ECE-298-JA: Advanced Mathematical Engineering: Fall 2015

Instructors: Jont Allen, Steve Levinson, John D'Angelo (Math) and others from Physics

Course Coordinator: Jont Allen

Prerequisites: AP in Calc I, II and/or III, Diff Eq, Linear Algebra

Target Audience: Sophomores (and some precocious freshmores) in Engineering, Physics and Mathematics

Text: Stillwell, John, (2002), *Mathematics and its History,* 3^d Ed.; Boas, R., (2009), *Invitation to complex analysis.*

Outline: A significant fraction of Engineering students come to Illinois and place out of the mathematics courses. Having done this they believe they are ready for their physics and engineering class. In fact, they are not prepared. This math course will fully orient them in mathematical physics and its scientific applications, via the 2000+ year history of math. An historical presentation of the material makes it accessible to almost everyone.

Existing math requirements, as taught in the standard Freshman and Sophomore math classes, fall short of providing a comprehensive understand of classical Engineering-Physics mathematics. It is a common observation that many students that AP may have mastered the mechanics, but not the concepts or context. This math course fills these gaps.

ECE attempted to address this problem with ECE-493/MATH-437. While highly successful, it is offered too late to provide the basics required to deal with ECE-210, ECE-310, ECE-329 and ECE-340 & Physics 211-214, etc.

This 298 course covers math in a different way, via history, starting with number systems, complex analysis and ending with the review of systems of differential equations (e.g., Kirchhoff's laws; Maxwell's Equations).

This proposal is a way of introducing multi-variate complex variable analysis (i.e., linear algebra with simple Euclidean and non-Euclidean geometry) to high-school students, in preparation for the UIUC Engineering curriculum.

Course outline by topic:[1]

W	L	c.	Description
			Part I: Number systems
1	3	(50)	The discovery of Number systems
			Introduction: integers, rationals, irrationals, real, complex, vectors, (1 vs. 2 sided)
			Why do we need vector functions of complex numbers in Engineering and Physics?
			Complex-analytic series representation: Why is convergence necessary?
			Sets and Fourier-like (i.e., z, Laplace, DFT, etc) Transforms;
2	1		Aristotle, Pythagoras' genius (& some $\sqrt{2}$ nonsense)
			Role of Music and acoustics to mathematics
	1		Ruler and Compass constructions, Conic sections
	1		On the way to analytic Geometry: Geometry and "roots"
3	1		Greek number theory: Euclid
	1		Pell's Equation; Diophantus (600 BCE)
			Fall of Rome and Rise of Islam (p 49)
	1		Eudoxus Theory of Proportions (p 56)
			Role of the integer in early Mathematics [Dedekind cut (p. 57)]
			Method of Exhaustion; Archimedes (200 BCE)
4	1		Number theory in Asia: Euclidean Algor and GED (p. 66)
			Chinese Remainder Theorem (300 BCE)
	1		Asian Bell's Eq (p. 72)
	1		Ch. 6: Polynomials (p. 82) and "algebra" or al-jabr

[1] W: Week; L: Lecture; c.: century (BCE), CE; Page numbers are for Stillwell 2^d edition.

