Chapter 1

Assignments with solutions: Number systems

1.2 Exercises NS-1

Topic of this homework:

Introduction to MATLAB/OCTAVE (see the Matlab or Octave tutorial for help).

Deliverable: Report with charts and answers to questions. Hint: Use LATEX. ¹

Plotting complex quantities in Octave/Matlab

Problem #1: Consider the functions \( f(s) = s^2 + 6s + 25 \) and \( g(s) = s^2 + 6s + 5 \).

–Q 1.1
Find the zeros of functions \( f(s) \) and \( g(s) \) using the command \texttt{roots}(). \textbf{Sol:} The roots of \( f(s) \) are \(-3 \pm 4i \) (in Matlab: \texttt{roots([1 6 25])}). The roots of \( g(s) \) are \(-1 \) and \(-5 \) (in Matlab: \texttt{roots([1 6 5])}). You will find the program that generates all these figures at \url{http://jontalle.web.engr.illinois.edu/uploads/298.17/NS1.m}.

–Q 1.2: Show the roots of \( f(s) \) as red circles and of \( g(s) \) as blue plus signs.
The x-axis should display the real part of each root, and the y-axis should display the imaginary part. Use \texttt{hold on} and \texttt{grid on} when plotting the roots. \textbf{Sol:}

¹\url{http://www.overleaf.com}
- Q 1.3 Give your figure the title ‘Complex Roots of f(s) and g(s)’ Label the x- and y-axis ‘Real Part’ and ‘Imaginary Part.’

Hint: use xlabel and ylabel. Type ylim([-10 10]) and xlim([-10 10]), to expand the axes.

Problem # 2: Consider the function \( h(t) = e^{j2\pi ft} \) for \( f = 5 \) and \( t = [0:0.01:2] \).

- Q 2.1: Use subplot to show the real and imaginary parts of \( h(t) \)

Make two graphs in one figure. Label the x-axes ‘Time (s)’ and the y-axes ‘Real Part’ and ‘Imaginary Part’.

Sol: Breaking \( h(t) \) into real and imaginary parts gives \( e^{j2\pi 5t} = \cos(10\pi t) + j\sin(10\pi t) \).

- Q 2.2: Use subplot to plot the magnitude and phase parts of \( h(t) \).

Use the command angle or unwrap(angle()) to plot the phase. Label the x-axes ‘Time (s)’ and the y-axes ‘Magnitude’ and ‘Phase (radians)’. Sol:
Prime numbers, infinity, etc. in Octave/Matlab

**Problem # 3: Prime numbers, infinity, etc.**

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**Q 3.1:** *Use the Matlab function* `factor` *to find the prime factors of 123, 248, 1767, and 999,999.*

**Sol:** Factors: 123 (3, 41), 248 (2,2,2,31), 1767 (3,19,31), 999999 (3,3,3,7,11,13,37)

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**Q 3.2:** *Use the Matlab function* `isprime` *to check if 2, 3 and 4 are prime numbers.*

What does the function `isprime` return when a number is prime, or not prime? Why?

**Sol:** Function `isprime(2)` returns 1, `isprime(3)` returns 1, and `isprime(4)` returns 0. 1 means ‘yes’ and 0 means ‘no’

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**Q 3.3:** *Use the Matlab/Octave function* `primes.m` *to generate prime numbers between 1 and $10^6$.*

Save them in a vector $x$. Plot this result using the command `hist(x)`. **Sol:**

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**Q 3.4:** *Now try* `[n,bincenters] = hist(x).*

Use `length(n)` to find the number of bins. **Sol:** `length(n)` is 10

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**Q 3.5:** *Set the number of bins to 100 by using an extra input argument to the function hist.*

Show the resulting figure and give it a title and axes labels. **Sol:**
Problem #4: Inf, NaN and logarithms in Octave/Matlab

–Q 4.1: Try 1/0 and 0/0 in the Octave/Matlab command window.
What are the results? What do these ‘numbers’ mean in Matlab?  Sol: 1/0 returns Inf (infinity) and 0/0 returns NaN (‘not a number’).

–Q 4.2: Try log(0) in the command window. In Matlab, the natural logarithm ln(·) is computed using the function log (log10, and log2 are computed using log10 and log2). Sol: log(0) is -Inf

–Q 4.3: Try log(-1) in the command window. Do you get what you expect for ln(-1)? Show how Matlab arrives at the answer by considering -1 = eiπ. Sol: log(-1) is 0 + iπ

–Q 4.4: Try log(exp(j*sqrt(pi))) in the command window. What do you expect? Show how Octave/Matlab arrives at the answer by considering -1 = e(iπ). Sol: log e^√πi = √πi because ln(·) is the inverse of e(·).

–Q 4.5: What does inverse mean in this context? What is the inverse of ln f(x)? Sol: ln f(x) = e(ln f(x)). Conclusion: eG and ln G are mutual inverses: that is: ln() of e(·) and e(·) of ln(). Or said another way: G = e^ln(G), G = ln e^G.

Problem #5: Very large primes on Intel computers

–Q 5.1: Find the largest prime number that can be stored on an Intel 64 bit computer, which we call πmax
Hint: As explained in the Matlab/Octave command help flintmax, the largest positive integer is 2^53, however the largest integer that can be factored is 2^32 = 2^53. Explain the logic of your answer. Hint: help isprime(). Sol: Using Matlab/Octave, start with the largest integer 2^32 and check if its prime. Then work down by subtracting 1, and again check. Stop when you get to the first prime below the largest integer. The answer I get is: 2^32 - 5 = 4, 294, 967, 291, is the first prime below 2^32 prime.

Problem #6: Suppose you are interested in primes that are greater than πmax. How can you find them on an Intel computer (i.e., one using IEEE-floating point)?

–Q 6.1: Thus consider a sieve containing only odd numbers, starting from 3 (not 2).
Hint 1: Since every prime number greater than 2 is odd, there is no reason to check the even numbers. Sol: At this time, I don’t see any way to do this, due to the Matlab limitation that it cannot factor numbers larger that 2^32.
Problem # 7: The following identity is interesting:

\[ 1 = 1^2 \]
\[ 1 + 3 = 2^2 \]
\[ 1 + 3 + 5 = 3^2 \]
\[ 1 + 3 + 5 + 7 = 4^2 \]
\[ 1 + 3 + 5 + 7 + 9 = 5^2 \]
\[ \cdots \]
\[ \sum_{n=0}^{N-1} 2n + 1 = N^2. \]

–Q 7.1: Can you find a proof?²

Sol: Subtracting any line from the line following it, gives:

\[ (1 - 1) + 3 = 2^2 - 1^2 \]
\[ 5 = 3^2 - 2^2 \]
\[ 7 = 4^2 - 3^2 \]
\[ 9 = 5^2 - 4^2 \]
\[ \cdots \]
\[ \sum_{n=0}^{N-1} 2n + 1 - \sum_{n=0}^{N-2} 2n + 1 = N^2 - (N - 1)^2 \]
\[ 2N - 1 = N^2 - (N^2 - 2N + 1) \]
\[ 2N - 1 = 2N - 1. \]

Thus the two sides are equal, as suggested by the above formula.

Can you find a simpler more constructive “proof?” Hint: assuming you know what integration by parts is, can you devise a concept called Summation by parts? ■

²This problem came from an exam problem for Math 213, Fall 2016.
1.3 Exercises NS-2

**Topic of this homework:**
Prime numbers, greatest common divisors, the continued fraction algorithm

**Deliverable:** Answers to questions.

**Prime numbers**

**Problem # 1:** Every integer may be written as a product of primes.

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**Q 1.1:** Put the numbers $1,000,000$, $1,000,004$ and $999,999$ in the form $N = \prod_k \pi_k^{\beta_k}$ (Hint: Use Matlab/Octave to find the prime factors).

**Sol:**
- $1,000,000 = 2^6 \cdot 5^6$
- $1,000,004 = 2^2 \cdot 53^2 \cdot 89$
- $999,999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$

---

**Q 1.2:** Give a generalized formula for the natural logarithm of a number, $\ln(N)$, in terms of its primes $\pi_k$ and their multiplicities $\beta_k$. Express your answer as a sum of terms.

**Sol:** $\ln N = \sum_k \beta_k \ln(\pi_k)$

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**Problem # 2:** Using the computer

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**Q 2.1:** Explain why the following brief Matlab/Octave program returns the prime numbers $\pi_k$ between 1 and 100.

```matlab
n=2:100;
k = isprime(n);
n(k)
```

**Sol:** The first line $n = 2:100$ defines the row vector $n = [2, 3, 4, \cdots, 100]$. The second line creates a row vector the same length as $n$, with entries of 1 if the element is prime and zero if the element is not prime. The third line $n(k)$ prints out $n()$ if $k = 1$, namely it is a list of all the primes from 2 to 100. Run this program without the ';' at the end of each line, and to see what it is doing.

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**Q 2.2:** How many primes are there between 2 and $N = 100$?

**Sol:** $\text{length}(n(k))$ returns 25. Thus there are 25 primes less than 100 ($N/4$, on average).

---

**Problem # 3:** Prime numbers may be identified using a 'sieve.'

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**Q 3.1:** By hand, perform the sieve of Eratosthenes for $n = 1 \ldots 49$. Circle each prime $p$, then cross out each number which is a multiple of $p$.

**Sol:** Note: 1 should not be circled as it is not a prime.
1.3. EXERCISES NS-2

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- **Q 3.2**: What is the largest number you need to consider before only primes remain?
  
  **Sol:** $\lfloor \sqrt{50} \rfloor = \lfloor 7.0711 \rfloor = 7$. ■

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- **Q 3.3**: Generalize: for $n = 1 \ldots N$, what is the highest number you need to consider before only the primes remain?
  
  **Sol:** $\text{floor}(\sqrt{N})$ ■

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- **Q 3.4**: Write each of these numbers as a product of primes:
  
  - 22 = $2 \cdot 11 = \pi_1 \pi_5$ ■
  - 30 = $2 \cdot 3 \cdot 5 = \pi_1 \pi_2 \pi_3$ ■
  - 34 = $2 \cdot 17 = \pi_1 \pi_7$ ■
  - 43 = $\pi_{14}$ ■
  - 44 = $4 \cdot 11 = \pi_2 \pi_5$ ■
  - 48 = $4 \cdot 12 = 4^2 \cdot 3 = \pi_1^4 \pi_2$ ■
  - 49 = $7^2 = \pi_4^2$ ■

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- **Q 3.5**: Find the largest prime $\pi_k \leq 100$? Hint: Do not use Matlab/Octave other than to check your answer. Hint: Write out the numbers starting with 100 and counting backwards: 100, 99, 98, 97, ···. Cross off the even numbers, leaving 99, 97, 95, ···. Pull out a factor (only 1 is necessary to show that it is not prime).
  
  **Sol:** 99 = $11 \cdot 9$, $\pi_{25} = 97$. ■

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- **Q 3.6**: Find the largest prime $\pi_k \leq 1000$? Hint: Do not use Matlab/Octave other than to check your answer.
  
  **Sol:** Write out the numbers starting with 1000 and counting backwards: 1000, 999, 998, 997, ···. Cross off the even numbers, leaving 999, 997, 995, ···. Pull out a factor (only 1 is necessary to show that it is not prime). $9 \cdot 111, 997 = \pi_{168}, 5 \cdot 199 = \pi_3 \cdot \pi_{46}$. ■

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- **Q 3.7**: Explain why $\pi_k^{-s} = e^{-s \ln \pi_k}$.
  
  **Sol:** This follows from the identity $z^a = e^{a \ln z}$ with $a, z \in \mathbb{C}$. ■

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### Greatest common divisors

Consider the Euclidean algorithm to find the greatest common divisor (GCD; the largest common prime factor) of two numbers. Note this algorithm may be performed using one of two methods:

<table>
<thead>
<tr>
<th>Method</th>
<th>Division</th>
<th>Subtraction</th>
</tr>
</thead>
</table>
| On each iteration... | $a_{i+1} = b_i$  
$b_{i+1} = a_i - b_i \cdot \text{floor}(a_i/b_i)$ | $a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$  
$b_{i+1} = \min(a_i, b_i)$ |
| Terminates when... | $b = 0 \ (\text{gcd}= a)$                      | $b = 0 \ (\text{gcd}= a)$                    |

The division method (Eq. 2.1, Sect. 2.1.2, Lec 5, Ch. 2) is preferred because the subtraction method is much slower.

**Problem # 4: Understanding the Euclidean (GCD) algorithm**
–Q 4.1: Use the Octave/Matlab command `factor` to find the prime factors of $a = 85$ and $b = 15$.

Sol: From Octave’s `factor()` we find $85 = 17 \cdot 5$, $15 = 3 \cdot 5$. ■

–Q 4.2: What is the greatest common prime factor of these two numbers?

Sol: The largest common factor $\text{gcd}(85,15)$ is $5$. ■

–Q 4.3: By hand, perform the Euclidean algorithm for $a = 85$ and $b = 15$.

Sol: Division method:

\[
\begin{align*}
a_1 &= 15 \\
b_1 &= 85 - 15 \frac{85}{15} = 10 \\
a_2 &= 10 \\
b_2 &= 15 - 10 \frac{15}{10} = 5 \\
a_3 &= 5 \\
b_3 &= 10 - 5 \frac{10}{5} = 0
\end{align*}
\]

∴ $\text{gcd} = 5$

Subtraction method:

\[
\begin{align*}
a_1 &= 85 - 15 = 70 \\
b_1 &= 15 \\
a_2 &= 70 - 15 = 55 \\
b_2 &= 15 \\
a_3 &= 55 - 15 = 40 \\
b_3 &= 15 \\
a_4 &= 40 - 15 = 25 \\
b_4 &= 15 \\
a_5 &= 25 - 15 = 10 \\
b_5 &= 15 \\
\text{swap} \\
a_6 &= 15 - 10 = 5 \\
b_6 &= 10 \\
a_7 &= 10 - 5 = 5 \\
b_7 &= 5
\end{align*}
\]

∴ $\text{gcd} = 5$ ■

–Q 4.4: By hand, perform the Euclidean algorithm for $a = 75$ and $b = 25$. Is the result a prime number?

Sol: Division method:

\[
\begin{align*}
a_1 &= 25 \\
b_1 &= 75 - 25 \frac{75}{25} = 0
\end{align*}
\]

Subtraction method:

\[
\begin{align*}
a_1 &= 75 - 25 = 50 \\
b_1 &= 25 \\
a_2 &= 50 - 25 = 25 \\
b_2 &= 25
\end{align*}
\]

∴ $\text{gcd} = 25$

The result is $25 = 5^2$, the square of a prime number. ■

–Q 4.5: Consider the first step of the GCD division algorithm when $a < b$ (e.g. $a = 25$ and $b = 75$). What happens to $a$ and $b$ in the first step? Does it matter if you begin the algorithm with $a < b$ vs. $b < a$?

Sol: If $a < b$, the first step of the division algorithm swaps the terms $(a \rightarrow b$ and $b \rightarrow a)$. ■
Q 4.6: Describe in your own words how the GCD algorithm works. Try the algorithm using numbers which have already been separated into factors (e.g. $a = 5 \cdot 3$ and $b = 7 \cdot 3$).

**Sol:** Division method:

\[
\begin{align*}
a_1 &= 5 \cdot 3 & b_1 &= 7 \cdot 3 - 5 \cdot 3 \left\lfloor \frac{7}{5} \right\rfloor &= 2 \cdot 3 \\
a_2 &= 2 \cdot 3 & b_2 &= 5 \cdot 3 - 2 \cdot 3 \left\lfloor \frac{5}{2} \right\rfloor &= 1 \cdot 3 \\
a_3 &= 1 \cdot 3 & b_3 &= 2 \cdot 3 - 1 \cdot 3 \left\lfloor \frac{2}{1} \right\rfloor &= 0
\end{align*}
\]

Subtraction method:

\[
\begin{align*}
a_1 &= 7 \cdot 3 - 5 \cdot 3 = 2 \cdot 3 & b_1 &= 5 \cdot 3 \\
a_2 &= 5 \cdot 3 - 2 \cdot 3 = 3 \cdot 3 & b_2 &= 2 \cdot 3 \\
a_3 &= 3 \cdot 3 - 2 \cdot 3 = 1 \cdot 3 & b_3 &= 2 \cdot 3 \\
a_4 &= 2 \cdot 3 - 1 \cdot 3 = 1 \cdot 3 & b_4 &= 1 \cdot 3
\end{align*}
\]

The algorithm iteratively converges on the GCD by subtracting out multiples of the GCD until only the GCD is left.

**Problem # 5: Coprimes**

1. Define the term *coprime*. **Sol:** when two integers have no common factors they are said to be *coprime*.

2. How can the Euclidean algorithm be used to identify coprimes? **Sol:** If $\text{gcd}(a,b) = 1$ they only have 1 as a common factor, thus they are coprime.

3. Give at least one application of the Euclidean algorithm. **Sol:** Given two integers $n, d \in \mathbb{Z}$, if we wish to reduce the fraction $n/d$, we must cancel the common factors. Example: If $n = 9, d = 6$ then $9/6 = (3 \cdot 3)/(2 \cdot 3) = 3/2$, where the GCD, 3, may be identified using the Euclidean algorithm. While this fraction may be easily simplified via inspection, the GCD algorithm could be very helpful for larger numbers $n, d$.

Q 5.1: Write a Matlab function, \texttt{function x = mygcd(a,b)}, which uses the Euclidean algorithm to find the GCD of any two inputs $a$ and $b$. Test your function on the $(a,b)$ combinations from parts (a) and (b). Include a printout (or handwrite) your algorithm to turn in.

**Hints and advice:**

- Don’t give your variables the same names as Matlab functions! Since \texttt{gcd} is an existing Matlab/Octave function, if you use it as a variable or function name, you won’t be able to use \texttt{gcd} to check your \texttt{gcd()} function. Try \texttt{clear all} to recover from this problem.

- Try using a ‘while’ loop for this exercise (see Matlab documentation for help).

- You may need to make some temporary variables for $a$ and $b$ in order to perform the algorithm.

**Sol:** Division method:

\[
\text{function x = mygcd(a,b)}
\]

\[
\text{while b>0}
\]

\[
\text{atmp = a; btmp = b;}
\]

\[
\text{a = btmp; b = atmp-btmp*floor(atmp/btmp);}
\]

\[
\text{end}
\]
Subtraction method:

```matlab
function x = my_gcd(a, b)
    while a ~= b
        atmp = a; btmp = b;
        a = max(atmp, btmp) - min(atmp, btmp); b = min(atmp, btmp);
    end
end
```

Alegbraic generalization of the GCD (Euclidean) algorithm

Problem # 6: In this problem we are looking for integer solutions \((m, n) \in \mathbb{Z}\) to the equations \(ma + nb = \gcd(a, b)\) and \(ma + nb = 0\) given positive integers \((a, b) \in \mathbb{Z}^+\).

Note that this requires that either \(m\) or \(n\) be negative. These solutions may be found using the Euclidean algorithm only if \((a, b)\) are coprime \((a \perp b)\). Note that integer (whole number) polynomial relations such as these are known as ‘Diophantine equations.’ The above equations are linear Diophantine equations, possibly the simplest form of such relations.

Example: \(\gcd(2, 3)=1\): For \((a, b) = (2, 3)\), the result is as follows:

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & -2
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
2 \\
3
\end{bmatrix} = \begin{bmatrix}
-1 \\
3
\end{bmatrix} \begin{bmatrix}
1 \\
-2
\end{bmatrix} \begin{bmatrix}
2 \\
3
\end{bmatrix}
\]

Thus from the above equation we find the solution \((m, n)\) to the integer equation

\[2m + 3n = \gcd(2, 3) = 1,
\]

namely \((m, n) = (-1, 1)\) (i.e., \(-2+3 = 1\)). There is also a second solution \((3, -2)\) (i.e., \(3 \cdot 2 - 2 \cdot 3 = 0\)), which represents the terminating condition. Thus these two solutions are a pair and the solution only exists if \((a, b)\) are coprime \((a \perp b)\).

Subtraction method: This method is more complicated than the division algorithm, because at each stage we must check if \(a < b\). Define

\[
\begin{bmatrix}
a_i \\
b_i
\end{bmatrix} = Q \begin{bmatrix}
a_i \\
b_i
\end{bmatrix} \quad \text{for } a_i > b_i,
\]

\[
\begin{bmatrix}
a_i \\
b_i
\end{bmatrix} = QS \begin{bmatrix}
a_i \\
b_i
\end{bmatrix} \quad \text{for } a_i < b_i.
\]

where \(Q\) sets \(a_i+1 = a_i - b_i\) and \(b_i+1 = b_i\) assuming \(a_i > b_i\), and \(S\) is a ‘swap-matrix’ which swaps \(a_i\) and \(b_i\) if \(a_i < b_i\). Using these matrices, the algorithm is implemented by assigning

\[
\begin{bmatrix}
a_{i+1} \\
b_{i+1}
\end{bmatrix} = Q \begin{bmatrix}
a_i \\
b_i
\end{bmatrix} \quad \text{for } a_i > b_i,
\]

\[
\begin{bmatrix}
a_{i+1} \\
b_{i+1}
\end{bmatrix} = QS \begin{bmatrix}
a_i \\
b_i
\end{bmatrix} \quad \text{for } a_i < b_i.
\]

The result of this method is a cascade of \(Q\) and \(S\) matrices. For \((a, b) = (2, 3)\), the result is as follows:

\[
\begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
2 \\
3
\end{bmatrix} = \begin{bmatrix}
2 \\
-1
\end{bmatrix} \begin{bmatrix}
1 \\
1
\end{bmatrix} \begin{bmatrix}
3
\end{bmatrix}.
\]

Thus we find two solutions \((m, n)\) to the integer equation \(2m + 3n = \gcd(2, 3) = 1\).

–Q 6.1: By inspection, find at least one integer pair \((m, n)\) that satisfies \(12m + 15n = 3\).

Sol: By inspection, \((m, n) = (-1, 1)\) is one solution.
Using matrix methods for the Euclidean algorithm, find integer pairs \((m, n)\) that satisfy \(12m + 15n = 3\) and \(12m + 15n = 0\). Show your work!!

**Sol:** Division method:

\[
\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 12 \\ 15 \end{bmatrix}
\]

Subtraction method:

\[
\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 12 \\ 15 \end{bmatrix}
\]

**–Q 6.2:** Does the equation \(12m + 15n = 1\) have integer solutions for \(n\) and \(m\)? Why, or why not?

**Sol:** No, because \(\gcd(12, 15) = \gcd(3 \cdot 4, 3 \cdot 5) = 3\), not 1. Thus there are no Diophantine solutions to this equation.

**Problem # 7: Matrix approach:**

It can be difficult to keep track of the a’s and b’s when the algorithm has many steps. We need an alternative way to run the Euclidean algorithm, using matrix algebra. Matrix methods provide a more transparent approach to the operations on \((a, b)\). Thus the Euclidean algorithm can be classified in terms of standard matrix operations.

**–Q 7.1:** Write out the matrix approach, discussed at the end of Sect. 2.4.1 (Eq. 2.8, p. 51).

**Sol:** Division method:

Define

\[
\begin{bmatrix} a \\ b \end{bmatrix}_0 = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}, \quad \begin{bmatrix} a \\ b \end{bmatrix}_{i+1} = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix}_i \begin{bmatrix} a \\ b \end{bmatrix}_i
\]

**Continued fractions**

**Problem # 8:** Here we explore the continued fraction algorithm (CFA), Sect. 2.4.2, (p. 52).

In its simplest form the CFA starts with a real number, which we denote as \(\alpha \in \mathbb{R}\). Let us work with an irrational real number, \(\pi \in \mathbb{I}\), as an example, because its CFA representation will be infinitely long. We can represent the CFA coefficients \(\alpha\) as a vector of integers \(n_k, k = 1, 2 \cdots \infty\)

\[
\alpha = [n_1; n_2, n_3, n_4, \cdots]
\]

\[
= n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \cdots}}}
\]

As discussed in Section 2.4.1 (p. 48), the CFA is recursive, with three steps per iteration:

For \(\alpha_1 = \pi, n_1 = 3, r_1 = \pi - 3\) and \(\alpha_2 \equiv 1/r_1\).

\[
\alpha_2 = 1/0.1416 = 7.0625 \cdots
\]

\[
\alpha_1 = n_1 + \frac{1}{\alpha_2} = n_1 + \frac{1}{n_2 + \frac{1}{\alpha_3}} = \cdots
\]

In terms of a Matlab/Octave script
\[
\alpha_0 = \pi; \\
K=10; \\
n=\text{zeros}(1,K); \alpha=\text{zeros}(1,K); \\
\alpha(1)=\alpha_0; \\
\text{for } k=2:K \text{ \%k=1 to K} \\
n(k)=\text{round}(\alpha(k-1)); \\
%n(k)=\text{fix}(\alpha(k-1)); \\
\alpha(k)=1/(\alpha(k-1)-n(k)); \\
%\text{disp([fix(k), round(n(k)), alpha(k)]); pause(1)} \\
\text{end} \\
\text{disp([n; alpha]);} \\
\text{\%Now compare this to matlab's rat() function} \\
\text{rat(}\alpha_0,1e-20) \\
\]

---

\textbf{Q 8.1:} By hand (you may use Matlab/Octave as a calculator), find the first 3 values of \( n_k \) for \( \alpha = e^{\pi} \).

**Sol:** The CFA for this is: \( e^{\pi} = 23.1407\cdots = [23; 7, 9, 4, \cdots] \).

---

\textbf{Q 8.2:} For part (1), what is the error (remainder) when you truncate the continued fraction after \( n_1, \ldots, n_3 \)? Give the absolute value of the error, and the percentage error relative to the original \( \alpha \).

**Sol:** The remainder is \( e^{\pi} - (23 + 1/(7 + (1/9))) \) which gives an error of \( \epsilon = |e^{\pi} - (23 + 1/(7 + (1/9)))|/e^{\pi} = 2.92 \cdot 10^{-6} = 0.0003\% \).

---

\textbf{Q 8.3:} Use the Matlab/Octave program provided to find the first 10 values of \( n_k \) for \( \alpha = e^{\pi} \), and verify your result using the Matlab/Octave command \texttt{rat()}.

**Sol:** \( e^{\pi} = 23.1407\cdots = [23; 7, 9, 4, -2, -591, -2, -10, 3, -2, \cdots] \).

---

\textbf{Q 8.4:} Discuss the similarities and differences between the Euclidean algorithm (EA) and CFA.

**Sol:**

1. Both are recursive, meaning that the steps are repeated one after another.

2. The EA starts from two numbers (a,b). The output of the gca(a,b) is the GCD. The CFA starts with a single number and the output is a sequence of integers. If the sequence terminates the number was rational. If the sequence does not terminate, the number is irrational.

3. The EA works with the difference between the minimum and maximum of the two numbers whereas the CFA works with the rounding function and the reciprocal of the error.

4. It would seem that the goals of the two algorithms, the starting point, and the results are totally different. Both are very useful and powerful. Both generalize to more difficult situations than working with simple numbers.

---

\textbf{Q 8.5: Extra Credit} Show that the CFA is the inverse operation (i.e., the CFA is the GCD, run in reverse) (Hint: see Sect. 2.4.1 (p. 48)). NOT ON EXAM

The full solution to this problem needs further analysis. **Sol:**

Starting from Eq. 2.7

\[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-m} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{m}{n} \\ 0 & 1 + \frac{m}{n} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{m}{n} \end{bmatrix}.
\]
The matrix equation for the CFA is derived in Sec. F, p. 285. We conclude that taking Eq. 2.7 to the \( \left[ \frac{m}{n} \right] \) power, and swapping rows, results in a CFA matrix. However, I believe this must be iterated. It follows that the GCD and CFA are inverses because the matrix formulations are inverses.

**Continued fraction algorithm (CFA) (8 pts)**

- **(4 pts)** Expand \( \frac{23}{7} \) as a continued fraction. Express your answer in bracket notation (e.g., \( \pi = [3, 7, 16, \ldots] \)). Show your work. **Sol:** \( \frac{23}{7} = \frac{21 + 2}{7} = 3 + \frac{2}{7} = 3 + \frac{1}{(6 + 1)/2} = 3 + 1/(3 + 1/2) \). In bracket notation \( \frac{23}{7} = [3, 3, 2] \). Matlab gives \( \text{rat}(23/7) = 3 + 1/(4 + 1/(-2)) \), or \([1, 4, -2]\) because rounding \( 7/2 \) can be taken as either \( 3+1/2 \) or \( 4-1/2 \).

- **(2 pts)** Can \( \sqrt{2} \) be represented as a finite continued fraction? Why or why not? **Sol:** No, because it is irrational.

- **(2 pts)** What is the CFA for \( \sqrt{2} - 1 \)? **Hint:** \( \sqrt{2} + 1 = \frac{1}{\sqrt{2} - 1} = [2, 2, 2, 2, \ldots] \).

**Sol:** \( 1 + \sqrt{2} = 2 + 1/(2 + 1/(2 + \cdots)) \) or \([2, 2, 2, 2, \ldots]\), thus
\[
\sqrt{2} - 1 = [2, 2, 2, 2, \cdots] - 2 = 0 + 1/(2 + 1/(2 + 1/(2 + \cdots))).
\]

- Find the CFA for \( 1 + \sqrt{3j} \)
\[
\text{rat}(1 + \text{sqrt}(3j)) = [2, 2, 2, 2, \ldots].
\]

- Show that
\[
\frac{1}{1 - \sqrt{a}} = a^{1/2} + a^{3/2} + a^{5/2} + a^2 + \sqrt{a} + a^3 + a^4 + a^5 + a^6 + a + 1 = 1 - a^6
\]
syms a
taylor(1/(1-sqrt(a)))
simplify((1-sqrt(a))*ans) = 1-a^6

**Sol:** I used symbolic analysis to show this. This is begging for an explanation.
1.4 Exercises NS-3

Topic of this homework: Pythagorean triples, Pell’s equation, Fibonacci sequence

Deliverable: Answers to problems

Pythagorean triplets

Problem # 1: Euclid’s formula for the Pythagorean triplets \(a, b, c\) is: \(a = p^2 - q^2\), \(b = 2pq\), and \(c = p^2 + q^2\).

