Models and Measurements of Desalination project
Math 487, Fall 2021

Allen, Jont B.

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Abstract This document is Allen's log for the modeling effort and data collection results for the ocean desalination project, ME-470, for the dates between May 25 and Aug 20, 2021.

Contents

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1 Introduction

The basic concept of solar desalination: The goal of the research presented here is to verify that the concepts and methods previously proposed by the author, for the solar desalination of cold ocean water, are valid. In a few words, the concept is to start with cold seawater (10-15°C having 3.5% salts consisting mostly (i.e., ≈85%) of NaCl), heating it by the sun as the sole energy source, to at least 40°C, thereby evaporating 5% by mass of the heated seawater ater. This pure water vapor is then continuously transported to a second chamber, where it is condensed back to pure water and dry air, by cooling it to ≈ room temperature, or below (e.g. 15-22°C), by the influx of the cold seawater. The cooling of the vapor warms up the sea water to well above 15°C due to the transfer of the heat energy in the vapor. The warmed seawater is then transported to the first chamber where it is further heated by the solar input. One important goal is to experimentally determine the equilibrium conditions of this energy conserving system of chambers. The final goal then is to demonstrate that these conditions may be met within a two chamber prototype. This work was done over the summer break of 2021 (May 25-Aug 15), with the helpful assistance of Professor Blake Johnson, and to a lesser extent Prof. Quin Brewster, both members of the UIUC MetSci-Eng dept.

Goals of the present research: This document discusses the results of my 2021 summer research, in support of a third semester of ME-470 for Fall 2021. The research goal for the May-Aug 2021 was to build a updated version of the ME-470 Spring 2021 prototype, for the conversion of ocean water into potable pure water, as described in the proceeding paragraph. A primary feature of this system is to capture the latent heat of evaporation by heating the cold (≈12°C) inlet sea water in the water vapor condensation phase, so that the solar energy is not lost to the environment, but recovered to further heat the 40°C seawater. During May-Aug 2021 these concepts were explored by building a two-stage physical working prototype of the system, to demonstrates the validity of the concepts. During the design, construction and implementation of the two stage physical prototype, several basic errors in the previous design were discovered, and fixed. By Aug 1 the prototype was ready for evaluation.

This document details the preliminary findings of the Aug. 1 version of the prototype. It is envisioned that this system can be used by a third semester of ME-470 (Fall 2021) to both run further experiments as well as fine tune the parameters of the system. These parameters require the quantification of the *entropy* (thermal heat energy) of the incoming *cold seawater*, the input heat energy, the capture of the *condensation entropy* of the hot (40°C) water vapor, and the *input solar energy*, due to the sunlight. In other words, an energy analysis of the two chambers, given equilibrium (i.e., steady-state) operating conditions.

The design of the system discussed here is based on the Spring 2021 semester of ME-370. Several modifications have been made as it became clear when two important "errors" were relieved (see discussion below). The design consists of three chambers, the *evaporation chamber* (EC), the *condensation chamber* (CC), and the *water chamber* (WC). All times are on using the 24 [hr] clock.

2 The problem with thermodynamics

Among students, there is a widely accepted difficulty in understanding the methods and concepts of thermodynamics. There are simple explanations for this difficulty, as explained in Appendix E of (Allen, 2020). First, the topic is complex. Take the case of water for example, with it's three states: liquid, ice and vapor. These forms coexist at water's *triple* point. The transition from one state to another can be mystifying, but is both important and relevant to this research.

But it is more than the complexity that is the source of this problem. There is a second difficulty, due to the mathematical description and analysis of the various states. For valid historical reasons, thermodynamics is characterized mathematically in terms of energy. However valid, this is not a good choice of variables, because power, the product of force and flow, is quadratic in the state variables, thus is nonlinear. Power, the energy rate, is the product of the two *conjugate variables*, temperature (a force-potential) and the entropy-rate (the heat flow).

State variables: The basic state variables that describe thermodynamics are density (ρ), pressure (P) and temperature (T). While the pressure (like temperature, pressure is a potential) is considered by most people to be a force per unit area (Pascals = Newton/meter^2), it is actually the difference, or the gradient of the pressure $-\nabla P$, that defines the force. Further complicating this scenario is that

mass and force are not the same thing. One kilogram is a mass which, which when placed on a scale, is measured by its weight (a force). However if 1 kilogram is taken to Mars it weighs about 38% of what it does on earth, while in empty space, it has zero weight. Of course the mass remains the same.

Thermodynamic temperature, like voltage and pressure, is also a potentials. For each of these, the gradient determines the driving force, resulting in a flow.

A fourth potentials is the gravitational potential. In every case (pressure, temperature, voltage, and the unnamed gravity potential), it is the change in potential that determines the linear component of the flow. For pressure the flow is the *volume velocity*. For temperature the flow is the *entropy rate*, or entropy velocity. For voltage the flow is the current, a measure of the charge transport. In fact electrons in a wire do not actually "flow" like water does, rather they transfer momentum by "bumping" each other. In the case of gravity, the equations are well define, at least on the macroscopic scale, but the jury seems to be out on what is actually going on with gravity. Since gravity waves travel at the speed of light, gravity must be an electromagnetic based force.

Conjugate variables: Every pair of variables (force, flow) are called *conjugate variables*. In every modality described above, the power is define as the *product* of the force and flow variables, while the impedance is defined as the *ratio* of force over flow. For example, the acoustics power is the product of the pressure difference times the volume velocity. For the case of thermodynamics, the thermal power is the product of the gradient of temperature times the entropy-rate. For electricity the power is the product of the gradient of the voltage times the current. The same holds true for gravity, but as best I know, these variables have not been given names, but were well described, first by Galileo, and some few years later by Newton.

To simplify the description of thermodynamic systems, they need to be reformulated in terms of linear variables (Allen, 2020, Appendix E). Power, the product in its conjugate variables, is quadratic. The equations may be made linear by defining an impedance, the ratio of conjugate variables. In the analysis of electrical circuits this is taught early in the curriculum, where it is called *circuit theory*. An impedance is expressed as a function of the *Laplace frequency* $s = \sigma + \jmath\omega$. For example, and inductor has an impedance $Z_l(s) = sL$ and a capacitor has impedance $Z_c(s) = 1/sC$, where L and C are called the *inductance* and the *capacitance*.

The problem with thermodynamics is that it is formulated in terms of energy. The product of temperature times entropy is the thermal energy. Entropy-rate is the time rate of change of the thermal charge, analogous to the time rate of change of the electric charge, which is the current. When the equations for thermodynamics are written in terms of the impedance, the equations become linear, and may be easily solved by the use of eigen-analysis, i.e., linear algebra (Allen, 2020).

3 Ocean water desalination method: Overview

Our basic idea for desalination is presented in a YouTube video.[1] Seawater is drawn from the ocean using electricity derived from an ocean wind-farm, and pumped into solar heated platforms, where it is directly warmed by the sunlight during daylight hours, assuming 1 [kW/m^2] for 5 [hr/day]. Sunlight has sufficient energy to heat water to 40°C, where it evaporates to its partial pressure/dew-point of 5% of the water's mass (See Fig. 1). The rate of the production of the water vapor thus depends on the input solar power [500 W/m^2] and the volume of water being heated. Other variables such as the light-absorbance, its spectrum and the heat capacity of water, as taken as their optimum values. For the rest of this document we conservatively assume half of that (500 W/m^2), which typically available for about 5 hr per day in North America. Temperature data across the globe is widely available in the open literature (Ambaum, 2010).

