooth maps between smooth manifolds.

Definition 1.3. Suppose $f : M \longrightarrow N$ is a continuous map from one smooth manifold M to another, N, where $\dim M = m$ and $\dim N = n$. If there exist compatible[b] coordinate charts (U, φ_U) at the point $p \in M$ and (V, ψ_V) at $f(p) \in N$ such that the map

$$\psi_V \circ f \circ \varphi_U^{-1} : \varphi_U(U) \longrightarrow \psi_V(V)$$

[b] That is, contained in the smooth structures of the respective manifolds.

is C^∞ at the point $\varphi_U(p)$, then the map f is called C^∞ at p. If the map f is C^∞ at every point p in M, then we say that f is a **smooth map** from M to N.

Remark. Since $\psi_V \circ f \circ \varphi_U^{-1}$ is a continuous map from an open set $\varphi_U(U) \subset \mathbb{R}^m$ to another open set $\psi_V(V) \subset \mathbb{R}^n$, its differentiability at the point $\varphi_U(p)$ is defined. Obviously the differentiability of f at p is independent of the choice of compatible coordinate charts (U, φ_U) and (V, φ_V).

In the case $\dim M = \dim N$, if $f : M \longrightarrow N$ is a homeomorphism and f, f^{-1} are both smooth maps, then we call $f : M \longrightarrow N$ a **diffeomorphism**. If the smooth manifolds M and N are diffeomorphic, then we say that the corresponding smooth structures of the manifolds are **isomorphic**. In the above example, the Milnor sphere Σ^7 is homeomorphic but not diffeomorphic to S^7. Hence their smooth structures are not isomorphic.

Another important special case of smooth maps between smooth manifolds is that of **parametrized curves** on manifolds, in which M is an open interval $(a, b) \subset \mathbb{R}^1$. A smooth map $f : (a, b) \longrightarrow N$ from M to the manifold N is a parametrized curve in the manifold N.

Now suppose M and N are m-dimensional and n-dimensional manifolds with differentiable structures $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in\mathcal{B}}$, respectively. We can construct a new $(m+n)$-dimensional smooth manifold $M \times N$ by the following method. First, we see that $\{U_\alpha \times V_\beta\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}$ forms an open covering of the topological product space $M \times N$. Then we define maps $\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \longrightarrow \mathbb{R}^{m+n}$ such that

$$\varphi_\alpha \times \psi_\beta(p, q) = (\varphi_\alpha(p), \psi_\beta(q)), \quad (p, q) \in U_\alpha \times V_\beta. \tag{1.16}$$

Thus $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)$ is a coordinate chart of $M \times N$. It is easy to prove that all the coordinate charts obtained in this way are C^∞-compatible, and hence they determine a smooth differentiable structure on $M \times N$.

Definition 1.4. The smooth differentiable structure determined by the C^∞-compatible coordinate covering $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}}$ of the topological product space $M \times N$ makes $M \times N$ an $(m+n)$-dimensional smooth manifold, called the **product manifold** of M and N.

The natural projections of the product manifold $M \times N$ onto its factors are denoted by

$$\pi_1 : M \times N \longrightarrow M, \qquad \pi_2 : M \times N \longrightarrow N,$$

where, for any $(x, y) \in M \times N$,

$$\pi_1(x, y) = x, \qquad \pi_2(x, y) = y.$$

Obviously these are both smooth maps.

§1–2 Tangent Spaces

At every point on a regular curve (or surface), we have the notion of the tangent line (or tangent plane). Similarly, given a differentiable structure on a topological manifold, we can approximate a neighborhood of any point by a linear space. More precisely, the concepts of the tangent space and the cotangent space can be introduced. We begin with the cotangent space.

Suppose M is an m-dimensional smooth manifold. Fix a point $p \in M$, and let f be a C^∞ function[c] defined in a neighborhood of p. Denote the set of all these functions by C_p^∞. Naturally, the domains of two different functions in C_p^∞ may be different, but addition and multiplication in the function space C_p^∞ are still well-defined. Suppose $f,\ g \in C_p^\infty$ with domains U and V respectively. Then $U \cap V$ is also a neighborhood containing p. Thus $f+g$ and $f \cdot g$ can be defined as functions on $U \cap V$, that is, $f+g$ and $f \cdot g \in C_p^\infty$.