-Q 1.1: What condition(s) must hold for \(p\) and \(q\) such that \(a\), \(b\), and \(c\) are always positive and nonzero?
Sol: \(p > q > 0\) (strictly greater than)

-Q 1.2: Solve for \(p\) and \(q\) in terms of \(a\), \(b\) and \(c\).
Sol:

Method 1: Given \(a, c\), one may find \(p, q\) via matrix operations by solving the nonlinear system of equations for \(p, q\).
First solve linear system of equations for \(p^2, q^2\):

\[
\begin{bmatrix}
a \\
c
\end{bmatrix} =
\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
p^2 \\
q^2
\end{bmatrix}
\]

Inverting this 2x2 matrix gives (the determinant \(\Delta = 2\))

\[
\begin{bmatrix}
p^2 \\
q^2
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
c
\end{bmatrix}.
\]

Thus \(p = \pm \sqrt{(a+c)/2}\), \(q = \pm \sqrt{(c-a)/2}\).

Method 2: The algebraic approach is:

\[a + c = (p^2 - q^2) + (p^2 + q^2) = 2p^2\]
\[-a + c = -(p^2 - q^2) + (p^2 + q^2) = 2q^2,\]

Thus \(p = \sqrt{(a+c)/2}\), \(q = \sqrt{(c-a)/2}\), where \(p, q \in \mathbb{N}\).

Method 1 seems more “transparent” than Method 2.

Problem # 2: The ancient Babylonians (c2000BEC) cryptically recorded \((a, c)\) pairs of numbers on a clay tablet, archeologically denoted Plimpton-322.
–Q 2.1: Find \( p \) and \( q \) for the first five pairs of \( a \) and \( c \) from the tablet entries. Table 1: First five \((a,c)\) pairs of Plimpton-322.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>169</td>
</tr>
<tr>
<td>3367</td>
<td>4825</td>
</tr>
<tr>
<td>4601</td>
<td>6649</td>
</tr>
<tr>
<td>12709</td>
<td>18541</td>
</tr>
<tr>
<td>65</td>
<td>97</td>
</tr>
</tbody>
</table>

Find a formula for \( a \) in terms of \( p \) and \( q \).

\[
(a, c) = (119, 169) \quad (p, q) = \pm (12, 5)
\]

\[
(a, c) = (3367, 4825) \quad (p, q) = \pm (64, 27)
\]

\[
(a, c) = (4601, 6649) \quad (p, q) = \pm (75, 32)
\]

\[
(a, c) = (12709, 18541) \quad (p, q) = \pm (125, 54)
\]

\[
(a, c) = (65, 97) \quad (p, q) = \pm (9, 4)
\]

–Q 2.2: Based on Euclid’s formula, show that \( c > (a, b) \).

\[
(c - a) - (p^2 + q^2) - (p^2 - q^2) = 2q^2
\]

Because \( 2q^2 \) is always positive, \( c > a \)
\[
c - b = (p^2 + q^2) - 2pq = (p - q)^2 > 0
\]

Note that by the definition of \( p, q \in \mathbb{N}, p > q \). ●

–Q 2.3: What happens when \( c = a \)?

\[
\text{Sol: Then its not a triangle since } b = 0. \text{ The triangle is degenerate. ●}
\]

–Q 2.4: Is \( b + c \) a perfect square? Discuss.

\[
\text{Sol: } b + c = p^2 + 2pq + q^2 = (p + q)^2. \text{ Since } p \text{ and } q \text{ are integers, } b + c \text{ will always be a perfect square (} \sqrt{b + c} \text{ will always be an integer).}
\]

Pell’s equation:

Problem # 3: Pell’s equation is one of the most historic \( i.e., \) important equations of Greek number theory because it was used to show that \( \sqrt{2} \in \mathbb{I} \). We seek integer solutions of

\[
x^2 - Ny^2 = 1.
\]

As shown in Section 2.4.4 (p. 57) of the lecture notes, the solutions \( x_n, y_n \) for the case of \( N = 2 \) are given by the linear 2x2 matrix recursion

\[
\begin{bmatrix}
x_{n+1} \\
y_{n+1}
\end{bmatrix} = 1J \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\
y_n
\end{bmatrix}
\]

with \([x_0, y_0]^T = [1, 0]^T \) and \( 1J = \sqrt{-1} = e^{\pi i/2} \). It follows that the general solution to Pell’s equation for \( N = 2 \) is

\[
\begin{bmatrix} x_n \\
y_n
\end{bmatrix} = (e^{\pi i/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\
y_0
\end{bmatrix}
\]

To calculate solutions to Pell’s equation using the matrix equation above, we must calculate

\[
A^n = e^{\pi i n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = e^{\pi i n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\]

which becomes tedious for \( n > 2 \), since it requires \( n \times 2 \times 2 \) matrix multiplications.
Diagonalization of a matrix ("eigenvalue/eigenvector decomposition"):

As derived in Appendix C of the lecture notes, the most efficient way to compute \( A^n \) is to diagonalize the matrix \( A \), by finding its eigenvalues and eigenvectors.

The eigenvalues \( \lambda_k \) and eigenvectors \( e_k \) of a square matrix \( A \) are related by

\[
A e_k = \lambda_k e_k,
\]

such that multiplying an eigenvector \( e_k \) of \( A \) by the matrix \( A \) is the same as multiplying by a scalar, \( \lambda_k \in \mathbb{C} \) (the corresponding eigenvalue). The complete eigenvalue problem may be written as

\[
A E = E \Lambda.
\]

If \( A \) is a 2 \times 2 matrix,\(^3\) the matrices \( E \) and \( \Lambda \) (of eigenvectors and eigenvalues, respectively) are

\[
E = \begin{bmatrix} e_1 & e_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.
\]

Thus, the matrix equation \( A E = [A e_1 \, A e_2] = [\lambda_1 e_1 \, \lambda_2 e_2] = E \Lambda \) contains Eq. 1.1 for each eigenvalue-eigenvector pair.

The diagonalization of the matrix \( A \) refers to the fact that the matrix of eigenvalues, \( \Lambda \), has non-zero elements only on the diagonal. The key result is found by post-multiplication of the eigenvalue matrix by \( E^{-1} \), giving

\[
A E E^{-1} = A = E \Lambda E^{-1}.
\]

If we now take powers of \( A \), the \( n \)th power of \( A \) is

\[
A^n = (E \Lambda E^{-1})^n
= E \Lambda E^{-1} E \Lambda E^{-1} \cdots E \Lambda E^{-1}
= E \Lambda^n E^{-1}.
\]

This means that the two equations have the same slope (the equation is degenerate).

The concepts may be easily extended to higher dimensions.
Finding the eigenvectors:

An eigenvector $e_k$ can be found for each eigenvalue $\lambda_k$ from Eq. 1.1,

$$(A - \lambda_k I)e_k = 0.$$ 

The left side of the above equation becomes a column vector, where each element is an equation in the elements of $e_k$, set equal to 0 on the right side. These equations are always degenerate, since the determinant is zero. Thus the two equations have the same slope.

Solving for the eigenvectors is often confusing, because they have arbitrary magnitudes,

$$||e_k|| = \sqrt{e_{k,1}^2 + e_{k,2}^2} = d.$$ 

From Eq. 1.1, you can only determine the relative magnitudes and signs of the elements of $e_k$, so you will have to choose a magnitude $d$. It is common practice to normalize each eigenvector to have unit magnitude ($d = 1$).

–Q 3.1: Find the companion matrix, and thus the matrix $A$, having the same eigenvalues as Pell’s equation.

**Hint:** Use Matlab’s function `[E, Lambda] = eig(A)` to check your results!

**Sol:** The companion matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

–Q 3.2: Solutions to Pell’s equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell’s equation is relevant to $\sqrt{2}$.

**Sol:** As discussed in the notes (Lec 9 of Chapter 2), as the iteration $n$ increases, the ratio of the $x_n/y_n$ approaches $\sqrt{2}$.

–Q 3.3: Find the first 3 values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell’s equation for $N = 2$.

**Sol:** See class notes (slide 9.4.2) for this calculation. By hand, find the eigenvalues $\lambda_{\pm}$ of the $2 \times 2$ Pell’s equation matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

**Sol:** The eigenvalues are given by the roots of the equation $(1 - \lambda_{\pm})^2 = 2$. Thus $\lambda_{\pm} = 1 \pm \sqrt{2} = \{2.4142, -0.4142\}$

–Q 3.4: By hand, show that the matrix of eigenvectors, $E$, is

$$E = \begin{bmatrix} e_+ & e_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$

**Sol:** The eigenvectors $e_{\pm}$ may be found by solving

$$A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda_{\pm} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \rightarrow (A - \lambda_{\pm} I) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$$

For $\lambda_{+}$, this gives

$$0 = \begin{bmatrix} 1 - (1 + \sqrt{2}) & 2 \\ 1 & 1 - (1 + \sqrt{2}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

which gives the relation between the elements of $e_+, e_1, e_2$, as $e_1 = \sqrt{2}e_2$.

The eigenvectors are defined to be unit length and orthogonal, namely
1. \|e_k\|^2 = e_k \cdot e_k = 1

2. \(e_+ \cdot e_- = 0\).

Once we normalize \(e_+\) to have unit length, we obtain the first eigenvector

\[ e_+ = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \]

Repeating this for \(\lambda_-\) gives

\[ e_- = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \]

Thus, the matrix of eigenvalues is

\[
E = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}
\]

\(--Q 3.5: Using the eigenvalues and eigenvectors you found for \(A\), verify that \(E^{-1}AE = \Lambda\)\)

\[
E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}
\]

\[
\text{Sol:} \text{ Using the formula for a matrix inverse, we find}

\[
E^{-1} = \frac{1}{\det(E)} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} = \frac{3}{-2\sqrt{2}\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} \\ 1 & -\sqrt{2} \end{bmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}
\]

Thus

\[
E^{-1}AE = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \frac{-\sqrt{2}}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

\[
= \frac{-1}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} (\sqrt{2} + 2) \\ (\sqrt{2} + 1) \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 - \sqrt{2} & 0 \\ 0 & 1 + \sqrt{2} \end{bmatrix} = \Lambda
\]

\(--Q 3.6: Now that you have diagonalized \(A\) (Equation 1.3), use your results for \(E\) and \(\Lambda\) to solve for the \(n = 10\) solution \((x_{10}, y_{10})^T\) to Pell’s equation with \(N = 2\).\)

\[
\text{Sol:} x_{10} = -3363 \text{ and } y_{10} = -2378. \text{ Note this formulation gives the negative solution, but since the values for } n = 10 \text{ are real, when they are squared in Pell’s equation, it makes no difference whether they are negative or positive.} \]

The Fibonacci sequence

The Fibonacci sequence is famous in mathematics, and has been observed to play a role in the mathematics of genetics. Let \(x_n\) represent the Fibonacci sequence,

\[
x_{n+1} = x_n + x_{n-1}, \tag{1.5}
\]

where the current input sample \(x_n\) is equal to the sum of the previous two inputs. This is a ‘discrete time’ recurrence relation. To solve for \(x_n\), we require some initial conditions. In this exercise, let
us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence \{1, 1, 2, 3, 5, 8, 13, \ldots\} for $n = 0, 1, 2, 3, \ldots$.

Equation 1.5 is equivalent to the $2 \times 2$ matrix equation

$$
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix}
= A
\begin{bmatrix}
x_{n-1} \\
y_{n-1}
\end{bmatrix},
$$

(1.6)

**Problem # 4:** Here we seek the general formula for $x_n$. Like the Pell’s equation, Eq. 1.5 has a recursive, eigen-decomposition solution. To find it we must recast $x_n$ as a 2x2 matrix relation, and then proceed as we did for the Pell case.

--Q 4.1: By example, show that the Fibonacci sequence $x_n$ as described above may be generated by

$$
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix}
= A^n
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix},
$$

(1.7)

--Q 4.2: What is the relationship between $y_n$ and $x_n$?

**Sol:** This equation says that $x_n = x_{n-1} + y_{n-1}$ and $y_n = x_{n-1}$. The latter equation may be rewritten as $y_n = x_{n-1}$. Thus

$$
x_n = x_{n-1} + x_{n-2},
$$

which is Eq. 1.5.

--Q 4.3: Write a Matlab/Octave program to compute $x_n$ using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is $x_{40}$?

**Note:** Consider using the eigen-decomposition of $A$, described by Eq. 1.3 (p. 380).

**Sol:** You can try something like:

```matlab
function xn = fib(n)
A = [1 1; 1 0]; [E,D] = eig(A); xy = E*D^n*inv(E)*[1; 0];
xn = xy(1);
end
```

Given the initial conditions we defined, $x_{40} = 165,580,141$.

--Q 4.4: Using the eigen-decomposition of the matrix $A$ (and a lot of algebra), it is possible to obtain the general formula for the Fibonacci sequence

$$
x_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].
$$

(1.8)

--Q 4.5: What are the eigenvalues $\lambda_{\pm}$ of the matrix $A$?

**Sol:** The eigenvalues of the Fibonacci matrix are given by

$$
det
\begin{bmatrix}
1 - \lambda & 1 \\
1 & -\lambda
\end{bmatrix}
= \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0,
$$

thus $\lambda_{\pm} = \frac{1 \pm \sqrt{5}}{2} = [1.618, -0.618]$.

--Q 4.6: How is the formula for $x_n$ related to these eigenvalues? Hint: find the eigenvectors.

**Sol:** The eigenvectors (determined from the equation $(A - \lambda_{\pm}I)e_{\pm} = 0$, and normalized to 1) are given by

$$
e_+ = \begin{bmatrix}
\frac{\sqrt{\lambda^2 + 1}}{1} \\
\frac{\sqrt{\lambda^2 + 1}}{1} \\
\frac{\sqrt{\lambda^2 + 1}}{1}
\end{bmatrix},
e_- = \begin{bmatrix}
\frac{\sqrt{\lambda^2 + 1}}{1} \\
\frac{\sqrt{\lambda^2 + 1}}{1} \\
\frac{\sqrt{\lambda^2 + 1}}{1}
\end{bmatrix},
E = \begin{bmatrix} e_+ & e_- \end{bmatrix}
$$
From the eigen-decomposition, we find that
\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = E \begin{bmatrix} \lambda_n^+ & 0 \\ 0 & \lambda_n^- \end{bmatrix} E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} \lambda_n^+ & 0 \\ 0 & \lambda_n^- \end{bmatrix} \frac{1}{(e_{11}e_{22} - e_{12}e_{21})} \begin{bmatrix} e_{22} \\ -e_{21} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Solving for \( x_n \) we find that
\[
x_n = \frac{1}{(e_{11}e_{22} - e_{12}e_{21})} \left( \lambda_n^+ e_{11} e_{22} - \lambda_n^- e_{12} e_{21} \right)
\]
\[
= \frac{1}{\sqrt{5}} \left[ \lambda_n^+ \left( \frac{\lambda_n^+}{\sqrt{(\lambda_n^+ + 1)(\lambda_n^- + 1)}} \right) - \lambda_n^- \left( \frac{\lambda_n^-}{\sqrt{(\lambda_n^+ + 1)(\lambda_n^- + 1)}} \right) \right]
\]
\[
= \frac{1}{\sqrt{5}} \left[ \lambda_n^{n+1} - \lambda_n^{n+1} \right]
\]

\[\Box\]

- **Q 4.7:** What happens to each of the two terms \([(1 \pm \sqrt{5})/2]^{n+1}\). **Sol:** \([(1 - \sqrt{5})/2]^{n+1} \to 0 \text{ and } [(1 + \sqrt{5})/2]^{n+1} \to \infty \]

- **Q 4.8:** What happens to the ratio \( x_{n+1}/x_n \)? **Sol:** \( x_{n+1}/x_n \to (1 + \sqrt{5})/2 \), because \( (1 - \sqrt{5})/2 \to 0 \) as \( n \to \infty \) thus for large \( n \), \( x_n \approx [(1 + \sqrt{5})/2]^{n+1} \).

**Problem # 5:** Replace the Fibonacci sequence with
\[
x_n = \frac{x_{n-1} + x_{n-2}}{2},
\]
such that the value \( x_n \) is the average of the previous two values in the sequence.

- **Q 5.1:** What matrix \( A \) is used to calculate this sequence? **Sol:**
\[
A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}
\]

- **Q 5.2:** Modify your computer program to calculate the new sequence \( x_n \). What happens as \( n \to \infty \)? **Sol:** As \( n \to \infty \), \( x_n \to 2/3 \).

- **Q 5.3:** What are the eigenvalues of your new \( A \)? How do they relate to the behavior of \( x_n \) as \( n \to \infty \)? Hint: you can expect the closed-form expression for \( x_n \) to be similar to Eq. 1.8. **Sol:** The eigenvalues are \( \lambda_+ = 1 \) and \( \lambda_- = -0.5 \). From Eq. 1.3, the expression for \( A^n \) is
\[
A^n = (E \Lambda E^{-1})^n = E \Lambda^n E^{-1} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n = \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix}.
\]
The solution is the sum of two sequences, one a constant and the other an oscillation that quickly fades. As \( n \to \infty \), \( \lambda_+^n = 1^n \to 1 \) and \( \lambda_-^n = (-1/2)^n \to 0 \). The solution becomes
\[
x_n = \frac{2}{3} [\lambda_+^n - \lambda_-^n] = \frac{2}{3} [1^n - (-1)^n] \to \frac{2}{3}.
\]

\[\Box\]
–Q 5.4: What matrix $A$ is used to calculate this sequence?

**Sol:**

$$ A = \begin{bmatrix} 1 & 1.01 \\ 1 & 0 \end{bmatrix} $$

–Q 5.5: Modify your computer program to calculate the new sequence $x_n$. What happens as $n \to \infty$?

**Sol:** As $n \to \infty$, $x_n \to \infty$.

–Q 5.6: What are the eigenvalues of your new $A$? How do they relate to the behavior of $x_n$ as $n \to \infty$? Hint: you can expect the closed-form expression for $x_n$ to be similar to Eq. 1.8.

**Sol:** The eigenvalues are $\lambda_+ = 1.0033$ and $\lambda_- = -0.5033$. As $n \to \infty$, $\lambda_+^n \to \infty$ and $\lambda_-^n \to 0$. Because $\lambda_+^n$ ‘blows up,’ the expression for $x_n$ also ‘blows up.’

**Problem # 6:** Consider the expression

$$ \sum_{n=1}^{N} f_n^2 = f_N f_{N+1}. $$

–Q 6.1: Prove this expression is always true.

**Sol:** Write this out for $N$ and $N - 1$:

$$ f_1^2 + f_2^2 + \cdots + f_{N-1}^2 + f_N^2 = f_N f_{N+1} $$

$$ f_1^2 + f_2^2 + \cdots + f_{N-1}^2 = f_{N-1} f_N $$

Subtracting gives

$$ f_N^2 = f_N f_{N+1} - f_{N-1} f_N = f_N (f_{N+1} - f_{N-1}) $$

$$ f_N = f_{N+1} - f_{N-1} $$

This last equation is exactly the Fibinocci equation we started from: $f_{N+1} = f_N + f_{N-1}$, hence the equation is true.

---

*4I found this problem on a worksheet for Math 213 midterm (213practice.pdf).*
Chapter 2
Algebraic Equations

2.1 Exercises AE-1

Topic of this homework: Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, roots.

Deliverable: Answers to problems

Note: The term ‘analytic’ is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

Problem #1: A polynomial of degree \( N \) is defined as

\[ P_N(x) = a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N \]

–Q 1.1: How many coefficients \( a_n \) does a polynomial of degree \( N \) have?
Sol: \( N + 1 \) ■

–Q 1.2: How many roots does \( P_N(x) \) have?
Sol: \( N \) ■

Problem #2: The fundamental theorem of algebra (FTA)

–Q 2.1: State and then explain the Fundamental Theorem of Algebra.
Sol: The FTA says that every polynomial has at least one root \( x = x_r \). ■

–Q 2.2: Using the FTA, prove your answer to the question (2) above.
Hint: Apply the FTA to prove how many roots a polynomial \( P_N(x) \) of order \( N \) has. Sol: When a root is determined, it may be factored out, leaving a new polynomial of degree one less than the first. Specifically

\[ P_{N-1}(x) = \frac{P_N(x)}{x - x_r}. \]

Thus it follows that by a recursive application of this theorem, a polynomial has a number of roots equal to its degree. All the roots must be counted, including repeated and complex roots, and roots at \( \infty \). ■
Problem #3: Consider the polynomial function $P_2(x) = 1 + x^2$ of degree $N = 2$, and the related function $F(x) = 1/P_2(x)$.

—Q 3.1: What are the roots (e.g. ‘zeros’) $x_{\pm}$ of $P_2(x)$?

Solution: Completing the square on $P_2(x) = 1 + x^2$ of degree 2, and find the roots. **Sol:** Solving for the roots by setting $P_2(x) = 0$ gives $x_{\pm}^2 = -1$ leading to $x_{\pm} = \pm 1j$. ■

Problem #4: $F(x)$ may be expressed as $(A, B, x_{\pm} \in \mathbb{C})$

$$F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}, \quad (2.1)$$

where $x_{\pm}$ are the roots (zeros) of $P_2(x)$, which become the poles of $F(x)$, and $A, B$ are the residues. The expression for $F(x)$ is sometimes called a ‘partial fraction expansion’ or ‘residue expansion’, and it appears in many engineering applications.

—Q 4.1

Find $A, B \in \mathbb{C}$ in terms of the roots $x_{\pm}$ of $P_2(x)$.

**Sol:** The fastest (i.e., easiest) way to find the constants $A, B$ is to cross-multiply

$$\frac{1}{1 + x^2} = \frac{A(x - x_-) + B(x - x_+)}{(x - x_+)(x - x_-)} = \frac{(A + B)x - (Ax_- + Bx_+)}{(x - x_+)(x - x_-)}$$

Since the numerator must equal 1, $B = -A$ and $A = 1/(x_+ - x_1)$.

In summary, in terms of the roots of Eq. 2.1

$$A = -B = \frac{1}{(x_+ - x_-)}, \quad \text{thus} \quad F(x) = \frac{1}{1 + x^2} = \frac{1}{2j} \left( \frac{1}{x - 1j} - \frac{1}{x + 1j} \right).$$

Verify your answers for $A, B$ by showing that this expression for $F(x)$ is indeed equal to $1/P_2(x)$.

—Q 4.2

Give the values of the poles and zeros of $P_2(x)$. **Sol:** The zeros are at $x_z = \pm j$, and the poles are at $x_p = \pm \infty$. ■

—Q 4.3

Give the values of the poles and zeros of $F(x) = 1/P_2(x)$. **Sol:** The poles are at $x_p = \pm j$, and the zeros are at $x_z = \pm \infty$. ■

Analytic functions

**Overview:** Analytic functions are defined by infinite (power) series. The function $f(x)$ is analytic at any value of $x = x_0$ where there exists a convergent power series

$$P(x) = \sum_{n=0}^{\infty} a_n x^n$$

such that $P(x_0) = f(x_0)$. The local power series for $f(x)$ near $x = x_0$ is often obtained by finding the Taylor series:

$$f(x) \approx f(x_0) + \frac{df}{dx} \bigg|_{x=x_0} (x - x_0) + \frac{1}{2!} \frac{d^2 f}{dx^2} \bigg|_{x=x_0} (x - x_0)^2 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n} \bigg|_{x=x_0} (x - x_0)^n.$$
The point \( x = x_0 \) is called the series expansion point.

When the expansion point is at \( x_0 = 0 \), the series is denoted a MacLaurin series. Two classic examples are the geometric series\(^1\) where \( a_n = 1 \)

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \tag{2.1}
\]

and the exponential function where \( a_n = 1/n! \)

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{2.2}
\]

The coefficients for both series may be derived from the Taylor formula (or MacLaurin formula, when the expansion point is zero).

**Problem # 1: The geometric series**

\[ -Q \, 1.1: \text{What is the region of convergence (RoC) for the power series of } 1/(1 - x) \text{ given above (e.g. where does the power series } P(x) \text{ converge to the function value } f(x)? \text{ State your answer as a condition on } x. \]

\[ \text{Hint: What happens to the power series when } x > 1? \]

\[ \text{Sol: } |x| < 1, \text{ because for } |x| \geq 1 \text{ the power series diverges to infinity.} \]

\[ -Q \, 1.2: \text{In terms of the pole, what is the RoC for the geometric series (Eq. 2.1)?} \]

\[ \text{Sol: The nearest pole relative to the expansion point, at } x = 0 \text{ is at the nearest pole } x_p = 1 \text{ to the expansion point at } x = 0. \text{ Namely the RoC is } 1 \text{ re } 0. \]

\[ -Q \, 1.3: \text{How does the RoC relate to the location of the pole of } 1/(1 - x)? \]

\[ \text{Sol: The pole is at } x = 1, \text{ on the border of the RoC. The nearest pole relative to the expansion point, at } x = 0 \text{ is at } x = 1. \text{ Thus the RoC is } 1. \]

\[ -Q \, 1.4: \text{Where are the zeros, if any, in Eq. 2.1?} \]

\[ \text{Sol: There is a single zero at } x = \infty. \]

\[ -Q \, 1.5: \text{Assuming } x \text{ is in the RoC, prove that the geometric series correctly represents } 1/(1 - x) \text{ by multiplying both sides of Eq. 2.1 by } (1 - x). \]

\[ \text{Sol:} \]

\[
1 = \frac{1 - x}{1 - x} \quad \text{for all } x \neq 1 \\
= (1 - x)(1 + x + x^2 + x^3 \ldots), \quad |x| < 1 \\
= (1 + x + x^2 + x^3 \ldots) - x(1 + x + x^2 \ldots) \\
= (1 + x + x^2 + x^3 \ldots) - (x + x^2 + x^3 \ldots) \\
= (1 + (x - x) + (x^2 - x^2) + (x^3 - x^3) \ldots), \quad |x| < 1 \\
= 1
\]

The introduction of a removable singularity voids the solution at \( x = 1 \). When one expands the denominator in a series, the relation is good for \( |x| < 1 \) (not for \( x = 1 \)).

If one lets \( z = 1/x \) the relation becomes

\[
1 = \frac{1 - z}{1 - z},
\]

which is valid for \( z \neq 1 \), which when expanded the RoC is \( |z| < 1, \text{ or } x > 1. \)

\(^1\)The geometric series is not defined as the function \( 1/(1 - x) \), it is defined as the series \( 1 + x + x^2 + x^3 + \ldots \), such that the ratio of consecutive terms is \( x \).
Q 1.6: Use the geometric series to study the degree $N$ polynomial (It is very important to note that all the coefficients of this polynomial are

$$P_N(x) = 1 + x + x^2 + \ldots + x^N = \sum_{n=0}^{N} x^n. \quad (2.3)$$

Q 1.7: Prove that

$$P_N(x) = \frac{1 - x^{N+1}}{1 - x} \quad (2.4)$$

Sol:

$$P_N(x) = 1 + x + x^2 \ldots x^N$$

$$= \sum_{n=0}^{\infty} x^n - \sum_{n=N+1}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} x^n - x^{N+1} \sum_{n=0}^{\infty} x^n$$

$$= (1 - x^{N+1}) \sum_{n=0}^{\infty} x^n$$

$$= \frac{1 - x^{N+1}}{1 - x}$$

Q 1.8: What is the RoC for Eq. 2.2?

Sol: This series converges everywhere (The RoC is the entire plane $|z| < \infty$), due to the strong convergence effect of the factorial ($a_n = 1/n!$) of the coefficients of the series.

Q 1.9: What is the RoC for Eq. 2.3?

Sol: There is no pole, thus the RoC is $\infty$. The polynomial only has zeros.

Q 1.10: What is the RoC for Eq. 2.4?

Sol: The pole nearest the expansion point is at $x = 1$. Thus the RoC is 1.

Q 1.11: Evaluate $P_N(x)$ at $x = 0$ and $x = 9$ for the case of $N = 100$, and compare the result to that from Matlab.

Sol: $P_N(0) = 1$ and $P_N(0.9) = \frac{1 - 0.9^{N+1}}{1 - 0.9} = 9.999760947410010$. According to Matlab $P_{100}(0) = 1$ and $P_{100}(0.9) = 9.999760947410014$, with a difference of $-3.55271 \times 10^{-15}$ (i.e., $16 \times \text{eps}$).

Q 1.12: How many poles does $P_N(x)$ have? Where are they?

Sol: Since $P_N(x)$ is defined by Eq. 2.3, there is no poles at $x = 1$. However it still has a pole of order $N$ at $x = \infty$. To show this, define $z = 1/x$ and study the zeros.

Q 1.13: How many zeros does $P_N(x)$ have? State where are they in the complex plane?

Sol: There are zeros at $x = \frac{-1 \pm \sqrt{1 - 4N^2}}{2N}$. Total = $N + 1$ zeros. However, the zero at 1 is removable because it is on top of the pole at 1. This is refered to as a removable singularity.
-Q 1.14: Does the above expression have both poles and zeros? Explain. Sol: Written in this way, every \( N \)th degree polynomial, with \( N \) zeros, has a single pole and \( N + 1 \) zeros.

