Roots of the Derivative of a Polynomial

Dan Romik

1. INTRODUCTION. The object of this note is to show a simple but amusing result on the roots of the polynomial $d/dx(x(x-1)(x-2)(x-3)\dots(x-n))$ that I discovered while working on explicit formulas for the Markov transform. I describe the result and then mention briefly how it is related to this beautiful subject. I hope that the interested reader will consult [**2**] or [**3**] for additional information.

2. THE RESULT. Consider the polynomial $p_n(x) = x(x-1)(x-2)\dots(x-n)$. Its derivative has a root between each two adjacent roots of p_n, so we may write

$$p_n'(x) = n \prod_{k=0}^{n-1} \bigl(x - (k + \alpha_{n,k})\bigr),$$

where the $\alpha_{n,k}$, the fractional parts of the roots of p_n', are between 0 and 1. A plot of the $\alpha_{n,k}$ as a function of k reveals the following picture (in this example, $n = 30$):

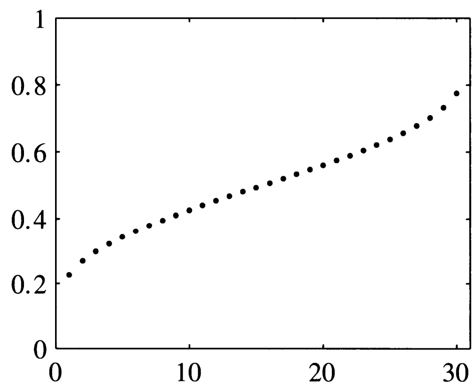

Figure 1. Fractional parts of the roots of p_{30}'.

Figure 1 suggests that for k properly scaled the $\alpha_{n,k}$ approach the graph of some continuous function as $n \to \infty$. Indeed, this is true, and the function is given by Theorem 1:

Theorem 1. *For all t in* $(0, 1)$,

$$\lim_{n\to\infty} \alpha_{n,\lfloor t\cdot n\rfloor} = \frac{1}{\pi} \operatorname{arccot}\left(\frac{1}{\pi}\log\left(\frac{1-t}{t}\right)\right). \tag{1}$$

Here, and later, arccot signifies the branch of the inverse cotangent function taking values between 0 and π, and $\lfloor x\rfloor$ denotes the largest integer not greater than x. Figure 2 shows the superposition of the function on the right-hand side of (1) on the roots.

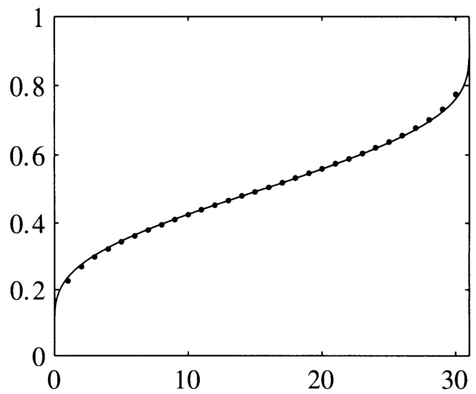

Figure 2. Fractional parts of the roots of p'_{30} and the limiting curve.

Proof. Let t satisfy $0 < t < 1$, and write $k = \lfloor t \cdot n \rfloor$. The $k + \alpha_{n,k}$ are the solutions of the equation

$$\frac{p'_n(x)}{p_n(x)} = \sum_{j=0}^{n} \frac{1}{x-j} = 0.$$

In other words, we have

$$\sum_{j=0}^{k} \frac{1}{\alpha_{n,k} + k - j} - \sum_{j=k+1}^{n} \frac{1}{-\alpha_{n,k} + j - k} = 0,$$

or, transforming the indices,

$$\sum_{j=0}^{k} \frac{1}{\alpha_{n,k} + j} - \sum_{j=0}^{n-k-1} \frac{1}{(1-\alpha_{n,k}) + j} = 0. \tag{2}$$