Define a relation $\sim$ in C_p^∞ as follows. Suppose $f, g \in C_p^\infty$. Then $f \sim g$ if and only if there exists an open neighborhood H of the point p such that $f|_H = g|_H$. Obviously $\sim$ is an equivalence relation in C_p^∞. We will denote the equivalence class of f by $[f]$, which is called a C^∞-**germ** at p on M. Let

$$\mathscr{F}_p = C_p^\infty / \sim = \left\{ [f] \,\middle|\, f \in C_p^\infty \right\}.$$

Then, by defining addition and scalar multiplication, $\mathscr{F}_p$ becomes a linear space over $\mathbb{R}$: for $[f], [g] \in \mathscr{F}_p, a \in \mathbb{R}$, define

$$\begin{cases} [f] + [g] = [f+g], \\ \quad a[f] = [af]. \end{cases} \tag{2.2}$$

In this definition, the right hand sides of (2.2) are independent of the choices of $f \in [f]$ and $g \in [g]$. The reader should verify that $\mathscr{F}_p$ is an infinite-dimensional real linear space.

Suppose γ is a parametrized curve on M through a point p. Then there exists a positive number δ such that $\gamma : (-\delta, \delta) \longrightarrow M$ is a C^∞ map and $\gamma(0) = p$. Denote the set of all these parametrized curves by Γ_p.

For $\gamma \in \Gamma_p, [f] \in \mathscr{F}_p$, let (see Figure 3)

$$\ll \gamma, [f] \gg = \left. \frac{d(f \circ \gamma)}{dt} \right|_{t=0}, \qquad -\delta < t < \delta. \tag{2.3}$$

[c]Suppose f is a function defined on an open set $V \subset M$. If the function $f \circ \varphi_U^{-1}$ is C^∞ on the open set $\varphi_U(U \cap V) \subset \mathbb{R}^m$ for any admissible coordinate chart (U, φ_U), where $U \cap V \neq \varnothing$, then we say f is a C^∞ function defined on V. In fact, V has a differentiable structure induced from M (see section §1–3). Thus f is a C^∞ function on the differentiable manifold V.

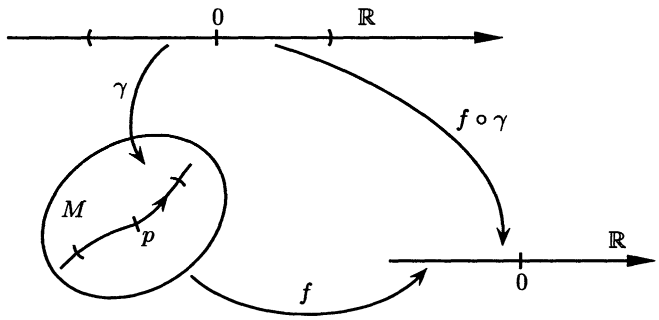

FIGURE 3.

Obviously, for a fixed γ, the value on the right hand side above is determined by $[f]$ and independent of the choice of $f \in [f]$. Also, $\ll\ ,\ \gg$ is linear in the second variable, i.e., for arbitrary $\gamma \in \Gamma_p$, $[f], [g] \in \mathcal{F}_p$, $a \in \mathbb{R}$, we have

$$\begin{aligned} \ll \gamma, [f] + [g] \gg &= \ll \gamma, [f] \gg + \ll \gamma, [g] \gg, \\ \ll \gamma, a[f] \gg &= a \ll \gamma, [f] \gg . \end{aligned} \tag{2.4}$$

Let

$$\mathcal{H}_p = \{[f] \in \mathcal{F}_p \mid \ll \gamma, [f] \gg = 0, \quad \forall \gamma \in \Gamma_p\}. \tag{2.5}$$

Then $\mathcal{H}_p$ is a linear subspace of $\mathcal{F}_p$.