4 Experimental methods

In this section we present the various proposals for modeling the evaporation of the sea-water, and the condensation of the pure water vapor.

The URL[2] is titled *Build a Temperature and/or Humidity Controlled Fan with a Picaxe 08M2 Micro-controller.*

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[1] `https://www.youtube.com/watch?v=tQe0NvnCEGg`

[2] `https://www.allaboutcircuits.com/projects/build-temperature-humidity-controlled-fan-Picaxe-08M2-microcontroller/`

5 Modeling methods

We hope to design experimental methods, that are best implemented in the lab, that verify that we know the sea-water evaporation rates for a given input power and output water temperature. While the system power input is to be solar, for experimental purposes we use a fish-tank heater, since they are easily calibrated for power input.

5.1 Percentage of water after desalination

Assuming a water temperature of 40°C and 3.5% of salts, by mass, as the sea-water evaporates, the percentage of salt will increase, until it reaches around 30-35% of the brine, at which point the salt begins to precipitate out of the brine, thus defining upper limit of the evaporation phase.

It is best to work on an example numerically: Starting with 1 [kgm][3] of ocean water. Following the desalination process, the 35 [gm] of salt stays the same, along with 70 [gm] of water (twice the mass of the salt),[4] for a total of 35+70=105 [gm].

In terms of the % of the 1 [kg] we started with, the water has gone from 1000-35=965 [gm] to 70 [gm]. In % this is $100 \cdot 965/1000 = 96.5\%$. The water remaining is therefore 65% (100-35), for a total of 35+65=100%.

While the salt mass remains at 35 [gm], in terms of % it is now 1/3 (33.33%) while the water is 2/3 (66.66%), or 70 [gm].

Thus 895=965-70 [gm] (91.795%) of pure water will have been harvested from the 1 [L] of ocean water.

5.1.1 Algebraic analysis of the water harvested

ck!!! If the salt content S increased from 3.5% to 33% ($\approx$10x), then the pure (potable) water W_1 would decrease from W_1 (96.6%) to W_2 (2x33% = 66%), because the % sum is 100%. Let

$$O = W_1 + S, \quad [\text{kg}]$$

where W_1 is the relative mass of the pure water (96.5%) and S is the relative mass of the salt (3.5%) in the total seawater mass (O [kgm]).

Assuming the salt increases in salinity by 10x, from 3.5% to 35%, the pure water component W_1 drops from 96.5 to 65%. In terms of mass, if we start with 1 [kg]= 965 [gm] H_2O + 35 [gm] $NaCl$ of sea water, as the H_2O evaporates, the salt remains at 35 [gm/kgm]. Thus the water drops to $W_2 = 65$ [gm/kgm] (65%), because $B = O/10 = 100$ [gm] $(S/B = 10S/O)$.

For example, if two parts pure water W_1 and 1 part salt S make up the final brine B,

$$B = W_2 + S$$

then S/B=1/3 (33%) and $W_2/B = 2/3$ (67%), and $W_2/S = 2$. In this example the salinity goes from 3.5% to 33%, slightly less than the assumed factor of 10.

Initialization to steady state:

1. When heating a mass M [kgm] of $T_{in} = 12$°C water, the equilibrium (final) temperature T_{equi} depends on the input power P_{in} [watts], the thermal load M [kgm], and of course the iso-pressure heat capacity (the capacitance) of the load c_p, which depends on its state (see Table 1). The capacity depends on the state of the water because the degrees of freedom depends on the state of the molecule (Liquid water has twice the degrees of freedom compared with vapor and ice).

   (a) When the heater is turned on the temperature continues to rise from T_{in} until the dew-point is reached, at which time the water vapor is at its maximum, called the *dew point*, at a rate determined P_{in}.

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[3] For those who prefer to think in US units, 1 [kg] of pure water is 4 [cups].

[4] At this point we must not remove more water, because when the salt is 1/3 of the brine, it will precipitate, which would clog the system.

Table 1 *Table of the heat capacity of water in its three common states. This property is similar to that of an electrical capacitor to hold charge Q, given a voltage V (the potential), where Q = CV. In the case of heat, the potential is the temperature T and the amount of heat-charge Q is called the* entropy, *where Q = $c_p T$.*

state	c_p	Units
liquid	4.218	[kJ/kg°C]
vapor	1.859	[kJ/kg°C]
ice	2.050	[kJ/kg°C]

(b) If the ratio P_{in}/M is properly chosen, the dew point temperature will be 40 °C. Of course this number depends on any thermal losses within the system, which we assume are minimal due to the system design (lots of Styrofoam thermal barriers). The main source of loss is the transport of the water vapor from the *evaporation chamber* (EC) to the *condensation chamber* (CC).

(c) The present design recovers this thermal loss by heating the cold seawater by the high vapor entropy, thus condensing the vapor. Of course this recover is not 100%. A key goal of this research is to see how close to 100% we come.

The flow of water vapor depends on the input flux (60 [l/min]) of room temperature air injected into the water by the air pump into the submerged *air stones* (see online videos[5]). Watching the videos is helpful in order to following the presentation.

Thus given $P_{in} = 500$ [watts] of input power (the rated heater output), how much water M do we need to bring the dew-point to T_{equi}?

(d) We need to find the mass of the water that is required to come to 40 °C given 500 [w] of input power from the fish-tank heater.

(e) Finally if we are replacing the evaporated water (at 40 °C) with 30 °C water, slight more energy will be required to warm the replacement water to bring it to the bath temperature. I suspect this effect will be small, but not zero. If we replaced the evaporated water with seawater, at 15 °C, the effect would be much larger. It would be nice to show that the multistage-warming is advantages.

2. Ideally we must to calibrate the fish-tank heater, to determine the actual heater power it injects, when the thermostat temperature is set to 40C.[6]

3. We can cap the temperature at the desired 40 °C by increasing the mass of the water, so that the evaporation rate exactly cancels the heat input form the 500 [w] heater.

6 Prototype results, Aug. 2021

6.1 Allen's Thermodynamic modeling suggestion

The basic modeling method uses the state-variable transmission matrix approach, a method well documented in the Electrical Engineering literature. For an extended discussion of this modeling methods, please consult Allen (2020, p. 224). A specific example of a related thermodynamic calculation is derived in Appendix B.
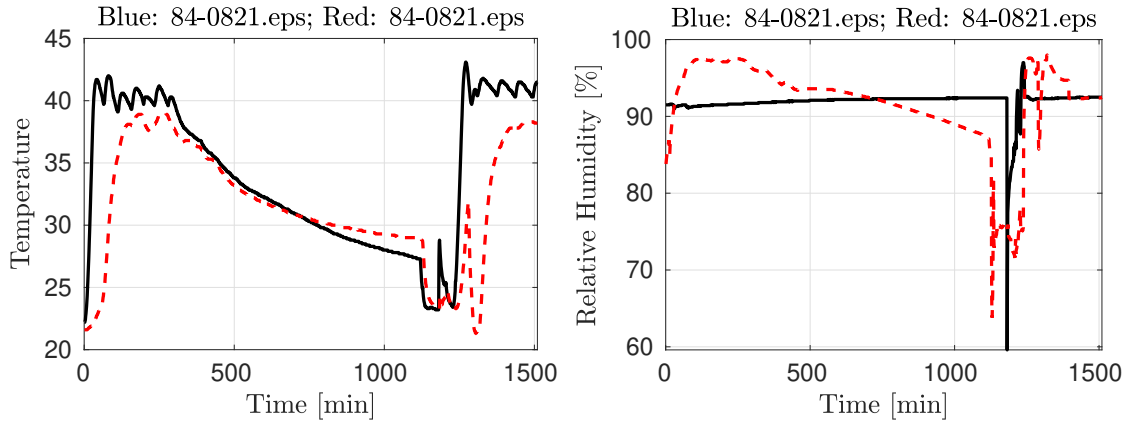
6.1.1 Experiment of Aug 20

During the development of the summer 2021 prototype (May 25-Aug 20), several design errors were discovered, and work-arounds were developed. The first problem was that the outside air pumped into the EC was causing the 40°C vapor to condense.