-Q 1.15: Explain why Eq. 2.3 and 2.4 have different numbers of poles and zeros. Sol: Eq. 2.4 has \( N + 1 \), known as the roots of unity, which exactly cancel the pole. Exactly how this happens is very interesting. This is similar to the GCD problem we saw previously where common factors in a rational number may be removed, but in this case, it is a common root that cancels.

-Q 1.16: Is the function \( \frac{1}{1-x} \) analytic outside of the RoC stated in part (a)? Hint: Can it be represented by a different power series outside this RoC? Sol: Yes, and we can use the geometric series to prove this. Consider \( x = \frac{1}{r} > 1 \), meaning \( r < 1 \).

\[
\frac{1}{1-x} = \frac{-r}{1-r} = -r \sum_{n=0}^{\infty} r^n = -\sum_{n=1}^{\infty} r^n = -\sum_{n=1}^{\infty} \frac{1^n}{x}
\]

Problem # 2 The exponential series.

-Q 2.1: What is the region of convergence (RoC) for the exponential series given above (e.g. where does the power series \( P(x) \) converge to the function value \( f(x) \))? Sol: The exponential is convergent everywhere on the open real line.

-Q 2.2: Let \( x = j \) in Eq. 2.2, and write out the series expansion of \( e^x \) in terms of its real and imaginary parts. Sol:

\[
e^j = \sum_{n=0}^{\infty} \frac{j^n}{n!} = 1 + j - \frac{1}{2!} - j\frac{1}{3!} + \frac{1}{4!} + j\frac{1}{5!} - \frac{1}{6!} \cdots
\]

\[
= \left(1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots\right) + j\left(\frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \cdots\right)
\]

\[
= \sum_{n=0,2,\cdots}^{n \text{ even}} \frac{(-1)^n}{n!} + j \sum_{n=1,3,\cdots}^{n \text{ odd}} \frac{(-1)^n}{n!}.
\]

-Q 2.3: Let \( x = j\theta \) in Eq. 2.2, and write out the series expansion of \( e^x \) in terms of its real and imaginary parts. How does your result relate to Euler’s identity \( e^{i\theta} = \cos(\theta) + j\sin(\theta) \)? Sol:

\[
e^{j\theta} = \sum_{n=0}^{\infty} \frac{j^n\theta^n}{n!}
\]

\[
= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} \cdots
\]

\[
= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots\right) + j\theta\left(\frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)
\]

\[
= \cos(\theta) + j\sin(\theta).
\]
Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function \((x, y \in \mathbb{C})\)
\[ y(x) = \frac{1}{1 - x} \]
the inverse is
\[ x = \frac{y - 1}{y} = 1 - \frac{1}{y}. \]

**Problem #1:** Considering the inverse function described above

–**Q 1.1:** Where are the poles and zeros of \(x(y)\)?

**Sol:** Pole at \(y = 0\), zero at \(y = 1\). There are no poles or zeros at \(\infty\) because \(\lim_{y \to \pm\infty} (y - 1)/y = 1\).

–**Q 1.2:** Where (for what condition on \(y\)) is \(x(y)\) analytic?

**Sol:** Anywhere but the pole, at \(y = 0\).

**Problem #2** Considering the exponential function \(z(x) = e^x (x, z \in \mathbb{C})\).

–**Q 2.1:** Find the inverse \(x(z)\).

**Sol:** Taking the natural log (ln) of both sides gives \(x = \ln(z)\). Thus the natural log is the inverse of the exponential.

–**Q 2.2:** Where are the poles and zeros of \(x(z)\)?

**Sol:** Pole at \(z = 0\), zero at \(z = 1\). There is another pole at \(z = +\infty\) as well.

–**Q 2.3:** Compose these two functions \((y \circ z)(x)\)

Give the expression for \((y \circ z)(x) = y(z(x))\). **Sol:**
\[ (y \circ z)(x) = \frac{1}{1 - e^x} \]

–**Q 2.4:** Where are the poles and zeros of \((y \circ z)(x)\)?

**Sol:** Pole at \(x = 0\), zero at \(x = +\infty\).

–**Q 2.5:** Where (for what condition on \(x\)) is \((y \circ z)(x)\) analytic?

**Sol:** Everywhere except \(x = 0\).

Convolution

Multiplying two polynomials, when they are short or simple, is not demanding. However if they have many terms, it can become tedious. For example, multiplying two \(10^{th}\) degree polynomials is not something one would want to do every day.

An alternative is a method called convolution, as described in Lecture 3.4 (p. 97).

**Problem #1:** Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Show you work!!! Check your solution using Matlab.

–**Q 1.1:** Convolve the sequence \([0 1 1 1]\) with itself. **Sol:** The answer is \([0 0 1 2 3 4 3 2 1]\).
2.1. EXERCISES AE-1

–Q 1.2: Convolve \([1 1]\) with itself, then convolve the result with \([1 1]\) again (e.g., calculate \([1 1] \ast [1 1] \ast [1 1]\)). ] Sol:

\[
[1, 1] \ast [1, 1] = [1, 2, 1] \\
[1, 1] \ast [1, 1] \ast [1] = [1, 1] \ast [1, 2, 1] \\
= [1, 3, 3, 1]
\]

Problem # 2: Multiplying two polynomials is the same as convolving their coefficients.

\[
f(x) = x^3 + 3x^2 + 3x + 1 \\
g(x) = x^3 + 2x^2 + x + 2
\]

In Matlab, compute \(h(x) = f(x) \cdot g(x)\) two ways using (a) the commands \texttt{roots} and \texttt{poly}, and (b) the convolution command \texttt{conv}. Confirm that both methods give the same result. That is, compute the convolution \([1, 3, 3, 1] \ast [1, 2, 1, 2]\).

–Q 2.1: What is \(h(x)\)?

t Sol: \(h(x) = x^6 + 5x^5 + 10x^4 + 12x^3 + 11x^2 + 7x + 2\)

Newton’s root-finding method

Problem # 1: Use Newton’s iteration to find roots of the polynomial

\[
P_3(x) = 1 - x^3.
\]

–Q 1.1: Draw a graph describing the first step of the iteration starting with \(x_0 = (1/2, 0)\).
Sol: Start with an \((x, y)\) coordinate system and put points at and the vertex of \(P_3(x)\).

–Q 1.2: Calculate \(x_1\) and \(x_2\). What number is the algorithm approaching?
Sol: First we must find \(P_3'(x) = -2x^2\). Thus the equation we will iterate is

\[
x_{n+1} = x_n + \frac{1 - x_n^3}{3x_n}
\]

–Q 1.3: Here is a matlab script for the \(P_2(x)\) case. Modify it to find \(P_3(x)\):

```matlab
x(1)=1/2; %x(1)=0.9; %x(1)=-10
y(1)=x(1);
for n=2:10
    x(n) = x(n-1) + (1-x(n-1)^2) / (2*x(n-1));
    y(n) = (1+y(n-1)^2) / (2*x(n-1));
end
semilogy(abs(x)-1); hold on
semilogy(abs(7)-1,'or'); hold off
```

–Q 1.4: Write a Matlab script to check your answer for part (a).
Sol:

```matlab
x=1/2;
for n = 1:3
    x = x+(1-x*x) / (2*x);
end
```
Q 1.5: For \( n = 4 \), what is the absolute difference between the root and the estimate, \(|x_r - x_4|\)?

Sol: 4.6E-8 (very small!)

Q 1.6: What happens if \( x_0 = -1/2 \)?

Sol: You converge on the negative root, \( x = -1 \).

Q 1.7: Does Newton’s method work for \( P_2(x) = 1 + x^2 \)?

If so, why? Hint: What are the roots in this case?

Sol: Here \( P'_2(x) = +2x \) thus the iteration gives

\[
x_{n+1} = x_n - \frac{1 + x^2_n}{2x_n}.
\]

In this case the roots are \( x_{\pm} = \pm i \), namely purely imaginary. Obviously Newton’s method fails, because there is no way for the answer to become complex. Real in, real out.

Q 1.8: What if you let \( x_0 = (1 + i)/2 \) for the case of \( P_2(x) = 1 + x^2 \)?

Sol: By starting with a complex initial value, we fix the Real in = Real out problem.

Riemann zeta function \( \zeta(s) \)

Definitions and preliminary analysis:

The zeta function \( \zeta(s) \) is defined by the complex analytic power series

\[
\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.
\] (2.5)

This series converges, and thus is valid, only in the region of convergence (ROC) given by \( \Re s = \sigma > 1 \) since there \(|n^{-\sigma}| < 1 \). To determine its formula in other regions of the \( s \) plane one must extend the series via analytic continuation.

Euler product formula: As was first published by Euler in 1737, one may recursively factor out the leading prime term, resulting in Euler’s product formula.2 Multiplying \( \zeta(s) \) by the factor \( 1/2^s \), and subtracting from \( \zeta(s) \), removes all the terms \( 1/(2n)^s \) (e.g., \( 1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots \))

\[
\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\right),
\] (2.6)

which results in

\[
\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots.
\] (2.7)

Problem #1:

Q 1.1: What is the RoC for Eq. 2.7

Sol: \([2^s] > 1 \).

Q 1.2: Repeat this with a lead factor \( 1/3^s \) applied to Eq. 2.7.

Sol:

\[
\frac{1}{3^s} \left(1 - \frac{1}{2^s}\right) \zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \cdots.
\] (2.8)

Subtracting Eq. 2.8 from Eq. 2.7 cancels the RHS terms of Eq. 2.7

\[
\left(1 - \frac{1}{3^s}\right) \left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} + \cdots.
\]

\[2\]This is known as Euler’s sieve, as distinguish from the Eratosthenes sieve.
2.1. EXERCISES AE-1

-Q 1.3: Repeat this process, with all prime scale factors (i.e., $1/5^s, 1/7^s, \ldots, 1/\pi_k^s, \ldots$), and show that

$$\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1 - \frac{s}{\pi_k}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s)$$

where $\pi_p$ represents the $p^{th}$ prime.

**Sol:** The above defines each factor $\zeta_k(s)$ as the $k^{th}$ term of the product. Each recursive step in this construction assures that the lead term, along with all of its multiplicative factors, are subtracted out. ■

-Q 1.4: Given the product formula we may identify the poles of $\zeta_p(s)$ ($p \in \mathbb{Z}$), which is important for defining the ROC of each factor.

For example, the $p^{th}$ factor of Eq. 2.9, expressed as an exponential, is

$$\zeta_p(s) \equiv \frac{1}{1 - e^{-sT_p}}$$

where $T_p \equiv \ln \pi_p$. **Sol:** Thus factor $\zeta_p(s)$ has poles at $s_n(p)$ where $2\pi jn = s_n T_p$, giving

$$s_n(p) = \frac{2\pi jn}{\ln \pi_p}$$

with $-\infty < n \in \mathbb{C} < \infty$. These poles might be viewed as the eigen-modes of the zeta function. ■

-Q 1.5: Plot $\zeta_p(s)$ using zviz for $p = 1$. Describe what you see.

**Sol:** $\zeta_1(s)$ has poles at integral multiples of $T_1 = \log 2$, as shown in Fig. 2.1. ■

---

Figure 2.1: *Plot of $w(s) = \frac{1}{1 \cdot e^{-s \pi}}$ which is related to each factor $\zeta_p(s)$ (Eq. 2.9). Here $w_k(s)$ has poles where $1 = e^{-sT_1}$, namely at $s_n = n2\pi j$, as may be seen from the colorized plot.*

---

5 Each factor (i.e., $\zeta_p(s)$) has poles at $s_n = j2\pi n/T_p, n \in \mathbb{C}$ (i.e., $e^{-s T_p} = 1$).
2.2 Exercises AE-2

Topic of this homework:

Linear systems of equations; Gaussian elimination; Matrix permutations; Overspecified systems of equations; Analytic geometry; Ohm’s law; Two-port networks

Deliverable: Answers to problems

Nonlinear (quadratic) to linear equations

Problem #1: Nonlinear (quadratic) to linear equations

In the following problems we deal with algebraic equations in more than one variable, that are not linear equations. For example, the circle $x^2 + y^2 = 1$ is just such an equation. It may be solved for $y(x) = \pm \sqrt{1 - x^2}$.

Example: If we let $z = x + y = x + y\sqrt{1 - x^2} = e^{\theta\jmath}$, we obtain the equation for half a circle ($y > 0$). The entire circle is described by the magnitude of $z$, as $|z|^2 = (x + y\jmath)(x - y\jmath) = 1$.

--Q 1.1: Given the curve defined by the equation:

$$x^2 + xy + y^2 = 1$$

--Q 1.2: Find the function $y(x)$.

Sol: Completing the square in $y$ and solve for $y(x)$:

$$(y + x/2)^2 - x^2/4 + x^2 = 1$$

$$(y + x/2)^2 = 1 - \frac{3}{4}x^2$$

$$y + x/2 = \pm \sqrt{\frac{4 - 3x^2}{4}}$$

$$y = \frac{1}{2} \left( \pm \sqrt{4 - 3x^2} - x \right)$$

--Q 1.3: Using Matlab/Octave, plot $y(x)$, and describe the graph.

Sol:
Thus we find the equation is a rotated ellipse. ■

-Q 1.4: What is the name of this curve?
Sol: It is an ellipse, rotated by 45 degrees. ■

-Q 1.5: Find the solution (in x, p, and q) to the following equations:

\[
\begin{align*}
  x + y &= p \\
  xy &= q
\end{align*}
\]

Sol: Solve the first equation for y as \( y = p - x \), and then substitute it into the second equation

\[ x(p - x) = -x^2 + px = q. \]

Thus we find the quadratic

\[ x^2 - px + q = 0 \]

having roots given by completing the square

\[ (x - p/2)^2 = (p/2)^2 - q. \]

resulting in \( x = p/2 \pm \sqrt{(p/2)^2 - q}, y = p - x \).

Summary: Here we started with one linear and one quadratic (hyperbola). By the use of composition we found the roots. ■

-Q 1.6: Find an equation that is linear in y starting from equations that are quadratic (2nd degree) in the two unknowns

\( x, y \):

\[
\begin{align*}
  x^2 + xy + y^2 &= 1 \\
  4x^2 + 3xy + 2y^2 &= 3
\end{align*}
\]

(2.11) (2.12)

Sol: The goal is to obtain a linear equation in y. Thus we need to remove the quadratic term \( y^2 \). Scale the upper equation by 2 and subtract it from the lower equation:

\[ 2x^2 + xy = 1 \]

Solving for y gives \( y = (1 - 2x^2)/x \).

Note that matrix notation is not as useful for quadratic equations as it is for linear equations (but it can be used in some cases). ■

-Q 1.7: Compose the two quadratic equations

\[
\begin{align*}
  x^2 + xy + y^2 &= 1 \\
  2x^2 + xy &= 1
\end{align*}
\]

and describe the results. Sol: By isolating y from one of the two equations, we may remove it from the other equation, giving us a single 4th degree equation in x:

\[ x^2 + (1 - 2x^2) + (1 - 2x^2)^2/x^2 = 1 \]

\[ ^4 \text{This problem is taken from Stillwell, Exercise 6.2.1 (p. 91).} \]
or
\[ x^4 + x^2 - 2x^4 + 1 - 4x^2 + 4x^4 - x^2 = 0 \]

Collecting terms
\[ 3x^4 - 4x^2 + 1 = 0 \]

This is a quartic, but is a quadratic in \( x^2 \). Of course \( x \) may be complex, rendering this very difficult to deal with in any detail. **Conclusion**: We started with two ellipses (they have an \( xy \) term which can be removed by a rotation, as we showed in Problem 1.1) This again demonstrates that composition of \( m \times n \) gives degree of \( mn \). When one multiplies polynomials the degree is \( m + n \). Thus composition gives the product and multiplication gives the sum of the degrees of the individual polynomials. ■

**Gaussian elimination**

**Problem # 2: Gaussian elimination**

**–Q 2.1: Find the inverse of**

\[ A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}. \]

**Sol:**

\[ A^{-1} = \frac{1}{3 - 8} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}. \]

■

**–Q 2.2: Verify that** \( A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

**Sol:** Multiply them to show this. ■

**Problem # 3: Find the solution to the following 3x3 matrix equation** \( Ax = b \) **by Gaussian elimination.** **Show your intermediate steps.** **You can check your work at each step using Octave/Matlab.**

\[
\begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
1 \\
9 \\
8
\end{bmatrix}.
\]

**–Q 3.1 Show (i.e., verify) that the first GE matrix \( G_1 \), which zeros out all entries in the first column, is given by**

\[ G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \]

Identify the elementary row operations that this matrix performs. **Sol:** Operate with GE matrix on \( A \)

\[ G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 & \vdots & 1 & 1 & -1 & 1 \\
-3 & 1 & -0 & \vdots & 9 & 3 & 1 & 1 \\
-1 & 0 & 1 & \vdots & 8 & 1 & -1 & 4
\end{bmatrix}
\]

\[ = \begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & -2 & 4 & 6 \\
0 & -2 & 5 & 7
\end{bmatrix}. \]

It scales the first row by -3 and adds it to the second row, and scales the first row by -1 and adds it to the third row. ■
–Q 3.2 Find a second GE matrix, $G_2$, to put $G_1A$ in upper triangular form. Identify the elementary row operations that this matrix performs.

Sol:

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

which scales the second row by -1 and adds it to the third row. Thus we have

$$G_2G_1[A|b] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 3 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

–Q 3.3 Find a third GE matrix, $G_3$, which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.

Sol:

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which scales the second row by -1/2. Thus we have

$$G_3G_2G_1[A|b] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

–Q 3.4: Finally, find the last GE matrix, $G_4$, that subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ($G_4G_3G_2G_1A = I$).

Sol:

$$G_4 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus we find $G_4G_3G_2G_1[A|b]$ is

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 3 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

–Q 3.5: Solve for $\{x_1, x_2, x_3\}^T$ using the augmented matrix format $G_4G_3G_2G_1[A|b]$ (where $\{A|b\}$ is the augmented matrix). Note that if you’ve performed the preceding steps correctly, $x = G_4G_3G_2G_1b$.

Sol: From the preceding problems, we see that $[x_1, x_2, x_3]^T = [3, -1, 1]^T$. ■
Permutations and Pivots

Problem #4: Permutations and Pivots

1. Find the pivot matrix $G$ that rescales the third row of the augmented matrix $A|b$ by $1/3$.

Sol:

$$ G = \begin{bmatrix} 1 & 1/3 \\ 1 & 1 \\ 1 \\ 1 \\ 1/3 \end{bmatrix} $$

Sol:

$$ G_1 A = \begin{bmatrix} 1 & 1/3 \\ 1 & 1 \\ 1 & 1 \\ 1 \\ 1/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 1/3 & 3 \\ 1 & -1 & 4 & 8 \end{bmatrix} $$

---

Two linear equations

Problem #5 In this exercise we transition from a general pair of equations

\[
\begin{align*}
    f(x, y) &= 0 \\
    g(x, y) &= 0
\end{align*}
\]

to the important case of two linear equations

\[
\begin{align*}
    y &= ax + b \\
    y &= \alpha x + \beta.
\end{align*}
\]

Note that, to help keep track of the variables, Roman coefficients $(a, b)$ are used for the first equation and Greek $(\alpha, \beta)$ for the second.

Question 5.1: What does it mean, graphically, if these two linear equations have

1. a unique solution,
2. a non-unique solution, or
3. no solution?

Sol: There are three possibilities:

1. When they have different slopes, they meet at one $(x,y)$ point, which is the solution.
2. If the two lines are identical, any point on the line is a solution.
3. If they have the same slope but different intercepts (are parallel to each other) there is no solution.

Question 5.2: Assuming the two equations have a unique solution, find the solution for $x$ and $y$.

Sol: Since there must be one point where the two are equal, we may solve for that by setting the $y$ values equal to each other:

\[
a x + b = \alpha x + \beta
\]
2.2. EXERCISES AE-2

Thus

\[
\begin{align*}
x &= \frac{\beta - b}{a - \alpha} \\
y &= a\frac{\beta - b}{a - \alpha} + b
\end{align*}
\]

**Q 5.3:** When will this solution fail to exist (for what conditions on \(a, b, \alpha,\) and \(\beta\))?  
**Sol:** As stated above, if they have the same slope \(\alpha = a\) but different intercepts \(\beta \neq b\), there is no solution. When \(\beta = b\) and \(\alpha = a\) every point on the line is a solution.  

**Q 5.4:** Write the equations as a 2x2 matrix equation of the form \(Ax = b\), where \(x = \{x, y\}^T\).

**Sol:**

\[
\begin{bmatrix}
1 & -a \\
1 & -\alpha
\end{bmatrix}
\begin{bmatrix}
y \\
x
\end{bmatrix}
= 
\begin{bmatrix}
b \\
\beta
\end{bmatrix}
\]

**Q 5.5:** Finding the inverse of the 2x2 matrix, and solve the matrix equation for \(x\) and \(y\).

**Sol:**

\[
\begin{bmatrix}
y \\
x
\end{bmatrix}
= \frac{1}{\Delta}
\begin{bmatrix}
-\alpha & a \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
b \\
\beta
\end{bmatrix}
= \frac{1}{\Delta}
\begin{bmatrix}
-\alpha b + a\beta \\
-b + \beta
\end{bmatrix}
\]

where the determinant is \(\Delta \equiv a - \alpha\).  

**Q 5.6:** Discuss the properties of the determinant of the matrix \(\Delta\) in terms of the slopes of the two equations \((a\) and \(\alpha)\).  
**Sol:** When the slopes are the same there is no solution and \(\Delta = 0\). Thus the matrix solution is consistent with the geometry. This is our first result in analytic geometry.  

**Problem # 6:** The application of linear functional relationships between two variables:

2x2 matrices are used to describe 2-port networks, as will be discussed in Sect. 3.7. Transmission lines are a great example, where both voltage and current must be tracked as they travel along the line. Figure 3.9 shows an example segment of a transmission line.

![Figure 3.9](image)

**Figure 2.2:** This figure shows a cell from an LC transmission line. The index 1 is at the input on the left and 2 represents the output, on the right. This figure is discussed in Sect. 3.7.3 (p. 124).

Suppose you are given the following pair of linear relationships between the input (source) variables \(V_1\) and \(I_1\), and the output (load) variables \(V_2\) and \(I_2\) of the transmission line.

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix}
= 
\begin{bmatrix}
J & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}
.
\]
–Q 6.1: Let the output (the load) be \( V_2 = 1 \) and \( I_2 = 2 \) (i.e., \( V_2/I_2 = 1/2 \) \{\( \Omega \)\}). Find the input voltage and current, \( V_1 \) and \( I_1 \).

**Sol:** This case corresponds to

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
1 + 2 \\
1 - 2
\end{bmatrix}
\]

Thus \( V_1 = 2 + 1j \) and \( I_1 = -1 \).

–Q 6.2: Let the input (source) be \( V_1 = 1 \) and \( I_1 = 2 \). Find the output voltage and current \( V_2 \) and \( I_2 \).

**Sol:** With the input specified the two equations are

\[
\begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
V_2 \\
I_2
\end{bmatrix},
\]

To find the input we must invert the matrix (\( \Delta = -j - 1 \))

\[
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix} = \frac{1}{1 + j} \begin{bmatrix}
1 & 1 \\
1 & -j
\end{bmatrix} \begin{bmatrix}
1 \\
2
\end{bmatrix}.
\]

Thus \( V_2 = 3/(1 + j) = 3(1 - j)/2 \), \( I_2 = (1 - 2j)/(1 + j) = -(1 + 3j)/2 \). The point of this exercise is that the two lines have a complex intersection point, not easily visualized.

**Linear equations with three unknowns**

**Problem # 7:** This problem is similar to the previous problem, except we consider 3 dimensions. Consider two linear equations in unknowns \( x, y, z \), representing planes:

\[
a_1x + b_1y + z = c_1 \quad (2.13)
\]
\[
a_2x + b_2y + z = c_2 \quad (2.14)
\]

–Q 7.1: In terms of the geometry (i.e., think graphically), under what conditions do these two linear equations have (a) a unique solution, (b) a non-unique solution, or (c) no solution?

**Sol:** This problem is virtually identical to the previous problem, except the solution is for the intersection of two planes in three dimensions \( z(x, y) \) rather than the intersection of two lines \( y(x) \), in 2 dimensions. One might picture a plane as a line in two dimensions. That is, if you “sweep” a line along a third dimension, it forms a plane.

In this case, 2 equations with 3 unknowns, there is no unique solution in \( x, y, z \). There are three possibilities:

1. There is no unique solution - we require a third plane to have a single point of intersection.
2. There are two cases: (1) When the two planes have different slopes, they meet at along a line. The solution of the problem \( (x, y) \) is then a line rather than a point. (non-unique). (2) If the two planes are identical (same slope and same intercept), all points on the plane(s) are solutions. (non-unique)
3. If the planes have the same slope, but different intercepts (are parallel to each other) there is no solution. (no solution)
2.2. EXERCISES AE-2

---Q 7.2: Do you know what is meant by the slope of a plane? Can you define it?  
Sol: This is a concept that is natural in vector calculus. We have not gotten there yet, but this idea requires the concept of a gradient, which defines a vector perpendicular to the plane. We shall deal with this concept the third section of this course (i.e., stream 3). ■

---Q 7.3: Given 2 equations in 3 unknowns, the closest we can come to a ‘unique’ solution is a line (describing the intersection of the planes) rather than a single point. This line is an equation in $(x, y)$, $(y, z)$, or $(x, z)$. Find a solution in terms of $x$ and $y$ by substituting one equation into the other.  
Sol: $(a_1 - a_2)x + (b_1 - b_2)y + (c_2 - c_1) = 0$ ■

Problem # 8: Now consider the intersection of the planes at some arbitrary constant height, $z = z_0$.

---Q 8.1: Write the modified plane equations as a 2x2 matrix equation in the form $Ax = b$ where $x = \{x, y\}^T$, and find the unique solution in $x$ and $y$ using matrix operations.  
Sol: 

\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
cia_1 - z_0 \\
cia_2 - z_0
\end{bmatrix}
\]

---Q 8.2: Assuming the two equations have a unique solution, find the solution for $x$ and $y$.  
Sol: We are looking for the specification of a line, that is $x(y)$ or $y(x)$, that is determined by the two equations, which define two planes $(z_1(x, y), z_2(x, y))$. To find the solution solve for $z(x, y)$ and use Gaussian elimination (GE) on the system of equations. Setting ■

---Q 8.3: When will this solution fail to exist (for what conditions on $a_1$, $a_2$, $b_1$, $b_2$, etc.)?  
Sol: As stated above, if the planes have the same “slope,” but different intercepts, there is no solution. The problem is that we don’t know what the slope or intercept of the plane means. But we do know how to apply gaussian elimination. We have shown before that if the determinant $\Delta = a_1b_2 - a_2b_1 \neq 0$, then the ‘slopes’ of the planes are not equal (the planes are not parallel to each other at height $z_0$). Thus the geometry has the same meaning as for the case of lines, but in three rather than two dimensions. To prove this we need to apply GE (or the equivalent). ■

---Q 8.4: Now, write the system of equations as a 3x3 matrix equation in $x$, $y$, $z$ given the additional equation $z = z_0$ (e.g. put it in the form $Ax = b$ where $x = \{x, y, z\}^T$).  
Sol: 

\[
\begin{bmatrix}
a_1 & b_1 & 1 \\
a_2 & b_2 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
=
\begin{bmatrix}
cia_1 \\
cia_2 \\
cia_3
\end{bmatrix}
\]

Problem # 9: The determinant of a 3x3 matrix is given by

\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix}
= a_{11}
\begin{vmatrix}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{vmatrix}
- a_{12}
\begin{vmatrix}
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{vmatrix}
+ a_{13}
\begin{vmatrix}
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{vmatrix}
\]

---Q 9.1: For the 3x3 matrix equation you wrote in the previous part, find the determinant. How is the determinant related to the 2x2 case? Why?  
Sol: The determinants are the same. In both cases, we require $a_1b_2 - a_2b_1 \neq 0$ for the solution to exist. It makes sense that these answers are the same, as we haven’t changed the basic equations, just the format we presented them in. ■
**Problem # 10:** Put the following systems of equations in matrix form, and use Octave/Matlab to find (i) the determinant of the matrix, (ii) the matrix inverse, and (iii) the solution \((x, y, z)\). *If it is not possible to complete (i-iii), state why.*

---

**Q 10.1: Example 1**

\[
\begin{align*}
    &x + 3y + 2z = 1 \\
    &x + 4y + z = 1 \\
    &x + y = 1
\end{align*}
\]

**Sol:**

\[
\begin{bmatrix}
1 & 3 & 2 \\
1 & 4 & 1 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

(i) The matrix determinant is -4. (ii) The inverse matrix is

\[
\begin{bmatrix}
.25 & -.5 & 1.25 \\
-.25 & .5 & -.25 \\
.75 & -.5 & -.25
\end{bmatrix}
\]

(iii) The solution is \([1, 0, 0]^T\)

---

**Q 10.2: Example 2:**

\[
\begin{align*}
    &x + 3y + 2z = 1 \\
    &2x + 6y + 4z = 1 \\
    &x + y = 1
\end{align*}
\]

**Sol:**

\[
\begin{bmatrix}
1 & 3 & 2 \\
2 & 6 & 4 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

(i) The determinant is 0. (ii-iii) No solution, because the first two equations represent parallel planes. To show this, expand the determinant along the bottom row: \(1(12 - 12 - 4 - 4) + 0(6 - 6))\).

---

**Integer equations: applications and solutions**

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

**Problem # 11:** *A practical example of using a Diophantine equation:*
“A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?”

