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About the book

Science has evolved over thousands of years. It began out of curiosity about how the world around us works as well as a need to know how to make thing work better. From water management to space travel, science is essential for success.

The evolution of science is layered: Early science depended mostly on critical observation, thus the early scientist was considered a philosopher who created a theory. Soon experiments were designed to test these theories. This process is successful when a new principle is established, leading to a deeper and reproducible experimental observation. This typically takes years. The key component of science is the making and testing models that are designed to quantitatively evaluate the results of experiment in a mathematical framework. Good science is observation and experimentation. Great science is the art of making a model that explains the experimental results. Great science always results in a deeper question, suggesting new experiments. Each generation had its geniuses. One of these was Galileo, who was a philosopher, an experimentalist, and the ultimate mathematician.

An understanding of physics requires a knowledge of mathematics. The converse is not true. By definition, pure mathematics contains no physics. Yet historically, mathematics has a rich history filled with physical applications. Mathematics was developed by individuals with the intent of making things work. As an engineer, I see these creators of early mathematics as budding engineers. This book is an attempt to tell the story of the development of mathematical physics as viewed by an engineer.

There are two distinct ways to learn mathematics: by learning definitions and relationships, or by associating each mathematical concept with its physical counterpart. Students of physics and engineering best learn mathematics based on the underlying physical concepts. Students of pure mathematics are taught via definitions of abstract structures, not from the history of mathematical physics. These two teaching methods result in very different understandings of the material.

There is a deep common thread between physics and mathematics, which is the chronological development, i.e., the history of mathematics. This is because much of mathematics was developed to solve physical problems. Most early mathematics evolved from attempts to understand the world, with the goal of navigating it. Pure mathematics followed as generalizations of these physical concepts.

Around 1638 Galileo stated that, based on his experiments with balls rolling down inclined planes and pendulums, the height of a falling object is given by

\[ h(t) = \frac{1}{2} Gt^2, \]

where \( t \) is time and \( G \) is a constant. This formula leads to a constant acceleration \( a(t) \) of the object, since

\[ a(t) = \frac{d^2}{dt^2} h(t) = G \]

is independent of time. It follows that the force on a body is proportional to its acceleration \( a \), defined as \( G \) – namely, \( F = a \equiv G \). Thus \( G \) must be the object’s mass, which must be a constant. If the object has a constant forward velocity, then the object will have a parabolic trajectory. The relative mass may be measured using a balance scale. I believe Galileo understood all this.

Years later, following up on the observations from Galileo’s study of pendulums and falling objects, Newton showed that differential equations were necessary to explain gravity and that the force of gravity is proportional to the masses of the two objects divided by the square of the reciprocal of the distance between them:

\[ \frac{d^2}{dt^2} r(t) = G \frac{mM}{r^2(t)}. \]
To find \( r(t) \) we must integrate this equation. For an object at height \( h(t) \) above the surface of the earth, 
\[
    r(t) = R_e + h(t) \approx R_e,
\]
where \( R_e \) is the radius of earth. In this case, the force is effectively constant, since \( h \ll R_e \). 
Newton’s equation says the acceleration is constant, 
\[
    \frac{d^2h(t)}{dt^2} = G\frac{mM}{R_e^2},
\]
but different from Galileo’s \( G \) (a simple mass). Yet it seems clear that the physics behind Newton’s 
formula for the acceleration \( a(t) \) of two large masses (sun and earth, or earth and moon) and Galileo’s 
physics for balls rolling down inclined planes are the same.\(^1\) The difference is that Newton’s proportion-
ality constant is a significant generalization of Galileo’s. But, other than the constant, which defines the 
acceleration, the two formulas are the same.

This is not a typical mathematics book; rather, it is about the relationship of math and physics, 
presented roughly in chronological order via their history. To teach mathematical physics in an orderly 
way, our treatment requires a step backward in terms of the mathematics, but a step forward in terms 
of the physics. Historically speaking, mathematics was created by individuals such as Galileo who, by 
modern standards, may be viewed as engineers. This book contains the basic information that well-
formed engineers needs to know, as best I can provide.

Let the reader beware that engineering and physics texts do not intend to be rigorous, in the math-
ematical sense. In some ways, mathematics cleans up the mess by proving theorems, which frequently start with speculations in physics and even engineering. The cleanup is a slow, tedious process. Just 
because something seems obvious based on the known physical facts does not make it a fundamental 
thorem of mathematics.

Although while there are similarities between this book and that of Graham et al. (1994), the differ-
ences are notable. First, Graham’s Concrete Mathematics presents an impossible standard to be 
measured against. Second, is is clearly a math book, brilliantly written and targeted at computer science 
students. This book is not just a math book, it is a mathematical-physics text, which depends much 
underlying math. I would like to believe there are similarities in (1) the broad range of topics, (2) the 
in-depth discussion, and (3) the use of historical context.

**Organization:** I have revived a venerable and useful notation for Section (e.g., §2.1) and Subsection 
(e.g., §1.1.3). We refer to chapters and appendices by number (e.g., Chapter 3, Appendix C). This 
provides a convenient, simple, precise, and compact notation, which is easily decoded. It would be 
cumbersome and confusing to refer to sections and subsections simply by a two or three digit number. 
For example, Eq. 5.3.1 is universal, while (5.3.1) would be a confusing alternative.

As discussed in §1.2.2 and Table 1.1 (p. 24), the book is divided up into three mathematical themes, 
called streams, presented as five chapters: Introduction, Number Systems, Algebra Equations, Scalar 
Calculus, and Vector Calculus. Appendices are used to isolate complex self-contained topics, and Tables, 
such a those for Laplace Transforms, so they are easily (i.e., rapidly) accessible.

Chapter 2, Number Systems, is important because it introduces two key concepts, the greatest com-
mon divisor (GCD) and continued fractions (CFA). When we deal with simple electrical networks com-
posed of inductors, resistors, and capacitors (Fig. 3.6, page 93) or mechanical networks consisting of 
masses, dashpots and springs, or their equivalent, pendulums, as used by Galileo in his studies of gravity 
(Figs. 1.3, page 19 and 3.9, page 105), the system may be modeled as a Brune impedance, defined as 
the ratio of polynomials of Laplace frequency \( s = \sigma + \omega j \) (See §3.2.2, page 75 and §4.3.2, page 123). 
Of special importance is the development of ordinary differential equations (§3.3.2, and Eq. 3.3.2.4) 
which under generalized symmetry conditions, called postulates (§3.7.1, page 106), characterize Brune 
impedances (Brune, 1931a).

Using the CFA (§2.3.2), we can generalize the Brune impedance. This generalization results in a 
transmission line, that describes wave propagation in horns, dealt with in Chapters 4 and 5 (Cauer et al., 1958; Cauer, 1958). This topic is both physically and mathematically important (Cauer, 1932)

\(^1\)https://physicstoday.scitation.org/do/10.1063/PT.6.3.20191002a/full/
Figure 1: There is a natural symbiotic relationship among Mathematics, Engineering, and Physics (MEP), depicted in the Venn diagram. Mathematics provides the method and rigor. Engineering transforms the method into technology. Physics explores the boundaries. While these three disciplines work well together, there is poor communication, in part due to the vastly different vocabularies. But style may be more at issue. For example, mathematics rarely uses a system of units, whereas physics and engineering depend critically on them. Mathematics strives to abstract the ideas into proofs. Physics rarely uses a proof. When they attempt rigor, physicists and engineers typically get into difficulty. An important observation by Felix Klein about pure mathematicians, regarding the unavoidable inaccuracies in physics: “It may be said that the idea [of inaccuracy] is usually so repulsive to [mathematicians] that its recognition sooner or later spoils their interest in natural science” (Condon and Morse, 1929, p. 19)

The material is delivered in numbered sections (e.g., §1.1) spread out over a semester of 15 weeks, there lectures per week, with a three-lecture time-out for administrative duties. Eleven problem sets are provided for weekly assignments.

Many students have rated these assignments as the most important part of the course. There is a built-in interplay between these assignments and the lectures. There are four exams, one at the end of each of the three sections plus the final. The first exam is in class, two others and the final are evening exams. When a student returns the exam, the full solution is provided while the exam is fresh in their mind, resulting in a teaching moment. It is my personal belief that, in principle, students can see the exam in advance of taking it, since each exam is entirely based on the assignments.

Author’s personal statement

An expert is someone who has made all possible mistakes in a small field. I don’t know if I would be called an expert, but I certainly have made my share of mistakes. I openly state that I love making mistakes because I learn so much from them. One might call that the “expert’s corollary.”

This book has been written out of my love for the topic of mathematical physics, a topic that provides many insights, that lead to a deep understanding of important physical concepts. Over the years I have developed a physical sense of math along with a related mathematical sense of physics. While doing my research, I believe that math can be physics, and physics math. I have come across what I feel are certain conceptual holes that need filling, and, sense many deep relationships between math and physics that remain unidentified. What we presently teach is not wrong, but it is missing these relationships. What is lacking is an intuition for how math “works.” Good scientists “listen” to their data. In the same way, we need to start listening to the language of mathematics. We need to let mathematics guide us toward our engineering goals.

As summarized in Fig. 1, this marriage of math, engineering, and physics (MEP) helps us make progress in understanding the physical world. We must turn to mathematics and physics when trying to understand the universe. My views follow from a lifelong attempt to understand human communication – that is, the perception and decoding of human speech sounds. This research arose from my 32 years at Bell Labs in the Acoustics Research Department. There such lifelong pursuits were not only possible but were openly encouraged. The idea was that if you are successful at something, take it as far as you can, but, on the other hand, you should not do something well that’s not worth doing. People got fired

2https://auditorymodels.org/index.php/Main/Publications
3MEP is a focused alternative to STEM.
for the latter. I should have left for a university after a mere 20 years at Bell Labs, but the job was just too cushy.

In this text it is my goal to clarify conceptual errors while telling the story of physics and mathematics. My views have been inspired by classic works, as documented in the Bibliography. This book was inspired by my reading of Stillwell (2002) through his Chapter 21. Somewhere in Chapter 22 I switched to the third edition (Stillwell, 2010), at which point I realized I had much more to master. It became clear that by teaching this material to first-year engineers, I could absorb the advanced material at a reasonable pace. This book soon followed.

Summary

This is foremost a math book, but not the typical math book. First, this book is for the engineering minded, for those who need to understand math to do engineering, to learn how things work. In that sense the book is more about physics and engineering than mathematics. Math skills are essential for making progress in building things, be it pyramids or computers, as clearly shown by the great civilizations of the Chinese, Egyptians, Mesopotamians, Greeks, and Romans.

Second, this is a book about the math that developed to explain physics, to allow people to engineer complex things. To sail around the world, one needs to know how to navigate. This requires a model of the planets and stars. You can only know where you are on earth once you understand where earth is relative to the sun, planets, Milky Way, and distant stars. The answer to such a cosmic question depends strongly on who you ask. Who is qualified to answer such a question? It is best answered by those who study mathematics applied to the physical world. The utility and accuracy of that answer depend critically on the depth of understanding of the physics of the cosmic clock.

The English astronomer Edmond Halley (1656–1742) asked Isaac Newton (1643–1727) for the equation that describes the orbit of the planets. Halley was obviously interested in comets. Newton immediately answered, “an ellipse.” It is said that Halley was stunned by the response (Stillwell, 2010, p. 176), as this was what had been experimentally observed by Kepler (ca. 1619) and he knew that Newton must have some deeper insight. Both were eventually knighted.

When Halley asked Newton to explain how he knew, Newton responded, “I calculated it.” But when challenged to show the calculation, Newton was unable to reproduce it. This open challenge eventually led to Newton’s grand treatise, Philosophiae Naturalis Principia Mathematica (July 5, 1687). It had a humble beginning, as a letter to Halley explaining how to calculate the orbits of the planets. To do this Newton needed mathematics, a tool he had mastered. It is widely accepted that Newton and Gottfried Leibniz invented calculus. But the early record shows that perhaps Bhāskara II (1114–1185 CE) had mastered the art well before Newton.5

Third, the main goal of this book is to teach mathematics to motivated engineers, in a way that it can be understood, mastered, and remembered. How can this impossible goal be achieved? The answer is to fill in the gaps with Who did what, and when? Compared with the math, the historical record is easily mastered.

To be an expert in a field, one must know its history. This includes who the people were, what they did, and the credibility of their story. Do you believe the Pope or Galileo on the roles of the sun and the earth? The observables provided by science are clearly on Galileo’s side. Who were those first engineers? They are names we all know: Archimedes, Pythagoras, Leonardo da Vinci, Galileo, Newton, and so on. All of these individuals had mastered mathematics. This book presents the tools taught to every engineer. Rather than memorizing complex formulas, make the relationships “obvious” by mastering each simple underlying concept.

Fourth, when most educators look at this book, their immediate reactions are: Each lecture is a topic we spend a week on (in our math/physics/engineering class) and You have too much material crammed into one semester. The first sentence is correct, the second is not. Tracking the students who have

---

4I started around December 1970, fresh out of graduate school, and retired on December 5, 2002.

5https://www-history.mcs.st-and.ac.uk/Projects/Pearce/Chapters/Ch8_5.html
taken the course, looking at their grades, and interviewing them personally demonstrate that the material presented here is appropriate for one semester.\footnote{https://www.istem.illinois.edu/news/jont.allen.html}

To write this book I had to master the language of mathematics. I had already mastered the language of engineering and a good part of physics. One of my secondary goals are to build this scientific Tower of Babel by unifying the terminology and removing the jargon.