This was resolved by pumping the air into the water via *air-stones*. In this way the injected air was warmed to 40C before it hit the vapor, removing the problem.

The second error was reveled in early Aug while doing some preliminary experiments on the F21 prototype. In the prototype of S20, the vents were implemented using three bulk-heads between the EC and CC, which were at least an order of magnitude too small. I first discovered this problem when I measured the time constants of vapor diffusion, which had a time constant greater than one [hr].

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[5] http://jontalle.web.engr.illinois.edu/MISC/Desalination.20/VideosDesal-Jul26.21/

[6] It is sold as a 500 [W] heater. For our modeling we shall assume this value is correct.

Fig. 1 *Results of revised prototype with the modified "large vents" (6×1 [in²]) between the EC and CC. The experiment began on Aug 20 at 12:15. Left: (**Dark-solid line**) EC temperature as a function of time, from 0 to 1500 [min] (25 [hr].) (**Solid-red dashed line**) CC temperature over the same time period. Note the large difference in rise times between the two experiments. Right: This panel shows the relative humidity [%] for the water in EC (solid black curve), and in the CC (red-dashed curve). Note that the humidity goes above 91-93 [%]. Perhaps this is possible since the CC is at a lower temperature, thus the dew point can have a high relative humidity. I am not positive that this is the correct explanation. At 1080 [min] (18 [hr]) into the experiment the system was reset (chambers were opened) and at 1260 [min] (21 [hr]) the second experiment (Exp-II), described below. The temperature cycles up and down due to the thermostat that is keeping the vapor at 40°C. This data was taken using two GOVee Bluetooth transducers, one in the EC and one in the CC, as shown in the online videos.*

This was fixed by a redesign of the vents, which are now 6x1 [in²]. Follow up experiments indicate that the transport of water vapor is now reasonable (See Fig. 2). This is based on the time for the temperature to reach 40C. This was the first time to have such a large open vent between the EC and CC, and it appears to have solved the problem.

Exp-I: The time vapor diffusion time constant was measured by the experiment of Aug 20 @ 12:15 (Exp-I). The conditions of Exp-I were to leave the power and air pump **off**, and monitor how the temperature and humidity increase with time in the two chambers, due to vapor-diffusion alone (there is no vapor convection since the air-pump is turned off).

These results are shown in Fig. 1: the **black-solid line** rises quickly (20 [min]), while the **bold-red dashed line** rises slowly (150 [min] (2.5 [hr]) before it comes to the 90% the steady state. The abscissa gives the duration in [min] of the experiment. The air-pump is turned on at 1128 [min] (Exp II), the time must be faster due to convection rather (vs. diffusion with the pump off). The time (x axis) is in minutes relative to the start of the experiment. Thus 60 on this scale is 1 [hr].

For the first experiment (first hour) the EC rise time in on the order of 10-20 mins. The on/off cycling of the thermostat is seen as the temperature cycles around the set point of 40 °C. The rise time for the CC is much slower, and has a lower temperature at steady state (≈ 38°C).

Note the delay between the start of the experiment (Aug 20, Exp I) and the time when the vapor reaches 35C (107 [m]), verses the same delay of Aug21 (Exp II) (from the start of the experiment (1128 [min]), i.e., when it reaches 15C at 1367 [m]. This difference of 2.25 [hr] (1367-1128=240 [m] vs. 107 [m]) is because of the ON state of the 10C water.

Very interesting is the first temperature peak 31.8 C (at 1280-1128=152 [m]. This delay in the drop in temperature is likely due to the time for the 10C water to cool the condenser fins. Once cold, the vapor drops to 21.4 C and then starts rising once more. The temperature is not expected to rise unless the water stops flowing, which it did at 9:42. This seems to explain the unexpected rise in vapor temperature in the CC.

It remains important to measure the WC vapor temperature (which we did not do). To do this I purchased a third GROVee sensor. Thus we can repeat this experiment with a full data set.

Estimating the input power: From the shape of the temperature curve it is be possible to determine when when the 500 [W] heater is on or off (it rises when on, and decays when off). These times may be determine exactly by measuring the current in the heater power line, or by monitoring the light on the thermostat control panel, which turns on and off with the power.

Exp-II: Once EC steady state is established, at $t = 97$ (12:46), the air pump was turned on. The air-pump has a capacity of 60 [L/m] (1 L/s), and its output is attached to six *Air-stones*,[7] which "foam" the air into tiny bubbles (I suspect these bubbles are much smaller than 1 [mm] in diameter, As these bubbles reach the surface, the maximum diameter of the largest are > 2 [cm] (shown in video 2).

Around $t = 287$ (time of the last maximum temperature) and T_{ec} goes into free fall. According to my notebook I did not turn off the heater, but I do note that, based on LabView data, $T_{water} = 38.7 > T_{vapor} = 37.7$, which may account for the drop (Does this condition turn off the heat in the thermostat? The thermostat assumes it is the water it is measuring since the heater is in the water. I'm not exactly clear on this point.)

There is an small but obvious drop in the vapor temperature T_{ec} at ≈ 98 [m] in the left panel of Fig. 1. It is convenient to call this Exp. Aug20.2 and Exp. Aug20.1 the data from 0 to 97 [m].

As it turns out when I downloaded the GOVee data, I started the record at 11:46, which was the start time of the Aug2 experiment, not the start time of the Aug 20 experiment, of 12:15. The day (Aug 20) was entered correctly, only the time was off by 31 (or 32) [m]. GOVee returns the data in 1 [m] intervals, even though it takes it more frequently internally. This difference between 11:14 and 12:15 has been corrected, in these calculations and charts.

The condenser fan in the CC was turned on for experiment Aug20 starting at 1229 [m] (relative to 12:15 (0 [m]), and was terminate at 1369 [m].[8]

The notebook entries are in an appendix. In Appendix B we shows how to do a two-port analysis method for thermodynamics, and then compare it to the traditional method of thermodynamic analysis. A simple thermodynamics problem is proposed and solved using two different methods. Their advantages and disadvantages are compared.

6.1.2 Earlier Experiments for Aug 2 (Exp I)

Here we estimate the time constants for the original setup[9], where the vents between the EC and CC were and order of magnitude smaller (< 1.33 vs 6 [in^2]).