Theorem 2.1. *Suppose $[f] \in \mathcal{F}_p$. For an admissible coordinate chart (U, φ_U), let*

$$F\left(x^1, \dots, x^m\right) = f \circ {\varphi_U}^{-1}\left(x^1, \dots, x^m\right). \tag{2.6}$$

Then $[f] \in \mathcal{H}_p$ if and only if

$$\left.\frac{\partial F}{\partial x^i}\right|_{\varphi_U(p)} = 0, \qquad 1 \le i \le m.$$

Proof. Suppose $\gamma \in \Gamma_p$, with coordinate representation

$$\left(\varphi_U \circ \gamma(t)\right)^i = x^i(t), \qquad -\delta < t < \delta. \tag{2.7}$$

$$\begin{aligned}
\ll \gamma, [f] \gg &= \frac{d}{dt}(f \circ \gamma)\bigg|_{t=0} \\
&= \frac{d}{dt} F\left(x^1(t), \dots, x^m(t)\right)\bigg|_{t=0} \\
&= \sum_{i=1}^{m} \left(\frac{\partial F}{\partial x^i}\bigg|_{\varphi_U(p)} \cdot \frac{dx^i(t)}{dt}\bigg|_{t=0} \right).
\end{aligned} \tag{2.8}$$

Since we may choose the appropriate γ to get any real value for $\frac{dx^i(t)}{dt}\Big|_{t=0}$, a necessary and sufficient condition for $\ll \gamma, [f] \gg = 0$ for arbitrary $\gamma \in \Gamma_p$ is

$$\frac{\partial F}{\partial x^i}\bigg|_{\varphi_U(p)} = 0, \qquad 1 \leq i \leq m.$$

□

We can summarize Theorem 2.1 as follows. The subspace $\mathcal{H}_p$ is exactly the linear space of germs of smooth functions whose partial derivatives with respect to local coordinates all vanish at p.

Definition 2.1. The quotient space $\mathcal{F}_p/\mathcal{H}_p$ is called the **cotangent space** of M at p, denoted by T_p^* (or $T_p^*(M)$). The $\mathcal{H}_p$-equivalence class of the function germ $[f]$ is denoted by $\widetilde{[f]}$ or $(df)_p$, and is called a **cotangent vector** on M at p.

T_p^* is a linear space. It has a linear structure induced from the linear space $\mathcal{F}_p$, i.e. for $[f], [g] \in \mathcal{F}_p$, $a \in \mathbb{R}$ we have

$$\begin{cases} \widetilde{[f]} + \widetilde{[g]} = (\widetilde{[f] + [g]}), \\ a \cdot \widetilde{[f]} = (\widetilde{a[f]}). \end{cases} \tag{2.9}$$

Theorem 2.2. *Suppose $f^1, \cdots, f^s \in C_p^\infty$ and $F\left(y^1, \cdots, y^s\right)$ is a smooth function in a neighborhood of $\left(f^1(p), \cdots, f^s(p)\right) \in \mathbb{R}^s$. Then $f = F\left(f^1, \cdots, f^s\right) \in C_p^\infty$ and*

$$(df)_p = \sum_{k=1}^{s} \left[\left(\frac{\partial F}{\partial f^k}\left(f^1(p), \cdots, f^s(p)\right) \right) \cdot (df^k)_p \right]. \tag{2.10}$$

Proof. Suppose the domain of f^k containing p is U_k. Then f is defined in $\bigcap_{k=1}^{s} U_k$, and for $q \in \bigcap_{k=1}^{s} U_k$,

$$f(q) = F\left(f^1(q), \cdots, f^s(q)\right).$$

Since F is a smooth function, $f \in C_p^\infty$. Let $a_k = \dfrac{\partial F}{\partial f^k}\left(f^1(p), \cdots, f^s(p)\right)$. Then for any $\gamma \in \Gamma_p$,

$$\begin{aligned}
\ll \gamma, [f] \gg &= \left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma) \\
&= \left.\frac{d}{dt}\right|_{t=0} F\left(f^1 \circ \gamma(t), \cdots, f^s \circ \gamma(t)\right) \\
&= \sum_{k=1}^{s} a_k \left.\frac{d}{dt}\right|_{t=0} \left(f^k \circ \gamma(t)\right) \\
&= \ll \gamma, \sum_{k=1}^{s} a_k \left[f^k\right] \gg .
\end{aligned}$$