Bachet de Béziriac (1623 CE)

Here, weighing is performed using a balance scale having two pans, with weights being put on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weight in the other pan. Then, the scale will be balanced. A solution to the four weights for Bachet’s problem is $1 + 3 + 9 + 27 = 40$ pounds.

A solution: $2 = -1 + 3$

---

**Q 11.1**: Show how the combination of 1, 3, 9, & 27 pound weights may be used to weigh 1, 2, 3, \ldots 8, 28, and 40 pounds of milk (or something else, such as flour). Assuming that the milk is in the left pan, provide the position of the weights using a negative sign ‘-’ to indicate the left pan and a positive sign ‘+’ to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan, ‘+1’ will balance the scales.

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]$T. The pan assignments matrix should only contain the values -1 (left pan), +1 (right pan), and 0 (leave out). You can indicate these using ‘-‘, ‘+‘, and blank spaces.

**Sol:** Any integer between 1 and 40 may be expanded using the weights 1, 3, 9, 27. Here is the problem stated in matrix form:

$$
\begin{bmatrix}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
\vdots \\
28 \\
\vdots \\
40
\end{bmatrix}
= \begin{bmatrix}
+ & + \\
- & + \\
+ & + \\
- & - & + \\
+ & + & - & + \\
- & - & + \\
\vdots \\
+ & + \\
\vdots \\
+ & + & + & + \\
\end{bmatrix}
\begin{bmatrix}
1 \\
3 \\
9 \\
27
\end{bmatrix}
$$

The left column is the weight of the milk. The right-most column are the four weights. It should be clear that these four weights span the integers from 1-40 with binary weights. Each weight may be computed recursively from twice the sum of the previous weights +1, that is $W_{n+1} = 2W_n + 1 = 2^{n+1}$ since $W_n = 2^n$.

For example to get 26 we place weights 9+3+1 in the pan with 26, and get 27-1. For example 27 = $2^7(9+3+1)+1$ is the next weight. Recursively, the weights are $3=2^1+1$, $9=2^3(3+1)+1$, $27=2^7(9+3+1)+1$. The next weight (not shown) would be: $81=2^4(27+9+3+1)+1 = 2^840+1$. ■
**Ohm’s Law**

In general, impedance is defined as the ratio of a force over a flow. For electrical circuits, the voltage is the ‘force’ and the current is the ‘flow.’ Ohm’s law states that the voltage across and the current through a circuit element are related by the impedance of that element (which may be a function of frequency).

For resistors, the voltage over the current is called the resistance, and is a constant (e.g. the simplest case, \( V/I = R \)). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g. \( V/I = Z(s) \), where \( s \) is the complex frequency \( s \in \mathbb{C} \)).

The impedance concept also holds in mechanics and acoustics. In mechanics, the ‘force’ is equal to the mechanical force on an element (e.g. a mass, dashpot, or spring), and the ‘flow’ is the velocity. In acoustics, the ‘force’ is pressure, and the ‘flow’ is the volume velocity or particle velocity of air molecules.

<table>
<thead>
<tr>
<th>Case</th>
<th>Force</th>
<th>Flow</th>
<th>Impedance</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>voltage (V)</td>
<td>current (I)</td>
<td>( Z )</td>
<td>Ohms [( \Omega )]</td>
</tr>
<tr>
<td>Mechanics</td>
<td>force (F)</td>
<td>velocity (V)</td>
<td>( Z )</td>
<td>Mechanical Ohms [( \Omega )]</td>
</tr>
<tr>
<td>Acoustics</td>
<td>pressure (P)</td>
<td>particle velocity (U)</td>
<td>( Z )</td>
<td>Acoustic Ohms [( \Omega )]</td>
</tr>
<tr>
<td>Thermal</td>
<td>temperature (T)</td>
<td>heat-flux (J)</td>
<td>( Z )</td>
<td>Thermal Ohms [( \Omega )]</td>
</tr>
</tbody>
</table>

**Problem #12:** The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As it lights up, the resistance of the metal filament increases.

Ohm’s law says that the current

\[
\frac{V}{I} = R(T).
\]

where \( T \) is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz].

---

**Q 12.1:** *Find the current when the light is first switched on.*

**Sol:** Thus the current is

\[
I = \frac{120}{R} = \frac{120}{100} = 1.2. \quad \text{[Amps]}
\]

As the bulb heats up, the current rapidly drops, and the resistance increases. This typically takes less than a millisecond [ms], which depends on the wattage of the light bulb. Such lightbulbs are nonlinear. These rules don’t apply to LED bulbs.

**Problem #13:** The power in Watts is the product of the force and the flow.

---

**Q 13.1:** *What is the power of the light bulb of this example?*

**Sol:** \( P = V \cdot I = 120 \times 1.2 = 120 + 24 = 144 \) [W].

**Problem #14:** State the impedance \( Z(s) \) of each of the following circuit elements:

---

**Q 14.1:** *A resistor with resistance R:*

**Sol:** \( Z = R \)

**Q 14.2:** *An inductor with inductance L:*

**Sol:** \( Z = sL \) with \( s = \sigma + \omega j \). Note the flux \( \psi(t) = Li(t) \). The voltage \( v(t) \) is the time derivative of the flux

\[
v(t) = \frac{d\psi(t)}{dt} = L \frac{di(t)}{dt}.
\]
2.2. EXERCISES AE-2

---Q 14.3: A capacitor with capacitance $C$:

**Sol:** $Z = 1/sC$. Note the charge $q(t) = Cv(t)$, thus the current $i(t)$ is the time derivative of the charge

$$i(t) = \frac{d}{dt}q(t) = C\frac{dv(t)}{dt}.$$  

Problem # 15: Consider what happens at the triple-point of water. As water freezes or thaws, the temperate remains constant at 0°C. Once all the water is frozen and more heat is removed, the temperature drops below 0 °. As heat is added, water thaws, but the temperature remains at 0°.

---Q 15.1: Once all the ice is melted, as more heat is added, find the temperature as more heat is added

Model the triple point using a zener diode, a resistor and a capacitor. A zener holds the voltage constant independent of current. For the case of water’s triple-point, the voltage represents the temperature of water at the triple point, clamped at 0 [C°]. The current represent the heat flux. The latent heat of water at the triple point is 32 [Cal/gm]. Thus as the temperature rises from below freezing, the water is clamped at 0°C once the triple point is reached. Once there adding more heat flux as no effect on the temperature until all the ice melts. Once melted, the temperature again begins to rise until it hits the boiling point, where it again stays at 100°, until all the water has evaporated. **Sol:** Need a figure here showing how to model the triple point of water. The Heat capacity may be modeled by a capacitor, which is fixed at 0° as the capacitor discharges. Once it is empty, the temperature again begins to rise as the heat $Q$ from the sun is added

$$T° = mcQ.$$  

Thus the required circuit needs to emulate this temperature behavior due to the latent heat of melting ice and boiling water into steam.

2-port network analysis

Problem # 16: Perform an analysis of electrical 2-port networks, shown in Fig. 2.3. This can be a mechanical system if the capacitors are taken to be springs, and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

![Figure 2.3: Left: A low-pass RC electrical filter. The circuit elements $R_1$, $R_2$, and $C$ are defined in the problems below. Right: A band-pass acoustic filter. Here, the pressure $P$ is analogous to voltage, and the velocity $U$ is analogous to current. The circuit elements are labeled with their $L$ and $C$ values as integers, to make the algebra simple.](image)

The definition of the ABCD transmission matrix ($T$) is

$$
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.
$$  

(2.15)
The impedance matrix, where the determinant \( \Delta_T = AD - BC \), is given by

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \frac{1}{C} \begin{bmatrix}
A & \Delta_T \\
1 & D
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\] (2.16)

---

**Q 16.1:** Derive the formula for the impedance matrix (Eq. 2.16) given the transmission matrix definition (Eq. 2.15).

Show your work. **Sol:** The formula may be easily derived by re-arranging the equations from the matrix (Eq. 2.16). Begin with

\[
V_1 = AV_2 - BI_2 \\
I_1 = CV_2 - DI_2
\]

From the second equation, we get

\[
V_2 = \frac{1}{C}I_1 + \frac{D}{C}I_2
\]

which gives (upon substitution)

\[
V_1 = \frac{A}{C}I_1 + \frac{AD}{C}I_2 - BI_2 = \frac{A}{C}I_1 + \left(\frac{AD}{C} - B\right)I_2
\]

which yields the matrix equation

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \begin{bmatrix}
A/C & (AD/C - B) \\
1/C & D/C
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix} = \frac{1}{C} \begin{bmatrix}
A & \Delta_T \\
1 & D
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\] (2.17)

---

**Problem #17:** Consider a single circuit element with impedance \( Z(s) \)

---

**Q 17.1:** What is the ABCD matrix for this element if it is in 'series'?

**Sol:**

\[
\begin{bmatrix}
1 & Z(s) \\
0 & 1
\end{bmatrix}
\]

---

**Q 17.2:** What is the ABCD matrix for this element if it is 'shunt'?

**Sol:**

\[
\begin{bmatrix}
1 & 0 \\
1/Z(s) & 1
\end{bmatrix}
\]

---

**Problem #18:** Find the ABCD matrix for each of the circuits of Figure 2.3.

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency \( s \in \mathbb{C} \), then (ii) substitute \( s = 1j \) and calculate the total transmission matrix at this single frequency.

---

**Q 18.1:** Left circuit (let \( R_1 = R_2 = 10 \text{ k}\Omega \) ‘kilo-ohms’, and \( C = 10 \text{ nF} \) ‘nano-farads’)

**Sol:** Write the system in chain matrix form:

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
1 & Z_1 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
Y_C & 1
\end{bmatrix}
\begin{bmatrix}
1 & Z_3 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
\]

Now we substitute the given values:

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
1 & 10^4 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
j10^{-8} & 1
\end{bmatrix}
\begin{bmatrix}
1 & 10^3 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix} = \begin{bmatrix}
1 + j10^{-4} & 2 \times 10^4 + j \\
-10^8 & 1 + j10^{-4}
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
\]

---
2.2. EXERCISES AE-2

–Q 18.2: Right circuit (use $L$ and $C$ values given in the figure), where the pressure $P$ is analogous to the voltage $V$, and the velocity $U$ is analogous to the current $I$.

**Sol:** Write the system in chain matrix form:

\[
\begin{bmatrix}
P_1 \\ U_1
\end{bmatrix} = \begin{bmatrix} 1 & sL_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sL_4 & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2
\end{bmatrix}
\]

Now we substitute the given values:

\[
\begin{bmatrix}
P_1 \\ U_1
\end{bmatrix} = \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2j & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{j} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/4j & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2
\end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & 4j \\ \frac{5}{3} & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2
\end{bmatrix}
\]

I used Matlab/Octave to evaluate this script:

\[
a=[1 \ j; \ 0 \ 1]; \ b=[1 \ 0; 2j \ 1]; \ c=[1 \ 1/3j; \ 0 \ 1]; \ d=[1 \ 0; 1/4j \ 1]; \ T=a*b*c*d.
\]

Finally I found $T(2,1)$ to be $19/12$ using the Matlab/Octave command: `rats(1.5833,6)`.

–Q 18.3: Convert both transmission (ABCD) matrices to impedance matrices using Equation 2.16. Do this for the specific frequency $s = 1j$, as in the previous part (feel free to use Matlab/Octave for your computation).

**Sol:** Left circuit: Using the previous solution, and matlab:

\[
\begin{bmatrix}
V_1 \\ V_2
\end{bmatrix} = \frac{1}{j10^{-8}} \begin{bmatrix} 1 + j10^{-4} & 1 \\ 1 & 1 + j10^{-4} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2
\end{bmatrix}
\]

–Q 18.4: Right circuit: Using the previous solution, and matlab:

**Sol:**

\[
\begin{bmatrix}
P_1 \\ P_2
\end{bmatrix} = \frac{1}{1.5833j} \begin{bmatrix} -\frac{2}{3} & 1 \\ \frac{5}{3} & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2
\end{bmatrix}
\]
2.3 Exercises AE-3

**Topic of this homework:**
Visualizing complex functions; Bilinear/Möbius transform; Riemann sphere.

**Algebra**

**Problem # 1: Fundamental theorem of algebra (FTA)**

–**Q 1.1:** State the fundamental theorem of algebra (FTA).

**Sol:** There are multiple definitions of the FTA, which of course must be equivalent.

Here are three (equivalent) answers from Wikipedia:

1. The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This may then be applied recursively till the degree is zero.

2. Every degree \( n \) polynomial with complex coefficients has, counted with multiplicity, exactly \( n \) roots. The equivalence of the two statements can be proven through the use of successive polynomial division.

3. The field of complex numbers is algebraically closed. Note: this one requires an understanding of the term *algebraically closed*.

Wikipedia warns:

In spite of its name, there is no purely algebraic proof of the theorem, since any proof must use the completeness of the reals (or some other equivalent formulation of completeness), which is not an algebraic concept.

**Problem # 1: Order and complex numbers:**

One can always say that 3<4, namely that real numbers have *order*. One way to view this is to take the difference, and compare to zero, as in \( 4 - 3 > 0 \). Here, we will explore how complex variables may be ordered. Define the complex variable \( z = x + iy \in \mathbb{C} \).

–**Q 1.1:** Explain the meaning of \( |z_1| > |z_2| \).

**Sol:** \( |z| = \sqrt{x^2 + y^2} \) is the length of \( z \), so the above expression says that a disk of radius \( |z_1| \) contains a second disk of radius \( |z_2| \).

–**Q 1.2:** If \( x_1, x_2 \in \mathbb{R} \) (are real numbers), define the meaning of \( x_1 > x_2 \).

**Hint:** Take the difference. **Sol:** This conditions is the same as \( x_1 - x_2 > 0 \). Order is meaningful on the real line, as a length.

–**Q 1.3:** Explain the meaning of \( z_1 > z_2 \).

**Sol:** It makes no sense to order complex numbers. A complex number has both a length and an angle (it is the same as a vector). The concept of an angle extends the sign of a real number, making order impossible. To show this, place to points on a plane, and ask which is larger than the other. The order of the x and y components, each have order. Thus order cannot be defined.
2.3. EXERCISES AE-3

–Q 1.4: If time were complex how might the world be different?
Sol: As best we know, time is real, thus it has the order property: the is a past, present and future. If time were complex this would not be the case. Thus if time were complex, the past could be the future. ■
(not graded)

Problem # 2: It is sometimes necessary to consider a function \( w(z) = u + iv \) in terms of the real functions, \( u(x, y) \) and \( v(x, y) \) (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse \( z(w) = x + iy \) where \( x(u, v) \) and \( y(u, v) \) are real functions.

–Q 2.1: Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = 1/z \).
Sol: Multiply by the complex conjugate \( x - iy \)

\[
w = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}
\]

Therefore \( u(x, y) = \frac{x}{x^2 + y^2} \) and \( v(x, y) = \frac{-y}{x^2 + y^2} \). ■

Problem # 3: Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = c z \) with complex constant \( c \in \mathbb{C} \) for the following cases:

–Q 3.1: \( c = e \)
Sol: Since \( u + iv = e^z = e^{x+iy} = e^x (\cos y + i \sin y) \),

\[
u = e^x \cos y
\]

and

\[
v = e^x \sin y.
\]

–Q 3.2: \( c = 1 \) (recall that \( 1 = e^{2\pi k} \) for \( k = 0, 1, 2, \ldots \))
Sol: From the general formula with \( c = 1 \)

\[
1^z = e^{z \log 1} = e^{ik2\pi} = e^0 = 1
\]

where \( k \) is any integer. Thus \( u = \cos(2\pi k) = 1 \) and \( v = \sin(2\pi k) = 0 \) ■

–Q 3.3: \( c = j \). Hint: \( j = e^{\pi/2} \).
Sol: \( j^3 = \left(e^{\pi/2}\right)^3 = e^{3\pi/2} = e^{-\pi/2} = 0.2079 \).

Thus \( j^z = \left(e^{\pi/2}\right)^z = e^{iz\pi/2} = \cos(z\pi/2) + j \sin(z\pi/2) \). ■

–Q 3.4: Find \( u(x, y) \) for \( w(z) = \sqrt{z} \). Hint: Begin with the inverse function \( z = w^2 \).
Sol: Expanding we find \( x + iy = u^2 - v^2 + 2uvj \). Breaking into real and imaginary parts we find \( x = u^2 - v^2 \) and \( y = 2uv \). Removing \( v = y/2u \) from the first equation gives

\[
u^2 = x + (y/2u)^2
\]

\[
u^2 = x + (y/2)^2 / u^2
\]

\[
u^4 - xu^2 = (y/2)^2
\]

\[
(u^2 - x/2)^2 = (x/2)^2 + (y/2)^2
\]

\[
u^2 = \frac{x}{2} \pm \sqrt{\frac{x^2 + y^2}{2} \cdot \frac{y}{2u}}
\]

Thus \( u, v \) should be the equation that shows as the contour lines for the Matlab command \( W=\sqrt{Z} \). This is more informative if you scale \( Z \) by 5 or even 10. ■

–Q 2.3 AE-4
Möbius transforms and infinity

**Problem #1: The bilinear transform:**
The bilinear $z$ transform is used in signal processing to design a digital (discrete-time) filter $H(z)$ starting from analog (continuous time) filter design $H(s)$. The goal of the bilinear transform is to take a function of “analog frequency” $\omega_a$, where $\omega_a \in (-\infty, \infty)$, and map it to a finite “digital frequency” range, $\omega_d \in [-\pi, \pi]$. 

$\omega$ relationship between $H$.

**Hint:** Use the Octave/Matlab command `help bilinear`.

Express your final result using a tangent function. **Hint:** Try to form sine and cosine terms!

The bilinear transform is given by

$$
H(z) = \frac{1}{\alpha \frac{1-z^{-1}}{1+z^{-1}} + 1} = \frac{1 + z^{-1}}{(1 + \alpha)(1 - \alpha)z^{-1}} = \frac{z + 1}{(1 + \alpha)z + (1 - \alpha)}.
$$

This has a zero at $z = -1$ and a pole at $z = \frac{1 - \alpha}{1 + \alpha}$. The poles in the $z^{-1}$ plane must be inside the unit circle for the filter to be stable, similar to the poles being in the left half plane in the $s$ plane. However most text books work in the $z$ plane rather than in the $z^{-1}$ plane. Thats crazy. Everyone knows your inside is left (Ha Ha). Seriously, if you define $z^{-1}$, then work in that plane. Would you define the $s$ plane and then work in the $s^{-1}$ plane? I hope not. What could it mean? 

**Problem #2: You are given the analog low-pass filter $h(t) = e^{-t}u(t)$**

It has a frequency response given by

$$
H(s) = \frac{1}{s + 1} = \int_0^\infty h(t)e^{-st}dt.
$$

$\omega$ starts from $\omega_a$(analog) filter design $H$.

**Hint:** Use the Octave/Matlab command `help bilinear` to find the discrete time filter $H(z)$.

**Sol:** Compose $H(s)$ with $s(z)$ to get $H(z) = H \circ s(z)$

$$
H(z) = \frac{1}{\alpha \frac{1-z^{-1}}{1+z^{-1}} + 1} = \frac{1 + z^{-1}}{(1 + \alpha)(1 - \alpha)z^{-1}} = \frac{z + 1}{(1 + \alpha)z + (1 - \alpha)}.
$$

**Sol:** For simplicity let $\alpha = 1/2$.

$$
\omega_d = 2 \tan^{-1}(2 \omega_a)
$$

$\omega_d$ range, $\omega_d \in [-\pi, \pi]$. Seriously, if you define $z^{-1}$, then work in that plane. Would you define the $s$ plane and then work in the $s^{-1}$ plane? I hope not. What could it mean? 

**Problem #2: Substitute $s = j\omega_a$ and $z = e^{j\omega_d}$ ($\sigma_a, \sigma_d = 0$) into the Eq. 2.18 to determine the relationship between $\omega_a$, $\omega_d$.**

Express your final result using a tangent function. **Hint:** Try to form sine and cosine terms! Recall that $\sin(\omega) = (e^{j\omega} - e^{-j\omega})/2j$ and $\cos(\omega) = (e^{j\omega} + e^{-j\omega})/2$.

$$
s = \frac{1}{2} \frac{1 - z^{-1}}{1 + z^{-1}}
$$

$$
\omega_a = \frac{1}{2} \frac{e^{j\omega_d/2} - e^{-j\omega_d/2}}{e^{j\omega_d/2} + e^{-j\omega_d/2}} = \frac{1}{2} \frac{2j \sin(\omega_d/2)}{2 \cos(\omega_d/2)}
$$

$$
\omega_d = 2 \tan^{-1}(2 \omega_a)
$$

$\omega_d$ range, $\omega_d \in [-\pi, \pi]$. Seriously, if you define $z^{-1}$, then work in that plane. Would you define the $s$ plane and then work in the $s^{-1}$ plane? I hope not. What could it mean? 

$\omega_d$ range, $\omega_d \in [-\pi, \pi]$. Seriously, if you define $z^{-1}$, then work in that plane. Would you define the $s$ plane and then work in the $s^{-1}$ plane? I hope not. What could it mean?
2.3. EXERCISES AE-3

–Q 2.3: By hand, draw a graph of the relationship you found the previous part, \( \omega_a = f(\omega_d) \). Make sure to specify the behavior of \( \omega_a \) at \( \omega_d = 0, \pm \pi/2, \pm \pi \). Sol: The graph is shown in Fig. 2.4 should show that \( \omega_a \to \pm \infty \) for \( \omega_d \pm \pi \), \( \omega_a = \pm \frac{1}{2} \) at \( \omega_d = \pm \pi/2 \), and \( \omega_a = 0 \) for \( \omega_d = 0 \).

–Q 2.4: Explain how this relationship maps the analog frequency \( \omega_a \to \pm \infty \) to a the digital frequency \( \omega_d \).
Sol: The analog frequencies \( \omega_a \to \pm \infty \) are mapped to \( \omega_d \to \pm \pi \).

–Q 2.5: Now consider the complex frequency planes, \( s = \sigma_a + j\omega_a \) and \( z = e^{\sigma_d + j\omega_d} \). To map \( \omega_a = f(\omega_d) \), we set \( \sigma_a, \sigma_d = 0 \). Draw the \( s \) and \( z \) planes, showing the real parts on the horizontal axes and the imaginary parts on the vertical axes. Mark (e.g. using thick lines) which sets of values are considered when \( \sigma_a, \sigma_d = 0 \). Sol: On the \( s \)-plane, draw a vertical line at \( \sigma_a = 0 \). On the \( z \)-plane, draw the unit circle \( z = e^{j\omega_d} \).

–Q 2.6: Geometrically, what is the effect of this Möbius transform? Consider your drawing in the previous part.
Sol: It maps a line in the \( s \) plane (\( j\omega_a \)) to a circle in the \( z \) plane (\( e^{j\omega_d} \)).

Probability

Many things in life follow rules we don’t understand, that are not predictable, yet have structure due to some underlying physics. Unlike mathematicians, engineers are taught to learn to deal with such uncertainty, in terms of random processes and probability theory. For many this starts out as a large set of boresing incomprehensible definitions, but once you begin to understand it, becomes interesting mathematics. It needs to be in your skin. If you don’t have an intuition for it, either keep working on it, or else find another job. Don’t memorize a bunch of formulae because that won’t work over the long run.

Some view probability as combinatorics. This is wrong. It is much more than that. From my auditory view of speech in noise, probability is about the signal processing of noise and signals (i.e., not combinatorics).

Definitions:

1. An event is the term to describe an unpredictable outcome.

Example: Measuring the temperature \( T(x, t) \in \mathbb{R} \) with \( x \in \mathbb{R}^3 \) at time \( t \) [s] is an event.

Example: Measuring the temperature every hour gives 24 events per day, with degrees as the units.

Example: The single toss of a coin, resulting in \( \{H, T\} \), is an unpredictable event.

–Q 0.7: What are the units of a coin toss?
Sol: It doesn’t seem to have any, which seems like an issue.
414

CHAPTER 2. ALGEBRAIC EQUATIONS

–Q 0.8: What are the units of \( H, T \)?

Sol: It doesn’t seem to have any, which seems like an issue.

2. A trial is \( N \) events.

3. An experiment \( \{M, N\} \) is defined as \( M \) trials of \( N \) events.

4. Number of events: One must always keep track of the number of events so that one can compute the mean (i.e., average) and the uncertainty of an observable outcome.

5. The mean of many trials is the average.

6. A random variable \( X \) is defined as the outcome from an experiment. A random variable rarely has stated units.

Example: Flipping a coin \( N = 8 \) times defines the number of trials.

–Q 0.9: Give the units of coin flips?

\[ X \equiv \{H, H, H, T, H, T, T, T\}. \]

Sol: The random variable \( \{H, T\} \) does not have units, it has random outcomes.

–Q 0.10: How do you give mean to something that doesn’t have units?

Sol: One must get creative. Let \( H = 1 \) and \( T = -1 \) so that the mean can be zero.

7. The expected value is the mean of \( N \) events.

–Q 0.11: What is the difference between the mean, expected value and the average?

Sol: These all mean the same thing. Having several words that mean the same thing is one of the many things that makes probability theory so arbitrary. It is sloppy to have unnecessary terminology.

–Q 0.12: How to assign mean outcomes \( \{H, T\} \) using numbers.

Sol: If we let \( H = 1 \) and \( T = 0 \) then the mean is

\[ \mu = (1 + 1 + 1 + 0 + 1 + 0 + 0 + 0)/8 = 1/2. \]

8. Note: It is critically important to keep track of the number of events \( (N = 8 \) in the above example) because \( N \) is more important than the actual measured sequence.

9. Note: It is helpful to think of \( N \) as the independent variable and \( X \) as the dependent variable. For example, think of \( X(N) \) rather than \( N(X) \). Concentrate on the fixed \( N \), not the random sequence.

Example: To form a trial by flipping a coin \( N = 10 \) times. We form an experiment by \( M \) repeated trials \( (M = 1000) \).

Problem # 1: Basic terminology of experiments

–Q 1.1: What is the mean of a trial, and what is the average over all trials?

Sol: The mean and average are the same. What is different in these two questions is the size of the set being averaged.
Q 1.2: What is the “expected value” of a random variable $X$?
**Sol:** This is a mathematical definition of how to compute the mean, as an inner product.

Q 1.3: What is the “standard deviation” about the “mean”?
**Sol:** This is the expected value of the second moment of the random variable.

Q 1.4: What is the definition of “information” of a random variable?
**Sol:** The information is $I = 1/P(X_k)$.

Q 1.5: What is the “entropy” of a random variable?
**Sol:** The expected value of the log of the information: $I$.

Q 1.6: What is the “sampling noise” of a random variable?
**Sol:** Sampling noise is the mean error due to a finite number of samples $N \in \mathbb{N}$. The magnitude of the mean sampling noise is $\sigma = 1/\sqrt{N}$.
Chapter 3
Differential equations

3.1 Exercises DE-1

Topic of this homework:
Complex numbers and functions (ordering and algebra); Complex power series; Fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions; Multivalued functions (branch cuts and Riemann sheets)

Complex Power Series

Problem # 1: In each case derive (e.g. using Taylor’s formula) the power series of \( w(s) \) about \( s = 0 \) and state the ROC of your series. If the power series doesn’t exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at \( s = 0 \).

- Q 1.1: \( 1/\left(1 - s\right) \)
  Sol: \( 1/(1 - s) = \sum_{n=0}^{\infty} s^n \) which converges for \( |s| < 1 \) (e.g., the ROC is \( |s| < 1 \)).

- Q 1.2: \( 1/(1 - s^2) \)
  Sol: \( 1/(1 - s^2) = \sum_{n=0}^{\infty} s^{2n} \) which converges for \( |s^2| < 1 \). (e.g., the ROC is \( |s| < 1 \)). One can also factor the polynomial, thus write it as: \( \frac{1}{(1-s)(1+s)} \). There are two poles, at \( s = \pm 1 \), and each has an ROC of 1.

- Q 1.3: \( 1/(1 - s)^2 \)
  Sol: To show this note that \( -\frac{d}{ds}(1-s)^{-1} = \frac{1}{(1-s)^2} \). Expanding this gives \( \frac{1}{(1-s)^2} = -\frac{d}{ds} \sum_{n=0}^{\infty} s^n = \sum_{n=1}^{\infty} ns^{n-1} = \sum_{n=0}^{\infty}(n+1)s^n \), which converges for \( |s| < 1 \). A second way is to factor \( 1 - s^2 \) and then convolve the coefficients of the \( \infty \) series of \( 1/(1 \pm is) \).

- Q 1.4: \( 1/(1 + s^2) \). Hint: This series will be very ugly to derive if you try to take the derivatives \( \frac{d}{ds}[1/(1 + s^2)] \). Using the results of our previous homework, you should represent this function as \( w(s) = -0.5i/(s - i) + 0.5i/(s + i) \).
  Sol: The resulting series is \( 1/(1 + s^2) = 0.5 \sum_{n=0}^{\infty} s^n((-i)^n + (i)^n) \). The ROC is \( |s| < 1 \). We can see this by considering the poles of the function at \( s = \pm i \); both poles are 1 from \( s = 0 \), the point of expansion. An alternative is to write the function as \( 1/(1 - (is)^2) = \sum (is)^n \).
CHAPTER 3. DIFFERENTIAL EQUATIONS

Q 1.5: \(1/s\)
Sol: If you try to do a Taylor expansion at \(s = 0\), the first term, \(w(0) \to \infty\). Thus, the Taylor series expansion in \(s\) does not exist.

Q 1.6: \(1/(1 - |s|^2)\)
Sol: The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed.