Describe how it conserves the latent heat of evaporation (see Fig. 2).
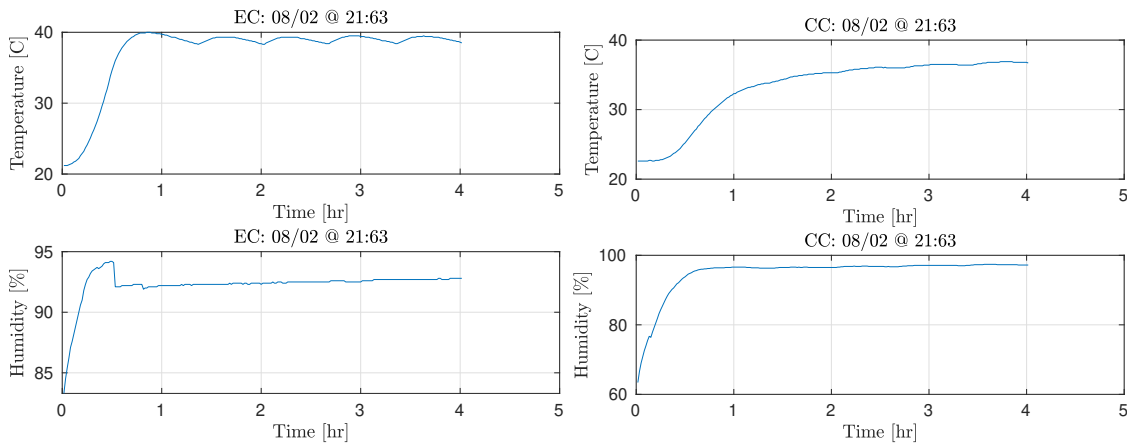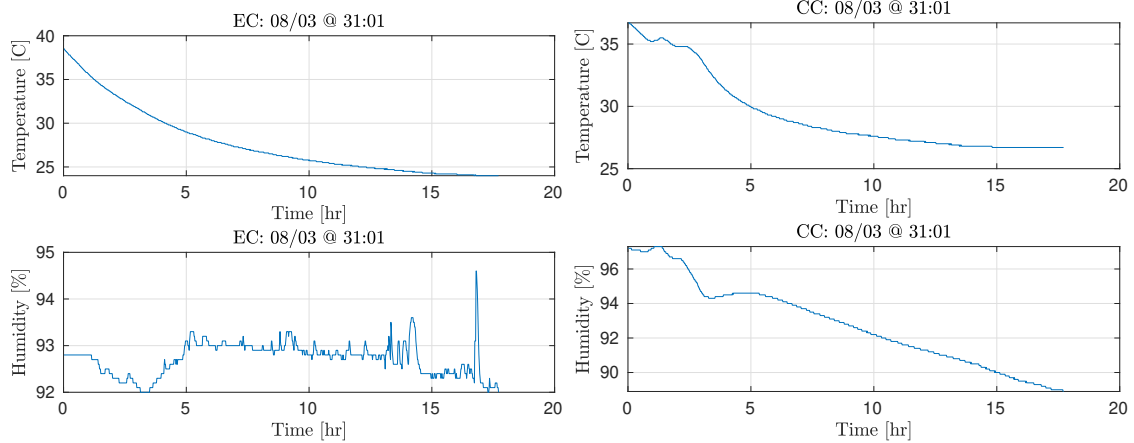

Fig. 2 *Data of Experiment-I collected on Aug 2, 2021 with power on, from 12:30 to 16:30, at which time the 500 [W] power was turned off. Left: Temperature and Humidity in EC starting from a quiescent state of at room temperature (typically 25C), with the heat and air pump on. Right: Same as the left figure but for the CC. At 4 [hr] into the experiment, the heat was switched off, allowing the temperature to decay. The data for this trial is given in Fig. 3. The decay is ≈ 15 [hr] for both the EC and the CC. Clearly this is determined by the EC time constant. Also important are the very large difference in the onset rise time. For the EC this is approximately 20 (1/2 [hr]) to rise from room temperature to the thermostat set point, of 40 [C]. The rise time of the CC is quite different, and is closer to NNN [hr].*

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[7] http://jontalle.web.engr.illinois.edu/MISC/Desalination.20/VideosDesal-Jul26.21/AirPumpInAction.mp4

[8] Thus the footnote above is correct, as assumed, that the experiment started with the fan off.

[9] Note title of graph which gives the date the expiment began. Time 21:63 is when the experiment was terminated.