Thus

$$[f] - \sum_{k=1}^{s} a_k \left[f^k\right] \in \mathcal{H}_p,$$

i.e.,

$$(df)_p = \sum_{k=1}^{s} a_k (df^k)_p.$$

□

Corollary 1. *For any $f, g \in C_p^\infty$, $a \in \mathbb{R}$, we have*

$$d(f+g)_p = (df)_p + (dg)_p, \tag{2.11}$$
$$d(af)_p = a \cdot (df)_p, \tag{2.12}$$
$$d(fg)_p = f(p) \cdot (dg)_p + g(p) \cdot (df)_p. \tag{2.13}$$

We see that (2.11) and (2.12) are the same as (2.9), and (2.13) follows directly from Theorem 2.2. □

Corollary 2. $\dim T_p^* = m$.

Proof. Choose an admissible coordinate chart (U, φ_U), and define local coordinates u^i by

$$u^i(q) = (\varphi_U(q))^i = x^i \circ \varphi_U(q), \quad q \in U, \tag{2.14}$$

where x^i is a given coordinate system in $\mathbb{R}^m$. Then $u^i \in C_p^\infty, (du^i)_p \in T_p^*$. We will prove that $\{(du^i)_p, 1 \le i \le m\}$ is a basis for T_p^*.

Suppose $(df)_p \in T_p^*$. Then $f \circ \varphi_U^{-1}$ is a smooth function defined on an open set of $\mathbb{R}^m$. Let $F(x^1, \dots, x^m) = f \circ \varphi_U^{-1}(x^1, \dots, x^m)$. Thus

$$f = F(u^1, \dots u^m). \tag{2.15}$$

By Theorem 2.2,

$$(df)_p = \sum_{i=1}^m \left[\left(\frac{\partial F}{\partial u^i} \left(u^1(p), \cdots, u^m(p)\right) \right) \cdot (du^i)_p \right]. \tag{2.16}$$

Thus $(df)_p$ is a linear combination of the $(du^i)_p, i \le i \le m$.

If there exist real numbers $a_i, 1 \le i \le m$, such that

$$\sum_{i=1}^m a_i (du^i)_p = 0, \tag{2.17}$$

i.e.

$$\sum_{i=1}^m a_i \left[u^i\right] \in \mathcal{H}_p,$$

then for any $\gamma \in \Gamma_p$, we have

$$\ll \gamma, \sum_{i=1}^m a_i \left[u^i\right] \gg = \sum_{i=1}^m a_i \left. \frac{d(u^i \circ \gamma(t))}{dt} \right|_{t=0} = 0. \tag{2.18}$$

Choose $\lambda_k \in \Gamma_p, 1 \le k \le m$ such that

$$u^i \circ \lambda_k(t) = u^i(p) + \delta_k^i t, \tag{2.19}$$

where

$$\delta_k^i = \begin{cases} 1, & i = k, \\ 0, & i \ne k. \end{cases}$$

Then

$$\left. \frac{d(u^i \circ \lambda_k(t))}{dt} \right|_{t=0} = \delta_k^i.$$

Let $\gamma = \lambda_k$. By (2.18) $a_k = 0, 1 \le k \le m$, i.e., $\{(du^i)_p, 1 \le i \le m\}$ is linearly independent. Therefore it forms a basis for T_p^*, called the **natural basis** of T_p^* with respect to the local coordinate system u^i. Thus T_p^* is an m-dimensional linear space. □

By definition, $[f]-[g]\in\mathcal{H}_p$ if and only if $\ll\gamma,[f]\gg\,=\,\ll\gamma,[g]\gg$ for all $\gamma\in\Gamma_p$, so we can define

$$\ll\gamma,(df)_p\gg\,=\,\ll\gamma,[f]\gg,\qquad \gamma\in\Gamma_p,\qquad (df)_p\in T_p^*. \tag{2.20}$$