Problem # 2: Consider the function \(w(s) = 1/s\)

Q 2.1: Expand this function as a power series about \(s = 1\). Hint: Let \(1/s = 1/(1-1+s) = 1/(1 - (1 - s))\).
Sol: The power series is
\[
 w(s) = \sum_{n=0}^{\infty} (-1)^n(s - 1)^n
\]
which converges for \(|s - 1| < 1\).

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s,s,’ExpansionPoint’,1)` which is equivalent to the shorthand `syms s; taylor(1/s,s,1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to \(z = -1\) via the Möbius “translation” \(s = z + 1\), and expand using the Taylor series
\[
 \frac{1}{s} = \frac{1}{1 + z} = \sum_{n=0}^{\infty} (-z)^n.
\]
Next back-substitute \(z = s - 1\) giving
\[
 \frac{1}{s} = \sum_{n=0}^{\infty} (-1)^n(s - 1)^n.
\]
It follows that the RoC is \(|z| = |s - 1| < 1\), as provided by Matlab/Octave.

Q 2.2: What is the ROC?
Sol: \(-s\)\(\to\)1

Q 2.3: Expand \(w(s) = 1/s\) as a power series in \(s^{-1} = 1/s\) about \(s^{-1} = 1\).
Sol: Let \(z = s^{-1}\) and expand about 1:
\[
 \frac{1}{1 - s^{-1}} = \frac{s}{s - 1} = s(1 + s + s^2 + s^3 \ldots) = s + s^2 + s^3 \ldots.
\]
which has a zero at 0 and a pole at 1.

Q 2.4: What is the ROC?
Sol: \(|s| < 1\)

Q 2.5: What is the residue of the pole?
Sol: -0.

Problem # 3: Consider the function \(w(s) = 1/(2 - s)\)

Q 3.1: Expand \(w(s)\) as a power series in \(s^{-1} = 1/s\). State the ROC as a condition on \(|s^{-1}|\). Hint: Multiply top and bottom by \(s^{-1}\).
Sol: \(1/(2 - s) = -s^{-1}/(1 - 2s^{-1}) = -s^{-1} \sum_{n=0}^{\infty} 2^n s^{-n}\). The ROC is \(|2/s| < 1\), or \(|s| > 2\)
3.1. EXERCISES DE-1

–Q 3.2: Find the inverse function \( s(w) \). Where are the poles and zeros of \( s(w) \), and where is it analytic?

**Sol:** Solving for \( s(w) \) we find \( 2 - s = 1/w \) and \( s = 2 - 1/w = (2w - 1)/w \). This has a pole at 0 and a zero at \( w = 1/2 \). The ROC is therefore from the expansion point out to, but not including \( w = 0 \). ■

–Q 3.3: If \( a = 0.1 \) what is the value of
\[
x = 1 + a + a^2 + a^3 \cdots?
\]
Show your work. **Sol:** To sum this series, use the fact that
\[
x - ax = (1 + a + a^2 + a^3 \cdots) - a(1 + a + a^2) = 1 + a(1 - 1) + a^2(1 - 1) + \cdots
\]
This gives \( x(1 - a) = 1 \), or \( x = 1/(1 - a) \). Now since \( a = .1 \), the sum is \( 1/(1 - 0.1) = 1.11 \). ■

–Q 3.4: If \( a = 10 \) what is the value of
\[
x = 1 + a + a^2 + a^3 \cdots?
\]
**Sol:** In this case the series clearly does not converge. To make it converge we need to write a formula for \( y = 1/x \) rather than for \( x \).
\[
y - y/a = (1 + 1/a + 1/a^2 + 1/a^3 \cdots) - 1/a(1 + 1/a + 1/a^2) = 1 + (1 - 1)/a + (1 - 1)/a^2 + \cdots
\]
This gives \( y(1 - 1/a) = 1 \), or \( y = 1/(1 - 1/a) \). Now since \( a = 10 \), the sum is \( y = 1/(1 - 0.1) = 9 \). We might conclude that since \( x = 1/y \), \( x = 1/9 \). Does this make sense? ■

Two fundamental theorems of calculus

**Fundamental Theorem of Calculus (Leibniz):**

According to the Fundamental Theorem of (Real) Calculus (FTC)
\[
F(x) = F(a) + \int_a^x f(\xi)d\xi,
\]
where \( x, a, \xi, F \in \mathbb{R} \). This is an **indefinite integral** (since the upper limit is unspecified). It follows that
\[
\frac{dF(x)}{dx} = \frac{d}{dx} \int_a^x f(x)dx = f(x).
\]
This justifies also calling the indefinite integral the **anti-derivative**.

For a closed interval \([a, b]\), the FTC is
\[
\int_a^b f(x)dx = F(b) - F(a),
\]
thus the integral is independent of the path from \( x = a \) to \( x = b \).

**Fundamental Theorem of Complex Calculus:**

According to the Fundamental Theorem of Complex Calculus (FTCC)
\[
f(z) = f(z_0) + \int_{z_0}^z F(\zeta)d\zeta,
\]
where \( z_0, z, \zeta, F \in \mathbb{C} \). It follows that
\[
\frac{df(z)}{dz} = \frac{d}{dz} \int_{z_0}^z F(\zeta)d\zeta = F(z).
\]
For a closed interval \([s, s_0]\), the FTCC is
\[
\int_{s_0}^{s} f(\zeta) d\zeta = F(s) - F(s_0),
\] (3.4)
thus the integral is independent of the path from \(x = a\) to \(x = b\).

**Problem # 4**

**-Q 4.1:** (1 pts)
Consider Equation 3.1. What is the condition on \(f(x)\) for which this formula is true? 

**Sol:** The sufficient condition is that the integrand \(f(x)\) is be analytic, namely \(f(x) = \sum_{n=0}^{\infty} a_n x^n\). This assures that \(f(x)\) is single valued and may be integrated, since it may integrated term by term. It follows that as long as \(x < \text{ROC}\), this integral exists. Thus the integral equals \(F(x) - F(a)\). Note that if the integrand has a Taylor series, all of its derivatives exist within the ROC, because the coefficients depend on derivatives of \(f(x)\).

**-Q 4.2:** (1 pts)
Consider Equation 3.3. What is the condition on \(f(z)\) for which this formula is true? 

**Sol:** The sufficient condition is that the integrand \(f(z)\) must be complex analytic, namely \(f(z) = \sum_{n=0}^{\infty} c_n z^n\), with \(c_n \in \mathbb{C}\).

**-Q 4.3:** (3 pts)
Perform the following integrals (\(z = x + iy \in \mathbb{C}\)):
1. \(I = \int_{0}^{1+j} zdz\) 
   **Sol:** 
   \[
   I = \int_{x=0}^{1} (x + 0y) dx + \int_{y=0}^{1} (1 + yj) dy j \\
   = \frac{1}{2} x^2\big|_{0}^{1} + \int_{y=0}^{1} (1 - y) dy \\
   = \frac{1}{2} + \left(y - \frac{1}{2} y^2\right)\big|_{0}^{1} \\
   = \frac{1}{2} + j - \frac{1}{2} = j
   \]
   We conclude that the integration of \(z\) is independent of the path. This is true for any integrand \(z^n\) with \(n \in \mathbb{Z}\).

2. \(I = \int_{0}^{1+j} zdz\), but this time make the path explicit: from 0 to 1, with \(y=0\), and then to \(y=1\), with \(x=1\). 
   **Sol:**
   \[
   I = \int_{x=0}^{1} (x + 0y) dx + \int_{y=0}^{1} (1 + yj) dy j \\
   = \frac{1}{2} x^2\big|_{0}^{1} + \int_{y=0}^{1} (1 - y) dy \\
   = \frac{1}{2} + \left(y - \frac{1}{2} y^2\right)\big|_{0}^{1} \\
   = \frac{1}{2} + j - \frac{1}{2} = j
   \]
   We conclude that the integration of \(z\) is independent of the path. This is true for any integrand \(z^n\) with \(n \in \mathbb{Z}\).

3. Discuss whether your results agree with Equation 3.2? 
   **Sol:** Yes the two integrals must agree, because the function is analytic, and the integral must be the same, independent of the path.

**-Q 4.4:** (3 pts)
Perform the following integrals on the closed path \(C\), which we define to be the unit circle. You should substitute \(z = e^{i\theta}\) and \(dz = ie^{i\theta} d\theta\), and integrate from \(-\pi, \pi\) to go once around the unit circle.
1. \(f_C zdz\) 
   **Sol:** 
   \[
   f_C zdz = \int_{-\pi}^{\pi} e^{i\theta} de^{i\theta} = \int_{-\pi}^{\pi} e^{i2\theta} i d\theta = e^{i2\theta}\big|_{-\pi}^{\pi} = 0.
   \]
2. \(f_C \frac{1}{z} dz\) 
   **Sol:** 
   \[
   f_C \frac{1}{z} dz = \int_{-\pi}^{\pi} i d\theta = 2\pi i.
   \]
3. Discuss whether your results agree with Equation 3.4? 
   **Sol:** (a) obeys the FTCC because \(f(z) = z\) is analytic everywhere, (b) does not obey the FTCC because \(f(z) = 1/z\) is not analytic at \(z=0\) (inside \(C\)).
Cauchy-Riemann Equations

For the following problem: \( i = \sqrt{-1}, s = \sigma + i\omega, \) and \( F(s) = u(\sigma, \omega) + iv(\sigma, \omega). \)

**Problem # 5:** According to the Fundamental theorem of complex calculus the integration of a complex analytic function is independent of the path. It follows that the derivative of \( F(s) \) is defined as

\[
\frac{df}{ds} = \frac{d}{ds} [u(\sigma, \omega) + iv(\sigma, \omega)].
\]  

(3.5)

If the integral is independent of the path, then the derivative must also be independent of direction

\[
\frac{df}{ds} = \frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial \omega}.
\]  

(3.6)

The Cauchy-Riemann (CR) conditions

\[
\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}
\]

may be used to show where Equation 3.6 holds.

--Q 5.1: Assuming Equation 3.6 is true, use it to derive the CR equations.

**Sol:** This was derived in class. First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations.

--Q 5.2: Merge the CR equations to show that \( u \) and \( v \) obey Laplace’s equation (\( \nabla^2 u(\sigma, \omega) = 0 \) and \( \nabla^2 v(\sigma, \omega) = 0 \)). One may conclude that the real and imaginary parts of complex analytic functions must obey these conditions.

**Sol:** Take partial derivatives with respect to \( \sigma \) and \( \omega \) and solve for one equation in each of \( u \) and \( v \).

**Problem # 6:** Apply the CR equations to the following functions. State for which values of \( s = \sigma + i\omega \) the CR conditions do or do not hold (e.g. where the function \( F(s) \) is or is not analytic). Hint: Recall your answers to problem 1.2 of this assignment.

--Q 6.1: \( F(s) = e^s \)

**Sol:** CR conditions hold everywhere.

--Q 6.2: \( F(s) = 1/s \)

**Sol:** CR conditions are violated at \( s = 0 \). The function is analytic everywhere except \( s = 0 \).

**Branch cuts and Riemann sheets**

**Problem # 7:** Consider the function \( w^2(z) = z \). This function can also be written as \( w(z) = \sqrt{z} \).

Define \( z = re^{i\phi} \) and \( w(z) = \rho e^{i\phi/2} \).

--Q 7.1: How many Riemann sheets do you need in the domain \( (z) \) and the range \( (w) \) to fully represent this function as single valued?

**Sol:** There are two sheets for \( z \) and one sheet for \( w = \sqrt{z} \). When the point in domain \( z \) (being mapped to \( w(z) \)) crosses the \( z \) branch cut, the range \( (w) \) switches from the \( z_+ \) sheet to the \( z_- \) sheet. \( w(z) \) remains analytic on the cut, since it is analytic everywhere. The function \( w(z) = \sqrt{z} \) is analytic everywhere, even at \( z = 0 \).

--Q 7.2: Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

**Sol:** Above we show the mapping for the square root function \( w(z) = \sqrt{z} = \sqrt{re^{i\phi}/2} \).
CHAPTER 3. DIFFERENTIAL EQUATIONS

– Q 7.3: Use zviz.m to plot the positive and negative square roots \( \pm \sqrt{z} \) and \( -\sqrt{z} \). Describe what you see.

Sol: The sheet for the positive root is shown in Fig 3.2 (page 106 of the Oct 24 version of the class notes.) Two view the two sheets use Matlab command `zviz sqrt(W) -sqrt(W)`. ■

– Q 7.4: Where does zviz.m place the branch cut for this function?

Sol: Typically the cut is placed along the negative real axis \( \phi = \pm \pi \). This is Matlab's/Octave's default location. In the figure above, it has been placed along the positive real axis, \( \phi = 0 = 2\pi \). ■

– Q 7.5: Must it necessarily be in this location?

Sol: No, it can be moved, at will. It must start from \( z = 0 \) and end at \( |z| \to \infty \). The cut may be moved when using zviz.m by multiplying \( z \) by \( e^{\phi_j} \). For example, `zviz W = sqrt(j*Z)` rotates the cut by \( \pi/2 \). The colors of \( w(z) \) (angle maps to color) always ‘jump’ at the branch cut, as you make the transition across the cut. ■

Problem # 8: Consider the function \( w(z) = \log(z) \). As before define \( z = re^{\phi_j} \) and \( w(z) = \rho e^{\theta_j} \).

– Q 8.1: Describe with a sketch, and then discuss the branch cut for \( f(z) \).

Sol: From the plot of zviz \( w(z) = \log(z) \) of Lecture 18, we see a branch cut going from \( w = 0 \) to \( w = -\infty \). If we express \( z \) in polar coordinates \( z = re^{\phi_j} \), then

\[
w(z) = \log(r) + \phi_j = u(x, y) + v(x, y)j,
\]

where \( r(x, y) = |z| = \sqrt{x^2 + y^2} \) and \( \phi = \angle z = \phi(x, y) \). Thus a zero in \( w(z) \) appears at \( z = 1 + 0j \), and only appears on the principle sheet of \( z \) (between \( [-\pi < \angle z = \phi < \pi] \)), because this is the only place where \( \phi = 0 \). As the angle \( \phi \) increases, the imaginary part of \( w(z) = \angle z \), which increases without bound. Thus \( w \) is like a spiral stair case, or cork-screw. If \( \rho = 1 \) and \( \phi \neq 0 \), \( w(r) = \log(1) + \phi_j \) is not zero, since the angle is not zero. ■

– Q 8.2: What is the inverse of this function, \( z(f) \)? Does this function have a branch cut (if so, where is it)?

Sol: \( z(w) = e^w \) is a single valued function, so a branch cut is not appropriate. Only multi-valued functions require a branch cut. ■

– Q 8.3: Using zviz.m, show that

\[
\tan^{-1}(z) = -\frac{j}{2} \log \frac{j - z}{j + z}.
\]  

(3.7)

In Fig. 4.2 (p. 145) these two functions are shown to be identical. Sol: Use the Matlab commands `atan(Z)` and `- (j/2) * log((j+Z) ./ (j-Z))`. ■

– Q 8.4: Algebraically justify Eq. 4.2. Hint: Let \( w(z) = \tan^{-1}(z), z(w) = \tan w = \sin w / \cos w \), then solve for \( e^{wz} \).

Sol: Following the hint gives

\[
z(w) = -je^{wz} - e^{-wz} = -je^{2wz} - 1.
\]

Solving this bilinear equation for \( e^{2wz} \) gives

\[
e^{2wz} = \frac{1 + zj}{1 - zj} = \frac{j - z}{j + z}.
\]

Taking the log and using our definition of \( w(z) \) we find

\[
w(z) = \tan^{-1}(z) = -\frac{j}{2} \log \frac{j - z}{j + z}.
\]

■
Cauer synthesis given an impedance

This section needs further development. Transcribed from page 162 of version 0.97.06

Problem # 9: This is new material: Appendix D, page 277 The continued fraction method can be generalized from a residue expansion of \( Z(s) = \frac{N(s)}{D(s)} \), by a transmission line network synthesis. In this problem we shall explore this method. It seems this proves that Cauer synthesis may be implemented with the residue expansion replacing the floor function in the CFA. Very cool. This seems to solve Burne’s network synthesis problem.

\[ -Q \text{ 9.1: Starting from the impedance } Z(s) = \frac{1}{s + 1}, \text{ find the impedance network as a ladder network.} \]

Sol: Taking the reciprocal we find the sum of two admittances

\[ Y(s) = s + 1. \]

The the impedance is \( Z(s) = 1/(s + 1) \).

\[ -Q \text{ 9.2: Use a residue expansion to mimic the CFA floor function for polynomial expansions. Find the expansion of } H(s) = \frac{s^2}{1 + s}. \]

Sol:

\[ Z(s) = A + Bs + C/(1 + s) = -1 + s + \frac{1}{1 + s}. \]

Thus the Cauer synthesis is a series \(-1 + s\) and a shunt \(1||s\) (i.e., \(Y(s) = 1 + s\)).

\[ -Q \text{ 9.3} \]

Discuss how the series impedance \(Z(s, x)\) and shunt admittance \(Y(s, x)\) determine the wave velocity \(\kappa(s, x)\) and the characteristic impedance \(z_o(s, x)\) when

1. \(Z(s)\) and \(Y(s)\) are both independent of \(x\)

Sol: In the most general case

\[ z_o(s, x) = \sqrt{Z(s, x)/Y(s, x)} \]

and

\[ \kappa(s, x) = \sqrt{Z(s, x)Y(s, x)}. \]

The general equations for \(z_o(s, s)\) and \(\kappa(s, x)\) are given in Mason (1927), and discussed in Appendix E (p. 281).

2. \(Z(s, x)\) and \(Y(s)\) (\(Y(s)\) is independent of \(x\), \(Z(s, x)\) depends on \(x\))

3. \(Z(s)\) and \(Y(s, x)\) (\(Z(s)\) is independent of \(x\), \(Y(s, x)\) depends on \(x\))

4. \(Z(s, x)\) and \(Y(s, x)\) (both \(Y(s, x), Z(s, x)\) depend on \(x\))
3.2 Exercises DE-2

Topic of this homework:
Integration of complex functions; Cauchy’s theorem, integral formula, residue theorem; power series; Riemann sheets and branch cuts; inverse Laplace transforms

Problem # 1: FTCC and integration in the complex plane
Recall that, according to the Fundamental Theorem of Complex Calculus (FTCC),
\[ f(z) = f(z_0) + \int_{z_0}^{z} F(\zeta)d\zeta, \tag{3.9} \]
where \( z_0, z, \zeta, F \in \mathbb{C} \). It follows that
\[ F(z) = \frac{d}{dz} f(z). \tag{3.10} \]
Theorem 3.3 is also known as the anti-derivative of \( f(z) \).

\[ \text{Q 1.1: For a closed interval \( \{a, b\} \), the FTCC can be stated as} \]
\[ \int_{a}^{b} F(z)dz = f(b) - f(a), \tag{3.11} \]
meaning that the result of the integral is independent of the path from \( x = a \) to \( x = b \). What condition(s) on the integrand \( f(z) \) is (are) sufficient to assure that Eq. 3.11 holds? \textbf{Sol:} If the integrand is complex analytic for all \( z \in \mathbb{C} \), meaning it can be represented as a convergent power series,
\[ f(z) = \sum_{k=0}^{\infty} c_k z^k, \]
then the theorem holds. For the integrand to have a convergent power series, it must satisfy the Cauchy-Riemann (CR) equations, which assure that the derivatives are independent of the path (i.e., direction) of integration.

\[ \text{Q 1.2: For the function} \ f(z) = c^z, \ \text{where} \ c \in \mathbb{C} \ \text{is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that} \ f(z) \ \text{is analytic for all} \ z \in \mathbb{C}. \ \textbf{Sol:} \ 
\text{We may rewrite this function as} \ f(z) = e^{\ln(c)z}, \ \text{where} \ z = x + iy \ \text{and} \ f = u + iv. \ \text{Thus}
\[ u(x, y) = e^{\ln(c)x} \cos(\ln(c)y), \]
\[ v(x, y) = e^{\ln(c)x} \sin(\ln(c)y) \]
\[ \frac{\partial u}{\partial x} = \ln(c) e^{\ln(c)x} \cos(\ln(c)y) = \frac{\partial v}{\partial y} = \ln(c) e^{\ln(c)x} \sin(\ln(c)y) \]
\[ \frac{\partial u}{\partial y} = -\ln(c) e^{\ln(c)x} \sin(\ln(c)y) = -\frac{\partial v}{\partial x} = -\ln(c) e^{\ln(c)x} \sin(\ln(c)y) \]
Thus the CR conditions are satisfied everywhere and the function is analytic for all \( z \in \mathbb{C} \).
Problem #2: In the following problems, solve the integral

\[ I = \int_C F(z)dz \]

for a given path \( C \). In some cases this might be the definite integral (Eq. 3.11).

Let the function \( F(z) = c^z \), where \( c \in \mathbb{C} \) is given for each problem below. **Hint: Can you apply the FTCC?**

**Q 2.1: Find the anti-derivative of \( F(z) \).**

**Sol:** Since \( c^z = e^{\ln(c)z} \), the indefinite integral (anti-derivative) is

\[
I(z) = \frac{1}{\ln c} e^{\ln(c)z} \quad \text{since} \quad \frac{d}{dz} I(z) = \frac{d}{dz} \frac{1}{\ln c} e^{\ln(c)z} = \frac{\ln(c)}{\ln c} e^{\ln(c)z} = F(z).
\]

**Q 2.2: \( c = 1/e = 1/2.7183 \ldots \) where \( C \) is \( \zeta = 0 \to i \to z \)**

**Sol:** The integrand is \( F(z) = e^{-z} \), which is entire. Thus the the integral is independent of the path (i.e., \( C \) is not relevant to the final answer).

\[
I(z) = \int_0^z e^{-\zeta}d\zeta + \int_z^z e^{-\zeta}d\zeta = F(z) - F(i) + F(i) - F(0)
\]

\[
= \int_0^z e^{-\zeta}d\zeta = -e^z \bigg|_0 = -e^{-z} - 1
\]

**Q 2.3: \( c = 2 \) where \( C \) is \( \zeta = 0 \to (1+i) \to z \)**

**Sol:** The integrand is \( F(z) = 2^z \), where \( 2 = e^{\ln 2} \). The path \( C \) is not relevant to the final answer.

\[
I(z) = \int_0^z 2^\zeta d\zeta = \int_0^z e^{\ln 2} d\zeta = \left. \frac{e^{\ln 2}^z}{\ln 2} \right|_0 = \frac{(e^{\ln 2} - 1)}{\ln 2} \approx 1.443(e^{0.693z} - 1)
\]

**Q 2.4: \( c = i \) where the path \( C \) is an inward spiral described by \( z(t) = 0.99t e^{i2\pi t} \) for \( t = 0 \to t_0 \to \infty \)**

**Sol:** \( i = e^{\pi/2} e^{i2\pi t} \). We have already proved that the path doesn’t matter for any \( F(z) = c^z \), so we just need to evaluated \( z(t) \) for \( t = 0 \) and \( t \to \infty \). This gives \( z(0) = 1 \) and \( z(t \to \infty) = 0 \).

\[
I = \int_{z(0)}^{z(t \to \infty)} i^2dz = \int_{z(0)}^{z(t \to \infty)} e^{i\pi z/2}dz = \left. \frac{2e^{i\pi z/2}}{i\pi} \right|_0^1 = \frac{2}{i\pi}(1 - e^{i\pi/2}) = \frac{-2(i + 1)}{\pi}
\]

**Q 2.5: \( c = e^{t-t_0} \) where \( t_0 > 0 \) is a real number, and \( C \) is \( z = (1 - i\infty) \to (1 + i\infty) \).**

**Hint:** Do you recognize this integral? If you do not recognize the integral, please do not spend a lot of time trying to solve it via the ‘brute force’ method.

**Sol:** This is the basically the inverse Laplace transform of \( e^{-\tau_0 t} \), we are just missing the scale factor \( \frac{1}{2\pi} \).

\[
I(t) = \int_{1-i\infty}^{1+i\infty} e^{(t-t_0)z}dz = \int_{1-i\infty}^{1+i\infty} e^{-\tau_0 z}e^{zt}dz = 2\pi i\delta(t - \tau_0)
\]

**Problem #3: Cauchy’s theorems for integration in the complex plane**

There are three basic definitions related to Cauchy’s integral formula. They are all related, and can greatly simplify integration in the complex plane. When a function depends on a complex variable we shall use uppercase notation, consistent with the engineering literature for the Laplace transform.
1. **Cauchy’s (Integral) Theorem** (Stillwell, p. 319; Boas, p. 45)

\[ \oint_C F(z)\,dz = 0, \]

if and only if \( F(z) \) is complex-analytic inside of \( C \).

This is related to the Fundamental Theorem of Complex Calculus (FTCC)

\[ f(z) = f(a) + \int_a^z F(z)\,dz, \]

where \( f(z) \) is the **anti-derivative** of \( F(z) \), namely \( F(z) = df/dz \). The FTCC requires \( F(z) \) to be complex-analytic for all \( z \in \mathbb{C} \). By closing the path (contour \( C \)), Cauchy’s theorem (and the following theorems) allows us to integrate functions that may not be complex-analytic for all \( z \in \mathbb{C} \).

2. **Cauchy’s Integral Formula** (Boas, p. 51; Stillwell, p. 220)

\[ \frac{1}{2\pi j} \oint_C \frac{F(z)}{z - z_0}\,dz = \begin{cases} F(z_0), & z_0 \in C \text{ (inside)} \\ 0, & z_0 \notin C \text{ (outside)} \end{cases} \]

Here \( F(z) \) is required to be analytic everywhere within (and on) the contour \( C \). \( F(z_0) \) is called the **residue** of the pole.

3. **(Cauchy’s) Residue Theorem** (Boas, p. 72)

\[ \oint_C F(z)\,dz = 2\pi j \sum_{k=1}^K \text{Res}_k, \]

where \( \text{Res}_k \) are the **residues** of all poles of \( F(z) \) enclosed by the contour \( C \).

**How to calculate the residues**: The residues can be rigorously defined as

\[ \text{Res}_k = \lim_{z \to z_k} \left[ (z - z_k)f(z) \right]. \]

This can be related to **Cauchy’s integral formula**: Consider the function \( F(z) = w(z)/(z - z_k) \), where we have factored \( F(z) \) to isolate the first-order pole at \( z = z_k \). If the remaining factor \( w(z) \) is analytic at \( z_k \), then the residue of the pole at \( z = z_k \) is \( w(z_k) \).

---

**Q 3.1**: Describe the relationships between the three theorems:

1. (1) and (2) **Sol**: When \( z_0 \) falls outside of \( C \), (2) reduces to (1).

2. (1) and (3) **Sol**: When there are no poles inside \( C \), all the residues are zero, and (3) reduces to (1).

3. (2) and (3) **Sol**: Case (2) has only one induced pole at \( z = z_0 \), having residue \( F(z_0) \). Thus (3) is the same as (2) when \( K = 1 \), the pole at \( z_0 \) is within contour \( C \), and the single residue is \( F(z_0) \).

---

**Q 3.2**: Consider the function with poles at \( z = \pm j \)

\[ F(z) = \frac{1}{1 + z^2} = \frac{1}{(z - j)(z + j)} \]

Find the residue expansion.

**Sol**:

\[ F(z) = \frac{j}{2} \left( \frac{1}{z + j} - \frac{1}{z - j} \right). \]
3.2. EXERCISES DE-2

–Q 3.3: Apply Cauchy’s theorems to solve the following integrals. State which theorem(s) you used, and show your work.

1. \( \oint_C F(z)\,dz \) where \( C \) is a circle centered at \( z = 0 \) with a radius of \( \frac{1}{2} \).
   \[ \text{Sol:} \] Because the contour \( C \) does not include the poles, \( F(z) \) is analytic everywhere inside \( C \). Using Cauchy’s integral theorem, the integral is 0.

2. \( \oint_C F(z)\,dz \) where \( C \) is a circle centered at \( z = j \) with a radius of 1.
   \[ \text{Sol:} \] Since we only enclose the pole at \( z = j \), use the integral formula with \( F(z) = \frac{1}{z+j} \):
   \[
   \oint_C F(z)\,dz = 2\pi j \text{Res}_j = 2\pi j \left( \frac{1}{z+j} \right)_{z=j} = 2\pi j \frac{1}{2j} = \pi
   \]

3. \( \oint_C F(z)\,dz \) where \( C \) is a circle centered at \( z = 0 \) with a radius of 2.
   \[ \text{Sol:} \] Since we enclose both poles, using the residue theorem:
   \[
   \oint_C F(z)\,dz = 2\pi j (\text{Res}_j + \text{Res}_{-j}) = 2\pi j \left( \frac{1}{2j} - \frac{1}{2j} \right) = 0
   \]
   As a side note, the inverse Laplace transform for \( F(z) \) is \( \sin(t) \), which is zero for \( t = 0 \), consistent with this result.

Problem #4: Integration in the complex plane

In the following questions, you’ll be asked to integrate \( F(s) = u(\sigma, \omega) + iv(\sigma, \omega) \) around the contour \( C \) for complex \( s = \sigma + i\omega \),

\[ \oint_C F(s)\,ds. \]

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations (but you should if you can’t answer the question ‘by inspection’).

–Q 4.1: \( F(s) = \sin(s) \)
   \[ \text{Sol:} \] Analytic everywhere. \( -\cos(s) = \int_0^{2\pi} \sin(s)\,ds = 0. \) This function is entire (i.e., has no poles) so the integral must be zero.