**Acknowledgments**

I would like to acknowledge John Stillwell for his brilliant and constructive historical summary of mathematics as well as my close friend and long-time (40 years) colleague Steve Levinson, who somehow drew me into this project without my even knowing it. Next, my brilliant graduate student Sarah Robinson was constantly at my side, first repairing blunders in my first-draft homeworks and then grading these and the exams and tutoring the students. Without her, I would never have survived the first semester the material I taught. Her proofreading skills are amazing. Thank you, Sarah, for your infinite help. Without Kevin Pitts this work never could have been started, as he provided early funding when the project was a germ of an idea. Matt Ando’s (math) and Michael Stone’s (physics) encouragement was psychologically important in helping me think I might actually write a book. Finally, I would like to thank John D’Angelo for his highly critical comments, in response to my thousands of silly math questions. When it comes to the heavy hitting, John was always there to provide a brilliant explanation that I could easily translate into engineereese (matheering?) (i.e., engineer language).

My delightful friend Robert Fossum Emeritus Professor of Mathematics from the University of Illinois, kindly pointed out flawed mathematical terminology. James (Jamie) Hutchinson’s precise use of the English language dramatically raised the bar on my more than occasionally casual writing style. To each of you, thank you!

Finally I would like to thank my wife Sheau Feng Jeng, (Patricia Allen), for her unbelievable support and love. She delivered constant peace of mind, without which this project could never have been started, much less finished. Many others, including many students, played important roles, but given their large numbers, sadly they must remain anonymous.

–Jont Allen, Mahomet, IL, May, 12, 2019
Chapter 1

Introduction

Much of early mathematics dating before 1600 BCE centered around the love of art and music, due to the sensations of light and sound. Our psychological senses of color and pitch are determined by the frequencies (i.e., wavelengths) of light and sound. The Chinese and later the Pythagoreans are well known for their early contributions to music theory. We are largely ignorant of exactly what the Chinese scholars knew. The best record comes from Euclid, who lived in the 3rd century after Pythagoras. Thus we can only trace the early mathematics back to the Pythagoreans in the 6th century (580-500 BCE), which is centered around the Pythagorean theorem and early music theory.

Pythagoras strongly believed that “all is number,” meaning that every number, and every mathematical and physical concept, could be explained by integral (integer) relationships, mostly based on either ratios, or the Pythagorean theorem. It is likely that his belief was based on Chinese mathematics from thousands of years earlier. It is also believed that his ideas about the importance of integers followed from the theory of music. The musical notes (pitches) obey natural integral ratio relationships, based on the octave (a factor of two in frequency). The western 12-tone scale breaks the octave into 12 ratios called semitones. Today this has been rationalized to be the 12 root of 2, which is approximately equal to $18/17 \approx 1.06$ or 0.0833 octaves. This innate sense of frequency ratios comes from the physiology of the auditory organ (the cochlea) which represents a fix distance along the Organ of Corti, the sensory organ of the inner ear.

As acknowledged by Stillwell (2010, p. 16), the Pythagorean view is relevant today:

> With the digital computer, digital audio, and digital video coding everything, at least approximately, into sequences of whole numbers, we are closer than ever to a world in which “all is number.”

1.1 Early science and mathematics

While early Asian mathematics has been lost, it clearly defined the course for math for at least several thousand years. The first recorded mathematics were those of the Chinese (5000-1200 BCE) and the Egyptians (3,300 BCE). Some of the best early records were left by the people of Mesopotamia (Iraq, 1800 BCE).\(^1\) While the first 5,000 years of math are not well documented, but the basic record is clear, as outlined in Fig. 1.1 (p. 15).

Thanks to Euclid, and later Diophantus (c250 CE), we have some basic (but vague) understanding of Chinese mathematics. For example, *Euclid’s formula* (Eq. 2.3.3.6, p. 49) provides a method for computing Pythagorean triplets, a formula believed to be due to the Chinese.\(^2\)

---

\(^1\)See Fig. 2.8, p. 50.
\(^2\)One might reasonably view Euclid’s role as that of a mathematical messenger.
Chinese bells and stringed musical instruments were exquisitely developed with tonal quality, as
documented by ancient physical artifacts (Fletcher and Rossing, 2008). In fact this development was
so rich that one must ask why the Chinese failed to initiate the industrial revolution. Specifically, why
did Europe eventually dominate with its innovation when it was the Chinese who did the extensive early
invention?

It could have been for the wrong reasons, but perhaps our best insight into the scientific history
from China may have come from an American chemist and scholar from Cambridge England, Joseph
Needham, who learned Chinese from a colleague, who he later married, and ended up researching early
Chinese science and technology for the US government (Winchester, 2009).

According to Lin (1995), the Needham question is:

“Why did modern science, the mathematization of hypotheses about Nature, with all its
implications for advanced technology, take its meteoric rise only in the West at the time of
Galileo, but had not developed in Chinese civilization or Indian civilization?”

Needham (2013) cites the many developments in China:

“Gunpowder, the magnetic compass, and paper and printing, which Francis Bacon consid-
ered as the three most important inventions facilitating the West’s transformation from the
Dark Ages to the modern world, were invented in China.” (Lin, 1995; Apte, 2009)

“Needham’s works attribute significant weight to the impact of Confucianism and Taoism
on the pace of Chinese scientific discovery, and emphasize what it describes as the ‘diffu-
sionist’ approach of Chinese science as opposed to a perceived independent inventiveness
in the western world. Needham held that the notion that the Chinese script had inhibited
scientific thought was ‘grossly overrated’ ” (Grosswiler, 2004).

Lin (1995) was focused on military applications, missing the importance of non-military contribu-
tions. A large fraction of mathematics was developed to better understand the solar system, acoustics,
musical instruments and the theory of sound and light. Eventually the universe became a popular topic,
as it still is today.

Regarding the “Needham question,” I suspect the resolution is now clear. In the end, China withdrew
itself from its several earlier expansions, based on internal politics (Menzies, 2004, 2008).

**Chronological history pre 16th century**

<table>
<thead>
<tr>
<th>Century</th>
<th>BCE</th>
<th>CE</th>
</tr>
</thead>
<tbody>
<tr>
<td>20th</td>
<td>Chinese (Primes; quadratic equation; Euclidean algorithm (GCD))</td>
<td></td>
</tr>
<tr>
<td>18th</td>
<td>Babylonia (Mesopotamia/Iraq) (quadratic solution)</td>
<td></td>
</tr>
<tr>
<td>6th</td>
<td>Thales of Miletus (first Greek geometry (624));</td>
<td></td>
</tr>
<tr>
<td>5th</td>
<td>Pythagoras and the Pythagorean tribe (570)</td>
<td></td>
</tr>
<tr>
<td>4th</td>
<td>Euclid; Archimedes</td>
<td></td>
</tr>
<tr>
<td>3rd</td>
<td>Eratosthenes (276-194) BCE</td>
<td></td>
</tr>
<tr>
<td>3rd</td>
<td>Diophantus (c250) CE</td>
<td></td>
</tr>
<tr>
<td>4th</td>
<td>Alexandria Library destroyed by fire (391)</td>
<td></td>
</tr>
<tr>
<td>6th</td>
<td>Brahmagupta (negative numbers; quadratic equation) (598-670)</td>
<td></td>
</tr>
<tr>
<td>10th</td>
<td>al-Khwârizmi (algebra) (830); Hasan Ibn al-Haytham (Alhazen) (965-1040)</td>
<td></td>
</tr>
<tr>
<td>12th</td>
<td>Bhaskara (calculus) (1114-1183); Marco Polo (1254-1324)</td>
<td></td>
</tr>
<tr>
<td>15th</td>
<td>Leonardo da Vinci (1452-1519); Michelangelo (1475-1564); Copernicus (1473-1543)</td>
<td></td>
</tr>
<tr>
<td>16th</td>
<td>Tartaglia (cubic solution); Bombelli (1526-1572); Galileo Galilei (1564-1642)</td>
<td></td>
</tr>
</tbody>
</table>
1.1. EARLY SCIENCE AND MATHEMATICS

1.1.1 The Pythagorean theorem

Thanks to Euclid’s *Elements* (c323 BCE) we have an historical record, tracing the progress in geometry, as established by the Pythagorean theorem, which states that for any right triangle having sides of lengths \((a, b, c) \in \mathbb{R}\) that are either positive real numbers, or more interesting, integers \(c > [a, b] \in \mathbb{N}\) such that \(a + b > c\),
\[
c^2 = a^2 + b^2.
\] (1.1.1.1)

Early integer solutions were likely found by trial and error rather than by an algorithm.

<table>
<thead>
<tr>
<th>1500BCE</th>
<th>BCE</th>
<th>500</th>
<th>1000</th>
<th>1400</th>
<th>1650</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chinese</td>
<td>Babylonian</td>
<td>Euclid</td>
<td>Archimedes</td>
<td>Brahmagupta</td>
<td>Bombelli</td>
</tr>
<tr>
<td>Caeser</td>
<td>Pythagoreans</td>
<td>Diophantus</td>
<td>Bhaskara</td>
<td>Copernicus</td>
<td>Marco Polo</td>
</tr>
</tbody>
</table>

Figure 1.1: Mathematical timeline between 1500 BCE and 1650 CE. The western renaissance is considered to have occurred between the 15th and 17th centuries. However, the Asian “renaissance” was likely well before the 1st century (i.e., 1500 BCE). There is significant evidence that a Chinese ‘treasure ship’ visited Italy in 1434, initiating the Italian renaissance (Menzies, 2008). This was not the first encounter between the Italians and the Chinese, as documented in ‘The travels of Marco Polo’ (c1300 CE).

If \(a, b, c\) are lengths, then \(a^2, b^2, c^2\) are each the area of a square. Equation 1.1.1.1 says that the area \(a^2\) plus the area \(b^2\) equals the area \(c^2\). Today a simple way to prove this is to compute the magnitude of the complex number \(c = a + bj\), which forces the right angle
\[
|c|^2 = (a + bj)(a - bj) = a^2 + b^2.
\] (1.1.1.2)

However, complex arithmetic was not an option for the Greek mathematicians, since complex numbers and algebra had yet to be discovered.

Almost 700 years after Euclid’s *Elements*, the Library of Alexandria was destroyed by fire (391 CE), taking with it much of the accumulated Greek knowledge. As a result, one of the best technical records remaining is Euclid’s *Elements*, along with some sparse mathematics due to Archimedes (c300 BCE) on geometrical series, computing the volume of a sphere, the area of the parabola, and elementary hydrostatics. In c1572 a copy Diophantus’s *Arithmetic* was discovered by Bombelli in the Vatican library (Burton, 1985; Stillwell, 2010, p. 51). This book became an inspiration for Galileo, Descartes, Fermat and Newton.

Early number theory: Well before Pythagoras, the Babylonians (c1,800 BCE) had tables of triplets of integers \([a, b, c]\) that obey Eq. 1.1.1.1, such as \([3, 4, 5]\). However, the triplets from the Babylonians were larger numbers, the largest being \(a = 12709, c = 18541\). A stone tablet (Plimpton-322) dating back to 1800 BCE was found with integers for \([a, c]\). Given such sets of two numbers, which determined a third positive integer \(b = 13500\) such that \(b = \sqrt{c^2 - a^2}\), this table is more than convincing that the Babylonians were well aware of Pythagorean triplets (PTs), but less convincing that they had access to Euclid’s formula, a formula for PTs. (Eq. 2.3.3.6 p. 49).

It seems likely that Euclid’s *Elements* was largely the source of the fruitful era due to the Greek mathematician Diophantus (215-285) (Fig. 1.1), who developed the field of discrete mathematics now known as Diophantine analysis. The term means that the solution, not the equation, is integer. The work of Diophantus was followed by fundamental change in mathematics, possibly leading to the development of algebra, but at least including the discovery of
1. negative numbers,
2. quadratic equation (Brahmagupta, 7th CE),
3. algebra (al-Khwārizmī, 9th CE), and
4. complex arithmetic (Bombelli, 15th CE).

These discoveries overlapped with the European middle (aka, dark) ages. While Europe went “dark,” presumably European intellectuals did not stop working during these many centuries.³

### 1.1.2 What is science?

Science is a process to quantify hypotheses to build truths.⁴ Today it has evolved from the early Greek philosophers, Plato and Aristotle, to a statistical method to either validate or prove wrong, the null hypothesis, using statistical tests. Scientists use the term “null hypothesis” to describe the supposition that there is no difference between the two intervention groups or “no effect” of a treatment on some measured outcome. This measure of the likelihood that an obtained value occurred by chance is called the “p-value,” which when small gives confidence that the null hypothesis is either true (no difference induced by the treatment variables) or faults (above chance effect induced by the treatment variables of probability p). While the present standard of scientific truth, it is not iron clad, and must be used with caution. For example, not all experimental tests may be reduced to a single binary test. Does the sun rotate around the moon, or around the earth? There is no test of this question, as it is nonsense. To even say that the earth rotates around the sun is, in some sense nonsense, because all the planets are involved in the many motions.

Yet science works quite well. We have learned many deep secrets regarding the universe over the last 5,000 years.

### 1.1.3 What is mathematics?

It seems strange when people complain they “can’t learn math,”⁵ but then claim to be good at languages. Pre-high-school students tend to confuse arithmetic with math. One does not need to be good at arithmetic to be good at math (but it doesn’t hurt). Gauss made his career based on numbers, especially primes.

Math is a language, with the symbols taken from various languages, not so different from other languages. Today’s mathematics is a written language with an emphasis on symbols and glyphs, biased toward Greek letters, obviously due to the popularity of Euclid’s Elements. The specific evolution of these symbols is interesting (Mazur, 2014). Each symbol is dynamically assigned a meaning, appropriate for the problem being described. These symbols are then assembled to make sentences. It is similar to Chinese in that the spoken and written versions are different across dialects. Like Chinese, the sentences may be read out loud in any language (dialect), while the mathematical sentence (like Chinese characters) is universal.