Fig. 3 *Data of Experiment-I collected on Aug 2, 2021, from 16:30 when the power was turned off until Aug 3 @ 22:50 when the run ended. Left: Temperature and Humidity in EC starting from a quiescent state of at room temperature (typically 25C), with the heat and air pump on. Right: Same as the left figure but for the CC. At 4 [hr] into the experiment, the heat was switched off, allowing the temperature to decay. The decay is XXX [hr] for both the EC and the CC. Clearly this is determined by the EC time constant. Also important are the very large difference in the onset rise time. For the EC this is approximately 20 (1/2 [hr]) to rise from room temperature to the thermostat set point, of 40 [C]. The rise time of the CC is quite different, and is closer to NNN [hr].*
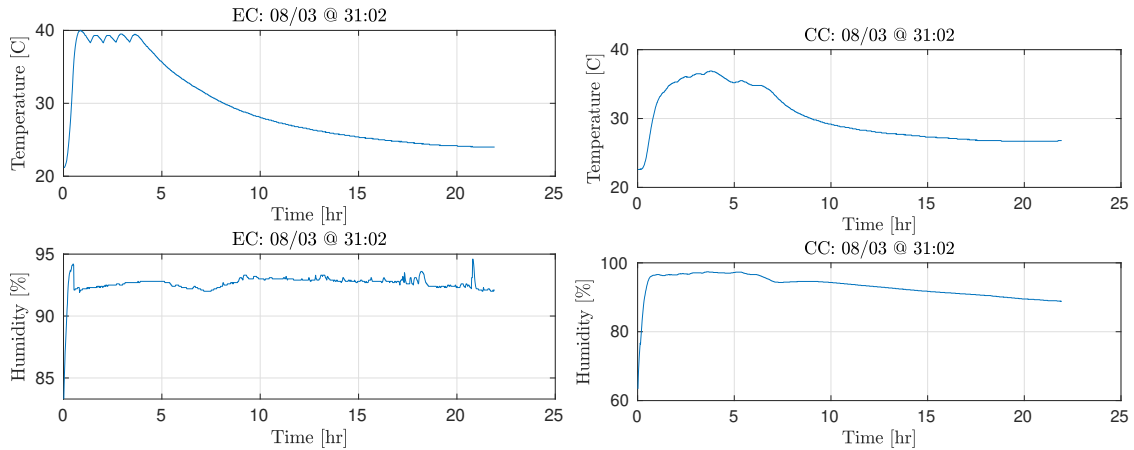
Fig. 4 *Data of Experiment-I collected on Aug 2, 2021, entire run from 12:30 to Aug 3, 10:50. Left: Temperature and Humidity in EC starting from a quiescent state of at room temperature (typically 25C), with the heat and air pump on. Right: Same as the left figure but for the CC. At 4 [hr] into the experiment, the heat was switched off, allowing the temperature to decay. The decay is XXX [hr] for both the EC and the CC. Clearly this is determined by the EC time constant. Also important are the very large difference in the onset rise time. For the EC this is approximately 20 (1/2 [hr]) to rise from room temperature to the thermostat set point, of 40 [C]. The rise time of the CC is quite different, and is closer to NNN [hr].*
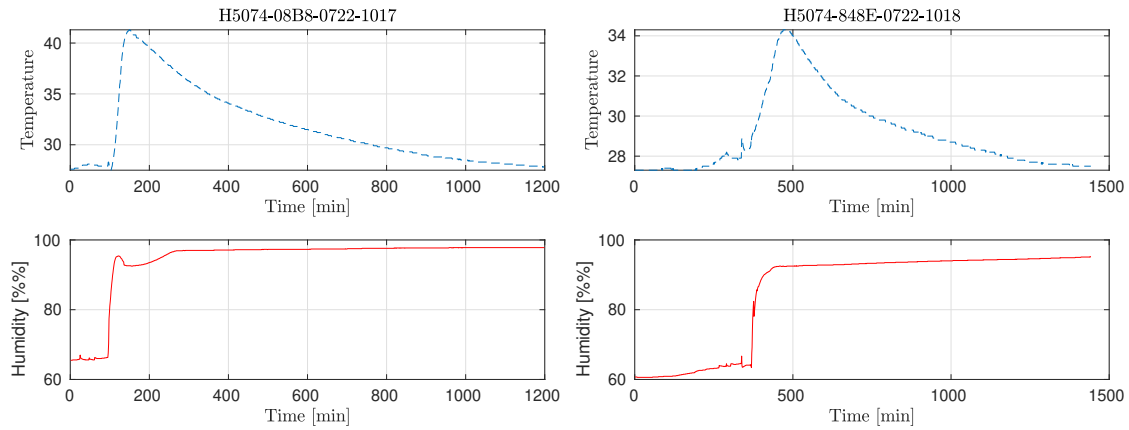
Fig. 5 *Left: Temperature in EC starting from a steady state of 40C, with the heat switched off, and a pump on. Right: Temperature in CC starting from a steady state of 34C for the same time period. Note that the decay is 1000 [min] (16.5 [hr]) for both the EC and the CC. Clearly this is determined by the EC time constant. Also note the large difference in the rise time. For the EC this is approximately 20 [min], to rise from room temperature to the thermostat set point of 40 [C]. The rise time of the CC is quite different, and is closer to 100 [min] (1 [hr], 40 [min]).*
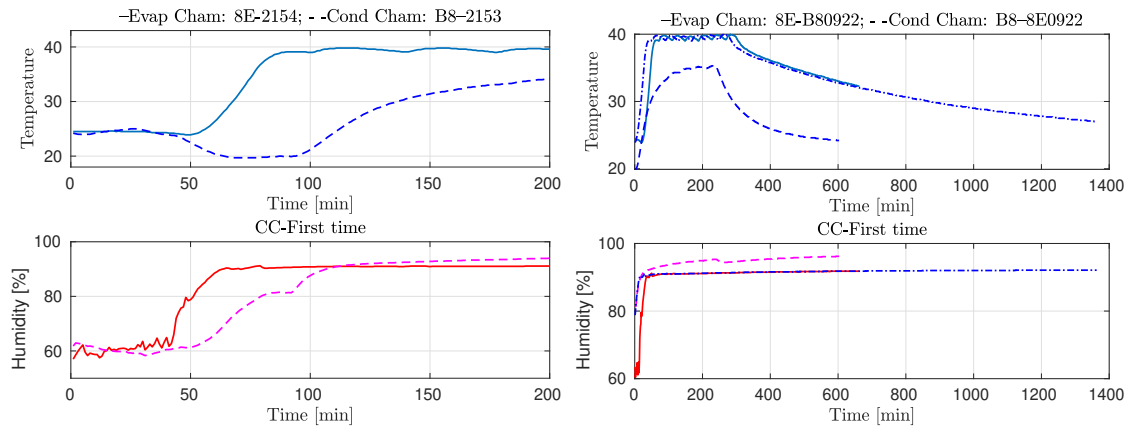
Fig. 6 *Exp-4 (Left): In an attempt to evaluate the present state of the Desal-prototype for the the first time, with the CC installed. For this experiment all systems were on: 500 [W] heater and the 60 [l/m] air pump. The data are from the two GOVee Temperature/Humidity wireless transducers. The experiment began on July 27 at 17:13 and terminated at 21:45. Exp-5 (Right): This shows the results in the two chambers for experimental run July 7 @ 17:22, to track how the two chambers recovered from Exp 4 once the power was turned off (no heat and now air pump). The time constants for the two chambers are quit different. The CC returned to room temperature in 1 [hr] while the EC too much longer. Even after 1300 [m] it still was well above the room temperature. This figure needs more work as the two curves do not appear to be properly aligned (with an error of up to 20 [m]).*
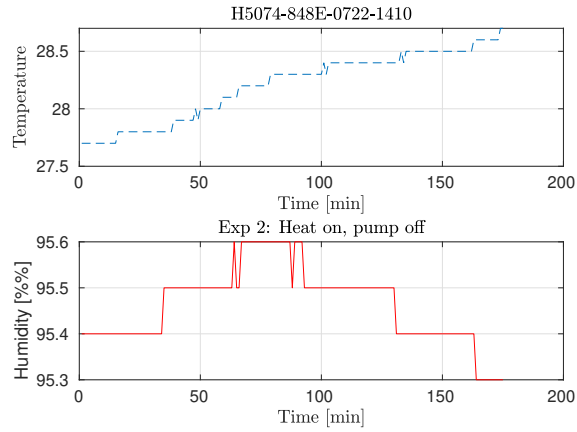
Fig. 7 *In this case we reversed the conditions of Fig. 2 by turning on the heater and off the air pump. The EC vapor was at 40 [C] and 98 [%] humidity while the CC was at 30 [C] and 95.4 [%]. Since the heat was off, the EC temperature decreased from 40 [C]. But since the air pump was on, the CC temperature was still increasing. However the humidity went through an inflection point, first rising and then falling. This difference in humidity was very small, as shown on the lower panel. While the effect is small, it seems interesting and need to be explained (and of course repeated several more times.*

A Entries from my notebook (Aug 20 @ 5:00):

All times are in HR:MIN, during the day. Due to an error in reading the GOVee data, I starting at 11:14. Thus their are 31 extra samples (1/m) until the 12:15 experiment actually started. I corrected this error in the Matlab program (ReadGOVee.m) by starting on data sample 31 (@ 12:15). The times in the graphs are in minutes starting from 12:16 (sample 31).   5

5:AM Returned to the lab to fix the blocked 10C water flow. Download GOVEE data @5:03.
9:49 GOVEE download
12:15 Start Exp Aug20.1
12:39 EC Vapor has saturated at between 39 and 40 C. $T_{water} \approx 35C$.
12:46 Air pump turned on, water still off. Not sure why T_{water} (LabView data) slightly increased. This could possibly   10
      be due to the colder (room temperature air) injected by the air pump.
12:50 EC $T_{water} > T_{vapor}$.
12:56 GOVee temperature shows changes. Is this due to the much larger EC to CC vent size?
1:01 EC T vapor $\approx$ water (LabView data). CC T still slowly rising. Note: Water (still) off.
1:10 Upload data from GOVee   15
1:14 Turn on condenser fans (Hum, I thought I turned them on earlier, but maybe not? Else why this note. The
      large Labview T just is due to my moving the Thermocouple in the CC air vent, causing Thermocouple #5 to
      jump to 31°C.
1:17 Left the lab.
2:50 Ret to lab. Turn off air pump.   20
3:20 Turn air pump on.
4:00 LabView crashes when I try to copy log file using cut/past in Windows. The error message said the file was not
      available for writing.
4:05 I held Thermocouple 5 in my fingers for a few seconds, just for fun, to prove it was really #5.
4:30 Left system on with heat off, and went home for the night (for a very long 3 hr nap).   25

More event times from my notebook: These are expressed in minutes as taken from the graphical output
of Matlab, from the GOVee data charts (Exact time in [m] from the GOVee data of Fig. 1).