–Q 4.2: Given function \( F(s) = \frac{1}{s} \)

1. State where the function is and is not analytic. \[ \text{Sol:} \] Analytic everywhere except at \( s = 0 \), where it has a pole.

2. Explicitly evaluate the integral when \( C \) is the unit circle, defined as \( s = e^{i\theta}, 0 \leq \theta \leq 2\pi. \)
   \[ \text{Sol:} \]
   \[
   \oint_C F(s)\,ds = \int_0^{2\pi} \frac{1}{e^{i\theta}} e^{i\theta} \,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i
   \]

3. Evaluate the same integral using Cauchy’s theorem and/or the residue theorem. \[ \text{Sol:} \] The residue is 1 so the integral is \( 2\pi i \).

–Q 4.3: \( F(s) = \frac{1}{s^2} \)
1. State where the function is and is not analytic. **Sol:** Analytic everywhere except at \( s = 0 \), where it has a 2\textsuperscript{nd} order pole. ■

2. Explicitly evaluate the integral when \( C \) is the unit circle, defined as \( s = e^{i\theta}, \theta \leq 2\pi \). **Sol:**

\[
\oint_C F(s)ds = \int_0^{2\pi} \frac{1}{e^{2\theta}} ie^{i\theta} d\theta = \int_0^{2\pi} ie^{-i\theta} d\theta = i \frac{1-e^{-i2\pi}}{0} = 1(e^{-i2\pi} - 0) = 0
\]

3. What does your result imply about the residue of the 2\textsuperscript{nd} order pole at \( s = 0 \)? **Sol:** The residue is 0. ■

---

**Q 4.4:** \( F(s) = e^{st} \)

1. State where the function is and is not analytic. **Sol:** Analytic everywhere. ■

2. Explicitly evaluate the integral when \( C \) is the square \((\sigma, \omega) = (1, 1) \rightarrow (1, -1) \rightarrow (-1, -1) \rightarrow (-1, 1) \rightarrow (1, 1)\). **Sol:** When you perform this integral piece-wise, you will find that all terms cancel out and the result is 0. ■

3. Evaluate the same integral using Cauchy’s theorem and/or the residue theorem. **Sol:** The function is analytic everywhere, so the integral is 0 by Cauchy’s theorem. ■

---

**Q 4.5:** \( F(s) = \frac{1}{s+2} \)

1. State where the function is and is not analytic. **Sol:** Analytic everywhere except at \( s = -2 \), where it has a pole. ■

2. Let \( C \) be the unit circle, defined as \( s = e^{i\theta}, \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem. **Sol:** The function is analytic everywhere inside \( C \), so the integral is 0 by Cauchy’s theorem. ■

3. Let \( C \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem. **Sol:** This contour contains the pole. The residue is 1, therefore the integral is equal to \( 2\pi i \). ■

---

**Q 4.6:** \( F(s) = \frac{1}{2\pi i} e^{st} \)

1. State where the function is and is not analytic. **Sol:** Analytic everywhere except at \( s = -2 \), where it has a pole. ■

2. Let \( C \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem. **Sol:** This contour contains the pole. The residue is \( \frac{1}{2\pi i} e^{-2t} \), therefore the integral is equal to \( e^{-2t} \). ■

3. Let \( C \) contain the entire left-half \( s \)-plane. Evaluate the integral using Cauchy’s theorem and/or the residue theorem. Do you recognize this integral? **Sol:** This contour contains the pole. The residue is \( \frac{1}{2\pi i} e^{-2t} \), therefore the integral is equal to \( e^{-2t} \). This contour is the inverse Laplace transform. ■

---

**Q 4.7:** \( F(s) = \pm \frac{1}{\sqrt{s}} \) (e.g. \( F^2 = \frac{1}{s} \))
1. State where the function is and is not analytic. **Sol:** Analytic everywhere except \( s = 0 \), where there is a pole. ■

2. This function is multivalued. How many Riemann sheets do you need in the domain \( (s) \) and the range \( (f) \) to fully represent this function? Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range. **Sol:** There are 2 sheets in the domain (for the \( \pm \) square root) which map to 1 sheet in the range. ■

3. Explicitly evaluate the integral 
\[
\int_C \frac{1}{\sqrt{z}} \, dz
\]
when \( C \) is the unit circle, defined as \( s = e^{i\theta} \), \( 0 \leq \theta \leq 2\pi \). Is this contour ‘closed’? State why or why not. **Sol:** The solution is 
\[
2\sqrt{z} \bigg|_{\theta=0}^{2\pi} = 2e^{j\theta/2} \bigg|_{0}^{2\pi} = 2(e^{j\pi} - e^0) = -4.
\]
In polar coordinates
\[
\int_0^{2\pi} \frac{ds}{\sqrt{s}} = \int_0^{2\pi} \frac{de^{i\theta}}{e^{i\theta/2}} = i \int_0^{2\pi} e^{i\theta/2} d\theta = i \int_0^{2\pi} e^{i\theta/2} d\theta = 2 e^{i\theta/2} \bigg|_{0}^{2\pi} = 2[e^{i\pi} - 1] = 2(-2) = -4.
\]
This contour is not closed. One way to determine this is to see if going once around the unit circle returns \( F(s) \) to its original value. 
\[
F(e^{i\theta}) = 1 \neq F(e^{i2\pi}) = e^{-i\pi} = -1.
\]
■

4. Explicitly evaluate the integral 
\[
\int_C \frac{1}{\sqrt{z}} \, dz
\]
when \( C \) is twice around the unit circle, defined as \( s = e^{i\theta} \), \( 0 \leq \theta \leq 4\pi \). Is this contour ‘closed’? State why or why not. **Hint:** Note that 
\[
\sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi}e^{i\theta}} = e^{i\pi} \sqrt{e^{i\theta}} = -1 \sqrt{e^{i\theta}}
\]
**Sol:**
\[
\int_0^{4\pi} \frac{ds}{\sqrt{s}} = \int_0^{4\pi} \frac{de^{i\theta}}{e^{i\theta/2}} = i \int_0^{4\pi} e^{i\theta/2} d\theta = i \int_0^{4\pi} e^{i\theta/2} d\theta = 2 e^{i\theta/2} \bigg|_{0}^{4\pi} = 2[e^{i2\pi} - 1] = 2(0) = 0.
\]
This contour is closed. One way to determine this is to see if going twice around the unit circle returns \( F(s) \) to its original value. 
\[
F(e^{i\theta}) = 1 = F(e^{i4\pi}) = e^{-i2\pi} = 1.
\]
■
5. What does your result imply about the residue of the (twice-around $\frac{1}{2}$ order) pole at $s = 0$?
   Sol: The residue is 0.

6. Show that the residue is zero. Hint: apply the definition of the residue. Sol: $c_{-1} = \lim_{z \to z_k} \frac{z}{\sqrt{z}} = \lim_{z \to z_k} \sqrt{z} = 0.$

Problem #5: A two-port network application for the Laplace transform

Recall that the Laplace transform ($\mathcal{L}T$) $f(t) \leftrightarrow F(s)$\(^1\) of a causal function $f(t)$ is

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}dt,$$

where $s = \sigma + j\omega$ is complex frequency\(^2\) in [radians] and $t$ is time in [seconds]. Causal functions and the Laplace transform are particularly useful for describing systems, which have no response until a signal enters the system (e.g. at $t = 0$).

The definition of the inverse Laplace transform ($\mathcal{L}T^{-1}$) requires integration in the complex plane:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} F(s)e^{st}ds = \frac{1}{2\pi j} \oint_{\mathcal{C}} F(s)e^{st}ds.$$

The Laplace contour $\mathcal{C}$ actually includes two pieces

$$\oint_{\mathcal{C}} = \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} + \int_{C_{\infty}},$$

where the path represented by $‘C_{\infty}’$ is a semicircle of infinite radius with $\sigma \to -\infty$. It is somewhat tricky to do, but it may be proved that the integral over the contour $C_{\infty}$ goes to zero. For a causal, ‘stable’ (e.g. doesn’t blow up over time) signal, all of the poles of $F(s)$ must be inside of the Laplace contour, in the left-half $s$-plane.

Transfer functions Linear, time-invariant systems are described by an ordinary differential equations. For example, consider the first-order linear differential equation

$$a_1 \frac{dy(t)}{dt} = b_1 \frac{dx(t)}{dt} + b_0 x(t). \quad (3.12)$$

This equation describes the relationship between the input ($x(t)$) and output ($y(t)$) of the system. If we define Laplace transforms $y(t) \leftrightarrow Y(s)$ and $x(t) \leftrightarrow X(s)$, then this equation may be written in the frequency domain as

$$a_1 sY(s) = b_1 sX(s) + b_0 X(s).$$

The transfer function for this system is defined as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_1 s + b_0}{a_1 s} = \frac{b_1}{a_1} + \frac{b_0}{a_1 s}.$$ \(\text{In this problem, we will look at the transfer function of a simple two-port network, shown in Figure 3.1. This network is an example of a RC low-pass filter, which acts as a leaky integrator.}\)

Problem #6: ABCD method

\(^1\)Many loosely adhere to the convention that the frequency domain uses upper-case [e.g. $F(s)$] while the time domain uses lower case [$f(t)$].

\(^2\)While radians are useful units for calculations, when providing physical insight in discussions of problem solutions, it is easier to work with Hertz, since frequency in [Hz] and time in [s] are mentally more more natural units than radians. The same is true of degrees vs. radians. Boas (p. 10) recommends the use degrees over radians. He gives the example of $3\pi/5$ [radians], which is more easily visualize as $108^\circ$.\)
3.2. EXERCISES DE-2

This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$.

---Q 6.1: Low-pass RC filter

1. Use the ABCD method to find the matrix representation of Fig. 3.1.

**Sol:**

$$
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
(1 + R_1Cs) & R_1 \\
\frac{1}{sC} & 1
\end{bmatrix}
\begin{bmatrix}
R_2 \\
0 \\
1
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
= \begin{bmatrix}
(1 + R_1Cs) & (R_1 + R_2 + R_1R_2Cs) \\
\frac{1}{sC} & (1 + R_2Cs)
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
$$

2. Assuming that $I_2 = 0$, find the transfer function $H(s) \equiv V_2/V_1$. From the results of the ABCD matrix you determined above, show that

$$H(s) = \frac{1}{1 + R_1Cs}.$$

**Sol:** Since $I_2 = 0$ the upper row of the ABCD matrix gives the relationship between $V_1$ and $V_2$ as

$$V_1 = (1 + R_1C)V_2$$

Thus the ratio is as desired.

3. The transfer function $H(s)$ has one pole. Where is the pole? Find the residue of this pole. **Sol:** If we rewrite $H(s)$ in the standard form, the pole $s_p$ and residue $A$ may be easily identified:

$$H(s) = \frac{A}{s - s_p} = \frac{1}{1 + R_1Cs} = \frac{1/(R_1C)}{s + 1/(R_1C)}$$

Thus the pole is $s_p = -1/R_1C$ and the residue is $A = 1/R_1C$.

4. Find $h(t)$, the inverse Laplace transform of $H(s)$. **Sol:**

$$h(t) = \frac{1}{R_1C} e^{-t/R_1C}u(t)$$

The integral follows from the Residue Theorem. The pole is at $s_p = -1/RC$ and the residue is $1/RC$.

5. Assuming that $V_2 = 0$ find $Y_{12}(s) \equiv I_2/V_1$. **Sol:** Setting $V_2 = 0$ we may easily read off the requested function as

$$V_1 = -(R_1 + R_2 + R_1R_2Cs)I_2$$

thus

$$Y_{12}(s) = -\frac{1}{R_1 + R_2 + R_1R_2Cs} = \frac{A}{s - s_p}$$

with residue $A = 1/R_1R_2C$ and pole $s_p = -(R_1 + R_2)/(R_1R_2C)$.
6. Find the input impedance to the right-hand side of the system, $Z_{22}(s) \equiv V_2/I_2$ for two cases:

(a) $I_1 = 0$
(b) $V_1 = 0$

**Sol:** There are two cases. When $I_1 = 0$, $Z_{22} = R_2 + 1/sC$. When $V_1 = 0$

\[
Z_{22}(s) = R_2 + \frac{R_1}{sC} = R_2 + \frac{R_1}{1 + R_1Cs}
\]

Reading this second case off of our matrix solution gives

\[
0 = (1 + R_1C)V_2 - (R_1 + R_2 + R_1R_2Cs)I_2
\]

or solving for $Z_{22}$ gives the brute-force result.

5

7. Compute the determinant of the ABCD matrix. *Hint: It is always 1.*

**Sol:**

\[
\begin{vmatrix}
1 + R_1Cs & R_1 + R_2 + R_1R_2Cs \\
R_1 + R_2 & 1 + R_2Cs
\end{vmatrix}
\]

\[
= 1 + (R_1 + R_2)Cs + R_1R_2(Cs)^2 - (rR_1 + R_2)(Cs) - R_1R_2(Cs)^2
\]

\[
= 1
\]

8. Compute the derivative of $H(s) = \frac{V_2}{V_1} \bigg |_{I_2=0}$. **Sol:** From the result of the previous problem 2

\[
H(s) = \frac{1}{1 + R_1Cs}
\]

Thus we wish to find

\[
\frac{d}{ds} H(s) = \frac{d}{ds} (1 + R_1Cs)^{-1} = \frac{-R_1C}{(1 + R_1Cs)^2}
\]

Here is a slightly easier way, using the log function:

\[
\frac{1}{H(s)} \frac{dH(s)}{ds} = \frac{d}{ds} \ln H(s) = \frac{d}{ds} \ln(1 + R_1Cs) = \frac{-1}{1 + R_1Cs} \frac{d}{ds}(1 + R_1Cs) = \frac{-R_1C}{(1 + R_1Cs)}
\]

Thus

\[
\frac{dH(s)}{ds} = H(s) \frac{-R_1C}{(1 + R_1Cs)} = \frac{-R_1C}{(1 + R_1Cs)^2}
\]

Problem # 7: *With the help of a computer*

In the following problems, we will look at some of the concepts from this homework using Matlab/Octave. We are using the *syms* function which requires Matlab’s/Octave’s symbolic math toolbox. Or you may use the EWS lab’s Matlab. Alternative symbolic-math tool, such as Wolfram Alpha.\(^3\)

\(^3\)https://www.wolframalpha.com/
Example: To find the Taylor series expansion about $s = 0$ of

$$F(s) = -\log(1 - s),$$

first consider the derivative and its Taylor series (about $s = 0$)

$$F'(s) = \frac{1}{1 - s} = \sum_{n=0}^{\infty} s^n.$$

Then, integrate this series term by term

$$F(s) = -\log(1 - s) = \int s F'(s) ds = \sum_{n=0}^{\infty} \frac{s^n}{n}. $$

Alternatively you may use Matlab/Octave commands:

```matlab
sym s
taylor(-log(1-s),'order',7)
```

–Q 7.1: Use Octave’s taylor ($-\log(1-s)$) to 7th order, as in the example above.

1. Try the above Matlab/Octave commands. Give the first 7 terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above). Sol:

$$F(s) = \cdots + \frac{s^7}{7} + \frac{s^6}{6} + \frac{s^5}{5} + \frac{s^4}{4} + \frac{s^3}{3} + \frac{s^2}{2} + s$$

2. What is the inverse Laplace transform of this series? Consider the series term by term. 

Sol: $f(t) = \sum \delta^{(n)}/n$

–Q 7.2: The function $1/\sqrt{z}$ has a branch point at $z = 0$, thus it is singular there.

1. Can you apply Cauchy’s integral theorem when integrating around the unit circle? Sol: No, one cannot apply the Cauchy Theorem since it is not analytic at $z = 0$. But the integral may be evaluated.

2. Below is a Matlab/Octave code that computes $\int_0^{4\pi} \frac{dz}{\sqrt{z}}$ using Matlab’s/Octave’s symbolic analysis package:

```matlab
sym z
I=int(1/sqrt(z))
J = int(1/sqrt(z),exp(-j*pi),exp(j*pi))
eval(J)
```

Run this script. What answers do you get for $I$ and $J$?

Sol: This script returns the answers $I = 2\sqrt{\pi}$ and $J = 2.4493e - 16$, which is numerically the same as zero.

3. Modify this code to integrate $f(z) = 1/z^2$ once around the unit circle. What answers do you get for $I$ and $J$? Sol: This function has a 2d order pole at $s = 0$. Thus from the CIT, the integral evaluates to zero.

Proof:

$$I = \oint \frac{ds}{s^2} = \frac{1}{s} \bigg|_{0}^{2\pi} = -e^{-i\theta} \bigg|_{0}^{2\pi} = -(1 - 1) = 0$$

More generally $I = \oint \frac{ds}{n^2} = 0$ for $n \neq 1$. As best I know, this holds for any $n \in \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{R}, \mathbb{C}$. For $n = 1$ it has a value of $2\pi i$. “
---Q 7.3: Bessel functions can describe waves in a cylindrical geometry

The Bessel function has a Laplace transform with a branch cut

\[ J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1 + s^2}}. \]

Draw a hand sketch showing the nature of the branch cut. Hint: Use \texttt{zviz}. \textbf{Sol:} The roots are given by \( s_{\pm} = \pm j \). The branch cut connects the two roots, or can go from each root to \( \infty \). Either choice is valid.

**Problem #8: Matlab/Octave exercises:**

---Q 8.1: Comment on the following Matlab/Octave exercises

1. Try the following Matlab/Octave commands, and then comment on your findings.

\begin{verbatim}
%Take the inverse LT of 1/sqrt(1+s^2)
syms s
I=ilaplace(1/(sqrt((1+s^2))));
disp(I)

\textbf{Sol:} I = J_0(t)u(t).
\end{verbatim}

%Find the Taylor series of the LT
T = taylor(1/sqrt(1+s^2),10);
disp(T);

\textbf{Sol:}

\[ T = \cdots + \frac{3s^4}{8} - \frac{s^2}{2} + 1 \]

\begin{verbatim}
%Verify this
syms t
J=laplace(besselj(0,t));
disp(J);

\textbf{Sol:} I = \frac{1}{\sqrt{1+s^2}}.
\end{verbatim}

%plot the Bessel function
t=0:0.1:10*pi;
b=besselj(0,t);
plot(t/pi,b);
grid on;

\textbf{Sol:} Plot of \( J_0(t)u(t) \).

---Q 8.2: When did Friedrich Bessel live?

\textbf{Sol:} 1784-1846, in Königsberg Germany.

---Q 8.3: What did he use Bessel functions for?

\textbf{Sol:} Solving the Bessel equation, which is the wave equation in 2D. Bessel functions were first introduced by the Daniel Bernoulli.
Q 8.4: Using \textit{viz}, for each of the following functions

1. Describe the plot generated by \textit{viz} $S=Z$. \textbf{Sol:} It is a polar plot of the function, with intensity coding the magnitude and color coding the phase. Red is a positive real number while blue is a negative real number.

2. Are the functions defined below legal Brune impedances? (i.e., Do they function obey $\Re Z(\sigma > 0) \geq 0$)? \textit{Hint: Consider the phase (color). Plot \textit{viz} 2 for a reminder of the colormap.}
   
   1. $\text{viz} \frac{1}{\sqrt{1+S^{-2}}}$
   \textbf{Sol:} No. The RHP has blue near the branch cut, in the RHP.

   2. $\text{viz} \frac{1}{\sqrt{1-S^{-2}}}$
   \textbf{Sol:} NO, there is a branch cut in the RHP.

   3. $\text{viz} \frac{1}{1+\sqrt{S}}$
   \textbf{Sol:} Yes, its red almost everywhere even though it has a branch cut from $[-\infty < \sigma \leq -10]$. Since $1/\sqrt{s}$ has an LT$^{-1}$, this function must as well. Matlab found
   \[
   \frac{1}{\sqrt{1+s}} \leftrightarrow \frac{e^{-t}}{\sqrt{\pi} \sqrt{t}} u(t),
   \]
   however Octave failed to find the inverse transform, (but was able to find the forward transform).

\textbf{Problem # 9: Find the LT$^{-1}$ of the zeta function $\zeta_p(s)$}

$\zeta_p(s) \leftrightarrow z_p(t)$ (Eq. 2.10, p. 395) and describe the result in words.

\textit{Hint: Consider the geometric series representation}

\[
\zeta_p(s) = \frac{1}{1-e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p},
\]

for which you can easily look up (or may have memorized) the LT$^{-1}$ transform of each term.

\textbf{Sol:} Since each term in the series is a pure delay\footnote{Here we use a shorthand double-parentheses notation to define the infinite (one-sided) sum $f(t)_{T \equiv \sum_{k=0}^{\infty} f(t-kT)}$.}

\[
z_p(t) = \delta(t)T_p \equiv \sum_{k=0}^{\infty} \delta(t-kT_p) \leftrightarrow \frac{1}{1-e^{-sT_p}}.
\]

\textbf{Problem # 10: Inverse transform of products:}

The time domain version of Eq. 3.13 (p. 435) may be written as the convolution of all the $z_k(t)$ factors

\[
z(t) \equiv z_2(t) \ast z_3(t) \ast z_5(t) \ast z_7(t) \cdots \ast z_p(t) \cdots,
\]

where $\ast$ represents time convolution.

\textbf{Sol:} In terms of the physics, these transmission line equations are telling us that $\zeta(s)$ may be decomposed into an infinite cascade of transmission lines (Eq. 7.17), each having a delay given by $T_p = \ln \pi_p$. The input admittance of this cascade may be interpreted as an analytic continuation of $\zeta(s)$ which defines the eigen-modes of that cascaded impedance function.
CHAPTER 3. DIFFERENTIAL EQUATIONS

Physical interpretation: Such functions may be generated in the time domain as shown in Fig. 3.2 (p. 436), using a feedback delay of \( T_p \) seconds described by the two equations in the figure with a unity feedback gain \( \alpha = -1 \). Taking the Laplace transform of the system equation we see that the transfer function between the state variable \( q(t) \) and the input \( x(t) \) is given by \( \zeta_p(s) \), which is and all-pole function, since

\[
Q(s) = e^{-sT_p}Q(s) + V(s), \quad \text{or} \quad \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}. \tag{3.16}
\]

Closing the feed-forward path gives a second transfer function \( Y(s) = I(s)/V(s) \), namely

\[
Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}. \tag{3.17}
\]

If we take \( i(t) \) as the current and \( v(t) \) as the voltage at the input to the transmission line, then \( y_p(t) \leftrightarrow \zeta_p(s) \) represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the \( j\omega \) axis. By a slight modification \( \zeta_p(s) \) may alternatively be written as

\[
Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j\tan(sT_p/2). \tag{3.18}
\]

Every impedance \( Z(s) \) has a corresponding reflectance function given by a Möbius transformation, which may be read off of Eq. 3.17 as

\[
\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}. \tag{3.19}
\]

since impedance is also related to the round-trip delay \( T_p \) on the line. The inverse Laplace transform of \( \Gamma(s) \) is the round trip delay \( T_p \) on the line

\[
\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}. \tag{3.20}
\]

In terms of the physics, these transmission line equations are telling us that \( \zeta(s) \) may be decomposed into an infinite cascade of transmission lines (Eq. 7.17), each having a delay given by \( T_p = \ln \pi_p \). The input admittance of this cascade may be interpreted as an analytic continuation of \( \zeta(s) \) which defines the eigen-modes of that cascaded impedance function.

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations, that are not obvious in the frequency domain. Transforming to the time domain is a form of analytic continuation of \( \zeta(s) \), that depends on the assumption that \( z(t) \) is one-sided in time (causal).
3.3 Exercises DE-3

Brune Impedance

Problem # 1: Residue form
A Brune impedance is defined as the ratio of the force $F(s)$ over the flow $V(s)$, and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)} \tag{3.21}$$

with

$$D(s) = \prod_{k=1}^{K} (s - s_k) \quad \text{and} \quad c_k = \lim_{s \to s_k} (s - s_k) D(s) = \prod_{n'=1}^{K-1} (s - s_n).$$

The prime on index $n'$ means that $n = k$ is not included in the product.

–Q 1.1: Find the Laplace transform ($\mathcal{L}T$) of a 1) spring, 2) dashpot and 3) mass. Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance:

1. Hooke’s Law $f(t) = Kx(t)$. Sol: Taking the $\mathcal{L}T$ gives

$$F(s) = KX(s) = KV(s)/s \leftrightarrow f(t) = Ku(t) \ast v(t) = K \int_{0}^{t} v(t),$$

since

$$v(t) = \frac{d}{dt}x(t) \leftrightarrow V(s) = sX(s).$$

Thus the impedance of the spring is

$$Z_s(s) = \frac{K}{s} \leftrightarrow z(t) = Ku(t),$$

which is analogous to the impedance of an electrical capacitor. The relationship may be made tighter by specifying the compliance of the spring as $C = 1/K$. 

2. Dash-pot resistance $f(t) = Rv(t)$. Sol: From the $\mathcal{L}T$ this becomes

$$F(s) = RV(s)$$

and the impedance of the dash-pot is then

$$Z_r = R \leftrightarrow R\delta(t),$$

analogous to that of an electrical resistor.

3. Newton’s Law for Mass $f(t) = Mdv(t)/dt$. Sol: Taking the $\mathcal{L}T$ gives

$$f(t) = M \frac{d}{dt}v(t) \leftrightarrow F(s) = M sV(s),$$

thus

$$Z_m(s) = sM \leftrightarrow M \frac{d}{dt},$$

analogous to an electrical inductor.
-Q 1.2: Take the Laplace transform (LT) of Eq. 3.51 (p. 129), and find the total impedance \( Z(s) \) of the mechanical circuit.

**Sol:** From the properties of the LT that \( \frac{dx}{dt} \leftrightarrow sX(s) \), we find

\[
f(t) \leftrightarrow F(s) = Ms^2X(s) + RsX(s) + KX(s).
\]

In terms of velocity this is \((Ms + R + K/s)V(s) = F(s)\). Thus the circuit impedance is

\[
z(t) \leftrightarrow Z(s) = \frac{F}{V} = \frac{K + Rs + Ms^2}{s}.
\]

\[\blacksquare\]

- Q 1.3: What are \( N(s) \) and \( D(s) \) (e.g. Eq. 3.21)?

**Sol:** \( D(s) = s \) and \( N(s) = K + Rs + Ms^2 \).

\[\blacksquare\]

- Q 1.4: Assume that \( M = R = K = 1 \), find the residue form of the admittance \( Y(s) = 1/Z(s) \) (e.g. Eq. 3.21) in terms of the roots \( s_\pm \).

You may check your answer with the Matlab’s `residue` command.

**Sol:** First find the roots of the numerator of \( Z(s) \) (the denominator of \( Y(s) \)):

\[
s^2_\pm + s_\pm + 1 = (s_\pm + 1/2)^2 + 3/4 = 0,
\]

which is

\[
s_\pm = \frac{-1 \pm j\sqrt{3}}{2}.
\]

Second form a partial fraction expansion

\[
\frac{s}{1 + s + s^2} = c_0 + \frac{c_+}{s-s_+} + \frac{c_-}{s-s_-} = \frac{s(c_+ + c_-) - (c_+s_- + c_-s_+)}{1 + s + s^2}.
\]

Comparing the two sides shows that \( c_0 = 0 \). We also have two equations for the residues \( c_+ + c_- = 1 \) and \( c_+s_- + c_-s_+ = 0 \). The best way to solve this is to set up a matrix relation and take the inverse

\[
\begin{bmatrix}
1 & 1 \\
1 & s_-
\end{bmatrix}
\begin{bmatrix}
c_+ \\
c_-
\end{bmatrix} =
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

thus:

\[
\begin{bmatrix}
c_+ \\
c_-
\end{bmatrix} = \frac{1}{s_+ - s_-} 
\begin{bmatrix}
s_+ & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix},
\]

which gives \( c_\pm = \pm \frac{s_\pm}{s_+ - s_-} \). The denominator is \( s_+ - s_- = j\sqrt{3} \) and the numerator is \( \pm 1 + j\sqrt{3} \). Thus

\[
c_\pm = \pm \frac{s_\pm}{s_+ - s_-} = \frac{1}{2} \left( \pm \frac{1}{2} \right).
\]

As always, finding the coefficients is always the most difficult part. Using 2x2 matrix algebra automates the process. Always check your final result as correct.

\[\blacksquare\]

- Q 1.5: By applying the CRT, find the inverse Laplace transform \( (LT)^{-1} \). Use the residue form of the expression that you derived in the previous exercise.

**Sol:**

\[
z(t) = \frac{1}{2\pi j} \int_C Z(s)e^{st}ds.
\]

were \( C \) is the Laplace contour which encloses the entire left-half \( s \) plane. Applying the CRT

\[
z(t) = c_+e^{s_+t} + c_-e^{s_-t}.
\]

where \( s_\pm = -1/2 \pm j\sqrt{3}/2 \) and \( c_\pm = 1/2 \pm j/(2\sqrt{3}) \).

\[\blacksquare\]
3.3. EXERCISES DE-3

Train-mission-line

We wish to model the dynamics of a freight-train having \( N \) such cars, and study the velocity transfer function under various load conditions. As shown in Fig. 3.3, the train model consists of masses connected by springs.

Equations for eigenvalues, eigenvectors and eigenmatrix of \( T \). Given a \( T \) (ABCD) transmission matrix, the eigenvalues and vectors are (see Appendix F.4 for details).