W	L	c.	Description
			Part II: Systems of equations (Functions, their inverse & roots)
5	1		Theory of "linear equations" (Descartes) (p84)
	1		Quadratic Equation: why are there still no complex numbers?
			2000 BCE (no negative numbers allowed by Babylonians, Euclid and al-Khwarizmi!)
	1		Why Math is so difficult without 0, reals, complex numbers, & ∞?
			Why Math is so simple with complex numbers
			The importance of ∞ (Riemann's extended plane)
6	1	16	Solution of the cubic (c1545) p. 91
			Angle division: de Moivre's formula vs. Euler's equation
	1		Fourth and higher order equations (p. 96); Tartaglia et al.. misguided cannon balls.
	1	17	Analytic Geometry; Fermat and Descartes
7	1	17	Bezout's Thm; Descartes great discovery (p. 113): $[p_n(x,y), q_m(x,y)] = 0 \rightarrow r_{nm}(x) = 0$
	1	17	Newton's methods (irrational power series), *sans complex numbers*
	1	17	The wave equation and Newton; d'Alembert, Descartes
8	1		Geometry via prospective (3D) drawing;points at infinity;
	1		Cross-ratio; Homogeneous Coordinates (p. 134)
	1		Möbius and Genus in mathematics
9	1		Fundamental Thms of mathematics (Ch 9):
		17	Calculus (p. 146); Newton & Leibniz
			series (zeros), partial fractions (poles), products,
	1		implicit differentiation (p 151), rational fractions (p. 154),
	1		inverse of analytic functions;
10	2		Infinite power Series and analytic function theory (p 171) as an extension of the polynomial;
	1		The "complexity" of convergence: case of a root at $\sqrt{-1}$
			Part III: Analysis of Systems (i.e., Differential Equations)
11	1	17	Introduction to the Bernoulli family: Math moves to Switzerland.
		1	Generating functions and the Z transform. The introduction of *delay* into mathematics.
			Continued fractions [1+$\sqrt{5}$(5)]/2 = 1+1/1+1/1+/1... (p. 183)
	1	18	The role of the Zeta Function; Euler's formula (c 1748)
12	1	18	Euler, the most of everything: The start of modern mathematics?
	1		Convolution and the multiplication of polynomials
			Generalizations of Pascal's triangle (bias coin tossing)
	1	17	Fermat's last thm (for fun)
13	1	19	Complex analytic function; Genus 1 (p. 343, i.e., coffee time)
	1		Multi-valued "functions" (and their many inverses!)
	1		Riemann extended plane and the regular point at infinity
14	1		Elliptic functions
			The Möbius transformation and transmission lines
		19	Möbius composition and non-commuting operators
	1	19	Maxwell's equations, Einstein and causality
			The quasi-static approximation and Quantum Mechanics
	1		Recap of the Fundamental Thms of Mathematics & their application
			Final Exam

Summary of the organization of the proposed math course: In the present engineering curriculum mathematics is broken down into finer and finer subjects, Calculus I, II, III, differential equations, real analysis, linear algebra, complex analysis, etc. One problem with this approach is that the student forgets one topic by the time they learn the next. There is no *big picture*. In the ECE curriculum we have the introductory ECE110, that attempts to provide an overview of all of engineering. This is considered to be a useful approach for the introduction of engineering. The students come out of this course with some sort of *big picture.* A similar approach could be useful for the core subject of mathematics. This is the topic of this experimental course, which is an attempt to provide a big mathematics picture, in one 3 hr course.

The proposal is broken down into three chronological parts, based on the history of mathematics, as presented in the primary text (Stillwell, 2002).

Year	Culture	Number System
5000 BCE	Chinese	
500 BCE	Greeks	integers, rationals, fear of ∞ (Zeno's argument)
0	Roman	no 0
500-1000?	Arabic	0 and negative integers
1400	Newton	series of real numbers (complex invalid)
1500		
1600		Fermat; The Bernoulli family
1700		Euler, Fourier, Laplace (Transforms vs. Series)
1800		Riemann sphere (extended plane)
1900		Hilbert space; Fundamental Thms Mathematics:
		Arith, algebra, calculus, vector calculus, etc.
		The importance of the extended plane, i.e., including the point at ∞

Table 1: The evolution of number systems.

I) Number systems: Starting at least as early as the 50 century (BCE), the solution to the quadratic equation was discovered. This was an auspicious start as numbers had yet to be explored. As of Roman times (500-0 BCE) number systems consisted of positive integers. Even 0 was not counted. For example, the Roman positive integer number system (e.g., I, II, III, IV, V, VI, etc.) contains no symbol for 0.

Geometry, on the other hand, was well developed (p. 3), as well documented by the Greeks (Diophantus, 600 BEC; Pythagoras, 500BEC, Archimedes and Euclid 300 BEC), and before that, the Babylonians (1,300 BEC). This early Greek mathematics was dominated by Euclid's *Elements* $\Sigma\tau o\iota\chi\epsilon\tilde{\iota}\alpha$ (i.e., Stoicheia), a set of 13 texts.[2] Even Abraham Lincoln studied these, by the light of his fire, while Newton hated them! Euler's teaching turn out to be relevant today, but more in terms of the algebra of complex variables and linear systems of equations (see section II).