Learning languages is an advanced social skill. However, the social outcomes of learning a language and learning math are very different. Learning a new language is fun because it opens doors to other cultures. Math is different due to the rigor of the grammar (rules of the language), along with the way it is taught (e.g., not as a language). A third difference between math and language is that math evolved from physics, with important technical applications.

As with any language, the more mathematics you learn, the easier it is to understand, because mathematics is built from the bottom up. It’s a continuous set of concepts, much like the construction of a

---

³It would be interesting to search the archives of the monasteries, where the records were kept, to determine exactly what happened during this religious blackout.
⁵“It looks like Greek to me.”
1.1. EARLY SCIENCE AND MATHEMATICS

house. If you try to learn calculus and differential equations, while skipping simple number theory, the lessons will be more difficult to understand. You will end up memorizing instead of understanding, and as a result you will likely soon forget it. When you truly understand something, it can never be forgotten. A nice example is the solution to a quadratic equation: If you learn how to complete the square (p. 59), you will never forget the quadratic formula.

The topics need to be learned in order, just as in the case of building the house. You can’t build a house if you don’t know about screws or cement (plaster). Likewise in mathematics, you will not learn to integrate if you have failed to understand the difference between integers, complex numbers, polynomials, and their roots.

A short list of topics for mathematics are numbers (\(\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{C}\)), algebra, derivatives, anti-derivatives (i.e., integration), differential equations, vectors and the spaces they define, matrices, matrix algebra, eigenvalues and vectors, solutions of systems of equations, matrix differential equations and their eigen-solution. Learning is about understanding, not memorizing.

The rules of mathematics are formally defined by algebra. For example, the sentence \(a = b\) means that the number \(a\) has the same value as the number \(b\). The sentence is spoken as “\(a\) equals \(b\).” The numbers are nouns and the equal sign says they are equivalent, playing the role of a verb, or action symbol. Following the rules of algebra, this sentence may be rewritten as \(a - b = 0\). Here the symbols for minus and equal indicate two types of actions (verbs).

Sentences can become arbitrarily complex, such as the definition of the integral of a function, or a differential equation. But in each case, the mathematical sentence is written down, may be read out loud, has a well-defined meaning, and may be manipulated into equivalent forms following the rules of algebra and calculus. This language of mathematics is powerful, with deep consequences, first known as algorithms, but eventually as theorems.

The writer of an equation should always translate (explicitly summarize the meaning of the expression), so the reader will not miss the main point, as a simple matter of clear writing.

Just as math is a language, so language may be thought of as mathematics. To properly write correct English it is necessary to understand the construction of the sentence. It is important to identify the subject, verb, object, and various types of modifying phrases. Look up the interesting distinction between that and which.\(^6\) Thus, like math, language has rules. Most individuals use what “sounds right,” but if you’re learning English as a second language, it is necessary to understand the rules, which are arguably easier to master than the foreign speech sounds.

Context can be critical, and the most important context for mathematics is physics. Without a physical problem to solve, there can be no engineering mathematics. People needed to navigate the earth, and weigh things. This required an understand of gravity. Many questions about gravity were deep, such as “Where is the center of the universe?”\(^7\) But church dogma only goes so far. Mathematics, along with a heavy dose of physics, finally answered this huge question. Someone needed to perfect the telescope, and put satellites into space, and view the cosmos. Without mathematics none of this would have happened.

1.1.4 Early physics as mathematics: Back to Pythagoras

There is a second answer to the question What is mathematics? The answer comes from studying its history, which starts with the earliest record. This chronological view starts, of course, with the study of numbers. First there is the taxonomy of numbers. It took thousands of years to realize that numbers are more than the counting numbers \(\mathbb{N}\), and to create a symbol for nothing (i.e., zero), and to invent negative numbers. With the invention of the abacus, a memory aid for the manipulation of complex sets of real integers, one could do very detailed calculations. But this required the discovery of algorithms (procedures) to add, subtract, multiply (many adds of the same number) and divide (many subtracts

\(^6\)https://en.oxforddictionaries.com/usage/that-or-which

\(^7\)Actually this answer is simple: Ask the Pope and he will tell you. (I apologize for this inappropriate joke.)
<table>
<thead>
<tr>
<th>1526</th>
<th>1596</th>
<th>1650</th>
<th>1700</th>
<th>1750</th>
<th>1800</th>
<th>1850</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bombelli</td>
<td>Descartes</td>
<td>Johann Bernoulli</td>
<td>Gauss</td>
<td>Ferma</td>
<td>Euler</td>
<td>Cauchy</td>
</tr>
<tr>
<td>Galileo</td>
<td>Daniel Bernoulli</td>
<td>d'Alembert</td>
<td></td>
<td>Newton</td>
<td>Lagrange</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1.2: Timeline covering the two centuries from 1596CE to 1855CE, covering the development of the modern theories of analytic geometry, calculus, differential equations and linear algebra. Newton was born about 1 year after Galileo died and thus was heavily influenced by his many discoveries. The vertical red lines indicate mentor-student relationships. Note the significant overlap between Newton and Johann and his son Daniel Bernoulli, and Euler, a nucleation point for modern mathematics. Gauss had the advantage of input from Newton, Euler, d'Alembert and Lagrange. Lagrange had a key role in the development of linear algebra. Likely Cauchy had a significant contemporary influence on Gauss as well. Finally, note that Fig. 1.1 ends with Bombelli while this figure begins with him. He was a mathematician who famously discovered a copy of Diophantus' book in the Vatican library. This was the same book that Fermat wrote in, for which the margin was too small to hold the "proof" of his "last theorem."

The role of mathematics is to summarize algorithms (i.e., sets of rules), and formalize the idea as a theorem. Pythagoras and his followers, the Pythagoreans, believed that there was a fundamental relationship between mathematics and the physical world. The Asian civilizations were the first to capitalize on the relationship between science and mathematics, to use mathematics to design things for profit. This may have been the beginning of capitalizing technology (i.e., engineering), based on the relationship between physics and math. This impacted commerce in many ways, such as map making, tools, implements of war (the wheel, gunpowder), art (music), water transport, sanitation, secure communication, food, etc. Of course the Chinese were among the first to master many of these technologies.

Why is Eq. 1.1.1.1 called a theorem? Theorems require a proof. What exactly needs to be proved? We do not need to prove that \((a,b,c)\) obey this relationship, since this is a condition that is observed. We do not need to prove that \(a^2\) is the area of a square, as this is the definition of an area. What needs to be proved is that the relation \(c^2 = a^2 + b^2\) holds if, and only if, the angle between the two shorter sides is \(90^\circ\). The Pythagorean theorem (Eq. 1.1.1.1) did not begin with Euclid or Pythagoras, rather they appreciated its importance, and documented its proof.

In the end the Pythagoreans, who instilled fear in the neighborhood, were burned out, and murdered, likely the fate of mixing technology with politics:

"Whether the complete rule of number (integers) is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Mesopotamia in 497 BCE."

–Stillwell (2010, p. 16)

1.2 Modern mathematics

Modern mathematics (what we practice today) was born in the 15-16\textsuperscript{th} centuries, in the minds of Leonardo da Vinci, Bombelli, Galileo, Descartes, Fermat, and many others (Burton, 1985). Many of these early masters were, like the Pythagoreans, secretive to the extreme about how they solved problems. This soon changed due to Galileo, Mersenne, Descartes and Newton, causing mathematics to blossom. The developments during this time may seemed hectic and disconnected. But this is a wrong impression, as the development was dependent on new technologies such as the telescope (optics) and
more accurate time and frequency measurements, due to Galileo’s studies of the pendulum, and a better understanding of the relation $f \lambda = c_o$ between frequency $f$, wavelength $\lambda$ and the wave speed $c_o$.

### 1.2.1 Science meets mathematics

**Early studies of vision and hearing:** Since light and sound (music) played such a key role in the development of the early science, it was important to fully understand the mechanism of our perception of light and sound. There are many outstanding examples where physiology impacted mathematics. Leonardo da Vinci (1452–1519) is well known for his early studies of human anatomy, the knowledge of which was key when it came to drawing and painting the human form.

**Galileo:** In 1589 Galileo famously conceptualized an experiment where he suggested dropping two different weights from the Leaning Tower of Pisa, and he suggested that they must take the same time to hit the ground.

![Figure 1.3](image.png)

**Figure 1.3:** Depiction of the argument of Galileo (unpublished book of 1638) as to why weights of different masses (i.e., weight) must fall with the same velocity, contrary to what Archimedes had proposed c250 BCE.

Conceptually this is a mathematically sophisticated experiment, driven by a mathematical argument in which he considered the two weights to be connected by an elastic cord (a spring) or balls rolling down a friction-less incline plane. His studies resulted in the concept of *conservation of energy*, one of the cornerstones of physical theory since that time.

Being joined with an elastic cord, the masses become one. If the velocity were proportional to the mass, as believed by Archimedes, the sum of the two weights would necessarily fall even faster. This results in a logical fallacy: How can two masses fall faster than either? This also violates the concept of conservation of energy, as the total energy of two masses would be greater than that of the parts. In fact Galileo’s argument may have been the first time that the principle of *conservation of energy* was clearly stated.

It seems likely that Galileo was attracted to this model of two masses connected by a spring because he was also interested in planetary motion, which consist of masses (sun, earth, moon), also mutually attracted by gravity (i.e., the spring).

Galileo also performed related experiments on pendulums, where he varied the length $l$, mass $m$, and angle $\theta$ of the swing. By measuring the period (periods/unit time) he was able to formulate precise rules between the variables. This experiment also measured the force exerted by gravity, so the experiments were related, but in very different ways. The pendulum served as the ideal clock, as it needed very little energy to keep it going, due to its very low friction (energy loss).

In a related experiment, Galileo measured the duration of a day by counting the number of swings of the pendulum in 24 hours, measured precisely by the daily period of a star as it crossed the tip of a church steeple. The number of seconds in a day is $24 \times 60 \times 60 = 86400 = 2^7 \cdot 3^2 \cdot 5^2 \ [s/day]$. Since
86,400 is the product of the first three primes, it is highly composite, and thus may be reduced in many equivalent ways. For example, the day can be divided evenly into 2, 3, 5 or 6 parts, and remain exact in terms of the number of seconds that transpire. Factoring the number of days in a year (365=5*73) is not useful, since it may not be decomposed into many small primes. Galileo also extended work on the relationship of wavelength and frequency of a sound wave in musical instruments. On top of these impressive accomplishments, Galileo greatly improved the telescope, which he needed for his observations of the planets.

Many of Galileo’s contributions resulted in new mathematics, leading to Newton’s discovery of the wave equation (c1687), followed 60 years later by its one-dimensional general solution by d’Alembert (c1747).

**Mersenne:** Marin Mersenne (1588–1648) also contributed to our understanding of the relationship between the wavelength and the dimensions of musical instruments, and is said to be the first to measure the speed of sound. At first Mersenne strongly rejected Galileo’s views, partially due to errors in Galileo’s reports of his results. But once Mersenne saw the significance of Galileo’s conclusion, he became Galileo’s strongest advocate, helping to spread the word (Palmerino, 1999).

This incident invokes an important theorem of nature: Is data more like bread or wine? The answer is, it depends on the data. Galileo’s original experiments on pendulums and ball falling down slopes, were flawed by inaccurate data. This is likely because he didn’t have good clocks. But he soon solved that problem and the data became more accurate. We don’t know if Mersenne repeated Galileo’s experiments, and then appreciated his theory, or if he communicated with Galileo. But the final resolution was that the early data were like bread (it rots), but when the experimental method was improved, with a better clock, the corrected data was like wine (which improves with age). Galileo claimed that the time for the mass to drop a fixed distance was exactly proportional to the square of the time. This expression lead to \( F = ma \). One follows from the other. If the mass varies then you get Newton’s second law of motion (Eq. 3.1.0.2, p. 56).

He was also a decent mathematician, inventing (1644) the Mersenne primes (MP) \( \pi_m \) of the form

\[
\pi_m = 2^{\pi_k} - 1,
\]

where \( \pi_k \) \((k < m)\) denotes the \( k \)th prime (p. 28). As of Dec 2018 51 MPs are known\(^9\). The first MP is \( 3 = \pi_2 = 2^{\pi_1} - 1 \), and the largest known prime is a MP. Note that 127 = \( \pi_{31} = 2^{7} - 1 \) is the MP of the MP \( \pi_7 \).

**Newton:** With the closure of Cambridge University due to the plague of 1665, Issac Newton returned home to Woolsthorpe-by-Colsterworth (95 [mi] north of London), to work by himself for over a year. It was during this solitary time that he did his most creative work.

Exploring our physiological senses requires a scientific understanding of the physical processes of vision and hearing, first considered by Newton (1643–1727), but researched later in much greater detail, by Helmholtz (Stillwell, 2010, p. 261).

While Newton (1642–1726) may be best known for his studies on light and gravity, he was the first to predict the speed of sound. However, his theory was in error\(^10\) by \( \sqrt{c_p/c_v} = \sqrt{\frac{1}{3}} = 1.183 \). This famous error would not be resolved for 129 years, awaiting the formulation of thermodynamics and the equi-partition theorem by Laplace in 1816 (Britannica, 2004).

Just 11 years prior to Newton’s 1687 *Principia*, there was a basic understanding that sound and light traveled at very different speeds, due to the experiments of Ole Rømer (Feynman, 1968; Feynman: Speed of Light, 2019, google online Feynman videos)

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\(^8\)For example, if the year were 364 = \( 2^2 \cdot 7 \cdot 13 \) days, it would would make for shorter years (by 1 day), 13 months per year (e.g., 28 = \( 2 \cdot 2 \cdot 7 \) day vacation per year?), perfect quarters, and exactly 52 weeks. Every holiday would always fall on the same day, every year. It would be a calendar that humans could understand.