Cell matrix:

\[
T(s) = \begin{bmatrix}
A(s) & B(s) \\
C(s) & D(s)
\end{bmatrix}.
\]

Eigenvalues:

\[
\begin{bmatrix}
\lambda_+ \\
\lambda_-
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
(A + D) - \sqrt{(A - D)^2 + 4BC} \\
(A + D) + \sqrt{(A - D)^2 + 4BC}
\end{bmatrix}
\]

Eigenvectors: The eigenvectors simplify even more, especially when \( A = D \):

\[
E_\pm = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 - \sqrt{(A - D)^2 + 4BC} \\
1 + \sqrt{(A - D)^2 + 4BC}
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 \\
-1
\end{bmatrix}
\]

Physical description:

Figure 3.3: Depiction of a train consisting of cars, treated as a mass \( M \) and linkages, treated as springs of stiffness \( K \) or compliance \( C = 1/K \). Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force \( f_n(t) \) to the voltage \( v_n(t) \). The length of each cell is \( \Delta [m] \). The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL.

Problem # 1: Transfer functions

Use the ABCD method to find the matrix representation of Fig. 3.3. Define the force on the \( n \)th train car \( f_n(t) \leftrightarrow F_n(\omega) \), and velocity \( v_n(t) \leftrightarrow V_n(\omega) \).

Break the model into cells consisting of three elements: a series inductor representing half the mass \( (M/2) \), a shunt capacitor representing the spring \( (C = 1/K) \), and another series inductor representing half the mass \( (L = M/2) \). Making the model a cascade of symmetric \( A = D \) identical cell matrix \( T(s) \).

–Q 1.1: Find the elements of the ABCD matrix \( T \) for the single cell that relate the input node 1 to output node 2

\[
\begin{bmatrix}
F_1 \\
V_1
\end{bmatrix} = T \begin{bmatrix}
F_2(\omega) \\
-V_2(\omega)
\end{bmatrix}.
\]

Sol:

\[
T = \begin{bmatrix}
sM/2 \\
0
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
sM/2 \\
1
\end{bmatrix} = \begin{bmatrix}
1 + s^2MC/2 & (sM)(1 + s^2MC/4) \\
sC & 1 + s^2MC/2
\end{bmatrix}
\]

(3.23a)
Q 1.2: Express each element of $\mathbf{T}(s)$ in terms of the complex “Nyquist” ratio $s/s_c < 1$ ($s = 2\pi f$, $s_c = 2\pi f_c$). The Nyquist sampling cutoff frequency $f_c$ is defined in terms of the minimum number of cells (i.e., 2) of length $\Delta$ per wavelength:

The Nyquist sampling theorem says that there are at least two cars per wavelength (more than two time samples at the highest frequency). From the figure, the distance between cars $\Delta = c_o T_0 [\text{m}]$ where

$$c_o = \frac{1}{\sqrt{MC}} \quad [\text{m/s}]$$

The cutoff frequency obeys $f_c \lambda_c = c_o$ where the Nyquist wavelength is $\lambda_c = 2\Delta$. Therefore the Nyquist sampling condition is

$$\omega < 2\pi f_c \equiv \frac{2\pi c_o}{\lambda_c} = \frac{2\pi c_o}{2\Delta} = \frac{\pi}{\Delta \sqrt{MC}} \quad [\text{Hz}]$$


$\text{Sol:}$ Reiterating what was said above: the system in Fig. 3.3 represents a transmission line having a wave speed of $c_o = 1/\sqrt{MC}$ and characteristic impedance $r_o = \sqrt{M/C}$. Each cell, composed of 2 masses $M$ connected by one spring $K$, has length $\Delta$.

We wish to define the Nyquist frequency $f_c$ such that the wavelength $\lambda > 2\Delta$, where $\Delta$ is the cell length. Using the formula for the wavelength in terms of the wave velocity and frequency we find

$$\lambda = c_o / f_c = 2\Delta,$$

thus we conclude that

$$f_c < \frac{c_o}{2\Delta} = \frac{1}{2\Delta \sqrt{MC}}.$$  

If we wish to have the system be accurate for a given frequency we may make the cell length $\Delta$ smaller, while keeping the velocity constant ($MC$ is held constant). Thus the characteristic resistance [ohms/unit length] $r_o$ must change as $f_c \to \infty$ and $\Delta \to 0$. We can either let $M \to \infty$ and $C \to 0$ (their product remains constant), or the other way around. In one case $r_o \to \infty$ and in the other case it goes to 0.

Q 1.3: Use the property of the Nyquist sampling frequency (Eq. 3.24) to remove higher order powers of frequency

$$1 + \left(\frac{s}{s_c}\right)^2 \approx 1$$

(3.25)

to determine a band-limited approximation of $\mathbf{T}(s)$. Sol:

$$\mathbf{T} \approx \begin{bmatrix} 1 & 2(s/s_c)^2 & sM(1 + (s/s_c)^2) \\ sC & 1 + 2(s/s_c)^2 & 1 \end{bmatrix}$$

The approximation is highly accurate below the Nyquist cutoff frequency $s < s_c$. Given any desired frequency $f$, we can always make the cell size $\Delta$ smaller by decreasing $M$ and $C$. Thus the Nyquist condition is a computational bound (not a physical limitation).

Problem # 2

Now consider the cascade of $N$ such $\mathbf{T}(s)$ matrices, and perform an eigen analysis.

Q 2.1: Find the eigenvalues and eigenvectors of $\mathbf{T}(s)$, as functions of $s/s_c$. Note that the formulae for the eigenvalues, eigenvectors and eigenmatrix are given above the problem setup. Sol: Matrix $\mathbf{T}(s)$ has eigenvalues

$$\lambda_{\pm} = 1 \mp 2s/s_c \approx e^{\pm 2s/s_c} = e^{\mp sT_c}.$$
From this we can interpret the eigenvalues as the cell delay $T_c = 2/s_c$.

The corresponding unnormalized eigenvectors are

$$E_\pm = \begin{bmatrix} \mp \sqrt{M/C} \\ 1 \end{bmatrix},$$

where the characteristic impedance defined is $r_o = \sqrt{M/C}$.

**Problem #3**

Finally, find the velocity transfer function:

- **Q 3.1:** Assuming that $N = 2$ and that $F_3 = 0$ (two half-mass problem), find the transfer function $H(s) = V_2/V_1$. From the results of the $T$ matrix you determined above, find

$$H_{21}(s) = \frac{V_2}{V_1}_{|F_2=0}$$

Express $H_{12}$ in terms of a residue expansion. **Sol:** From Eq. 3.23a, $V_1 = sC F_2 - (s^2 M C/2 + 1)V_2$. Since $F_2 = 0$

$$\frac{V_2}{V_1} = -1 \frac{c_+}{s - s_+} + \frac{c_-}{s - s_-}$$

having eigenfrequencies $s_\pm = \pm j\sqrt{2MC} = \pm s_c$ and residues $c_\pm = \pm j/\sqrt{2MC} = \pm s_c$.

- **Q 3.2:** Find $h_{21}(t) \leftrightarrow H_{21}(s)$.

**Sol:**

$$h(t) = \int_{\sigma_0-j\infty}^{\sigma_0+j\infty} \frac{e^{st}}{s^2 M C/2 + 1} \frac{ds}{2\pi j} = c_+ e^{-s_+ t} u(t) + c_- e^{-s_- t} u(t).$$

The integral follows from the Cauchy Residue theorem (CRT).

- **Q 3.3:** What is the input impedance $Z_2 = F_2/V_2$ assuming $F_3 = -r_0 V_3$?

**Sol:** Starting from Eq. 3.23a find $Z_2$

$$Z_2(s) = \frac{F_2}{V_2} = T \begin{bmatrix} F_3 \\ -V_3 \end{bmatrix} = \frac{-(1 + s^2 M C/2) r_0 V_3 - s M (1 + s^2 M C/4) V_3}{-s r_0 V_3 - (1 + s^2 M C/2) V_3}$$

**Q 3.4:** Simplify the expression for $Z_2$ assuming:

1. $N \rightarrow \infty$
2. The characteristic impedance $r_0 = \sqrt{M/C}$
3. $F_3 = -r_0 V_3$ (i.e., $-V_3$ cancels),
4. Ignore higher order frequency terms ($|s/s_c| < 1$).

**Sol:** Applying the Nyquist approximation (i.e., ignore second order frequency terms $(s/s_c)^2 \ll 1$)

$$Z_2(s) = \frac{r_o (1 + s^2 M C/2)^0 + s M (1 + s^2 M C/4)^0}{r_o s C + (1 + s^2 M C/2)^0}
\approx \frac{r_o + s M}{1 + r_o s C} = \frac{M}{M + r_o s C} \cdot \frac{r_o C + s M}{M + r_o s M C} = \frac{r_o}{M + r_o s C}
\approx \frac{r_o}{M + r_o s C} \frac{r_o C + s / s_c}{M + r_o s / s_c}
= \frac{r_o}{M + r_o s C} \frac{r_o C}{M + r_o s C} = \frac{r_o}{M}$$

We conclude that below the Nyquist cutoff frequency, as $N \rightarrow \infty$ the system approximates a transmission line terminated by its characteristic impedance.
Q 3.5: State the ABCD matrix relationship between the first and Nth node in terms of the cell matrix. Write out the transfer function for one cell: \( H_{21} \)?

Sol:

\[
T = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

Now use the formulae for the eigenvalues and vectors to obtain \( T \) for \( N = 1 \):

\[
T = E \Lambda E^{-1} = E \begin{bmatrix}
\lambda_+ & 0 \\
0 & \lambda_-
\end{bmatrix} E^{-1}.
\]

Q 3.6: What is the velocity transfer function \( H_{N1} = \frac{V_N}{V_1} \)?

Sol:

\[
\begin{bmatrix}
F_1 \\
V_1
\end{bmatrix} = T^N \begin{bmatrix}
F_N(\omega) \\
-V_N(\omega)
\end{bmatrix}
\]

along with the eigenvalue expansion

\[
T^N = E \Lambda^N E^{-1} = E \begin{bmatrix}
\lambda_+^N & 0 \\
0 & \lambda_-^N
\end{bmatrix} E^{-1}.
\]

where \( \lambda_{\pm}^N = e^{\pm sNT_o} \). Recall that \( NT_o \) is the one way delay.

We conclude that as we add more cells, the delay linearly increases with \( N \), since each eigenvalue represents the delay of one cell, and delay adds. ■
Chapter 4

Vector differential equations

4.1 Exercises VC-1

Topic of this homework: Vector algebra and fields in $\mathbb{R}^3$; Gradient and scalar Laplacian operator; Definitions of Divergence and Curl; Gauss’s (divergence) & Stokes’ (Curl) Law; Schwarz inequality; Quadratic forms; System postulates

Vector algebra in $\mathbb{R}^3$.

Definitions of the vector scalar (aka dot) $A \cdot B$, cross $A \times B$ and triple product $A \cdot (B \times C)$ may be found in Appendix A (p. 253), where $A, B, C$ in $\mathbb{R}^3 \subset \mathbb{R}^3$. A fourth “double-cross” ($\mathcal{C}$) vector product is: $1$

$$A \times (B \times C) = \alpha_o B - \beta_o C.$$  

where $\alpha_o = A \cdot C$ and $\beta_o = A \cdot B$ (Note: $A \times (B \times C) \neq (A \times B) \times C$).

Figure 4.1: Definitions of vectors $A, B, C$ (vectors in $\mathbb{R}^3$) used in the definition of $A \cdot B, A \times B$ and $A \cdot (B \times C)$. There are two algebraic vector products, the scalar (dot) product $A \cdot B \in \mathbb{R}$ and the vector (cross) product $A \times B \in \mathbb{R}^3$. Note that the result of the dot product is a scalar, while the vector product yields a vector, which is perpendicular to the plane containing $A, B$. This is figure 3.4 (p. 102), Sect. 3.5.

Problem #1: Scalar product $A \cdot B$

$1$Greenberg p. 694, Eq. 8.
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CHAPTER 4. VECTOR DIFFERENTIAL EQUATIONS

\[ \text{Q 1.1: If } \mathbf{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \text{ and } \mathbf{B} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}, \text{ write out the definition of } \mathbf{A} \cdot \mathbf{B}. \]

Sol: See the definition in the above figure. \[ \mathbf{A} \cdot \mathbf{B} = a_x b_x + a_y b_y + a_z b_z. \text{ In general: } \mathbf{A} \cdot \mathbf{B} = \sum_k A_k B_k. \]

\[ \text{Q 1.2: The dot product is often defined as } ||\mathbf{A}|| ||\mathbf{B}|| \cos(\theta), \text{ where } ||\mathbf{A}|| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \text{ and } \theta \text{ is the angle between } \mathbf{A}, \mathbf{B}. \text{ If } ||\mathbf{A}|| = 1, \text{ describe how the dot product relates to the vector } \mathbf{B}. \]

Sol: See the definition in the above figure. The vector product is the portion of \( \mathbf{B} \) in the direction of \( \mathbf{A} \).

**Problem # 2: Vector (cross) product \( \mathbf{A} \times \mathbf{B} \)**

\[ \text{Q 2.1: If } \mathbf{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \text{ and } \mathbf{B} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z}, \text{ write out the definition of } \mathbf{A} \times \mathbf{B}. \]

Sol:

\[ \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{x} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{y} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{z} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}. \]

\[ \text{Q 2.2: Show that the cross product is equal to the area of the parallelogram formed by } \mathbf{A}, \mathbf{B}, \text{ namely } ||\mathbf{A}|| ||\mathbf{B}|| \sin(\theta), \text{ where } ||\mathbf{A}|| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \text{ and } \theta \text{ is the angle between } \mathbf{A} \text{ and } \mathbf{B}. \]

Sol: A parallelogram’s area is equal to its base times its height. Therefore, let’s say the base is length \( ||\mathbf{A}|| \), and the height \( ||\mathbf{B}|| \sin(\theta) \), which is the portion of \( \mathbf{B} \) that is perpendicular to \( \mathbf{A} \).

**Problem # 3: Triple product \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \)**

Let \( \mathbf{A} = [a_1, a_2, a_3]^T, \mathbf{B} = [b_1, b_2, b_3]^T, \mathbf{C} = [c_1, c_2, c_3]^T \) be three vectors in \( \mathbb{R}^3 \).

\[ \text{Q 3.1: Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: } \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \]

Sol: Using the determinate-definition of the cross product,

\[ \mathbf{B} \times \mathbf{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \hat{x} \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \hat{y} \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \hat{z} \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}. \]

Let \( \mathbf{D} = \mathbf{B} \times \mathbf{C} \) and compute \( \mathbf{A} \cdot \mathbf{D} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \). Finally compute the requested right-hand side, and compare the two. It should be clear that they are the same, because the dot product transfers the elements of vector \( \mathbf{A} \) to cross product and reduces the product to the scalar.

\[ \text{Q 3.2: Describe why } ||\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|| \text{ is the volume of parallelepiped generated by } \mathbf{A}, \mathbf{B} \text{ and } \mathbf{C}. \]

Sol: Note that the norm of \( \mathbf{B} \times \mathbf{C} \) is the area of the parallelogram generated by \( \mathbf{C} \) and \( \mathbf{B} \). Taking the dot product with \( \mathbf{A} \) results in the volume of the corresponding parallelepiped (prism). So the absolute value of triple product is volume of parallelepiped.

\[ \text{Q 3.3: Explain why three vectors } \mathbf{A}, \mathbf{B}, \mathbf{C} \text{ are in one plane if and only if the triple product } \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0. \]

Sol: (triple product is zero) if and only if: (volume is zero), if and only if: (they are in the same plane).

**Problem # 4: Given two vectors } \mathbf{A}, \mathbf{B} \text{ in the } \hat{x}, \hat{y} \text{ plane (see Fig. 1), with } \mathbf{B} = \hat{y} \text{ (i.e., } ||\mathbf{B}|| = 1).
4.1. EXERCISES VC-1

–Q 4.1: Show that \( A \) may be split into two orthogonal parts, one in the direction of \( B \) and the other perpendicular (\( \perp \)) to \( B \). Hint: Express the vector products of \( A \) and \( \hat{B} \) (dot and cross) in polar coordinates (Greenberg, 1988).

\[
A = (A \cdot \hat{B})\hat{B} + \hat{B} \times (A \times \hat{B})
\]

\[
= A_\parallel + A_\perp.
\]

Sol:

\[
A \cdot \hat{B} = ||A|| \cos(\theta) \quad \text{and} \quad A \times \hat{B} = ||A|| \sin(\theta)
\]

The first quantity is in the direction of \( \hat{B} \), while the second is in the direction \( A \times \hat{B} \), which is \( \perp \) to \( \hat{B} \). Thus

\[
A = ||A|| \left( \hat{B} \cos(\theta) + A \times \hat{B} \sin(\theta) \right)
\]

\[
= A_\parallel + A_\perp.
\]

Scalar fields and the \( \nabla \) operator

Problem # 5: Let \( T(x, y) = x^2 + y \) be an analytic scalar temperature field in 2 dimensions (single-valued \( \in \mathbb{R}^2 \)).

–Q 5.1: Find the gradient of \( T(x) \) and make a sketch of \( T \) and the gradient.

Sol: Forming this operation we find that

\[
\frac{\partial^2}{\partial x^2} x^2 + \frac{\partial^2}{\partial y^2} y = 2.
\]

So \( T(x) \) does not satisfy laplace’s equation, rather it satisfies the Poisson equation \( \nabla^2 T(x) = 2 \).

–Q 5.2: Compute \( \nabla^2 T(x) \), to determine if \( T(x) \) satisfies Laplace’s equation.

Sol: The iso-potential contours are the concave parabolas \( y = T_0 - x^2 \).

–Q 5.3: Sketch the iso-temperature contours at \( T = -10, 0, 10 \) degrees.

Sol: The iso-potential contours are the concave parabolas \( y = T_0 - x^2 \).

–Q 5.4: The heat flux \( \nabla^2 \) is defined as \( J(x, y) = -\kappa(x, y)\nabla T \) where \( \kappa(x, y) \) is a constant denoting thermal conductivity at the point \((x, y)\). Assuming \( \kappa = 1 \) everywhere (the medium is homogenous), plot the vector \( J(x, y) = -\nabla T \) at \( x = 2, y = 1 \). Be clear about the origin, direction and length of your result.

Sol: \( J = \nabla T = -2x\hat{x} + \hat{y} \) thus \( -\kappa \nabla T(2, 1) = J = -(4\hat{x} + \hat{y}) \), which has a length of \( \sqrt{17} \) and is pointed \( 1\sqrt{17} \) unit down and \( 4/\sqrt{17} \) units to the left.

\(^2\)The heat flux is proportional to the change in temperature times the thermal conductivity \( \kappa \) of the medium.
-Q 5.5: Find the vector $\perp$ to $\nabla T(x, y)$, namely tangent to the iso-temperature contours. Hint: Sketch it for one $(x, y)$ point (e.g., 2,1) and then generalize.

**Sol:** We may invoke the third dimension $\hat{z}$ to generate this vector: $\pm \hat{z} \times \nabla T = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \pm 1 \\ 2x & 1 & 0 \end{bmatrix} = \mp(1\hat{x} - 2x\hat{y} + 0\hat{z})$. Alternatively, rotate $\nabla T$ by $\pm \pi/2$ in the $(x, y)$ plane.

- Q 5.6: The thermal resistance $R_T$ is defined as the potential drop $\Delta T$ over the magnitude of the heat flux $|J|$. At a single point the thermal resistance is

$$R_T(x, y) = -\nabla T/|J|.$$  

How is $R_T(x, y)$ related to the thermal conductivity $\kappa(x, y)$?

**Sol:** $R_T(x, y) = 1/\kappa(x, y)$. In general, resistance is the reciprocal of conductivity (conductance). This is true for electrical and acoustic systems as well.

**Problem # 6: Acoustic wave equation:** Note: In the following problem, we will work in the frequency domain.

The basic equations of acoustics in 1 dimension are

$$-\partial_x P = \rho_0 s \Psi' \quad \text{and} \quad -\partial_x \Psi = \frac{s}{\eta_0 P_o} P.$$

Here $P(x, \omega)$ is the pressure (in the frequency domain), $\Psi'(x, \omega)$ is the volume velocity (integral of the velocity over the wave-front having area $A$), $s = \sigma + \omega \iota$, $\rho_0 = 1.2$ is the specific density of air, $\eta_0 = 1.4$ and $P_o$ is the atmospheric pressure (i.e., $10^5 \text{[Pa]}$) (see the handout Appendix F.2 for details). Note that the pressure field $P$ is a **scalar** (pressure does not have direction), while the volume velocity field $\Psi'$ is a **vector** (velocity has direction).

We can generalize these equations to 3 dimensions using the $\nabla$ operator

$$-\nabla P = \rho_0 s \Psi' \quad \text{and} \quad -\nabla \cdot \Psi' = \frac{s}{\eta_0 P_o} P.$$

- Q 6.1: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure $P$,

$$\nabla^2 P = \frac{s^2}{c_0^2} P$$

where $c_0$ is a constant representing the speed of sound. **Sol:** We wish to remove $\Psi'$ from the two equations, to obtain a single equation in pressure. If we take the partial wrt $x$ of the pressure equation, and then substitute the velocity equation, to remove the velocity:

$$\nabla^2 P = -\rho_0 s \nabla \cdot \Psi' = \frac{s^2 \rho_0}{\eta_0 P_o} P = \frac{s^2}{c_0^2} P$$

- Q 6.2: What is $c_0$ in terms of $\eta_0$, $\rho_0$, and $P_o$?

**Sol:** Comparing the last two terms from the previous solution we see that

$$c_0 = \sqrt{\eta_0 P_o / \rho_0}.$$ 

- Q 6.3: Rewrite the pressure wave equation in the time domain, using the time derivative property of the Laplace transform (e.g. $dx/dt \leftrightarrow sX(s)$). For your notation, define the time-domain signal using a lowercase letter, $p(x, y, z, t) \leftrightarrow \Phi$.

**Sol:**

$$\nabla^2 p(x, y, z, t) = \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} p(x, y, z, t)$$
Vector fields and the $\nabla$ operator

Vector Algebra

Problem #7: Let $\mathbf{R}(x, y, z) \equiv x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$:

–Q 7.1: If $a, b, c$ are constants, what is $\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c)$?
Sol: Using the formula for a scalar dot-product:

\[
\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c) \equiv [x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}] \cdot [a\hat{x} + b\hat{y} + c\hat{z}]
\]

\[
= x(t)a + y(t)b + z(t)c. \quad (4.1)
\]

–Q 7.2: If $a, b, c$ are constants, what is $\frac{d}{dt} (\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c))$?
Sol: $\left(\frac{d}{dt}a\hat{x}(t) + \frac{d}{dt}b\hat{y}(t) + \frac{d}{dt}c\hat{z}(t)\right)$.

Problem #8: Find the divergence and curl of the following vector fields:

–Q 8.1: $\mathbf{v} = \hat{x} + \hat{y} + 2\hat{z}$
Sol: $\nabla \cdot \mathbf{v} = 0$, $\nabla \times \mathbf{v} = 0$.

–Q 8.2: $\mathbf{v}(x, y, z) = x\hat{x} + xy\hat{y} + z^2\hat{z}$
Sol: $\nabla \cdot \mathbf{v} \equiv \partial_x x + \partial_y xy + \partial_z z^2 = 1 + x + 2z \nabla \times \mathbf{v} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ x & xy & z^2 \end{vmatrix} = (0-0)\hat{x} + (0-0)\hat{y} + (y-0)\hat{z} = y\hat{z}$.

–Q 8.3: $\mathbf{v}(x, y, z) = x\hat{x} + xy\hat{y} + \log(z)\hat{z}$
Sol: Divergence: $\partial_x x + \partial_y xy + \partial_z \log(z) = 1 + x + 1/z$, Curl: $\hat{x}(\partial_y \log(z) - \partial_z xy) + \hat{y}(\partial_z x - \partial_x \log(z)) + \hat{z}(\partial_x xy - \partial_y x) = \hat{y}$.

–Q 8.4: $\mathbf{v}(x, y, z) = \nabla \left(\frac{1}{x+1/y+1/z}\right)$
Sol: First find $\mathbf{v} = -(\hat{x}/x^2 + \hat{y}/y^2 + \hat{z}/z^2)$. Divergence of $\mathbf{v}$: $-(\partial_x 1/x^2 + \partial_y 1/y^2 + \partial_z 1/z^2) = 2(1/x^3 + 1/y^3 + 1/z^3)$, Curl of $\mathbf{v}$: 0, because the curl of the gradient is always zero.

Vector & scalar field identities

Problem #9: Find the divergence and curl of the following vector fields:

–Q 9.1: $\mathbf{v} = \nabla \phi$, where $\phi(x, y) = xe^y$
Sol: $\nabla \times \mathbf{v} = xe^y$, and $\nabla^2 \phi = 0$.

–Q 9.2: $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = x\hat{x} + y\hat{y} + z\hat{z}$
Sol: $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, and $\nabla \times (\nabla \times \mathbf{A}) = 0$.

–Q 9.3: $\mathbf{v} = \nabla \times \mathbf{A}$, where $\mathbf{A} = y\hat{x} + x^2\hat{y} + z\hat{z}$
Sol: $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, and $\nabla \times (\nabla \times \mathbf{A}) = -2\hat{y}$.

–Q 9.4: For any differentiable vector field $\mathbf{V}$, write down two vector-calculus identities that are equal to zero.
Sol: Curl of the gradient $\nabla \times \nabla \Phi(x, y, z) = 0$ and the divergence of the curl $\nabla \cdot \nabla \times \mathbf{V}(x, y, z) = 0$ are both zero. (Page 780, Stillwell).
–Q 9.5: What is the most general form of a vector field may be expressed in, in terms of scalar Φ and vector A potentials?
Sol: \( \mathbf{V} = \nabla \Phi(x, y, z) + \nabla \times \mathbf{A}(x, y, z) \), where \( \Phi \) is the scalar potential and \( \mathbf{A} \) is the vector potential.

Problem # 10: Perform the following calculations. If you can state the answer without doing the calculation, explain why.

–Q 10.1: Let \( \mathbf{v} = \sin(x) \mathbf{\hat{x}} + y \mathbf{\hat{y}} + z \mathbf{\hat{z}} \). Find \( \nabla \cdot (\nabla \times \mathbf{v}) \)

Hint: Look at Lec 41 on page 83 of the notes, Eq. 1.58, 59.
Sol: 0

–Q 10.2: Let \( \mathbf{v} = \sin(x) \mathbf{\hat{x}} + y \mathbf{\hat{y}} + z \mathbf{\hat{z}} \). Find \( \nabla \times (\nabla \sqrt{\mathbf{v} \cdot \mathbf{v}}) \)

Sol: 0

–Q 10.3: Let \( \mathbf{v}(x, y, z) = \nabla (x + y^2 + \sin(\log(z))) \). Find \( \nabla \times \mathbf{v}(x, y, z) \).

Sol: It is zero because \( \nabla \times \nabla f(x, y, z) \) is always zero.

Integral theorems

Problem # 11: Gauss’ and Stokes’ laws.

–Q 11.1: In a few words, identify the law, define what it means, and explain the following formula:

\[
\int_S \mathbf{n} \cdot \mathbf{v} \, dA = \int_V \nabla \cdot \mathbf{v} \, dV.
\]

Sol: This is the integral form of Gauss’ law. The unit normal vector is \( \perp \) to the surface \( S \) having area \( A \equiv \int_S dA \) The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field \( \nabla \cdot \mathbf{v} \) over the volume contained by the surface, and defined as \( \Psi' \).

–Q 11.2: What is the name of this formula?

\[
\int_S (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = \oint_C \mathbf{V} \cdot d\mathbf{R}
\]

Give one important application. Sol: Stokes Theorem, which relates the differential to the integral form of Maxwell’s equations.

–Q 11.3: Describe a key application of the vector identity

\[
\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}.
\]

Sol: When we wish to reduce Maxwell’s two curl equations to the vector wave equation, we must use this identity.

Schwarz inequality

Problem # 12: Below is a picture of these three vectors for an arbitrary value of \( a \) and a specific \( a = a^* \).

\[
\mathbf{V} + 0.5U
\]

\[
\mathbf{V} \cdot U = ||\mathbf{V}|| \alpha U
\]

\[
\alpha U
\]
4.1. EXERCISES VC-1

---

**Q 12.1:** Find the value of \( a^* \in \mathbb{R} \) such that the length (norm) of \( E \) (i.e., \( ||E|| \geq 0 \)) is minimum? Hint minimize

\[
||E||^2 = E \cdot E = (V + aV) \cdot (V + aU) \geq 0
\]  

(4.3)

with respect to \( a \).

**Sol:** In Fig. 12 we see vectors \( V, U \), and for reference, \( V + 0.5U \). Also shown are scaled values of \( U, aU \) and \( a^*U \). These point in the same direction, but are shorter by amounts \( a \) and \( a^* \). When \( U \) is scaled by \( a^* \), length \( ||E(a^*)|| \) is minimum, and \( (V - a^*U) \perp U \), namely vector \( E(a^*) \) is \( \perp \) to vector \( U \). This follows from

\[
\frac{\partial}{\partial a} ||E||^2 = \frac{\partial}{\partial a} ((V + aU) \cdot (V + aU)) = 2(V + aU) \cdot U = 0.
\]

Thus

\[
a^* = -\frac{V \cdot U}{||U||^2}
\]

---

**Q 12.2:** Find the formula for \( ||E(a^*)||^2 \geq 0 \). Hint: Substitute \( a^* \) into Eq. 4.3, and show that this results in the Schwarz inequality

\[
|U \cdot V| \leq ||U|| ||V||.
\]

**Sol:** From Eq. 4.3

\[
||V||^2 + 2a^*V \cdot U + (a^*)^2 ||U||^2 \geq 0
\]

Substituting \( a^* \) gives

\[
||V||^2 ||U||^2 - 2(V \cdot U)^2 + |U \cdot V|^2 \geq 0.
\]

Simplifying

\[
||V||^2 ||U||^2 \geq |U \cdot V|^2
\]

and taking the square root (and swap order), gives the Schwarz inequality

\[
|U \cdot V| \leq ||U|| ||V||.
\]

---

**Problem # 13:** What is the geometrical meaning of the dot product of two vectors? **Sol:** The dot product of two vectors is the length of the \( \perp \) projection of one vector on the other. According to the Schwarz inequality, this project length must be less than the product of the lengths of the two vectors.