0 min The heat and air pump are on, but the 500 [W] heater and the cold 10°C water is off.[10]
29 Labview is turned on 29 mins after the experiment began. Thus it failed to capture the rise in EC temperature.
    It will be necessary to repeat this experiment to further verify the conclusions. These data are more difficult   30
    to analyze due to the way they were stored by LabView (csv format), but will be added once this problem is
    resolved. I wrote down the 5 temperatures once LabView started. They are [40, 40, 19.6, 19.07, 21.98] °C (see
    p. 113 of notes).
53 T_{ec} reaches its saturation value of 41-42 °C. As described elsewhere the thermometer that drives the thermostatic
    control of the heater (ON/OFF) is in the vapor rather than in the water, thus the water temperature must be   35
    higher, which from the LabView experiments was between 41 to 42 °C).
150 T_{cc} reaches its equilibrium temperature of 37-38 °C, and stays within that range until the power is cut (other
    conditions remain the same).
≈320 The power is cut, in order to study the recover times of the temperature and relative humidity of the vapor.
    At this time both the EC and CC temperatures begin to fall at the same rate, with a $\tau = 800 - 320$ where   40
    $T_{ec} = (40-25)$ where τ is the time constant defined by $(t/\tau) + 25 = 30$. This describes the decay in temperature
    starting from when the power is cut (at 320 [s]) to the time it reaches 30 °C (at 800 [s]). It makes sense, given
    the large vents between EC and CC (6 in^2), that the two time constants are identical. In our experiments earl
    Aug, with 3 3/8" vents, the time constants for the two chambers were orders of magnitude different. Of course
    this evidence from early Aug led to the modifications of the present design.   45
1,140 Experiment-I was terminated and the two chambers were opened to allow them to return to equilibrium.

Exp of Aug 21 @ 8:09: I returned Saturday to evaluate the over-night run and to do follow up experiments.
      Exp-Aug21.1: The goal of this experiment is to see the effect of turning on the 10C water, which till now has
been off. Conditions: Air on, Heat on, 10C off. A second goal is to determine how long it takes for the EC to fill up
with water, at which point I want to show that the 10C is automatically turned of by Switch 2 (Chamber @ 14.5   50
[L]).
      LabView was also being used and the Thermocouple numbers are 0=EC water; 1 EC=Vapor; 2 WC air tem-
perature; 3 30C out; 4 WC Vent temperature.
      Finally note that I added a 1" styrofoam wall (removable) to the outside of EC. This was not in place for Aug
20 experiments. It can be removed to observe the depth of the water in the EC.   55

8:24 Labview temperatures [24.6, 23.5, 23.2, 23.6, 23.55] (All around room temperature).
8-1 Labview temps measured at intervals.
9:04 Turn on 10°C water. Moved LabView Channel 4 probe to input water valve (12.67 C)
9:42 Water turned off by Switch 2, due to EC water level at "FULL" capacity (14 [L]).
9:50 Turn off heat; leave air on. Leave for my office.   60
12:56 Ret to shut all systems shut down. Take GOVee units home, in case I wish to probe them more.

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[10] The CC-fan is off (Ck the notebook). It makes no sense to have the CC-fan on with the air pump off. This is
analogous to short-circuiting a battery.

A.1 Black-body radiation power input

The sun has a surface black-body temperature of approximately 7,000 [°C] (i.e., 7 [k°]). According to the Stefann-Boltzmann law, the power radiated by a black body is σT^4 [J], where the constant $\sigma = 5.67 \times 10^{-8}$ [W/m^2 K^4]. Thus at the surface of the sun the power per m^2 is

$$P_S = 5.6710^{-8} \cdot 7000^4 \text{ [W/m}^2\text{]},$$

or about 4 [mW/m^2]. This does not account for the surface of the sun shining on the earth. In terms of the power that is collected at the surface of the earth, using the average temperature of the earth of 25 [°C/m^2], we find

$$P_E = 5.6710^{-8} \cdot 25 + 273^4 \text{ [W/m}^2\text{]},$$

or 0.12 [μW/m^2].[11] The *solar constant* is the power delivered to earth by the sun per unit area, and that number is $\approx$ 1.366 [kW/m^2] (Ambaum, 2010, p.170).

If we account for the power lost to the atmosphere, a useful, and easy to remember estimate, but not very accurate, is 1 [kW/m^2].[12]

B Thermodynamic modeling

This section demonstrates that the two-port transmission-matrix analysis method may be used for thermodynamics calculations, and compares it to classic energy methods of thermodynamic analysis. A simple thermodynamics problem is proposed and solved using the two different methods, and their advantages and disadvantages are compared. We conclude that using using impedance methods linearizes thermodynamic energy relations, making linear algebra methods an applicable solution method.

C Introduction

The purpose of this correspondence is to investigate the possibility and utility of using the 2-port transmission matrix method to analyze thermodynamic problems, commonly used in modeling electrical and mechanical systems, such as LRC electrical circuits and spring-mass-damper systems (Allen, 2020).

Specifically, this report will investigate the use of the Laplace frequency domain to model thermodynamic systems, and to draw connections with components of electrical, mechanical and thermodynamic analysis, using the two-port transmission line methods.

Traditionally, thermodynamics is analyzed in the time domain using energy relationships (Ambaum, 2010). Energy relationships are nonlinear in the *conjugate variables*, the *product* of which define the power (energy-rate). For example, voltage times current (coulomb/sec) or temperature times entropy-rate are each a power, having units of watts (Allen, 2020, Appendix I). Unlike thermodynamics, which is formulated in terms of energy, electrical and mechanical circuits use impedance, defined as the *ratio of conjugate variables* (e.g., $Z(s) = $ `voltage`/`current`) when modeling electrical circuits, or $Z(s) = $ `force`/`velocity` for mechanical systems.

The definition of an impedance $Z(s)$ utilizes the Laplace frequency s. The Laplace transform replaces calculus with algebra in the Laplace frequency variable ($s = \sigma + \jmath\omega$). This is primarily because electrical and mechanical circuits are second order (or higher) systems, that benefit greatly from this type of analysis. Presently thermodynamics is modeled using only RC circuits (first order system).

C.1 Problem Statement

To show how the two-port transmission line analysis works with Thermodynamics, a simple and classic thermodynamic problem is proposed and solved, using the classic method (Ambaum, 2010), followed by a two-port analysis.
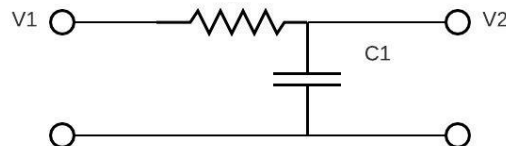
Fig. 8 *The two-port matrix representation of an RC Circuit*

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[11] See Fig. (9.7), p. 179 of (Ambaum, 2010) for the details.

[12] A more reasonable number, called the *insolation*, is 0.342 [kW/m^2] (Ambaum, 2010, p. 171). Note this is only valid during that part of the day when the sun is shining. So there are many confusing caveats that must be taken into consideration. It is important to not over-estimate the effective value for back of the envelope calculations, or in computer models, where we trying to match experimental results (Ambaum, 2010, §9.3).