Now define a relation $\sim$ in Γ_p as follows. Suppose $\gamma,\gamma'\in\Gamma_p$. Then $\gamma\sim\gamma'$ if and only if for any $(df)_p\in T_p^*$,

$$\ll\gamma,(df)_p\gg\,=\,\ll\gamma',(df)_p\gg. \tag{2.21}$$

Obviously this is an equivalence relation. Denote the equivalence class of γ by $[\gamma]$. Hence we can define

$$\langle[\gamma],(df)_p\rangle\,=\,\ll\gamma,(df)_p\gg. \tag{2.22}$$

We will prove that the $[\gamma],\gamma\in\Gamma_p$, form the dual space of T_p^*. For this purpose we will use local coordinate systems.

Under the local coordinates u^i, suppose $\gamma\in\Gamma_p$ is given by the functions

$$u^i=u^i(t),\qquad 1\le i\le m. \tag{2.23}$$

Then (2.22) can be written as

$$\langle[\gamma],(df)_p\rangle=\sum_{i=1}^m a_i\xi^i, \tag{2.24}$$

where

$$a_i=\left(\frac{\partial(f\circ\varphi_U^{-1})}{\partial u^i}\right)_{\varphi_U(p)},\qquad \xi^i=\left(\frac{du^i}{dt}\right)_{t=0}. \tag{2.25}$$

The coefficients a_i are exactly the components of the cotangent vector $(df)_p$ with respect to the natural basis $(du^i)_p$ [see (2.16)]. Obviously, $\langle[\gamma],(df)_p\rangle$ is a linear function on T_p^*, which is determined by the components ξ^i. Choose γ such that

$$u^i(t)=u^i(p)+\xi^i t \tag{2.26}$$

with ξ^i arbitrary. Thus the $\langle[\gamma],(df)_p\rangle,\gamma\in\Gamma_p$, represent the totality of linear functionals on T_p^* and form its dual space, T_p, called the **tangent space** of M at p. Elements in the tangent space are called **tangent vectors**.

The geometric meaning of tangent vectors is quite simple: if $\gamma'\in\Gamma_p$ is given by functions

$$u^i=u'^i(t),\qquad 1\le i\le m,$$

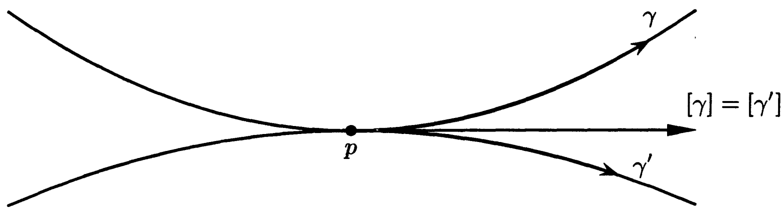

FIGURE 4.

then a necessary and sufficient condition for $[\gamma] = [\gamma']$ is

$$\left(\frac{du^i}{dt}\right)_{t=0} = \left(\frac{du'^i}{dt}\right)_{t=0}.$$

Hence the equivalence of γ and γ' means that these two parametrized curves have the same tangent vector at the point p (see Figure 4). Thus we identify a tangent vector X of M at p with the set of all parametrized curves through p with a common tangent vector.

By the discussion above, the function

$$\langle X, (df)_p \rangle, \qquad X = [\gamma] \in T_p, \ (df)_p \in T_p^*$$

is bilinear, i.e., linear in either variable. Suppose parametrized curves λ_k, $1 \leq k \leq m$, are given as in (2.19). Then

$$\langle [\lambda_k], (du^i)_p \rangle = \delta_k^i. \tag{2.27}$$

Therefore $\{[\lambda_k],\ 1 \leq k \leq m\}$ is the dual basis of $\{(du^i)_p,\ 1 \leq i \leq m\}$. (For the definition of dual basis, see section §2–1 of the next chapter.)