---

**Q 13.1:** Give the formula for the dot product between two vectors. Explain the meaning based on Fig. 12.

**Sol:** \( V \cdot U = ||V|| ||U|| \cos \theta_{V,U} \). It represents the amount of one vector going in the direction of the other. In a drawing, it is a projection of the one on the other, found by dropping the \( \perp \) from the tip of one, on the other.

---

**Q 13.2:** Write the formula for the “dot product” between two vectors: \( U \cdot V \) in \( \mathbb{R}^n \) in polar form (e.g., assume the angle between the vectors is equal to \( \theta \)).

**Sol:** \( U \cdot V = \sum_{i=1}^{n} a_i b_i (= ||U|| ||V|| \cos(\theta)) \). This last relationship defines the angle between two vectors.

---

**Q 13.3:** How is this related to the Pythagorean theorem?

**Sol:** It says that for a right triangle, the case when \( a = a^* \), the lengths of the two vectors must be greater than the projection of one on the other, unless they are co-linear (i.e., the angle between them is zero).
–Q 13.4: Starting from \(|U + V|\) derive the triangle inequality

\[ |U + V| \leq |U| + |V|. \]

**Sol:**
\[
|U + V|^2 = (U + V) \cdot (U + V) = |U|^2 + |V|^2 + 2U \cdot V \leq |U|^2 + |V|^2 + 2|U| \cdot |V| \]
Using the Schwarz inequality we find
\[
|U + V|^2 \leq (|U| + |V|)^2.
\]
Completing the square on the right gives
\[
|U + V|^2 \leq (|U| + |V|)|V|.
\]
Final taking the square root gives the triangle inequality.

–Q 13.5: The triangular inequality \(|U + V| \leq |U| + |V|\) is true for 2 and 3 dimensions: Does it hold for 5 dimensional vectors?

**Sol:** It is true in any number of dimensions.

### Quadratic forms

A matrix that has positive eigenvalues is said to be *positive-definite*. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy since the power is the voltage times the current. Given an impedance matrix

\[ V = ZI, \]

the power \(P\) is

\[ P = I \cdot V = I \cdot ZI, \]

which must be positive definite for the system to obey conservation of energy.

**Problem # 14:** For the following problems, consider the 2 \(\times\) 2 impedance matrix

\[ Z = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}. \]

–Q 14.1: Solve for the power \(P(i_1, i_2)\) by multiplying out the matrix equation below (which is in quadratic form) \((I \equiv \begin{bmatrix} i_1 & i_2 \end{bmatrix}^T)\)

\[ P(i_1, i_2) = I^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} I. \]

**Sol:**
\[
P(i_1, i_2) = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = i_1^2 + 2i_1i_2 + 4i_2^2 = 2i_1^2 + 2i_1i_2 + 4i_2^2.
\]

–Q 14.2: Is the impedance matrix positive definite? Show your work by finding the eigenvalues of the matrix \(Z\).

**Sol:** Yes, as it is positive definite if the eigenvalues are both positive. You need to show that the eigenvalues are positive (not zero or negative). They are, so it is. How to do all this is worked on in Example 3, page 593.

\[
\begin{vmatrix} 2 - \lambda & 1 \\ 1 & 4 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = 3 \pm \sqrt{2} > 0
\]

–Q 14.3: Should an impedance matrix always be positive definite? Explain.

**Sol:** Yes.
4.1. EXERCISES VC-1

System Classification

Problem # 15: Answer the following system classification questions about physical systems, in terms of the system postulates.

–Q 15.1: Provide a brief definition of the following properties:

- L/NL : linear(L)/nonlinear(NL):
  Sol: Superposition and scaling hold

- TI/TV : time-invariant(TI)/time varying(TV):
  Sol: The measurement time is irrelevant

- P/A : passive(P)/active(A):
  Sol: An active system has a power source, a passive system does not.

- C/NC : causal(C)/non-causal(NC):
  Sol: Responds only upon or after being driven.

- Re/Clx : real(Re)/complex(Clx):
  Sol: The time function is real (or complex).

–Q 15.2: Along the rows of the table, classify the following systems: In terms of a table having 5 columns, labeled with the abbreviations: L/NL, TI/TV, P/A, C/NC, Re/Clx:

<table>
<thead>
<tr>
<th>Category</th>
<th>Case:</th>
<th>Definition</th>
<th>L/NL</th>
<th>TI/TV</th>
<th>P/A</th>
<th>C/NC</th>
<th>Re/Clx</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Resistor</td>
<td>( v(t) = r_0 i(t) )</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>2</td>
<td>Inductor</td>
<td>( v(t) = L \frac{di}{dt} )</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>3</td>
<td>Switch</td>
<td>( v(t) = \begin{cases} 0 &amp; t \leq 0 \ V_b &amp; t &gt; 0 \end{cases} )</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>4</td>
<td>Transistor</td>
<td>( I_{out} = g_m(V_m) )</td>
<td>Sol: NL</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>5</td>
<td>“Resistor”</td>
<td>( v(t) = r_0 i(t + 3) )</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: NC</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>6</td>
<td>Modulator</td>
<td>( f(t) = e^{i2\pi t}g(t) )</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Clx</td>
</tr>
</tbody>
</table>

Sol: Notes:
1. is a nonlinear system and is active system only when it is connected to a battery, similar to a diode.
2. The current is non-causal since it has a 3 \( [s] \) negative time delay, specified in the time domain.
3. is 1 Hz complex-modulation, so it is both complex and time-varying (TV)

–Q 15.3: Using the same classification scheme, characterize the following equations:

<table>
<thead>
<tr>
<th>Category</th>
<th>Case:</th>
<th>L/NL</th>
<th>TI/TV</th>
<th>P/A</th>
<th>C/NC</th>
<th>Re/Clx</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( A(x) \frac{d^2 y(t)}{dt^2} + D(t)y(x, t) = 0 )</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{dy(t)}{dt} + \sqrt{t} y(t) = \sin(t) )</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: ?</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>3</td>
<td>( y^2(t) + y(t) = \sin(t) )</td>
<td>Sol: NL</td>
<td>Sol: TI</td>
<td>Sol: ?</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{dy^2}{dt} + xy(t + 1) + x^2 y = 0 )</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: NC</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{dy(t)}{dt} + (t - 1) y^2(t) = ie^t )</td>
<td>Sol: NL</td>
<td>Sol: TV</td>
<td>Sol: A?</td>
<td>Sol: C</td>
<td>Sol: Clx</td>
</tr>
</tbody>
</table>
4.2 Exercises VC-2

Topic of this homework:

Maxwell’s equations (ME) and variables (E, D; B, H); Compressible and rotational properties of vector fields; fundamental theorem of vector calculus (Helmholtz’ Theorem); Riemann zeta function; Wave equation.

Notation: The following notation is used in this assignment:

1. \( s = \sigma + j\omega \) is the Laplace frequency, as used in the Laplace transform.
2. A Laplace transform pair are indicated by the symbol \( \leftrightarrow \): e.g., \( f(t) \leftrightarrow F(s) \).
3. \( \pi_k \) is the \( k \)th prime (i.e., \( \pi_k \in \mathbb{P} \), e.g., \( \pi_k = [2, 3, 5, 7, 11, 13 \ldots] \) for \( k = 1..6 \)).

Partial differential equations (PDEs): Wave equation

Problem #1: Solve the wave equation in one dimension by defining \( \xi = t - x/c \).

- Q 1.1: Show that d’Alembert’s solution, \( \varrho(x, t) = f(t - x/c) + g(t + x/c) \), is a solution to the acoustic pressure wave equation, in 1-dimension:

\[
\frac{\partial^2 \varrho(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \varrho(x, t)}{\partial t^2},
\]

where \( f(\xi) \) and \( g(\xi) \) are arbitrary functions. Sol:

\[
\frac{\partial}{\partial x} \varrho(x, t) = \frac{\partial}{\partial x} f(t - x/c) + \frac{\partial}{\partial x} g(t + x/c) = -\frac{1}{c} f'(t - x/c) + \frac{1}{c} g'(t + x/c) \quad (4.4)
\]

\[
\frac{\partial^2}{\partial x^2} \varrho(x, t) = \frac{\partial^2}{\partial x^2} f(t - x/c) + \frac{\partial^2}{\partial x^2} g(t + x/c) = \frac{1}{c^2} f''(t - x/c) + \frac{1}{c^2} g''(t + x/c) \quad (4.5)
\]

\[
\frac{\partial^2}{\partial t^2} \varrho(x, t) = \frac{\partial^2}{\partial t^2} f(t - x/c) + \frac{\partial^2}{\partial t^2} g(t + x/c) = f''(t - x/c) + g''(t + x/c) \quad (4.6)
\]

Problem #2: Solution to the wave equation in spherical coordinates (i.e, 3-dimensions):

- Q 2.1: Write out the wave equation in spherical coordinates \( \varrho(r, \theta, \phi, t) \). Only consider the radial term \( r \) (i.e., dependence on angles \( \theta, \phi \) is assumed to be zero). Hint: The form of the Laplacian as a function of the number of dimensions is given in the last appendix on Transmission lines and Acoustic Horns. Alternatively, look it up on the internet or in a calculus book.

Sol: Given the formula for the Laplacian in spherical coordinates, the wave equation is

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \frac{1}{r^2} \frac{\partial}{\partial r} \varrho(r, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varrho(r, t)
\]
4.2. EXERCISES VC-2

–Q 2.2: Show that the following is true:

\[ \nabla^2 g(r) \equiv \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} g(r) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r g(r). \] (4.7)

Hint: Expand both sides of the equation. Sol: Both sides of the equation expand to

\[ \frac{\partial^2 R}{\partial r^2} + \frac{2}{r} \frac{\partial R}{\partial r}. \]

–Q 2.3: Use the results from Eq. 4.7 to show that the solution to the spherical wave equation is

\[ \nabla^2 g(r, t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} g(r, t) \]

\[ g(r, t) = \frac{f(t - r/c)}{r} + \frac{g(t + r/c)}{r}. \] (4.9)

Sol: This proceed exactly as in the rectangular case (see above) except one must first recognize that the Laplacian in spherical coordinates may be written as

\[ \frac{1}{r} \frac{\partial}{\partial r} r^2 g(r). \] (4.10)

One then may proceed to use the solution for the rectangular case, but for \( r g(r) \), and then divide that solution by \( r \).

–Q 2.4: With \( f(\xi) = \sin(\xi) \) and \( g(\xi) = e^{\xi} u(\xi) \) \( \text{[} u(\xi) \text{ is the step function (Eq. 4.9)} \text{]} \) write down the solutions to the spherical wave equation. \[ \text{Sol: In each case replace } \xi = t - x/c \text{ to obtain the solution to the wave equation for 1 dimensional waves. Thus} \]

\[ g(r, t) = \frac{f(t - r/c)}{r} + \frac{g(t + r/c)}{r} = \frac{\sin(t - r/c)}{t - r/c} + \frac{e^{(t+r/c)} u(t + r/c)}{t + r/c}. \]

–Q 2.5: Sketch this last case for several times (e.g., 0, 1 2 seconds), and describe the behavior of the pressure \( g(r, t) \) as a function of time \( t \) and radius \( r \).

Sol: Plot the functions at several times (e.g., 0, 1 2 seconds), as a function of \( x \). The first function becomes smaller as the radius grows. The second function becomes larger as the inbound waves approaches \( r = 0 \).

–Q 2.6: What happens when the inbound wave reaches the center at \( r = 0 \)?

Sol: Stand back. It blows up. The equations fail when the solution becomes so large that the linearity assumption fails. I’m not sure what actually happens, in practice. This seems to be how they detonate nuclear weapons.

Helmholtz formula

Every differentiable vector field may be written as the sum of a scalar potential \( \phi \) and vector potential \( w \). This relationship is best known as The Fundamental theorem of vector calculus (Helmholtz’ formula).

\[ \mathbf{v} = -\nabla \phi + \nabla \times \mathbf{w}, \] (4.11)
A conservative vector field is defined as the gradient of a scalar potential. This formula seems a natural extension of the algebraic $A \cdot B \perp A \times B$, since $A \cdot B \propto ||A|| ||B|| \cos(\theta)$ and $A \times B \propto ||A|| ||B|| \sin(\theta)$ as developed in the notes (Fig. A.1). Thus these orthogonal components have magnitude 1 when we take the norm, due to Euler’s identity ($\cos^2(\theta) + \sin^2(\theta) = 1$).

As described in Table 5.1 (p. 188), Helmholtz’ formula separates a vector field (i.e., $\mathbf{v}(\mathbf{x})$) into compressible and rotational parts:

1. The rotational (e.g. angular) part is defined by the vector potential $\mathbf{w}$, requiring $\nabla \times \nabla \times \mathbf{w} \neq 0$. A field is irrotational (conservative) when $\nabla \times \mathbf{v} = 0$, meaning that the field $\mathbf{v}$ can be generated using only a scalar potential, $\mathbf{v} = \nabla \phi$ (note this is how a conservative field is usually defined, by saying there exists some $\phi$ such that $\mathbf{v} = \nabla \phi$).

2. The compressible (e.g. radial) part of a field is defined by the scalar potential $\phi$, requiring $\nabla \cdot \nabla \phi = \nabla^2 \phi \neq 0$. A field is incompressible (solenoidal) when $\nabla \cdot \mathbf{v} = 0$, meaning that the field $\mathbf{v}$ can be generated using only a vector potential, $\mathbf{v} = \nabla \times \mathbf{w}$.

The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities:

1. $\nabla \cdot (\nabla \times \mathbf{w}) = 0$
2. $\nabla \times (\nabla \phi) = 0$

**Problem #3: Define the following:**

---

**Q 3.1: A conservative vector field**

**Sol:** A conservative vector field is defined as the gradient of a scalar potential $\mathbf{v} = \nabla \phi(x, y, z)$. Every conservative field is necessarily irrotational (the test for an irrotational field is $\nabla \times \mathbf{v} = 0$).

---

**Q 3.2: A irrotational vector field**

**Sol:** The vector field $\mathbf{v}$ is rotational if there exists a vector potential $\mathbf{w}$ such that $\mathbf{v} = \nabla \times \mathbf{w}(x, y, z)$. The for irrotational is $\nabla \times \mathbf{v} = 0$. A purely rotational field is not conservative.

---

**Q 3.3: An incompressible vector field**

**Sol:** A field $\mathbf{v}$ is incompressible if $\nabla \cdot \mathbf{v} = 0$.

---

**Q 3.4: A solenoidal vector field**

**Sol:** A rotational field is one having a divergence of zero, i.e., $\nabla \cdot \mathbf{v} = 0$, or alternatively, $\mathbf{v} \equiv \nabla \times \mathbf{w}(x, y, z)$, since any field defined by a curl is rotational, since the divergence of the curl is always zero.

---

**Q 3.5: When is a conservative field irrotational?**

**Sol:** Always!

---

**Q 3.6: When is an incompressible field irrotational?**

**Sol:** A field is incompressible if $\nabla \cdot \mathbf{v} = 0$ and irrotational if $\nabla \times \mathbf{v} = 0$. So, almost never. The only case is the trivial solution $\mathbf{v} = 0$, or a constant field $\mathbf{v} = x_0 \hat{x} + y_0 \hat{y} + z_0 \hat{z}$.

**Problem #4:** For each of the following, (i) compute $\nabla \cdot \mathbf{v}$, (ii) compute $\nabla \times \mathbf{v}$, (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.):

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4.2. EXERCISES VC-2

–Q 4.1: \( \mathbf{v}(x, y, z) = -\nabla (3yx^3 + y \log(xy)) \)

**Sol:** The field is conservative (or irrotational) because it is defined by a gradient. To test for irrotational, show that the curl is zero. But \( \nabla \times \nabla \phi(x, y, z) = 0 \) for any \( \phi(x, y, z) \). Thus you do not need to do any computation, just state the answer. 

–Q 4.2: \( \mathbf{v}(x, y, z) = xy \hat{x} - z \hat{y} + f(z) \hat{z} \)

**Sol:** To test for a irrotational field, take the curl, to see if it is zero:

\[
\nabla \times \mathbf{v} \equiv \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial_x & \partial_y & \partial_z \\
xy & -z & f(z)
\end{vmatrix} = \hat{x} - x \hat{z},
\]

which is not zero. We can also see by inspection that \( \nabla \cdot \mathbf{v} \neq 0 \). Thus the vector field is rotational and compressible.

–Q 4.3: \( \mathbf{v}(x, y, z) = \nabla \times (x \hat{x} - z \hat{y}) \)

**Sol:** \( \mathbf{v} = \hat{x} \). Therefore, \( \nabla \times \mathbf{v} = 0 \), and \( \nabla \cdot \mathbf{v} = 0 \). This field is technically incompressible and irrotational, but it is also very boring, since it is a constant.

**Maxwell’s Equations**

The variables have the following names and defining equations:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Equation</th>
<th>Name</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{E} )</td>
<td>( \nabla \times \mathbf{E} = -\mathbf{B} )</td>
<td>Electric Field strength</td>
<td>[Volts/m]</td>
</tr>
<tr>
<td>( \mathbf{D} = \varepsilon_0 \mathbf{E} )</td>
<td>( \nabla \cdot \mathbf{D} = \rho )</td>
<td>Electric Displacement (flux density)</td>
<td>[Col/m²]</td>
</tr>
<tr>
<td>( \mathbf{H} )</td>
<td>( \nabla \times \mathbf{H} = \mathbf{J} + \dot{\mathbf{D}} )</td>
<td>Magnetic Field strength</td>
<td>[Amps/m]</td>
</tr>
<tr>
<td>( \mathbf{B} = \mu_0 \mathbf{H} )</td>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
<td>Magnetic Induction (flux density)</td>
<td>[Webers/m²]</td>
</tr>
</tbody>
</table>

Note that \( \mathbf{J} = \sigma \mathbf{E} \) is the current density (which has units of [Amps/m²]). Furthermore the speed of light in vacuo is \( c_0 = 3 \times 10^{8} \) [m/s], and the characteristic resistance of light \( r_0 = 377 = \sqrt{\mu_0/\varepsilon_0} \) [Ω (i.e., ohms)].

**Speed of light**

**Problem # 5:** The speed of light in vacuo is \( c_0 = 1/\sqrt{\mu_0\varepsilon_0} \approx 3 \times 10^{8} \) [m/s]. The characteristic resistance in in-vacuo is \( r_0 = 1/\sqrt{\mu_0\varepsilon_0} \approx 377 \Omega \). [1]

–Q 5.1: Find a formula for the in-vacuo permittivity \( \varepsilon_o \) and permeability in terms of \( c_o \) and \( r_o \). **Sol:** \( \varepsilon_o = 1/r_o c_o \) and \( \mu_o = r_o/c_o \). Based on your formula, what are the numeric values of \( \varepsilon_o \) and \( \mu_o \)? **Sol:** \( \varepsilon_o \approx 10^{-8}/3 \cdot 377 \approx 8.84 \cdot 10^{-12} \) and \( \mu_o \approx 377/3 \cdot 10^{8} \approx 1.26 \cdot 10^{-6} \).

–Q 5.2: In a few words, identify the law, define what it means, and explain the following formula:

\[
\int_{S} \hat{n} \cdot \mathbf{v} \, dA = \int_{\mathcal{V}} \nabla \cdot \mathbf{v} \, dV
\]

**Sol:** This is the integral form of Gauss’ law. The unit normal vector is \( \perp \) to the surface \( S \) having area \( A \equiv \int_{S} dA \) The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field \( \nabla \cdot \mathbf{v} \) over the volume contained by the surface, and defined as \( \mathcal{V'} \).
Application of ME

Problem # 6: The electric Maxwell equation is $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}$, where $\mathbf{E}$ is the electric field strength and $\mathbf{B}$ is the time rate of change of the magnetic induction field, or simply the magnetic flux density. Consider this equation integrated over a two-dimensional surface $S$, where $\mathbf{n}$ is a unit vector normal to the surface (you may also find it useful to define the closed path $C$ around the surface):

$$\iint_S [\nabla \times \mathbf{E}] \cdot \mathbf{n} dS = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n} dS$$

--Q 6.1: Apply Stokes’ theorem to the left-hand side of the equation.
Sol: The surface $S$ must be open, with its edge $C$ defining the path for the line integral.]

$$\text{emf} \equiv \iint_S \nabla \times \mathbf{E} \cdot \mathbf{n} dS = \oint_C \mathbf{E} \cdot d\mathbf{R}. \quad (4.13)$$

From Stokes’ theorem: the electromotive force (emf) is the line integral of $\mathbf{E}$ around the rim of the open surface. Think of the flux change as the Thevenin source driving the voltage. ■

--Q 6.2: Consider the right-hand side of the equation. How is it related to the magnetic flux $\Psi$ through the surface $S$?
Sol: It is equal to the negative time rate of change of the flux, $-\frac{\partial}{\partial t} \Psi$. From Gauss’ Law the total magnetic flux $\Psi$ is the surface integral over the normal component of the magnetic flux density $\mathbf{B}$. After applying Gauss’ Laws, the surface integral becomes

$$\Psi = -\iint_S \mathbf{B} \cdot \mathbf{n} dS \quad (4.14)$$

■

--Q 6.3: Assume the right-hand side of the equation is zero. Can you relate your answer to part (a) to one of Kirchhoff’s laws?
Sol: This result is well known as Kirchhoff’s first (voltage) law (KVL), $\text{emf} = \sum_k V_k = -\frac{\partial}{\partial t} \Psi$. When the flux induced into the loop may be ignored (e.g., it is very small), the sum of the voltages around the loop is zero. In rectangular coordinates with a plane surface this is simply $\Phi = \mathbf{B}_n A$, where $A$ is the area and $\mathbf{B}_n$ the normal component of $\mathbf{B}$ (perpendicular to the surface $S$). ■

Problem #7: The magnetic Maxwell equation is $\nabla \times \mathbf{H} = \mathbf{C} \equiv \mathbf{J} + \dot{\mathbf{D}}$, where $\mathbf{H}$ is the magnetic field strength, $\mathbf{J} = \sigma \mathbf{E}$ is the conductive (resistive) current density and the displacement current $\dot{\mathbf{D}}$ is the time rate of change of the electric flux density $\mathbf{D}$. Here we defined a new variable $\mathbf{C}$ as the total current density.

--Q 7.1: First consider the equation over a two dimensional surface $S$,

$$\iint_S [\nabla \times \mathbf{H}] \cdot \mathbf{n} dS = \iint_S [\mathbf{J} + \dot{\mathbf{D}}] \cdot \mathbf{n} dS = \iint_S \mathbf{C} \cdot \mathbf{n} dS$$

Apply Stokes’ theorem to the left-hand side of this equation. In a sentence or two, explain the meaning of the resulting equation. Hint: What is the right-hand side of the equation? Sol: The surface $S$ must be open, with its edge $C$ prescribing the line integral, and its surface of $\mathbf{C}$ defines the total current $I(t)$. The normal component of the surface integral over the total current $\mathbf{C}$ gives total current $I(t)$. By Stokes theorem:

$$\text{mmf} \equiv \iint_S \nabla \times \mathbf{H} \cdot \mathbf{n} dS = \oint_C \mathbf{H} \cdot d\mathbf{R} = \iint_S \mathbf{C} \cdot \mathbf{n} dS = I(t)$$

This is Ampere’s Law. ■
Problem # 8: Now consider this equation in three dimensions. Take the divergence of both sides, and integrate over a volume \( V \) (closed surface \( S \)).

\[
\iiint_V \nabla \cdot (\nabla \times \mathbf{H}) dV = \iiint_V \nabla \cdot \mathbf{C} dV
\]

-Q 8.1: What happens to the left-hand side of this equation? Hint: Can you apply a vector identity?

Sol: It is 0.

Apply the divergence theorem (sometimes known as Gauss’s theorem) to the right-hand side of the equation, and interpret your result. Hint: Can you relate your result to one of Kirchhoff’s laws?

Sol: We get

\[
\iiint_V \nabla \cdot \mathbf{C} dV = \iint_S \mathbf{C} \cdot \mathbf{n} dS = 0
\]

This result is Kirchhoff’s second (current) law (KCL), \( \sum_k I_k = \iint \mathbf{D}(t) \cdot dS \). When the stray capacitance (\( \mathbf{D} \)) can be ignored the sum of the currents into the ‘node’ is zero. Generalizing, a ‘node’ to a volume \( V \), the total current \( f(t) \) flowing in/out of the volume is the integral of the normal component of the current density over the cross-sectional closed surface area, which equals 0.

Second-order differentials

Problem # 9: In this section we ask questions about second order vector differentials.

-Q 9.1: If \( \mathbf{v}(x, y, z) = \nabla \phi(x, y, z) \), then what is \( \nabla \cdot \mathbf{v}(x, y, z) \)?

Sol: Since \( \nabla \cdot \nabla = \nabla^2 \) this is \( \nabla^2 \phi(x, y, z) \).

-Q 9.2: Evaluate \( \nabla^2 \phi \) and \( \nabla \times \nabla \phi \) for \( \phi(x, y) = xe^y \).

Sol: \( \nabla \times \nabla \phi = 0 \), \( \nabla^2 \phi = xe^y \).

-Q 9.3: Evaluate \( \nabla \cdot (\nabla \times \mathbf{v}) \) and \( \nabla \times (\nabla \times \mathbf{v}) \) for \( \mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \).

Sol: \( \nabla \cdot (\nabla \times \mathbf{v}) = 0 \), \( \nabla \times (\nabla \times \mathbf{v}) = 0 \).

-Q 9.4: When \( \mathbf{V}(x, y, z) = \nabla(1/x + 1/y + 1/z) \) what is \( \nabla \times \mathbf{V}(x, y, z) \)?

Sol: This is always zero.

-Q 9.5: When was Maxwell born (and die)? How long did he live (within \( \pm 10 \) years)?

Sol: He lived 48 years, from 1831-1879.

Capacitor analysis

Problem # 10: Find the solution to the Laplace equation between two infinite \(^4\) parallel plates, separated by a distance of \( d \). Assume that the left plate, at \( x = 0 \), is at a voltage of \( V(0) = 0 \), and the right plate, at \( x = d \), is at a voltage of \( V_d \equiv V(d) \).

-Q 10.1: Write down Laplace’s equation in one dimension for \( V(x) \).

Sol: This is the Laplace equation for rectangular coordinates

\[
\frac{\partial^2 V(x)}{\partial x^2} = 0
\]

\(^4\)We study plates that are infinite because this means the electric field lines will be perpendicular to the plates, running directly from one plate to the other. However, we will solve for per-unit-area characteristics of the capacitor.
CHAPTE4. VECTOR DIFFERENTIAL EQUATIONS

–Q 10.2: Write down the general solution to your differential equation for $V(x)$.
Sol: Integration is trivial since the solution must be of the form $V(x) = A + Bx$. ■

–Q 10.3: Apply the boundary conditions $V(0) = 0$ and $V(d) = V_d$ to solve for the constants in your equation from the previous part.
Sol: From the BC $A = 0$ and $B = V_d/d$. Thus $V(x) = \frac{V_d}{d}x$. ■

–Q 10.4: Find the charge density per unit area ($\sigma = Q/A$, where $Q$ is charge and $A$ is area) on the surface of each plate. Hint: $\mathbf{E} = -\nabla V$, and Gauss’s Law states that $\iint_S \mathbf{D} \cdot \mathbf{n} dS = Q_{\text{enclosed}}$.
Sol: To find the charge, we must first compute the electric field from the voltage using $\mathbf{E} = -\nabla V(x)$

$$-\mathbf{E} \equiv \nabla V(x) = \hat{x} \frac{\partial}{\partial x} V(x) = \hat{x} V_d$$

Since $\mathbf{D} = \varepsilon_0 \mathbf{E}$ we find the normal component of the $\mathbf{D}$ field

$$\mathbf{D} = \varepsilon_0 \mathbf{E} = -\varepsilon_0 \nabla V$$
is just a constant Thus using Gauss’ law ($\sigma = -\frac{1}{A} \iint D_x dA = D_r$), the surface charge density $\sigma$ in farads per square-meter is

$$\sigma = \frac{\varepsilon_0}{d} V_d$$ ■

–Q 10.5: Determine the per-unit-area capacitance $C$ of the system.
Sol: Since $\sigma = CV_d$, the capacity $C$ per unit area is

$$C = \frac{\varepsilon_0}{d} \text{[F/m}^2\text{].}$$

The units are farads per square-meter. Note that the sign must work out so that $C > 0$.

Webster Horn Equation

Problem #11: Horns provide an important generalization of the solution of the 1D wave equation, in regions where the properties (i.e., area of the tube) vary along the axis of wave propagation. Classic applications of horns are vocal tract acoustics, loudspeaker design, cochlear mechanics, any case having wave propagation.

–Q 11.1: Write out the formula for the Webster horn equation, and explain the variables.
Sol: The horn equation may be written as

$$\frac{1}{A(x)} \frac{\partial}{\partial x} \left( A(x) \frac{\partial \varrho}{\partial x} \right) = \frac{1}{c^2} \frac{\partial^2 \varrho}{\partial t^2}. \tag{4.15}$$

where $A(x)$ is the area of the horn at $x$ (range variable). $\varrho(x, t)$ is the pressure and $c$ is the wave speed. ■