The example electrical (i.e., initial) problem is shown in Fig. 8. The thermodynamic version will include the heat generated in the resistor R_1, due to electrical current, causing it to produce heat energy.

Assume this RC circuit, with resistance (R_1), placed in an incompressible fluid (e.g., water) having mass (m_w) and specific heat capacity under constant pressure (c_p), which is otherwise isolated from the environment. Let the source voltage and current be $[V_1, I_1]$. After the RC circuit has been turned on and has reached equilibrium, we study the change in temperature $T(t)$ of the fluid as a function of time t. Stated another way, what is the time response of temperature of the fluid, as the capacitor is charging? Finally, what is the impact of the power lost to heating the water around the resistor, on the charging of the capacitor? The final voltage on C_1 will be different due to the power lost to the water.

C.2 Classic Solution

To determine the energy dissipated by the resistor into the fluid, the current passing through the resistor must be determined. To find this current the RC circuit may be analyzed as a two port transmission line. Figure 8 can then be analyzed using a 2x2 representation matrix relation

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix},$$

where the currents are defined into the ports and the voltages across the ports (Allen, 2020). This may be found by collapsing the matrix product,

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 + sRC & R_1 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$

which then provides the relations between the input and outputs

$$V_1 = 1 + sR_1C_1V_2 - RI_2$$
$$I_1 = sC_1V_2 - I_2.$$

Setting $I_2 = 0$ and combining the two equations, we find

$$V_2 = \frac{V_1}{1 + sR_1C_1}$$
$$I_1 = sC_1V_2,$$

or

$$I_1 = \frac{C_sV_1}{1 + sR_1C_1}$$
$$= C_1V_1\frac{s}{1 + sR_1C_1}$$
$$= \frac{V_1}{R_1}\frac{s}{s + 1/R_1C_1}. \tag{1}$$

From the initial condition ($t = 0$)

$$V_1(t) = V_0u(t) \leftrightarrow V_0/s. \tag{2}$$

Substituting $V_1(0)$ in into Eq. 1 gives

$$I_1 = \frac{V_0}{R_1}\frac{1}{s + 1/R_1C_1}.$$

This equation is in the Laplace frequency domain.

In order to convert this equation back to the time domain, the inverse Laplace transform must be taken, giving

$$I_1(s) = \frac{V_0}{R_1}\frac{1}{s + 1/R_1C_1} \quad \leftrightarrow \quad i_1(t) = \frac{V_0}{R_1}e^{-t/R_1C_1} \tag{3}$$
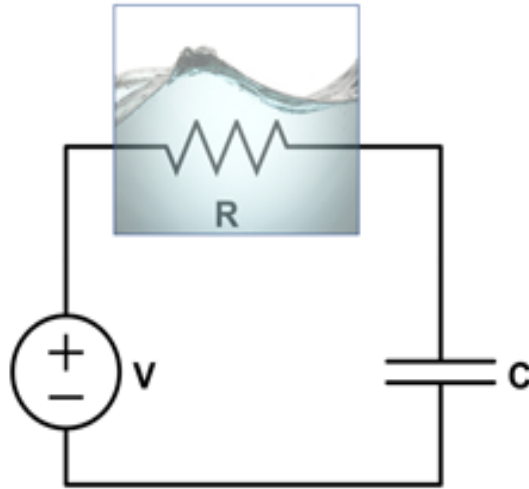
$$= I_0e^{-t/\tau}, \tag{4}$$

where $\tau = R_1C_1$ and $I_0 = \frac{V_0}{R_1}$.

C.3 Thermodynamic relations

The power dissipated by the resistor at time t is

$$P(t) = V_1(t)I_1(t) = I_0(t)^2R_1 = I_0^2e^{-2t/\tau}R_1.$$

Fig. 9 *Equivalent RC circuit of Fig. 8, including the thermal losses in the resistor immersed in a water bath.*

Table 2 *Table of parameters for the circuit of Fig. 9.*

Parameters	symbol	Value	Units
Voltage	V_o	10	[V]
Resistance	R_1	10	[Ω]
Capacitance	C_1	1	[F]
Mass of water	m_f	1	[g]
Specific Heat Capacity	c_f	4186	[$\frac{J}{kg \cdot K}$]

The total energy dissipated in the resistor is the time integral of $P(t)$

$$Q(t)i = \int_0^t P(t)dt$$
$$= I_0^2 R_1 \int_0^t e^{-2t/RC} dt$$
$$= I_0^2 R_1 \frac{\tau}{2} \left(1 - e^{-2t/\tau} \right).$$

Assuming all the energy dissipated by the resistor is absorbed by the fluid, the relationship between the energy absorbed by the fluid and the change in temperature (Ambaum, 2010)

$$Q = m_f c_p \Delta T.$$

Rearranging and substituting

$$\Delta T(t) = \frac{Q(t)}{m_f c_p} = \frac{I_0^2 R_1 \frac{\tau}{2}(1 - e^{-2t/\tau})}{m_f c_p} \tag{5}$$
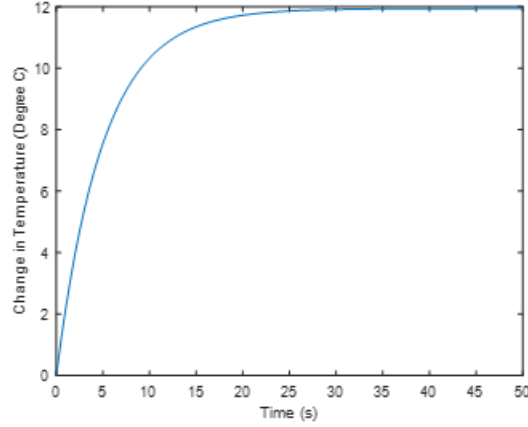
Using the values the constants as given in Table 2, we find

$$\tau = RC = 10 \cdot 1 = 10 \ \texttt{[sec]}$$

$$I_0 = \frac{V_0}{R_1} = \frac{10}{10} = 1 \ \texttt{[Amp]}$$

$$\Delta T(t) = \frac{I_0^2 R_1 \frac{\tau}{2}(1 - e^{-2t/\tau})}{m_f c_f}$$
$$= \frac{1^2 \cdot 10 \cdot \frac{10}{2} \cdot (1 - e^{-2t/10})}{0.001 \cdot 4186} \ \texttt{[°C]}, \tag{6}$$

as shown in Fig. 10, where we visualize $\Delta T(t)$.

Fig. 10 *Time response of the temperature $\Delta T(t)$ of the water.*

C.4 Discussion

While the classic analysis provides the current in the resistor, allowing us to calculate the power and total energy dissipated in the resistor as a function of time, it is not actually correct, since the energy absorbed by the water will change the energy balance relations. Thus the classic $I_1(t)$ is not the true current. To obtain the correct answer, we must include the energy dissipated in the water. This requires a thermodynamic calculation, which we shall provide in §D.