\(^9\)https://mathworld.wolfram.com/MersennePrime.html

\(^{10}\)The square root of the ratio of the specific heat capacity at constant pressure \( c_p \) to that at constant volume \( c_v \).
Ole Rømer first demonstrated in 1676 that light travels at a finite speed (as opposed to instantaneously) by studying the apparent motion of Jupiter’s moon Io. In 1865, James Clerk Maxwell proposed that light was an electromagnetic wave, and therefore traveled at the speed $c_o$ appearing in his theory of electromagnetism (Wikipedia: Speed of Light, 2019).

The idea behind Rømer’s discovery was that due to the large distance between Earth and Io, there was a difference between the period of the moon when Jupiter was closest to Earth vs. when it was farthest from Earth. This difference in distance caused a delay or advance in the observed eclipse of Io as it went behind Jupiter, delayed by the difference in time due to the difference in distance. It is like watching a video of a clock, delayed or sped up. When the video is either delayed or slowed down, the time will be inaccurate (it will indicate an earlier time).

The amazing Bernoulli family: The first individual who seems to have openly recognized the importance of mathematics, enough to actually teach it, was Jacob Bernoulli (1654–1705) (Fig. 1.4). Jacob worked on what is now viewed as the standard package of analytic “circular” (i.e., periodic) functions: $\sin(x)$, $\cos(x)$, $\exp(x)$, $\log(x)$.

Eventually the full details were developed (for real variables) by Euler (p. 78).

From Fig. 1.2 (p. 18) we see that Jacob (1654–1705) would have been strongly influenced by Newton, who in turn would have been influenced by Fermat, Descartes, and of course Galileo. Viète and Wallis, but mostly by Galileo, who died 1 year before Newton was born, on Christmas day 1642. Because the calendar was modified during Newton’s lifetime, his birth date is no longer given as Christmas (Stillwell, 2010, p. 175).

Jacob Bernoulli, like all successful mathematicians of the day, was largely self-taught. Yet Jacob was in a new category of mathematicians because he was an effective teacher. Jacob taught his sibling Johann (1667–1748), who then taught his sibling Daniel (1700–1782). But most importantly, Johann taught Leonhard Euler (1707–1783), the most prolific (thus influential) of all mathematicians. This teaching resulted in an explosion of new ideas and understanding. It is most significant that all four mathematicians published their methods and findings. Much later, Jacob studied with students of Descartes (Stillwell, 2010, p. 268-9).

Euler: Euler’s mathematical talent went far beyond that of the Bernoulli family (Burton, 1985). Another special strength of Euler was the degree to which he published. First he would master a topic, and then he would publish. His papers continued to appear long after his death (Calinger, 2015). It is also somewhat interesting that Leonhard Euler was a contemporary of Mozart (and James Clerk Maxwell of Abraham Lincoln Fig. 1.5, p. 23).

d’Alembert: Another individual of that time of special note, who also published extensively, was d’Alembert (Fig. 1.4). Some of the most innovative ideas were first proposed by d’Alembert. Unfortunately, and perhaps somewhat unfairly, his rigor was criticized by Euler, and later by Gauss (Stillwell, 2010). But once the tools of mathematics were finally openly published, largely by Euler, mathematics grew exponentially.\footnote{There are at least three useful exponential scales: Factors of 2, factors of $e \approx 2.7$, and factors of 10. The octave and decibel use factors of 2 (6 [dB]) and 10 (20 [dB]). Information theory uses powers of 2 (1 [bit]), 4 (2 [bits]). Circuit theory uses all three.}

\footnote{It seems clear that Descartes was also a teacher.}

\footnote{For a related timeline see https://www.famousscientists.org/joseph-louis-lagrange/}

\footnote{https://www-history.mcs.st-andrews.ac.uk/Biographies/Newton.html}

\footnote{The log and tan functions are related by Eq. 4.10.2 p. 116.}
Figure 1.4: Above left: Jakob (1655–1705) and right: Johann (1667–1748) Bernoulli, both painted by their portrait painter brother, Nicolaus. Below left: Leonhard Euler (1707-1783) and right: Jean le Rond d’Alembert (1717-1783). Euler was blind in his right eye, hence the left portrait view.
Gauss: Figure 1.2 (p. 18) shows the timeline of the most famous mathematicians. It was one of the most creative times in mathematics. Gauss was born at the end of Euler’s long and productive life. I suspect that Gauss owed a great debt to Euler: surely he must have been a scholar of Euler. One of Gauss’s most important achievements may have been his contribution to solving the open question about the density of prime numbers and his use of least-squares.16

Cauchy: Augustin-Louis Cauchy (1760–1848), Fig. 1.2 (p. 18), was the son of a well-to-do family, but had the misfortune of being born during the time of the French revolution, which perhaps started with the seven years war which began around 1756. Today the French celebrate Bestial day, July 14 1789, which is viewed as a celebration of the revolution. The French revolution left Cauchy with a lifetime stigma for French politics, that deeply influenced his life. But regardless of his scorn for French politics, he had an unmatched intellect for mathematics. His most obvious achievement was complex analysis, for which he proved many key theorems.

Hermann von Helmholtz: Perhaps starting with the deep work of Hermann von Helmholtz (1821-1894), Fig. 1.5 CH:START (p. 23), educated an experienced as a military surgeon, who mastered classical music, acoustics, physiology, vision, hearing (Helmholtz, 1863b), and, most important of all, mathematics. CH:END Kirchhoff frequently expanded on Helmholtz’s contributions. It is reported that Lord Rayleigh learned German so he could read Helmholtz’s great works.

CH:START Helmholtz’s studies and theories of music and the perception of sound are fundamental scientific contributions (Helmholtz, 1863a). His best known mathematical contribution is today known as the fundamental theorem of vector calculus, or simply Helmholtz theorem (p. 177). CH:END

The history during this time is complex. For example, Lord Kelvin wrote a letter to Stokes, suggesting that Stokes try to prove what is today known as “Stokes theorem.” As a result, Stokes posted a reward (the Smith Prize), searching for a prove of “Lord Kelvin’s theorem,” which was finally proved by Hankel (1839-73).17 Many new concepts were being proved and appreciated over this productive period. In 1863-65, Maxwell published his famous equations, followed by a reformating in modern vector notation by Heaviside, Gibbs and Hertz. The vertical red lines connect mentor-student relationships. This figure should put to rest the idea that ones best work is done in the early years. Many of

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16 https://www-history.mcs.st-andrews.ac.uk/Biographies/Gauss.html
17 https://en.wikipedia.org/wiki/Hermann_Hankel
these scientists were fully productive to the end of old age. Those that were not, died early, due to poor health or accidents. **CH:START**Remove Mozart and Beethoven ... **CH:END**

**Lord Kelvin:** Lord Kelvin (aka William Thompson),\textsuperscript{18} (1824-1907) was one of the first true engineer-scientists, equally acknowledged as a mathematical physicist, well known for his interdisciplinary research, and knighted by Queen Victoria in 1866. Lord Kelvin coined the term *thermodynamics*,\textsuperscript{19} a science more fully developed by Maxwell (the same Maxwell of electrodynamics).

**Lord Rayleigh (aka William Strutt):** (1842-1919). Rayleigh (1896) is a classic text, widely read even today by those who study acoustics. In 1904 he received the Nobel Prize in Physics for his investigations of the densities of the most important gases and for his discovery of argon in connection with these studies.

**1.2.2 Three Streams from the Pythagorean theorem**

From the outset of his presentation, Stillwell (2010, p. 1) defines “three great streams of mathematical thought: *Numbers, Geometry and Infinity*” that flow from the Pythagorean theorem, as summarized in Table 1.1. This is a useful concept, based on reasoning not as obvious as one might think. Many factors are in play here. One of these is the strongly held opinion of Pythagoras that all mathematics should be based on integers. The rest are tied up in the long, necessarily complex history of mathematics, as best summarized by the fundamental theorems (Box material §2.2.1, p. 37), each of which is discussed in detail in a relevant chapter.

<table>
<thead>
<tr>
<th>Table 1.1: Three streams followed from Pythagorean theorem: number systems, geometry and infinity.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) <strong>Numbers:</strong></td>
</tr>
<tr>
<td>6\textsuperscript{th} BCE (\mathbb{N}) counting numbers, (\mathbb{Q}) rationals, (\mathbb{P}) primes</td>
</tr>
<tr>
<td>5\textsuperscript{th} BCE (\mathbb{Z}) common integers, (\mathbb{I}) irrationals</td>
</tr>
<tr>
<td>7\textsuperscript{th} CE zero (\in\mathbb{Z})</td>
</tr>
<tr>
<td>2) <strong>Geometry:</strong> (e.g., lines, circles, spheres, toroids, ...)</td>
</tr>
<tr>
<td>17\textsuperscript{th} CE Composition of polynomials (Descartes, Fermat)</td>
</tr>
<tr>
<td>Euclid’s geometry &amp; algebra (\Rightarrow) analytic geometry</td>
</tr>
<tr>
<td>18\textsuperscript{th} CE Fundamental theorem of algebra</td>
</tr>
<tr>
<td>3) <strong>Infinity:</strong> ((\infty \rightarrow\text{Sets}))</td>
</tr>
<tr>
<td>17-18\textsuperscript{th} CE Taylor series, functions, calculus (Newton, Leibniz)</td>
</tr>
<tr>
<td>19\textsuperscript{th} CE (\mathbb{R}) real, (\mathbb{C}) complex 1851</td>
</tr>
<tr>
<td>20\textsuperscript{th} CE Set theory</td>
</tr>
</tbody>
</table>

As shown in the insert, Stillwell’s concept of three streams, following from the Pythagorean theorem, is the organizing principle behind this book. Broken down by chapter.

§1 The *Introduction* is an historical survey of pre-college mathematical physics, presented in terms of the three main Pythagorean streams S1-S3, leading to the book’s five chapters:

S1 §2 Number systems (p. 27)

\textsuperscript{18}Lord Kelvin was one of a half dozen interdisciplinary mathematical physicists, all working about the same time, who made a fundamental change in our scientific understanding. Others include Helmholtz, Stokes, Green, Heaviside, Rayleigh and Maxwell.

\textsuperscript{19}Thermodynamics is another case which warrants an analysis along historical lines (Kuhn, 1978).
S2 §3 Algebraic equations (p. 55)
S3a §4 Scalar calculus (p. 115)
S3b §5 Vector calculus (p. 139)

The third steam is broken into two parts.

§2 Number Systems: Some important ideas from number theory, starting with prime numbers, complex numbers, vectors and matrices. Four classic number theory problems are discussed, the Euclidean algorithm (GCD), continued fractions (CFA), Euclid’s formula, Pell’s equation and the Fibonacci difference equation. The general solution of these several problems leads to the concept of the eigenfunction analysis, which is first introduced in §2.3.5 (p. 52).

§3 Algebraic Equations: Algebra and its development, as we know it today. The theory of real and complex equations and functions of real and complex variables. Newton’s method for finding complex roots of polynomials. Poles vs. zeros and the Gauss-Lucas theorem (bounds on the root locations of the derivative of a polynomial). Complex impedance $Z(s) = \sigma + \omega \text{j}$ is covered with some care, developing the topic which is needed for engineering mathematics.

While the algebra of real and complex functions is identical, the calculus is fundamentally different. This will lead to the concepts of complex analytic functions, complex Taylor series and the Cauchy-Riemann conditions. These are fundamental concepts when dealing with impedance functions which describe the linear relations between force and flow in the complex frequency domain (i.e., complex impedance).


Chapter 2
Stream 1: Number Systems

Number theory (the study of numbers) was a starting point for many key ideas. For example, in Euclid’s geometrical constructions the Pythagorean theorem for real \([a, b, c]\) was accepted as true, but the emphasis in the early analysis was on integer constructions, such as Euclid’s formula for Pythagorean triplets (Eq. 2.3.3.6, Fig. 2.7, p. 49).

As we shall see, the derivation of the formula for \textit{Pythagorean triplets} is the first of a rich source of mathematical constructions, such as solutions of \textit{Pell’s equation} (p. 50),\(^1\) and the recursive difference equations, such as solutions of the Fibonacci recursion formula \(f_{n+1} = f_n + f_{n-1}\) (p. 52) which goes back at least to the Chinese (c2000 BCE). These are early pre-limit forms of calculus, best analyzed using an eigen-function (e.g., eigen-matrix) expansion, a geometrical concept from linear algebra, as an orthogonal set of normalized unit-length vectors (Appendix B.3, p. 201).

\textbf{The first use of zero and }\infty:\textbf{ It is hard to imagine that one would not appreciate the concept of zero and negative numbers when using an abacus. It does not take much imagination to go from counting numbers }\mathbb{N}\text{ to the set of all integers }\mathbb{Z},\text{ including zero. On an abacus, subtraction is obviously the inverse of addition. Subtraction, to obtain zero abacus beads, is no different than subtraction from zero, giving }\textit{negative}\text{ beads. To assume the Romans, who first developed counting sticks, or the Chinese who then deployed the concept using beads, did not understand negative numbers, is impossible. However, understanding the concept of zero (and negative numbers) is not the same as having a symbolic notation. The Roman number system has no such symbols. The first recorded use of a symbol for zero is said to be by Brahmagupta\(^2\) in 628 CE. However this is likely wrong, given the notation developed by the Mayan civilization which existed from 2000BCE to 900CE (https://www.storyofmathematics.com/mayan.html). There is speculation that the Mayans cut down so much of the Amazon jungle that it eventually resulted in a global warming anomaly, possibly resulting in their demise.}

Defining zero (c628 CE) depends on the concept of subtraction, which formally requires the creation of algebra (c830 CE, Fig. 1.1, p. 15). But apparently it takes more than 600 years, i.e., from the time Roman numerals were put into use, without any symbol for zero, to the time when the symbol for zero is first documented. Likely this delay is more about the political situation, such as government rulings, than mathematics.