During the so-called "dark ages" in the west ($\approx$ 500-1300 CE, the east was developing algebra (Brahmagupta, al-Khwārizmī, 830 CE, p. 82). The extent of this is best understood through the work of Newton, which may be some of the best documented history, only because of his work *Principia, 1687, p. 234*, which was the first to analyze gravity, sound and light, along with his investigations into the Calculus. Newton certainly did not work in a vacuum, as is well documented by Stillwell.

Arguably Principia was the beginning of engineering (not mathematics). The work of many investigators are documented during this critical period. More important perhaps are those that proceeded Newton, such as Fermat (1629), Huygens (1660). But others following Newton were just as important (e.g., d'Alembert (1751)).

Our students need to know these names, and understand their contributions, in greater detail than a footnote.

The theme in mathematics that really did *not* have much success before the 16 century was the extraction of the roots of polynomials. While the quadratic was solved by the Babylonians (2000 BEC) and perhaps even before, by the Chinese, the cubic was not cracked until the 16c, by Tartaglia in 1535, Feb. 12 (p. 92), as discussed Cardano. The quartic was reduced to a cubic by Cardano in 1545. The quintic was final proven to not have a solution by Able in 1826 (p. 96). In all of this analysis complex numbers (i.e., the roots) were not accepted as having any meaning, and were simply ignored. Finding the roots of polynomials "became the major goal of algebra for the next 250 years." (Stillwell, p. 95)

This is an amazing fact, that until the cubic equation was solved, complex numbers were not considered to be proper roots of equations. In my view, this limitation on number system was the source of much of the problem in both mathematics and engineering.

[2] `http://en.wikipedia.org/wiki/Euclid's_Elements`

Starting from the very first concepts of non-geometrical mathematics (i.e., Euclid excluded), numbers were the limitation. If one is limited to positive integers (1, 2, 3, ..., N) excluding ∞, life is pretty limited. Including zero seems to have been invented in East. Negative integers were soon to follow. While rational numbers were accepted by the Greeks (Pythagoras's system was based only on rational numbers). Irrational numbers were late in coming, maybe as late as the 17 c. Newton ignored complex roots.

For example, the first use of complex numbers to represent an impedance was not until 1893, by Arthur E. Kennelly, who worked with T.A. Edison and moved on to became a professor at Harvard (1902-1930) and MIT (1913-24).[3]

Limits and density of prime numbers, were a distance future, introduced by Gauss in the 18c. While number theory (manipulation of whole numbers) was appreciated by the Chinese, it took centuries to appreciate the real line. Euclid, with his geometrical methods, did not make these artificial distinctions. However, Euclid did not realize the complex numbers, first predicted by the roots of the quadratic equation, but not fully accepted until the roots of the cubic were discovered.

What is simply amazing, to me, is that without complex numbers, the fundamental theorem of algebra (Bezout's Thm) could not be "discovered," since they simply did not accept the counting of complex roots of a polynomial (p. 266). If you refuse to recognize the reality of a complex root, you cannot "count" it. This realization seems to fully first appreciated with the discoveries of short lived Bernhard Riemann (1826-1866), dying of TB at the age of 40 (p. 288).

In studying numbers, set theory is key. For example. the positive integers form an important set, which form the domain of the Z transform. Other sets we will consider are the set of all integers (Fourier Series), the real line (Fourier Transforms), the positive line (Laplace Transforms).

Invariant transformations of sets lead to groups. Important but trivial examples of groups are time invariance and periodic functions. We shall discuss the frequency domain representations in terms of such groups, to better understand Fourier-like transforms, which are critical to modern engineering.

II. Systems of equations: Riemann's introduction of the mapping from the plane to the sphere, also known as the *extended plane*,[4] was the missing step that addressed the fundamental problem of complex calculus. As I understand the argument, it is the regularization of the point at ∞ that fixed the calculation problem. While the complex plane is *open* at ∞, the extended plane is *closed*. The closing at ∞ (on the extended plane) makes that point *analytic* (unless, of course, there is a pole there). In other words, it is the extension of the complex plane that makes the point at ∞ like every other point.