This equation for $\alpha_{n,k}$ cannot be solved explicitly (otherwise we would have explicit expressions for $\alpha_{n,k}$), but an asymptotic solution is easily obtained using a well-known asymptotic formula that is related to Euler's product formula for the gamma function:

$$\sum_{j=0}^{m} \frac{1}{u+j} = -\frac{\Gamma'(u)}{\Gamma(u)} + \log(m) + o(1) \qquad (m \to \infty). \tag{3}$$

Relation (3), which holds for all u in $\mathbb{C} \setminus \mathbb{Z}$, transforms (2) to

$$-\frac{\Gamma'(\alpha_{n,k})}{\Gamma(\alpha_{n,k})} + \frac{\Gamma'(1-\alpha_{n,k})}{\Gamma(1-\alpha_{n,j})} = \log\left(\frac{n-k-1}{k}\right) + o(1) \qquad (n \to \infty). \tag{4}$$

The right-hand side of (4) is the same as $\log((1-t)/t) + o(1)$, since $k = \lfloor t \cdot n \rfloor$. The left-hand side is exactly $\pi \cot(\pi \alpha_{n,k})$ (to see this, take the logarithmic derivative of the identity $\Gamma(u)\Gamma(1-u) = \pi / \sin(\pi u)$). Thus, we have shown that

$$\pi \cot(\pi \alpha_{n,k}) = \log\left(\frac{1-t}{t}\right) + o(1) \qquad (n \to \infty).$$

The last statement can be rephrased

$$\alpha_{n,k} = \frac{1}{\pi} \operatorname{arccot}\left(\frac{1}{\pi} \log\left(\frac{1-t}{t}\right)\right) + o(1) \qquad (n \to \infty),$$

as claimed. ■

3. THE MARKOV TRANSFORM. Theorem 1 can be thought of as a special case of an *inversion formula* for the Markov transform. The *Markov transform* is a correspondence between measures τ and μ on $\mathbb{R}$ defined by the equation

$$\int_{\mathbb{R}} \frac{d\mu(u)}{z-u} = \exp\left(\int_{\mathbb{R}} \log \frac{1}{z-u} d\tau(u)\right) \qquad (\operatorname{Im} z \neq 0). \tag{5}$$

Here τ is an *interlacing measure* (i.e., τ is a signed measure of total measure $\tau(\mathbb{R}) = 1$ that satisfies $0 \leq \tau((-\infty, x]) \leq 1$ for each x in $\mathbb{R}$), and μ, the Markov transform of τ, is a probability measure, whose existence is guaranteed (see [**2**]).

Equation (5) is fascinating for the interplay it expresses between the additive and multiplicative structures on its two sides. A natural question that arises is how to calculate the transform explicitly in the important case where μ is an absolutely continuous measure. I recently obtained the following partial answer [**3**]: If μ (hence, also τ) is supported on an interval $[a, b]$, then under some fairly general conditions it is the case that

(i) $$\frac{d\mu(x)}{dx} = \frac{1}{\pi} \sin\big(\pi\tau([a,x])\big) \exp\left(\int_a^b \log \frac{1}{x-u} d\tau(u)\right),$$

(ii) $$\tau\big([a,x]\big) = \frac{1}{\pi} \operatorname{arccot}\left(\frac{1}{\pi \frac{d\mu(x)}{dx}} \int_a^b \frac{d\mu(u)}{u-x}\right),$$

where the integrals are principal-value integrals. The expression on the right side of (ii) reminds us of the limiting curve in Theorem 1. In fact, Theorem 1 is the special case in which μ is the uniform measure on [0, 1] (i.e., Lebesgue measure), so the limiting curve is simply the inverse Markov transform of the uniform law! It was while trying to find the inverse formula (ii) for the Markov transform that I arrived at the calculation in Theorem 1. Generalizing the same asymptotic calculation, it is not hard to obtain the general formula (ii) from there.

4. FURTHER REMARKS. Equation (5) was first studied by Markov, who considered it in the context of continued fraction expansions. It has since been applied to moment problems, means of Dirichlet processes, the growth of random Young diagrams, and in other places. Kerov's survey [**2**] is a good reference. Regarding the explicit formulas, (i) was proved by Cifarelli and Regazzini [**1**] in the case where τ is a probability measure and was conjectured by Kerov to hold in the general case. Note that it is not at all obvious that the expression on the right-hand side of (i) is a probability density. Thus, substituting various expressions for τ gives rise to some amusing and perhaps unknown integration identities.