The concept that caused much more difficulty was \(\infty\), first resolved by Riemann in 1851 with the development of the \textit{extended plane}, which mapped the plane to a sphere (Fig. 3.13 p. 113). His construction made it clear that the point at \(\infty\) is simply another point on the open complex plane, since rotating the sphere (extended plane) moves the point at \(\infty\) to a finite point on the plane, thereby closing

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\(^1\)Heisenberg, an inventor of the matrix algebra form of quantum mechanics, learned mathematics by studying Pell’s equation (p. 50) https://www.aip.org/history-programs/niels-bohr-library/oral-histories/4661-1 by eigen-vector and recursive analysis methods.


\(^3\)https://www.nytimes.com/2017/10/07/opinion/sunday/who-invented-zero.html
2.1 The taxonomy of numbers: \( \mathbb{P}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C} \)

Once symbols for zero and negative numbers were accepted, progress could be made. To fully understand numbers, a transparent notation was required. First one must differentiate between the different classes (genus) of numbers, providing a notation that defines each of these classes, along with their relationships. It is logical to start with the most basic counting numbers, which we indicate with the double-bold symbol \( \mathbb{N} \). For easy access, double-bold symbols and set-theory symbols, i.e., \( \{\} , \cup, \cap, \in, \varnothing, \perp \) etc., are summarized in Appendix A, p. 185.

Counting numbers \( \mathbb{N} \): These are known as the “natural numbers” \( \mathbb{N} = \{1, 2, 3, \cdots \} \), denoted by the double-bold symbol \( \mathbb{N} \). For clarity we shall refer to the natural numbers as counting numbers, since natural, which means integer, is vague. The mathematical sentence “2 \( \in \mathbb{N} \)” is read as 2 is a member of the set of counting numbers. The word set is defined as the collection of any objects that share a specific property. Typically the set may be defined either as a sentence, or by example.

Primes \( \mathbb{P} \): A number is prime \( \left( \pi_n \in \mathbb{P} \right) \) if its only factors are 1 and itself. The set of primes \( \mathbb{P} \) is a subset of the counting numbers \( \left( \mathbb{P} \subset \mathbb{N} \right) \). A somewhat amazing fact, well known to the earliest mathematicians, is that every integer may be written as a unique product of primes. A second key idea is that the density of primes \( \rho_e(N) \sim 1/\log(N) \), that is \( \rho_e(N) \) is inversely proportional to the log of \( N \), an observation first quantified by Gauss (Goldstein, 1973). A third is that there is a prime between every integer \( N \geq 2 \) and 2\( N \).

We shall use the convenient notation \( \pi_n \) for the prime numbers, indexed by \( n \in \mathbb{N} \). The first 12 primes \( \{n|1 \leq n \leq 12\} = \{\pi_n|2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\} \). Since \( 4 = 2^2 \) and \( 6 = 2 \cdot 3 \) may be factored, \( 4, 6 \not\in \mathbb{P} \) (read as: 4 and 6 are not in the set of primes). Given this definition, multiples of a prime, i.e., \( \{2, 3, 4, 5, \ldots \} \cdot \pi_k \) of any prime \( \pi_k \), cannot be prime. It follows that all primes except 2 must be odd and every integer \( \pi_n \) is unique in its prime factorization.

Exercise: Write out the first 10 to 20 integers in prime-factored form. Solution: 1, 2, 3, 2\( ^2 \), 5, 2\( \cdot \)3, 7, 2\( ^3 \), 3\( ^2 \), 2 \( \cdot \)5, 11, 3 \( \cdot \)2\( ^2 \), 13, 2 \( \cdot \)7, 3 \( \cdot \)5, 2\( ^3 \), 17, 2 \( \cdot \)3\( ^2 \), 19, 2\( ^2 \) \( \cdot \)5.

Exercise: Write integers 2 to 20 in terms of \( \pi_n \). Here is a table to assist you:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_n )</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>13</td>
<td>17</td>
<td>19</td>
<td>23</td>
<td>29</td>
<td>31</td>
<td>...</td>
</tr>
</tbody>
</table>

Solution:

<table>
<thead>
<tr>
<th>( n )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pi \pi_n )</td>
<td>( \pi_1 )</td>
<td>( \pi_2 )</td>
<td>( \pi_1 \pi_2 )</td>
<td>( \pi_3 )</td>
<td>( \pi_1 \pi_2 \pi_3 )</td>
<td>( \pi_4 )</td>
<td>( \pi_1 \pi_2 \pi_4 )</td>
<td>( \pi_5 )</td>
<td>( \pi_1 \pi_2 \pi_4 \pi_5 )</td>
<td>( \pi_6 )</td>
<td>( \pi_1 \pi_2 \pi_6 )</td>
<td>...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Coprimes are two relatively prime numbers having no common (i.e, prime) factors. For example, \( 21 = 3 \cdot 7 \) and \( 10 = 2 \cdot 5 \) are coprime whereas \( 4 = 2 \cdot 2 \) and \( 6 = 2 \cdot 3 \), which have \( 2 \) as a common factor, are not. By definition all unique pairs of primes are coprime. We shall use the notation \( m \perp n \) to indicate that \( m, n \) are coprime. The ratio of two coprimes is reduced, as it has no factors to cancel. The ratio of two numbers that are not coprime may always be reduced by canceling the common factors. This is called the reduced form, or an irreducible fraction. When doing numerical work, for computational accuracy it is always beneficial to work with coprimes. Generalizing this idea we could define tri-primes as three primes with no common factor, such as \( \{\pi_3, \pi_9, \pi_2\} \).
The fundamental theorem of arithmetic states that each integer may be uniquely expressed as a unique product of primes. The Prime Number Theorem estimates the mean density of primes over $\mathbb{N}$.

**Integers** $\mathbb{Z}$: These include positive and negative counting numbers and zero. Notionally we might indicate this using set notation as $\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$. Read this as The integers are in the set composed of the negative natural numbers $-\mathbb{N}$, zero, and $\mathbb{N}$.

**Rational numbers** $\mathbb{Q}$: These are defined as numbers formed from the ratio of two integers. Given two numbers $n, d \in \mathbb{N}$, then $n/d \in \mathbb{Q}$. Since $d$ may be 1, it follows that the rationals include the counting numbers as a subset. For example, the rational number $3/1 \in \mathbb{N}$.

The main utility of rational numbers is that they can efficiently approximate any number on the real line, to any precision. For example, the rational approximation $\pi \approx 22/7$ has a relative error of $\approx 0.04\%$ (p. 46).

**Fractional number** $\mathbb{F}$: A fractional number $\mathbb{F}$ is defined as the ratio of signed coprimes. If $n, d \in \pm \mathbb{F}$, then $n/d \in \mathbb{F}$. Given this definition, $\mathbb{F} \subset \mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$. Because of the powerful approximating power of rational numbers, the fractional set $\mathbb{F}$ has special utility. For example, $\pi \approx 22/7$, $1/\pi \approx 7/22$ (to 0.04%), $e = 19/7$ to 0.15%, and $\sqrt{2} \approx 7/5$ to 1%.

**Irrational numbers** $\mathbb{I}$: Every real number that is not rational is irrational ($\mathbb{I} \perp \mathbb{Q}$). Irrational numbers include $\pi, e$ and the square roots of primes. These are decimal numbers that never repeat, thus requiring infinite precision in their representation. Such numbers cannot be represented on a computer, as they would require an infinite number of bits (precision).

The rationals $\mathbb{Q}$ and irrationals $\mathbb{I}$ split the reals ($\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, $\mathbb{Q} \perp \mathbb{I}$), thus each is a subset of the reals ($\mathbb{Q} \subset \mathbb{R}$, $\mathbb{I} \subset \mathbb{R}$). This relation is analogous to that of the integers $\mathbb{Z}$ and fractionals $\mathbb{F}$, which split the rationals ($\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$, $\mathbb{Z} \perp \mathbb{F}$) (thus each is a subset of the rationals ($\mathbb{Z} \subset \mathbb{Q}$, $\mathbb{F} \subset \mathbb{Q}$)).

Irrational numbers ($\mathbb{I}$) were famously problematic for the Pythagoreans, who incorrectly theorized that all numbers were rational. Like $\infty$, irrational numbers required mastering a new and difficult concept before they could even be defined: It was essential to understand the factorization of counting numbers into primes (i.e., the fundamental theorem of arithmetic) before the concept of irrationals could be sorted out. Irrational numbers could only be understood once limits were mastered.

As discussed on p. 47, fractionals can approximate any irrational number with arbitrary accuracy. Integers are also important, but for a very different reason. All numerical computing today is done with $\mathbb{Q} = \mathbb{F} \cup \mathbb{Z}$. Indexing uses integers $\mathbb{Z}$, while the rest of computing (flow dynamics, differential equations, etc.) is done with fractionals $\mathbb{F}$ (i.e., IEEE-754). Computer scientists are trained on these topics, and computer engineers need to be at least conversant with them.

**Real numbers** $\mathbb{R}$: Reals are the union of rational and irrational numbers, namely $\mathbb{R} = \mathbb{Q} \cup \mathbb{I} = \mathbb{Z} \cup \mathbb{F} \cup \mathbb{I}$. Lengths in Euclidean geometry are reals. Many people assume that IEEE 754 floating point numbers (c1985) are real (i.e., $\in \mathbb{R}$). In fact they are rational ($\mathbb{Q} = \{\mathbb{F} \cup \mathbb{Z}\}$) approximations to real numbers, designed to have a very large dynamic range. The hallmark of fractional numbers ($\mathbb{F}$) is their power in making highly accurate approximations of any real number.

Using Euclid’s compass and ruler methods, one can make line length proportionally shorter or longer, or (approximately) the same. A line may be made be twice as long, or an angle can be bisected. However, the concept of an integer length in Euclid’s geometry was not defined. Nor can one construct an imaginary or complex line, as all lines are assumed to be real lengths. The development of analytic geometry was an analytic extension of Euclid’s simple (but important) geometrical methods.

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4 As best I know.
Real numbers were first fully accepted only after set theory was developed by Cantor (1874) (Stillwell, 2010, pp. 461). At first blush, this seems amazing, given how widely accepted real numbers are today. In some sense they were accepted by the Greeks, as lengths of real lines.

**Complex numbers** \( \mathbb{C} \): Complex numbers are best defined as ordered pairs of real numbers.\(^5\) For example, if \( a, b \in \mathbb{R} \) and \( j = -i = \pm \sqrt{-1} \), then \( c = a + bj \in \mathbb{C} \). The word “complex,” as used here, does not mean that the numbers are complicated or difficult. They are also known as “imaginary” numbers, but this does not mean the numbers disappear. Complex numbers are quite special in engineering mathematics, as roots of polynomials. The most obvious example is the quadratic formula, for the roots of polynomials of degree 2, having coefficients \( \in \mathbb{C} \). All real numbers have a natural order on the real line. Complex numbers do not have a natural order. For example, \( j > 1 \) makes no sense.

Today the common way to write a complex number is using the notation \( z = a + bj \in \mathbb{C} \), where \( a, b \in \mathbb{R} \). Here \( 1j = \sqrt{-1} \). We also define \( 1i = -1j \) to account for the two possible signs of the square root. Accordingly \( 1j^2 = 1i^2 = -1 \).

Cartesian multiplication of complex numbers follows the basic rules of real algebra, for example, the rules of multiplying two monomials and polynomials. Multiplication of two first-degree polynomials (i.e., monomials) gives

\[
(a + bx)(c + dx) = ac + (ad + bc)x + bdx^2.
\]

If we substitute \( 1j \) for \( x \), and use the definition \( 1j^2 = -1 \), we obtain the Cartesian product of the two complex numbers

\[
(a + bj)(c + dj) = ac - bd + (ad + bc)j.
\]

Thus multiplication and division of complex numbers obey the usual rules of algebra.

However, there is a critical extension: Cartesian multiplication only holds when the angles sum to less than \( \pm \pi \), namely the range of the complex plane. When the angles add to more that \( \pm \pi \), one must use polar coordinates, where the angles add for angles beyond \( \pm \pi \) (Boas, 1987, p. 8). This is particularly striking for the \( \mathcal{LT} \) transform of a delay (Table C.3, p. 208).

Complex numbers can be challenging, providing unexpected results. For example, it is not obvious that \( \sqrt{3 + 4j} = \pm(2 + j) \).

**Exercise:** Verify that \( \sqrt{3 + 4j} = \pm(2 + j) \). **Solution:** Square the left side gives \( \sqrt{3 + 4j}^2 = 3 + 4j \). Squaring the right side gives \( (2 + j)^2 = 4 - j^2 + 4j = 3 + 4j \). Thus the two are equal.\(^\ast\)

**Exercise:** What is special about this example? **Solution:** Note this is a \( \{3,4,5\} \) triangle. Can you find another example like this one? Namely how does one find integers that obey Eq. 1.1.1.1 (p. 15)?\(^\ast\)

An alternative to Cartesian multiplication of complex numbers is to work in polar coordinates. The polar form of complex number \( z = a + bj \) is written in terms of its magnitude \( \rho = \sqrt{a^2 + b^2} \) and angle \( \theta = \angle z = \tan^{-1} z = \arctan z \), as

\[
z = \rho e^{j\theta} = \rho(\cos \theta + j \sin \theta).
\]

From the definition of the complex natural log function

\[
\ln z = \ln \rho e^{j\theta} = \ln \rho + j\theta,
\]

which is important, even critical, in engineering calculations. When the angles of two complex numbers are greater that \( \pm \pi \), one must use polar coordinates. It follows that when computing the phase, this is much different than the single- and double-argument \( \angle \theta = \arctan(z) \) function.