However it seems that Cauchy, who was more formally trained, was the first to fully capitalize on this idea (p. 312). It seems that while Riemann had the insight, he was not successful in popularizing the concept. Rather it was Cauchy that got much of the credit for the understanding of the larger picture, including Riemann's insight.[5]

Fermat and Descartes (16c) were the first players in this field, using algebra to quantify Euclid's geometry, in terms of analysis, as first developed in the east during the dark-ages by al-Khwarizmi.[6]

Descartes (16c) (p. 113) was the first to articulate that the polynomials of order p and q, solve for a single equation, gave a polynomial of order $p + q$. This seems to have been the beginnings of the *Fundamental theorem of algebra*.[7] N linear equations lead to an N order equation. This theorem relates linear algebra to polynomials, with a simple construction. When the roots of the N linear equations are complex, then the linear equations take on the most general form. This insight needs to be taught to our students, as early as possible. Only by teaching the history can they fully appreciate the significance of this monumental discovery, from which starts with Descartes in the 17c. (p. 111).

Mathematics got is true start with the 17 century Swiss *Bernoulli family,* Jakob, Johann and Johann's sun Daniel, all beautifully painted by their brother Nicholas (p. 249). Euler (1707-1783), as a 13 year old student of Johann, soon followed (p. 188). It was Euler who really kick-started mathematics due to his massive influx of technique. Many of the modern techniques were introduced by his deep insight into analysis and manipulation of differential equations. Prior to Euclid, mathematics was an art, vs. a science.

On the whole, the Bernoulli was quite dysfunctional. With great consternation of his uncle, Daniel was the first to formulate fluid mechanics (Part III).

The field of mathematics was greatly extended by Euler, due to both his productivity, but even more due to his transparency (p. 191). Following in the tradition of Pythagoras, secrecy was the norm in the 17-18 century. Euler canceled all of this by openly disclosing his methods. Math became a commodity, but only for the brilliant mind. Newton use Latin, Euler Swiss-German (ck this). But at least the mathematical details were openly disclosed, for the first time. Thus the transition from the Bernoullis to Euler, defined the new era.

[3] Kennelly, A.E., (1893), *Impedance*. Transactions of the American Institute of Electrical Engineers, **10**: 172-232)

[4] Boas, R.P. (2009), p. 3

[5] Since I am not a mathematician, my summary needs formal conformation.

[6] Perhaps we need a graphical time-line, showing how this all unraveled?

[7] While our students all know this theorem, but not by name, and not by its history. Nor do they fully appreciate its significance.

III: Analysis of Systems: By the 19c the advancements begin to accelerate. Fourier analysis, Maxwell (Systems of equations, fluid mechanics), Impedance, signal processing, Probability theory, information theory, and so on. Building on the previous substantial developments in real analysis (but not necessarily in complex analysis), and mostly in the hands of the Bernoulli family, mathematics began to take on a much larger form. For example, Daniel Bernoulli developed an early form of fluid mechanics. The the deeper progress was made by the massive productivity of Euler (p. 188-191, 184). There are so many examples that they cannot be easily summarized here.

In my opinion many of the basic ideas were in place and Euler was the right person at the right time. If he had not been the one, others would have filled in the gaps. However he greatly accelerated the pace of advancement. He deserves, and has gotten, a lions share of the credit, for many of the advancements. There are limits on this credit however. An example is the best argument here. As is well know, it was Euler who first worked out the factored from of the zeta function (Euler's *product formula*, p. 184). However this formula was for the function of a real argument. It was Riemann who extended this to have a complex argument, thus first identifying the poles and zeros. This extension to the complex plane that gives this formula its full power. [8]

Finally it is the most power concept, again my view, of the *branch-cut* and the concept of the multi-valued function as the inverse of a periodic function. In my own work, in acoustics, this has turned out to be critical, for reasons that are far beyond the scope of this discussion. Let me just say that there are cases where we wish to compute something that takes the difference between a function as it crosses a branch cut. When this happens it is critical to see this has happened, so that the calculation may be properly formed. This requires moving the branch cut so that the difference is an analytic function. In signal processing this is sometimes referred to as the *phase unwrapping* problem. It is my experience this important concept can be highly confusing.

It is important to mention Newton's development of the *wave equation* (p. 242) (1687) which was then first theoretically explained by d'Alembert in 1747. The general solution by d'Alembert led to a long and deep controversy (p. 244) not resolved until at least 1872 (p. 456).

Even more important was Maxwell's development of his famous equations which were then more fully investigated by Einstein, in his theory of relativity. Perhaps this is, arguably, best left to the realm of physics rather than mathematics.

[8] As before, this needs vetting.