D Two-Port Analysis Method

The system including the heat lost can also be modeled as a two-port transmission line, with a resistor, an ideal transformer and two capacitors, as shown in Fig. 11.
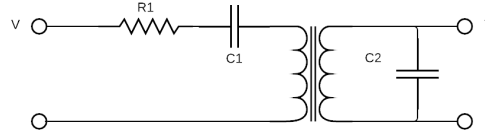
Fig. 11 *Two-port model including the iso-baric heat lost to resistor R_1 in the water bath. The turns ratio of the transformer (a) relates the voltage and current to the temperature and entropy–rate. For example $T = V/a$ and $\dot{S} = aI$. Thus the units on a are either [V/°C] or [entropy-rate/A].*

Evaluating the transmission matrix of Fig.11 gives

$$\mathcal{T}(s) = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_2 & 1 \end{bmatrix},$$

where

$$\begin{bmatrix} V(s) \\ I(s) \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} T(s) \\ -\dot{S} \end{bmatrix} = \mathcal{T}(s) \begin{bmatrix} T(s) \\ -\dot{S}(s) \end{bmatrix}.$$

Thus

$$\begin{bmatrix} V \\ I \end{bmatrix}(s) = \frac{1}{a} \begin{bmatrix} \frac{s + a^2 C_1 + C_2}{C_1} & \frac{sR_1 C_1 + 1}{sC_1} \\ sC_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ -\dot{S} \end{bmatrix}(s).$$

Since the system is isolated from the environment is adiabatic, the entropy flux out of the water (i.e., heat flow $\dot{S}$) is zero. This allows us to find the relationship between the input voltage and the temperature:

$$V(s) = \left(\frac{C_1 C_2 R_1 s + a^2 C_1 + C_2}{aC_1} \right) T(s).$$

Assuming that $V(s)$ is a unit step function (see Eq. 2),

$$T(s) = \frac{V_o}{s} \frac{aC_1}{C_1 C_2 R_1 s + a^2 C_1 + C_2}$$
$$= \frac{V_o}{s} \frac{a}{C_2 R_1 s + a^2 + C_2/C_1}.$$

Expressing this in pole-residue form (Allen, 2020)

$$T(s) = \frac{V_o a}{s} \frac{1}{C_2 R_1 s + a^2 + C_2/C_1}$$
$$= \frac{V_o a}{C_2 R_1} \frac{1}{s} \frac{1}{s + \frac{a^2 + C_2/C_1}{C_2 R_1}},$$

the inverse Laplace is then

$$T(s) \leftrightarrow T(t) = \frac{V_o a}{C_2 R_1} \int_0^t e^{\frac{-t(a^2 + C_2/C_1)}{C_2 R_1}} dt.$$

In this case we can define $\tau_2 = C_2 R_1/(a^2 + C_2/C_1)$. Evaluating the integral gives

$$T(t) = \frac{V_o a}{C_2 R_1} \tau_2 \left(e^{-t/\tau_2} \right) + T_o.$$

5   Since the temperature rise ΔT is of interest, the boundary condition is $T(t = 0) \equiv T_o = 0$. This can be seen in the following equation.

$$\Delta T(t) = \frac{V_o a}{C_2 R_1} \tau_2 e^{-t/\tau_2} + \frac{V_o a}{C_2 R_1} \tau_2,$$

or

$$\Delta T(t) = \frac{V_o a}{C_2 R_1} \tau_2 \left(1 - e^{-t/\tau_2} \right). \tag{7}$$

10   ## D.1 Discussion

Note that this is now of the same form as Eq. 5 (τ_2 is quite different), and is identical if we set $C_2 = 0$.

     Thus it is a matter of determining the value of a, C_1, and $C2$. Assuming that C_1 stays the same for the two solutions, a and C_2 can be determined by renormalizing the two solutions to have the same functional form. Given

$$T(t) = \frac{V_o a}{C_2 R_1} \tau_2 \left(1 - e^{\frac{-t}{\tau_2}} \right) = \frac{I_0^2 R_1 \frac{\tau_1}{2}}{m_f c_f} (1 - e^{-2t/\tau_1}), \tag{8}$$

15   where

$$\tau_1 = \tau = RC_1 \tag{9}$$

then if we reapply the definition of $\tau_2 = C_2 R_1/(a^2 + C_2/C_1)$

$$\frac{V_o a}{C_2 R_1} \tau_2 = \frac{V_o a}{C_2 R_1} \left(\frac{C_2 R_1}{a^2 + C_2/C_1} \right) = \frac{V_o a}{a^2 + C_2/C_1} \tag{10}$$

and if we equate the linear constants on both sides

20

$$\frac{V_o a}{a^2 + C_2/C_1} = \frac{I_0^2 R_1 \frac{\tau}{2}}{m_f c_f} \tag{11}$$

and if we equate the exponent

$$\frac{-1}{\tau_2} = \frac{-2}{\tau_1} \Rightarrow \frac{-(a^2 + C_2/C_1)}{C_2 R_1} = \frac{-2}{R_1 C_1}. \tag{12}$$

we have a linear set of two equations and two unknowns.

$$a^2 R_1 C_1 + R_1 C_2 = 2 R_1 C_2 \Rightarrow C_2 = a^2 C_1$$

25   Substituting Eq. 12 back into Eq. 11 gives

$$\frac{V_o a}{a^2 + (a^2 C_1)/C_1} = \frac{I_0^2 R_1 \frac{R_1 C_1}{2}}{m_f c_f}.$$

Simplifying

$$\frac{V_o a}{2a^2} = \frac{V_o^2 C_1}{2 m_f c_f}$$

$$a = \frac{m_f c_f}{V_o C_1}$$

Substituting back into Eq. 12 and solving for C_2

$$C_2 = \frac{m_f^2 c_f^2}{C_1} \tag{13}$$

Plotting Eq. 7 and comparing to Eq. 6, we see that the solution has the same functional form, but is numerically distinct, due to the added heat loss into the water, thus accounting for this important missing term in the classic solution. They are identical when $C_2 = 0$, thus decoupling the entropy-rate (heat loss) and the electrical current and voltage.
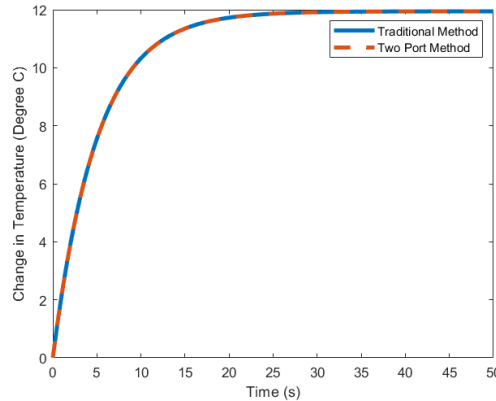
Fig. 12 *Time response of the water for both methods.*

E Conclusion

From the above demonstration of both methods, the advantages and disadvantages of both the classical method to thermodynamics become transparent. For trivial thermodynamics problems, such as the one demonstrated above, it is often easier to use the classic method of power and energy conversions. However this ignores the heat lost to the resistor during the charging of the capacitor.

The classical method lends itself to a more instinctual understanding of the problem, as most of the problem is solved in the time domain. However, the two-port representation naturally includes the heat lost to the water, and is an algorithmic approach to solving such problems. As an interesting example, consider the case where C_1 is replaced by an inductor. In this case the circuit's resonant frequency is dramatically reduced (becomes finite) by adding the heat capacity of the water.

The transmission matrix method lends itself to much more complex versions of the thermodynamic problem, where, for example, the voltage applied is not be a simple unit step function. This method would also be more useful in creating simulated environment algorithms that are more accurate and efficient compared to methods that are based around time integration such as modeling more complex thermodynamic phenomenons such as triple point and super cooling. By understanding this analysis and being able to apply this method to thermodynamics, it may open up new insights into the discipline of thermodynamics.

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