There is another meaning of the tangent vectors $[\lambda_k]$. We have

$$\begin{aligned} \langle [\lambda_k], (df)_p \rangle &= \left\langle [\lambda_k], \sum_{i=1}^{m} \left\{ \left(\frac{\partial f}{\partial u^i}\right)_p \cdot (du^i)_p \right\} \right\rangle \\ &= \left(\frac{\partial f}{\partial u^k}\right)_p, \end{aligned} \tag{2.28}$$

where $\partial f / \partial u^i$ means $\partial (f \circ {\varphi_U}^{-1}) / \partial u^i$. Thus the $[\lambda_k]$ are the partial differential operators $(\partial / \partial u^k)$ on the function germs $[f]$; and (2.27) can be written as

$$\left\langle \left.\frac{\partial}{\partial u^k}\right|_p, (du^i)_p \right\rangle = \delta_k^i. \tag{2.29}$$

We call the dual basis of $\{(du^i)_p,\ 1 \le i \le m\}$ in T_p the **natural basis** of the tangent space T_p under the local coordinate system (u^i). From (2.24) we have

$$[\gamma] = \sum_{i=1}^{m} \xi^i \left.\frac{\partial}{\partial u^i}\right|_p .$$

Thus ξ^i are the components of the tangent vector $[\gamma]$ with respect to the natural basis. If $[\gamma'] \in T_p$ has components ξ'^i, then $[\gamma] + [\gamma']$ is determined by the components $\xi^i + \xi'^i$. Similarly the tangent vector $a \cdot [\gamma]$ ($a \in \mathbb{R}$) has components $a\xi^i$.

For simplicity, we sometimes suppress the lower index p of tangent and cotangent vectors when there is no confusion.

Definition 2.2. Suppose f is a C^∞-function defined near p. Then $(df)_p \in T_p^*$ is also called the **differential** of f at the point p. If $(df)_p = 0$, then p is called a **critical point** of f.

The study of critical points of smooth functions on M is an important topic in differentiable manifolds, called **Morse Theory**. The reader can refer to Milnor, 1963.

Definition 2.3. Suppose $X \in T_p,\ f \in C_p^\infty$. Denote

$$Xf = \langle X, (df)_p \rangle. \tag{2.30}$$

Xf is called the **directional derivative** of the function f along the vector X.

The following theorem gives some properties of the directional derivative.

Theorem 2.3. *Suppose* $X \in T_p,\ f, g \in C_p^\infty,\ \alpha, \beta \in \mathbb{R}$. *Then*

1) $X(\alpha f + \beta g) = \alpha \cdot Xf + \beta \cdot Xg;$
2) $X(fg) = f(p) \cdot Xg + g(p) \cdot Xf.$

Proof. These follow from Corollary 1 of Theorem 2.2 directly. □

Remark 1. Statement 1) of Theorem 2.3 indicates that a tangent vector X can also be viewed as a linear operator on C_p^∞. Using 1) and 2), we see that the result of X operating on any constant function c is 0.

Remark 2. Frequently, in the literature[d], properties 1) and 2) are used to define tangent vectors. In fact, all the operators on C_p^∞ satisfying these two properties form a linear space dual to T_p^*, which must then be identical to T_p.

[d]For example, see Chevalley 1946.

Under local coordinates u^i, a tangent vector $X = [\gamma] \in T_p$ and a cotangent vector $a = df \in T_p^*$ have linear representations in terms of natural bases:

$$X = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial u^i}, \quad a = \sum_{i=1}^{m} a_i du^i , \tag{2.31}$$

where

$$\xi^i = \frac{d(u^i \circ \gamma)}{dt}, \quad a_i = \frac{\partial f}{\partial u^i}.$$

Under another local coordinate system u'^i, if the components of X and a with respect to the corresponding natural bases are ξ'^i and a'_i, respectively, then they satisfy the following transformation rules:

$$\xi'^j = \sum_{i=1}^{m} \xi^i \frac{\partial u'^j}{\partial u^i}, \tag{2.32}$$

$$a_i = \sum_{j=1}^{m} a'_j \frac{\partial u'^j}{\partial u^i}, \tag{2.33}$$

where

$$\frac{\partial u'^j}{\partial u^i} = \frac{\partial(\varphi'_U \circ \varphi_U^{-1})^j}{\partial u^i}$$

is the Jacobian matrix of the change of coordinates $\varphi'_U \circ \varphi_U^{-1}$. In classical tensor analysis, the vectors satisfying (2.32) are called **contravariant**, and those satisfying (2.33) are called **covariant**, vectors.