REFERENCES

1. D. M. Cifarelli and E. Regazzini, Distribution functions of means of a Dirichlet process, *Ann. Statist.* **18** (1990) 429–442.

2. S. V. Kerov, Interlacing measures, in *Kirillov's Seminar on Representation Theory*, Amer. Math. Soc. Transl. Ser. 2, no. 181, American Mathematical Society, Providence, 1998, pp. 35–83.
3. D. Romik, Explicit formulas for hook walks on continual Young diagrams, *Adv. Appl. Math.* **32** (2004) 625–654.

Laboratoire de Probabilités, Université Paris 6

Current address:
Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
romik@wisdom.weizmann.ac.il

Quaternions and Rotations in $\mathbb{E}^4$

Joel L. Weiner and George R. Wilkens

1. INTRODUCTION. In 1843, Sir William Rowan Hamilton invented the quaternion algebra, which is customarily denoted $\mathbb{H}$ in his honor. Soon after, people recognized that quaternions could be used to represent rotations in $\mathbb{E}^3$. In 1855, Arthur Cayley discovered that quaternions could also be used to represent rotations in $\mathbb{E}^4$. This note explores Cayley's representation. Ultimately we use it to show that any rotation in $\mathbb{E}^4$ is a product of rotations in a pair of orthogonal two-dimensional subspaces, a result first proved by Edouard Goursat [**3**].

In section 2 we review the algebraic structure of $\mathbb{H}$ and show that $\mathbb{H}$ has a natural inner product that allows us to identify it with four-dimensional Euclidean space $\mathbb{E}^4$. In section 3 we show that a pair $\mathbf{p}$ and $\mathbf{q}$ of unit vectors (also called unit quaternions) in $\mathbb{H}$ determines a rotation $C_{\mathbf{p},\mathbf{q}} : \mathbb{H} \to \mathbb{H}$. According to Goursat's result, $C_{\mathbf{p},\mathbf{q}}$ is a product of rotations in a pair of orthogonal planes. By this we mean the following: there exist rotations $R_1, R_2 : \mathbb{H} \to \mathbb{H}$ and a pair of orthogonal planes V_1 and V_2 in $\mathbb{H}$, such that the restrictions $R_1|_{V_2}$ and $R_2|_{V_1}$ are identities on their respective planes and

$$C_{\mathbf{p},\mathbf{q}} = R_1 \circ R_2 = R_2 \circ R_1.$$

Thus, $\mathbb{H} = V_1 \oplus V_2$, where $V_1 \perp V_2$, and $C_{\mathbf{p},\mathbf{q}}$ rotates vectors in the plane V_1 through a determined angle α_1 and vectors in the plane V_2 through a determined angle α_2.

The principal goals of this note are to prove Theorems 1 and 2, which are stated precisely in section 5. Theorem 1 not only proves Goursat's result for $C_{\mathbf{p},\mathbf{q}}$, but also shows that one can easily determine the planes V_1 and V_2 and the angles α_1 and α_2 in terms of $\mathbf{p}$ and $\mathbf{q}$. Theorem 2 establishes that every rotation in $\mathbb{E}^4$ can be represented by some $C_{\mathbf{p},\mathbf{q}}$. Together, these theorems prove Goursat's result for every four-dimensional rotation.

The observation that $C_{\mathbf{p},\mathbf{q}}(V_i) = V_i$ $(i = 1, 2)$ motivates the method of proof. The V_i are known as invariant subspaces for $C_{\mathbf{p},\mathbf{q}}$. If we wish to see that $C_{\mathbf{p},\mathbf{q}}$ is indeed a product of rotations, it is natural to look first for invariant subspaces of that transformation. In section 4 we recall some elementary results from the theory of ordinary differential equations that are related to subspaces and two-dimensional rotations. Finally, in section 5, we apply these results to find the $C_{\mathbf{p},\mathbf{q}}$-invariant subspaces and the rotation angles.