The polar representation makes clear the utility of a complex number: Its magnitude scales while its angle \( \Theta \) rotates. The property of scaling and rotating is what makes complex numbers useful in engineering calculations. This is especially obvious when dealing with impedances, which have complex roots with very special properties, as discussed on p. 120.

---

\(^5\)A polynomial \( a + bx \) and a 2-vector \( [a, b]^T = \begin{bmatrix} a \\ b \end{bmatrix} \) are also examples of ordered pairs.
### 2.1. THE TAXONOMY OF NUMBERS: P, N, Z, Q, F, I, R, C

**Matrix representation:** An alternative way to represent complex numbers is in terms of $2 \times 2$ matrices. This relationship is defined by the mapping from a complex number to a $2 \times 2$ matrix

\[
 a + bj \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad 1 \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad 1j \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e^{\theta j} \leftrightarrow \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.
\] (2.1.0.1)

The conjugate of $a + bj$ is then defined as $a - bj \leftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. By taking the inverse of the $2 \times 2$ matrix (assuming $|a + bj| \neq 0$), one can define the ratio of one complex number by another. Until you try out this representation, it may not seem obvious, or even that it could work.

This representation proves that $1j$ is not necessary when defining a complex number. What $1j$ can do is to conceptually simplify the algebra. It is worth your time to become familiar with the matrix representation, to clarify any possible confusions you might have about multiplication and division of complex numbers. This matrix representation can save you time, heartache and messy algebra. Once you have learned how to multiply two matrices, it’s a lot simpler than doing the complex algebra. In many cases we will leave the results of our analysis in matrix form, to avoid the algebra altogether. Thus both representations are important. More on this topic may be found on p. 27.

**History of complex numbers:** It is notable how long it took for complex numbers to be accepted (1851), relative to when they were first introduced by Bombelli (16th century CE). One might have thought that the solution of the quadratic, known to the Chinese, would have settled this question. It seems that complex integers (aka, Gaussian integers) were accepted before non-integral complex numbers. Perhaps this was because real numbers ($\mathbb{R}$) were not accepted (i.e., proved to exist, thus mathematically defined) until the development of real analysis in the late 19th century, thus sorting out a proper definition of the real number (due to the existence of irrational numbers).

**Exercise:** Using Matlab/Octave, verify that

\[
\frac{a + bj}{c + dj} = \frac{ac + bd + (bc - ad)j}{c^2 + d^2} \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2 + d^2}.
\] (2.1.0.2)

**Solution:** The typical way may be using numerical examples. A better solution is to use a symbolic code:

```matlab
syms a b c d A B
A=[a -b;b a];
B=[c -d;d c];
C=A*inv(B)
```

**2.1.1 Numerical taxonomy**

A simplified taxonomy of numbers is given by the mathematical sentence

\[
\pi_k \in \mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.
\]

This sentence says:

1. Every prime number $\pi_k$ is in the set of primes $\mathbb{P}$,
2. which is a subset of the set of counting numbers $\mathbb{N}$,

Sometimes we let the computer do the final algebra, numerically, as $2 \times 2$ matrix multiplications.
3. which is a subset of the set of integers $\mathbb{Z} = -\mathbb{N}, \{0\}, \mathbb{N}$,

4. which is a subset of the set of rationals $\mathbb{Q}$ (ratios of signed counting numbers $\pm \mathbb{N}$),

5. which is a subset of the set of reals $\mathbb{R}$,

6. which is a subset of the set of complex numbers $\mathbb{C}$.

The rationals $\mathbb{Q}$ may be further decomposed into the fractionals $\mathbb{F}$ and the integers $\mathbb{Z}$ ($\mathbb{Q} = \mathbb{F} \cup \mathbb{Z}$), and the reals $\mathbb{R}$ into the rationals $\mathbb{Q}$ and the irrationals $\mathbb{I}$ ($\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$). This classification nicely defines all the numbers (scalars) used in engineering and physics.

The taxonomy structure may be summarize with the single compact sentence, starting with the prime numbers $\pi_k$ and ending with complex numbers $\mathbb{C}$:

$$\pi_k \in \mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset (\mathbb{Z} \cup \mathbb{F} = \mathbb{Q}) \subset (\mathbb{Q} \cup \mathbb{I} = \mathbb{R}) \subset \mathbb{C}.$$}

As discussed in Appendix A (p. 185), all numbers may be viewed as complex. Namely, every real number is complex if we take the imaginary part to be zero (Boas, 1987). For example, $2 \in \mathbb{P} \subset \mathbb{C}$. Likewise every purely imaginary number (e.g., $0 + 1j$) is complex with zero real part.

Finally, note that complex numbers $\mathbb{C}$, much like vectors, do not have “rank-order,” meaning one complex number cannot be larger or smaller than another. It makes no sense to say that $j > 1$ or $j = 1$ (Boas, 1987). The real and imaginary parts, and the magnitude and phase, have order. Order seems restricted to $\mathbb{R}$. If time $t$ were complex, there could be no yesterday and tomorrow.\footnote{One can usefully define $\xi = x + 1j \, ct$ to be complex ($x, t \in \mathbb{R}$), with $x$ in meters $[m]$, $t$ is in seconds $[s]$, and the speed of light $c_0$ $[m/s]$.}

### 2.1.2 Applications of integers

The most relevant question at this point is “Why are integers important?” First, we count with them, so we can keep track of “how much.” But there is much more to numbers than counting: We use integers for any application where absolute accuracy is essential, such as banking transactions (making change), the precise computing of dates (Stillwell, 2010, p. 70) and locations (“I’ll meet you at 34th and Vine at noon on Jan. 1, 2034.”), or building roads or buildings out of bricks (objects built from a unit size).

To navigate we need to know how to predict the tides, the location of the moon and sun, etc. Integers are important, precisely because they are precise: Once a month there is a full moon, easily recognizable. The next day it’s slightly less than full. If one could represent our position as integers in time and space, we would know exactly where we are at all times. But such an integral representation of our position or time is not possible.

The Pythagoreans claimed that all was integer. From a practical point of view, it seems they were right. Today all computers compute floating point numbers as fractionals. However, in theory they were wrong. The error (difference) is a matter of precision.

**Numerical Representations of $\mathbb{I}, \mathbb{R}, \mathbb{C}$**: When doing numerical work, one must consider how we may compute within the set of reals (i.e., which contain irrationals). There can be no irrational number representation on a computer. The international standard of computation, IEEE floating point numbers,\footnote{IEEE 754: https://www.h-schmidt.net/FloatConverter/IEEE754.html.} is based on rational approximation. The mantissa and the exponent are both integers, having sign and magnitude. The size of each integer depends on the precision of the number being represented. An IEEE floating-point number is rational because it has a binary (integer) mantissa, multiplied by 2 raised to the power of a binary (integer) exponent. For example, $\pi \approx \pm a2^{\pm b}$ with $a, b \in \mathbb{Z}$. In summary, IEEE floating-point rational numbers cannot be irrational, because irrational representations would require an infinite number of bits. True floating point numbers contain irrational numbers, which must be approximated by fractional numbers. This leads to the concept of *fractional representation*, which requires the definition of the
mantis, base and exponent, where both the mantissa and the exponent are signed. Numerical results must not depend on the base. One could dramatically improve the resolution of the numerical representation by the use of the fundamental theorem of arithmetic (p. 37). For example, one could factor the exponent into its primes and then represent the number as $a2^b3^c5^d7^e \ (a,b,c,d,e \in \mathbb{Z})$, etc. Such a representation would improve the resolution of the representation. But even so, the irrational numbers would be approximate. For example, base ten $9^3$ is natural using this representation since $10^n = 2^n5^n$.

$$\pi \cdot 10^5 \approx 314 \, 159 \, 2\ldots = 3 \cdot 5^5 + 1 \cdot 2^45^4 + 4 \cdot 2^35^3 + \cdots + 9 \cdot 2^05^0 + 2 \cdot 2^{-5}1^{-1} \ldots$$

**Exercise:** If we work in base 2, and use the approximation $\pi \approx 22/7$, then according to the Matlab/Octave `dec2bin()` routine, show that the binary representation of $\hat{\pi}2^{17}$ is

$$\pi \cdot 2^{17} \approx 411, 940_{10} = 64, 924_{16} = 11001001001001001002.$$  

**Solution:** First we note that this must be an approximation since $\pi \in \mathbb{I}$, which cannot have an exact representation $\in \mathbb{F}$. We are asking for the fractional ($\in \mathbb{F}$) approximation to $\pi$$\hat{\pi}2 = 22/7 = 3 + 1/7 = [3; 7]$, (2.1.2.3)

where int64(fix($2^{17} \times 22/7$)) = 411,940 and dec2hex(int64(fix($2^{17} \times 22/7$))) = 64,924, where 1 and 0 are multipliers of powers of 2, which are then added together

$$411, 940_{10} = 2^{18} + 2^{17} + 2^{14} + 2^{11} + 2^8 + 2^5 + 2^2.$$  

Computers keep track of the decimal point using the exponent, which in this case is the factor $2^{17} = 131072_{10}$. The concept of the decimal point is replaced by an integer, having the desired precision, and a scale factor of any base (radix). This scale factor may be thought of as moving the decimal point to the right (larger number) or left (smaller number). The mantissa “fine-tunes” the value about a scale factor (the exponent). In all cases the number actually used is a positive integer. Negative numbers are represented by an extra sign bit.

**Exercise:** Using Matlab/Octave, use base 16 (i.e., hexadecimal) numbers, with $\hat{\pi}2 = 22/7$, find

1. $\hat{\pi}2 \cdot 10^5$ **Solution:** Using the command `dec2hex(fix(22/7*1e5))` we get $4cbd_{16}$, since $22/7 \times 10^5 = 314285.7\cdots$ and hex2dec(‘4cbd’) = 314285. 

2. $\hat{\pi}2 \cdot 2^{17}$ **Solution:** $2^{18} \cdot 11_{16}/7_{16}$. 

**Exercise:** Write out the first 11 primes, base 16. **Solution:** The Octave/Matlab command `dec2hex()` provides the answer:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\hat{\pi}n$ dec</th>
<th>$\hat{\pi}n$ hex</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>0B</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>0D</td>
</tr>
<tr>
<td>6</td>
<td>13</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>17</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>19</td>
<td>17</td>
</tr>
<tr>
<td>9</td>
<td>23</td>
<td>1D</td>
</tr>
<tr>
<td>10</td>
<td>29</td>
<td>1F</td>
</tr>
<tr>
<td>11</td>
<td>31</td>
<td></td>
</tr>
</tbody>
</table>

**Example:** $x = 2^{17} \times 22/7$, using IEEE-754 double precision,$^{10}$

$$x = 411, 940.5625_{10}$$

$$= 2^{54} \times 1198372$$

$$= 0, 10010, 00, 110010, 010010, 010010, 0100102$$

$$= 0x48c92492_{16}.$$  

---

$^9$Base 10 is the natural world wide standard simply because we have 10 fingers, that we count with.

$^{10}$https://www.h-schmidt.net/FloatConverter/IEEE754.html
The exponent is $2^{18}$ and the mantissa is $4,793,490_{10}$. Here the commas in the binary (0,1) string are to help visualize the quasi-periodic nature of the bit-stream. The numbers are stored in a 32 bit format, with 1 bit for sign, 8 bits for the exponent and 23 bits for the mantissa. Perhaps a more instructive number is

$$x = 4,793,490.0$$

$$= 0,100,1010,100,100,100,100,100,100,100,100_2$$

which has a repeating binary bit pattern of $((100))_3$, broken by the scale factor $0x4a$. Even more symmetrical is

$$x = 0x24,924,924_{16}$$

$$= 00,100,100,100,100,100,100,100,100,100_2$$

$$= 6.344,131,191,146,900 \times 10^{-17}.$$ 

In this example the repeating pattern is clear in the hex representation as a repeating $((942))_3$, as represented by the double brackets, with the subscript indicating the period, in this case, three digits. As before, the commas are to help with readability and have no other meaning.

The representation of numbers is not unique. For example, irrational complex numbers have approximate rational representations (i.e., $\pi \approx 22/7$). A better example is complex numbers $z \in \mathbb{C}$, which have many representations, as a pair of reals (i.e., $z = (x,y)$), or by Euler’s formula, and matrices ($\theta \in \mathbb{R}$)

$$e^{j\theta} = \cos \theta + j \sin \theta \leftrightarrow \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$ 

At a higher level, differentiable functions, aka, analytic functions, may be represented by a single-valued Taylor series expansion (p. 72), limited by its region of convergence (RoC).

**Pythagoreans and Integers:** The integer is the cornerstone of the Pythagorean doctrine, so much so that it caused a fracture within the Pythagoreans when it was discovered that not all numbers are rational. One famous proof of such irrational numbers comes from the spiral of Theodorus, as shown in Fig. 2.1, where the short radius of each triangle has length $b_n = \sqrt{n}$ with $n \in \mathbb{N}$, and the long radius (the hypotenuses) is $c_n = \sqrt{1+b_n^2} = \sqrt{1+n}$. This figure may be constructed using a compass and ruler by maintaining right triangles.

![Figure 2.1](https://en.wikipedia.org/wiki/spiral_of_Theodorus)

Figure 2.1: This construction is called the spiral of Theodorus, made from contiguous right triangles having lengths $a = 1$, $b_n = \sqrt{n}$, with $n \in \mathbb{N}$, and $c_n = \sqrt{1+n}$. In this way, each value of $c_n^2 = b_n^2 + 1 \in \mathbb{N}$. This sequence of triangles generate the set $\{\sqrt{n}\} \in \mathbb{N}$, and is easily generated using a compass and a ruler. (Adapted from https://en.wikipedia.org/wiki/spiral_of_Theodorus).