Smooth maps between smooth manifolds induce linear maps between tangent spaces and between cotangent spaces. Suppose $F : M \longrightarrow N$ is a smooth map, $p \in M$, and $q = F(p)$. Define the map $F^* : T_q^* \longrightarrow T_p^*$ as follows:

$$F^*(df) = d(f \circ F), \qquad df \in T_q^*. \tag{2.34}$$

Obviously this is a linear map, called the **differential** of the map F.

Consider next the adjoint of F^*, namely the map $F_* : T_p \longrightarrow T_q$ defined for $X \in T_p$, $a \in T_q^*$ as follows:

$$\langle F_* X, a \rangle = \langle X, F^* a \rangle. \tag{2.35}$$

F_* is called the **tangent map** induced by F.

Suppose u^i and v^α are local coordinates near p and q, respectively. Then the map F can be expressed near p by the functions

$$v^\alpha = F^\alpha\left(u^1, \dots, u^m\right), \qquad 1 \le \alpha \le n. \tag{2.36}$$

Thus the action of F^* on the natural basis $\{dv^\alpha, 1 \le \alpha \le n\}$ is given by

$$\begin{aligned} F^*(dv^\alpha) &= d(v^\alpha \circ F) \\ &= \sum_{i=1}^m \left(\frac{\partial F^\alpha}{\partial u^i}\right)_p du^i. \end{aligned} \tag{2.37}$$

The matrix representation of F^* in the natural bases $\{dv^\alpha\}$ and $\{du^i\}$ is exactly the Jacobian matrix $(\partial F^\alpha/\partial u^i)_p$.

Similarly, the action of F_* on the natural basis $\{\partial/\partial u^i\}$ is given by

$$\begin{aligned} \left\langle F_*\left(\frac{\partial}{\partial u^i}\right), dv^\alpha\right\rangle &= \left\langle \frac{\partial}{\partial u^i}, F^*(dv^\alpha)\right\rangle \\ &= \sum_{j=1}^m \left\langle \frac{\partial}{\partial u^i}, du^j\right\rangle \left(\frac{\partial F^\alpha}{\partial u^j}\right)_p \\ &= \left\langle \sum_{\beta=1}^n \left(\frac{\partial F^\beta}{\partial u^i}\right)_p \frac{\partial}{\partial v^\beta}, dv^\alpha\right\rangle, \end{aligned}$$

i.e.,

$$F_*\left(\frac{\partial}{\partial u^i}\right) = \sum_{\beta=1}^n \left(\frac{\partial F^\beta}{\partial u^i}\right)_p \frac{\partial}{\partial v^\beta}. \tag{2.38}$$

Hence the matrix representation of the tangent map F_* under the natural bases $\{\partial/\partial u^i\}$ and $\{\partial/\partial v^\alpha\}$ is still the Jacobian matrix $(\partial F^\alpha/\partial u^i)_p$.

§1–3 Submanifolds

Before discussing submanifolds, we will first study tangent maps induced by smooth maps between smooth manifolds. Given a smooth map $\varphi : M \longrightarrow N$, for any point $p \in M$ there exists an induced tangent map between the corresponding tangent spaces, $\varphi_* : T_p(M) \longrightarrow T_q(N)$, where $q = \varphi(p)$. The crucial point is that the properties of the tangent map φ_* at $p \in M$ determine the properties of the map φ in a neighborhood of p. A classical result in this regard is the **inverse function theorem** in calculus.

Theorem 3.1. *Suppose W is an open subset of $\mathbb{R}^n$ and $f : W \longrightarrow \mathbb{R}^n$ is a smooth map. If at a point $x_0 \in W$ the determinant of the Jacobian matrix is nonzero, i.e.,*

$$\det \left(\frac{\partial f^i}{\partial x^j}\right)\bigg|_{x_0} \ne 0,$$