**Public-key security:** An important application of prime numbers is public-key encryption, essential for internet security applications (e.g., online banking). Most people assume encryption is done by a
2.1. THE TAXONOMY OF NUMBERS: \( P, N, Z, Q, F, I, R, C \)

personal login and passwords. Passwords are fundamentally insecure, for many reasons. Decryption depends on factoring large integers, formed from products of primes having thousands of bits.\(^{11}\) The security is based on the relative ease of multiplying large primes, along with the virtual impossibility of factoring their products.

When a computation is easy in one direction, but its inverse is impossible, it is called a *trap-door function*. We shall explore trapdoor functions in Appendix 2. If everyone were to switch from passwords to public-key encryption, the internet would be much more secure.\(^{12}\)

**Puzzles:** Another application of integers is imaginative problems that use integers. An example is the classic Chinese *four stone problem*: Find the weight of four stones that can be used with a scale to weigh anything (e.g., salt, gold) between 0, 1, 2, \ldots, 40 [gm] (Fig. 8.1, p. 42 in Assignment AE-2). As with the other problems, the answer is not as interesting as the method, since the problem may be easily recast into a related one. This type of problem can be found in airline magazines as amusement on a long flight. This puzzle is best cast as a linear algebra problem, with integer solutions. Again, once you know the trick, it is “easy.”\(^{13}\)

---

\(^{11}\)It would seem that public-key encryption could work by having two numbers with a common prime, and then using the Euclidean algorithm, the greatest common divisor (GCD) could be worked out. One of the integers could be the public-key and the second could be the private key.

\(^{12}\)https://fas.org/irp/agency/dod/jason/cyber.pdf

\(^{13}\)Whenever someone tells you something is “easy,” you should immediately appreciate that it is very hard, but once you learn a concept, the difficulty evaporates.
CHAPTER 2. STREAM 1: NUMBER SYSTEMS

2.2 The role of physics in mathematics

Bells, chimes and eigenmodes: Integers naturally arose in art, music and science. Examples include the relations between musical notes, the natural eigenmodes (tones) of strings and other musical instruments. These relations were so common that Pythagoras believed that to explain the physical world (aka, the universe), one needed to understand integers. As discussed on p. 13, “all is integer” was a seductive song.

As will be discussed on p. 55, it is best to view the relationship between acoustics, music and mathematics as historical, since these topics inspired the development of mathematics. Today integers play a key role in quantum mechanics, again based on eigenmodes, but in this case, eigenmodes follow from solutions of Schrödinger equation, with the roots of the characteristic equation being purely imaginary. If there were a real part (i.e., damping), the modes would not be integers.

As discussed by Salmon (1946, p. 201), Schrödinger’s equation follows directly from the Webster horn equation. While Morse (1948, p. 281) (a student of Arnold Sommerfeld) fails to make the direct link, he comes close to the same view when he shows that the real part of the horn resistance goes exactly to zero below a cutoff frequency. He also discusses the trapped modes inside musical instruments due to the horn flaire (p. 268). One may assume Morse read Salmon’s paper on horns, since he cites Salmon (Morse, 1948, Footnote 1, p. 271).

Engineers are so accustomed to working with real (or complex) numbers, the distinction between real (i.e., irrational) and fractional numbers is rarely acknowledged. Integers, on the other-hand, arise in many contexts. One cannot master programming computers without understanding integer, hexadecimal, octal, and binary representations, since all numbers in a computer are represented in numerical computations in terms of rationals \( \mathbb{Q} = \mathbb{Z} \cup \mathbb{F} \).14

As discussed on p. 28, the primary reason integers are so important is their absolute precision. Every integer \( n \in \mathbb{Z} \) is unique,15 and has the indexing property, which is essential for making lists that are ordered, so that one can quickly look things up. The alphabet also has this property (e.g., a book’s index).

Because of the integer’s absolute precision, the digital computer quickly overtook the analog computer, once it was practical to make logic circuits that were fast. From 1946 the first digital computer was thought to be the University of Pennsylvania’s Eniac. We now know that the code-breaking effort in Bletchley Park, England, under the guidance of Alan Turing, created the first digital computer (The Colossus), used to break the WWII German “Enigma” code. Due to the high secrecy of this war effort, the credit was only acknowledged in the 1970s when the project was finally declassified.

There is zero possibility of analog computing displacing digital computing, due to the importance of precision (and speed). But even with binary representation, there is a non-zero probability of error, for example on a hard drive, due to physical noise. To deal with this, error correcting codes have been developed, reducing the error by many orders of magnitude. Today error correction is a science, and billions of dollars are invested to increase the density of bits per area, to increasingly larger factors. A few years ago the terabyte drive was unheard of; today it is standard. In a few years petabyte drives will certainly become available. It is hard to comprehend how these will be used by individuals, but they are essential for on-line (cloud) computing.

CH:START

2.2.1 The role of mathematics in physics: The three streams

CH:END

Modern mathematics is built on a hierarchical construct of fundamental theorems, as summarized in the boxed material (p. 37). The importance of such theorems cannot be overemphasized. Gauss’s and Stokes’s laws play a major role in understanding and manipulating Maxwell’s equations. Every engineering student needs to fully appreciate the significance of these key theorems. If necessary, memorize

14See Appendix A (p. 185) for a review of mathematical notation.
15Check out the history of \( 1729 = 1^3 + 12^3 = 9^3 + 10^3 \).
them. But that will not do over the long run, as each and every theorem must be fully understood. Fortunately most students already know several of these theorems, but perhaps not by name. In such cases, it is a matter of mastering the vocabulary.

The three streams of mathematics:

1. Number systems: Stream 1
   - arithmetic
   - prime number

2. Geometry: Stream 2
   - algebra

3. Calculus: Stream 3 (Flanders, 1973)
   - Leibniz $\mathbb{R}^1$
   - complex $\mathbb{C} \subset \mathbb{R}^2$
   - vectors $\mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^\infty$
     - Gauss’s law (divergence theorem)
     - Stokes’s law (curl theorem, or Green’s theorem)
     - Vector calculus (Helmholtz’s theorem)

These theorems are naturally organized, and may be thought of, in terms of Stillwell’s three streams. For Stream 1 there is the fundamental theorem of arithmetic and the prime number theorem. For Stream 2 there is the fundamental theorem of algebra while for Stream 3 there are a host of theorems on calculus, ordered by their dimensionality. Some of these theorems seem trivial (e.g., the fundamental theorem of arithmetic). Others are more challenging, such as the fundamental theorem of vector calculus and Green’s theorem.

Complexity should not be confused with importance. Each of these theorems, as stated, is fundamental. Taken as a whole, they are a powerful way of summarizing mathematics.

2.2.2 Stream 1: Prime number theorems

There are two easily described fundamental theorems about primes:

1. The fundamental theorem of arithmetic: This states that every integer $n \in \mathbb{Z}$ may be uniquely factored into prime numbers. This raises the question of the meaning of factor (split into a product). The product of two integers $m, n \in \mathbb{Z}$ is $mn = \sum m_n = \sum n_m$. For example, $2 \times 3 = 2 + 2 + 2 = 3 + 3$.

2. The prime number theorem: One would like to know how many primes there are. That is easy: $|\mathbb{P}| = \infty$ (the size of the set of primes is infinite). A better way of asking this question is What is the average density of primes, in the limit as $n \to \infty$? This question was answered, for all practical purposes, by Gauss, who in his free time computed the first three million primes by hand. He discovered that, to a good approximation, the primes are equally likely on a log scale. This is nicely summarized by the limerick:

   Chebyshev said, and I say it again: There is always a prime between $n$ and $2n$.

   attributed to the mathematician Pafnuty Chebyshev, which nicely sums up the prime number theorem (Stillwell, 2010, p. 585).

When the ratio of two frequencies (pitches) is 2, the relationship is called an octave. With a slight stretch of terminology, we could say there is at least one prime per octave. An interesting extention of
In modern western music the octave is further divided into 12 ratios called semitones equal to $\sqrt[12]{2}$. Twelve semitones is an octave. In the end, it is a question of the density of primes on a log-log (i.e., ratio) scale. One might wonder about the maximum number of primes per octave, as a function of $N$, or ask for the fractions of an octave (factors of 2) for $\pi_k$ as $k$ becomes large? The maximum value of $R_k < 0.5$ thus Chebyshev’s bound of 2 is quite conservative, by a factor of 0.6/2=0.3. As $k \to \infty$ the bound is exponentially tightened. The results of this calculation are show in Fig. 2.2. For reference when $k = 9592$, $\pi_k = 99991$, $\pi_{k-1} = 99989$, $\pi_{k+1} = 100003$, thus $1 - P(k+1)/P(k-1) = 1.4e-4$.

2.2.3 Stream 2: Fundamental theorem of algebra

This theorem states that every polynomial in $x$ of degree $N$

$$P_N(x) = \sum_{k=0}^{N} a_k x^k$$  \hspace{1cm} (2.2.3.1)

has at least one root (p. 79). When a common root is factored out, the degree of the polynomial is reduced by 1. Applied recursively, a polynomial of degree $N$ has $N$ roots. Note there are $N+1$ coefficients (i.e., $[a_N, a_{N-1}, \cdots, a_0]$). If we are only interested in the roots of $P_N(x)$ it is best to define $a_N = 1$, which defines the monic polynomial. If the roots are fractional numbers this might be possible. However if the roots are irrational numbers (likely), a perfect factorization is at least unlikely if not impossible.

2.2.4 Stream 3: Fundamental theorems of calculus

In §5.5 and §5.5.6 we will deal with each of the theorems for Stream 3, where we consider the several fundamental theorems of integration, starting with Leibniz’s formula for integration on the real line ($\mathbb{R}$), then progressing to complex integration in the complex plane ($\mathbb{C}$) (Cauchy’s theorem), which is required for computing the inverse Laplace transform. Gauss’s and Stokes’s laws for $\mathbb{R}^2$ require closed and open surfaces, respectively. One cannot manipulate Maxwell’s equations, fluid flow, or acoustics, without understanding these theorems. Any problem that deals with the wave equation in more than one dimension requires an understanding of these theorems. They are the basis of the derivation of the Kirchhoff voltage and current laws.
Finally we define the four basic vector operations based on the gradient operator differential vector operator, which have been given mnemonic abbreviations (see p. 169),

$$\nabla \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z},$$

(2.2.4.2)

pronounced as “del” (preferred) or “nabla,” which are the gradient $\nabla()$, divergence $\nabla \cdot ()$, curl $\nabla \times ()$.

Second order operators such as the scalar Laplacian $\nabla \cdot \nabla() = \nabla^2()$ may be constructed from first order operators. The most important of these is the vector Laplacian $\nabla^2()$ which is required when working with Maxwell’s Wave Equations.

The first three operations are defined in terms of integral operations on a surface in 1, 2 or 3 dimensions, by taking the limit as that surface, or the volume contained within, goes to zero. These three differential operators are essential to fully understand Maxwell’s equations, the crown jewel of mathematical physics. Hence mathematics plays a key role in physics, as does physics in math.

2.2.5 Other key mathematical theorems

Besides the widely recognized fundamental theorems for the three streams, there are a number of equally important theorems that have not yet been labeled as “fundamental.”

The widely recognized Cauchy integral theorem is an excellent example, since it is a stepping stone to Green’s theorem and the fundamental theorem of complex calculus. In §4.4 (p. 132) we clarify the contributions of each of these special theorems.

Once these fundamental theorems of integration (Stream 3) have been mastered, the student is ready for the complex frequency domain, which takes us back to Stream 2 and the complex frequency plane ($s = \sigma + \omega \in \mathbb{C}$). While the Fourier and LT’s are taught in mathematics courses, the concept of complex frequency is rarely mentioned. The complex frequency domain (p. 104) and causality are fundamentally related (pp. 133–135), and are critical for the analysis of signals and systems, and especially when dealing with the concept of impedance (p. 120).

Without the concept of time and frequency, one cannot develop an intuition for the Fourier and Laplace transforms, especially within the context of engineering and mathematical physics. The Fourier transform covers signals, while the Laplace transform describes systems. Separating these two concepts, based on their representations as Fourier and Laplace transforms, is an important starting place for understanding physics and the role of mathematics. However, these methods, by themselves, do not provide the insight into physical systems necessary to be productive, or better, creative with these tools. One needs to master the tools of differential equations, and then partial differential equations, to fully appreciate the world that they describe. Electrical and mechanical networks, composed of inductors, capacitors and resistors, are isomorphic to mechanical systems composed of masses, springs and dashpots. Newton’s laws are analogous to those of Kirchhoff, which are the rules needed to analyze simple physical systems composed of linear (and nonlinear) sub-components. When lumped-element systems are taken to the limit, in several dimensions, we obtain Maxwell’s partial differential equations, the laws of continuum mechanics, and beyond.

The ultimate goal of this book is to make you aware of and productive in using these tools. This material can be best absorbed by treating it chronologically through history, so you can see how this body of knowledge came into existence, through the minds and hands of Galileo, Newton, Maxwell and Einstein. Perhaps one day you too can stand on the shoulders of the giants who went before you.

2.3 Applications of prime numbers

If someone asked you for a theory of counting numbers, I suspect you would laugh and start counting. It sounds like either a stupid question, or a bad joke. Yet integers are a rich topic, so the question is not
even slightly dumb. It is somewhat amazing that even birds and bees can count. While I doubt birds
and bees can recognize primes, cicadas and other insects only crawl out of the ground in prime number
cycles, (e.g., 13 or 17 year cycles). If you have ever witnessed such an event (I have), you will never
forget it. Somehow they know. Finally, there is an analytic function, first introduced by Euler, based on
his analysis of the sieve, now known as the Riemann zeta function \( \zeta(s) \), which is complex analytic, with
its poles at the logs of the prime numbers. The properties of this function are truly amazing, even fun.
Many of the questions and answers about primes go back to at least the early Chinese (c1500 BCE).

1. Write out \( N - 1 \) integers \( n \), starting from 2: \( n \in \{2, 3, \ldots, N\} \). (e.g., \( N = 4, n \in \{2, 3, 4\} \)).
   Note that the first element \( \pi_1 = 2 \) is the first prime. Cross out all multiples of \( \pi_1 \). That is, cross
   out \( n \cdot \pi_1 = 4, 6, 8, 10, \ldots, 50 \), that is all \( n \) such that \( \mod(n, \pi_1) = 0 \).

2. Let \( k = 2 \) and note that \( \pi_2 = 3 \). Cross out \( n\pi_2 \cdot (2, 3, 4, 5, 6, 7, \ldots, 45) \), i.e., all \( n \) such that
   \( \mod(n, \pi_2) = 0 \).

3. Let \( k = 3, \pi_3 = 5 \). Cross out \( n\pi_3 \cdot (25, 35) \mod(n, 5) = 0 \).

4. Finally let \( k = 4, \pi_4 = 7 \mod(n, 7) = 0 \). Cross out \( n\pi_4 \cdot (49) \). Thus there are 15 primes less
   than \( N = 50 \): \( \pi_k = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 \) (highlighted in red). Above
   2, all end in odd numbers, and above 5, all end with 1, 3, 7, 9.

   Figure 2.3: Sieve of Eratosthenes for \( N = 50 \).

The importance of prime numbers: Each prime perfectly predicts multiples of that prime, but there
seems to be no regularity in predicting primes. It follows that prime numbers are the key to the theory
of numbers, because of the fundamental \( \text{CH:START} \) theorem of arithmetic \( \text{CH:END} \) (FTA).

Likely the first insight into the counting numbers started with the \( \text{sieve} \), shown in Fig. 2.3. A sieve
answers the question “How can one identify the prime numbers?” The answer comes from looking for
irregular patterns in the counting numbers, by playing the counting numbers against themselves.
2.3. APPLICATIONS OF PRIME NUMBERS

A recursive sieve method for finding primes was first devised by the Greek Eratosthenes (O’Neill, 2009).

For example, starting from $\pi_1 = 2$ one strikes out all even numbers $2 \cdot (2, 3, 4, 5, 6, \cdots)$, but not 2. By definition the multiples are products of the target prime (2 in our example) and every another integer ($n \geq 2$). In this way all the even numbers are removed in this first iteration. The next remaining integer (3 in our example) is identified as the second prime $\pi_2$. Then all the multiples of $\pi_2 = 3$ are removed. The next remaining number is $\pi_3 = 5$, so all multiples of $\pi_3 = 5$ are removed (i.e., $5\cdot 5, 15, 25$ etc., $\cdots$). This process is repeated until all the numbers of the list have either been canceled or identified as prime.

As the word *sieve* implies, this process takes a heavy toll on the integers, rapidly pruning the non-primes. In four iterations of the sieve algorithm, all the primes below $N = 50$ are identified in red. The final set of primes is displayed in step 4 of Fig. 2.3.

Once a prime greater than $N/2$ has been identified (25 in the example), the recursion stops, since twice that prime is greater than $N$, the maximum number under consideration. Thus once $\sqrt{49}$ has been reached, all the primes have been identified (this follows from the fact that the next prime $\pi_n$ is multiplied by an integer $n = 1, \ldots, N$).

When using a computer, memory efficiency and speed are the main considerations. There are various schemes for making the sieve more efficient. For example, the recursion $n\pi_k = (n-1)\pi_k + \pi_k$ will speed up the process by replacing the multiply with an addition of $\pi_k$.

**Two fundamental theorems of primes:** Early theories of numbers revealed two fundamental theorems (there are many more than two), as discussed in §2.2.3, §2.2.4 and Section 2, page 27. The first of these is the fundamental theorem of arithmetic, which says that every integer $n \in \mathbb{N}$ greater than 1 may be uniquely factored into a product of primes

$$n = \prod_{k=1}^{K} \pi_k^{\beta_k},$$  \hspace{1cm} (2.3.0.1)

where $k = 1, \ldots, K$ indexes the integer’s $K$ prime factors $\pi_k \in \mathbb{P}$. Typically prime factors appear more than once, for example 25 = $5^2$. To make the notation compact we define the multiplicity $\beta_k$ of each prime factor $\pi_k$. For example $2312 = 2^3 \cdot 17^2 = \pi_1^3 \pi_2^2$ (i.e., $\pi_1 = 2, \beta_1 = 3; \pi_2 = 17, \beta_2 = 2$) and $2313 = 3^2 \cdot 257 = \pi_3^2 \pi_{55}$ (i.e., $\pi_3 = 3, \beta_3 = 2; \pi_{55} = 257, \beta_{55} = 1$). Our demonstration of this is empirical, using the Matlab/Octave `factor(N)` routine, which factors $N$.\(^\dagger\)

What seems amazing is the unique nature of this theorem. Each counting number is uniquely represented as a product of primes. No two integers can share the same factorization. Once you multiply the factors out, the result is unique ($N$). Note that it’s easy to multiply integers (e.g., primes), but expensive to factor them. And factoring the product of three primes is significantly more difficult than factoring two.

Factoring is much more expensive than division. This is not due to the higher cost of division over multiplication, which is less than a factor of 2.\(^\ddagger\) Dividing the product of two primes, given one, is trivial, slightly more expensive that multiplying. Factoring the product of two primes is nearly impossible, as one needs to know what to divide by. Factoring means dividing by some integer and obtaining another integer with remainder zero.

This brings us to the prime number theorem (PNT). The security problem is the reason why these two theorems are so important: 1) Every integer has a unique representation as a product of primes, and 2) the density of primes is large (see the discussions on p. 37). Thus security reduces to the “needle in the haystack problem” due to the cost of a search.

Thus one could factor a product of primes $N = \pi_k \pi_l$ by doing $M$ divisions, where $M$ is the number of primes less than $N$. This assumes the list of primes less than $N$ are known. However, most integers are not a simple product of two primes

\(^\dagger\)If you wish to be a mathematician, you need to learn how to prove theorems. If you’re a physicist, you are happy that someone else has already proved them, so that you can use the result.

\(^\ddagger\)https://streamcomputing.eu/blog/2012-07-16/how-expensive-is-an-operation-on-a-cpu/
But the utility of using prime factorization has to do with their density. If we were simply looking up a few numbers from a short list of primes, it would be easy to factor them. But given that their density is logarithmic (≫1 per octave), factoring becomes at a very high computational cost.

### 2.3.1 Greatest common divisor (Euclidean algorithm)

The Euclidean algorithm is a systematic method to find the largest common integer factor \( k \) between two integers \( n, m \), denoted \( k = \gcd(n, m) \), where \( n, m, k \in \mathbb{N} \) (Graham et al., 1994). For example, \( 15 = \gcd(30, 105) \) since, when factored \((30, 105) = (2 \cdot 3 \cdot 5, 7 \cdot 3 \cdot 5) = 3 \cdot 5 \cdot (2, 7) = 15 \cdot (2, 7) \). The Euclidean algorithm was known to the Chinese (i.e., not discovered by Euclid) (Stillwell, 2010, p. 41). Two integers are said to be coprime if their gcd is 1 (i.e., they have no common factor).

#### Examples of the GCD: \( l = \gcd(m, n) \)

- Examples \((m, n, l \in \mathbb{Z})\):
  - \( 5 = \gcd(13 \cdot 5, 11 \cdot 5) \). The GCD is the common factor 5.
  - \( \gcd(13 \cdot 10, 11 \cdot 10) = 10 = \gcd(130, 110) = 10 = 2 \cdot 5, \text{ is not prime} \)
  - \( \gcd(1234, 1024) = 2 \text{ since } 1234 = 2 \cdot 617, 1024 = 2^{10} \)
  - \( \gcd(\pi_k \pi_m, \pi_n \pi_n) = \pi_k \)
  - \( l = \gcd(m, n) \) is the part that cancels in the fraction \( m/n \in F \)
  - \( m/\gcd(m, n) \in \mathbb{Z} \)
- Coprimes \((m \perp n)\) are numbers with no distinct common factors: i.e., \( \gcd(m, n) = 1 \)
  - The GCD of two primes is always 1: \( \gcd(13, 11) = 1 \), \( \gcd(\pi_m, \pi_n) = 1 \) \((m \neq n)\)
  - \( m = 7 \cdot 13, n = 5 \cdot 19 \Rightarrow (7 \cdot 13) \perp (5 \cdot 19) \)
  - If \( m \perp n \) then \( \gcd(m, n) = 1 \)
  - If \( \gcd(m, n) = 1 \) then \( m \perp n \)
- The GCD may be extended to polynomials: e.g., \( \gcd(ax^2 + bx + c, ax^2 + \beta x + \gamma) \)
  - \( \gcd((x - 3)(x - 4), (x - 3)(x - 5)) = (x - 3) \)
  - \( \gcd(x^2 - 7x + 12, 3(x^2 - 8x + 15)) = 3(x - 3) \)
  - \( \gcd(x^2 - 7x + 12, (3x^2 - 24x + 45) = 3(x - 3) \)
  - \( \gcd((x - 2\pi)(x - 4), (x - 2\pi)(x - 5)) = (x - 2\pi) \) (Needs long division)

The Euclidean algorithm is best explained by a trivial example: Let the two numbers be 6, 9. At each step the smaller number (6) is subtracted from the larger (9) and the smaller number and the difference (the remainder) are saved. This process continues until the two resulting numbers are equal, which is the GCD. For our example: \( 9 - 6 = 3 \), leaving the smaller number 6 and the difference 3. Repeating this we get \( 6 - 3 = 3 \), leaving the smaller number 3 and the difference 3. Since these two numbers are the same we are done, thus \( 3 = \gcd(9, 6) \). We can verify this result by factoring [e.g., \((9, 6) = 3(3, 2)\)]. The value may also be numerically verified using the Matlab/Octave GCD command \( \gcd(6, 9) \), which returns 3.

**Direct matrix method:** The GCD may be written as a matrix recursion given the starting vector \((m_0, n_0)^T\). The recursion is then

\[
\begin{bmatrix} m_{k+1} \\ n_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_k \\ n_k \end{bmatrix}.
\]  

(2.3.1.2)
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Figure 2.4: The Euclidean algorithm recursively subtracts $n$ from $m$ until the remainder $m - kn$ is either less than $n$ or zero. Note that this recursion is the same as $\text{mod}(m, n)$. Thus the GCD recursively computes $\text{mod}(m, n)$ until the remainder $\text{rem}(m, n)$ is less than $n$, which is denoted the GCD’s turning-point. It then swaps $m, n$, so that $n < m$. This repeats, until it terminates on the GCD. Due to its simplicity this is called the direct method for finding the GCD. For the case depicted here the value of $k$, that renders the remainder $m - 6n < n$. If one more step were taken beyond the turning-point ($k = 7$), the remainder would become negative. Thus the turning-point satisfies the linear relation $m - \alpha n = 0$ with \(\alpha \in \mathbb{R}\).

This recursion continues until $m_{k+1} < n$, at which point $m$ and $n$ must be swapped. The process is repeated until $m_k = n_k$, which equals the GCD. We call this the direct method (see Fig. 2.4). The direct method is inefficient because it recursively subtracts $n_k$ many times until the resulting $m_k$ is less than $n_k$. It also must test for $m \leq n$ after each subtraction, and then swap them if $m_k < n_k$. If they are equal we are done.

The GCD’s turning-point may be defined using the linear interpolation $m - \alpha n = 0, \alpha \in \mathbb{R}$, where the solid line cross the abscissa in Fig. 2.4. If, for example, $l = 6 + 43/97 \approx 6.443298 \cdots$, then $l = [m/n] < n$ and $\alpha \in \mathbb{F} \in \mathbb{R}$. This is nonlinear (truncation) arithmetic, which is a natural property of the GCD. The floor() functions finds the turning-point, where we swap the two numbers, since by definition, $m > n$. In this example $6 = \lfloor l \rfloor$.

Exercise: Show that

\[
\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}
\]

Solution: Let $n = 2$. Next show that the recursive multiplication add 1 to the upper right corner.

**Why is the GCD important?** The utility of the GCD algorithm arises directly from the fundamental difficulty in factoring large integers. Computing the GCD, using the Euclidean algorithm, is low cost, compared to factoring when finding the coprime factors, which is extremely expensive. The utility surfaces when the two numbers are composed of very large primes.

When two integers have no common factors they are said to be coprime and their GCD is 1. The ratio of two integers which are coprime is automatically in reduced form (they have no common factors). For example, $4/2 \in \mathbb{Q}$ is not reduced since $2 = \gcd(4, 2)$ (with a zero remainder). Canceling out the common factor 2 gives the reduced form $2/1 \in \mathbb{F}$. Thus if we wish to form the ratio of two integers, first compute the GCD, then remove it from the two numbers to form the ratio. This assures the rational number is in its reduced form ($\in \mathbb{F}$ rather than $\in \mathbb{Q}$). If the GCD were $10^3$ digits, it is obvious that any common factor would need to be removed, thus greatly simplifying further computation. This will make a huge difference when using IEEE-754.

The floor function and the GCD are related in an important way, as discussed next.

**Indirect matrix method:** A much more efficient method uses the floor() function, which is called division with rounding, or simply the indirect method. Specifically the GCD may be written in one step
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\[
\begin{bmatrix}
m \\
n \\
\end{bmatrix}_{k+1} = \begin{bmatrix}
0 & 1 \\
-\frac{m}{n} & 1 \\
\end{bmatrix} \begin{bmatrix}
m \\
n \\
\end{bmatrix}_{k}. \\
\tag{2.3.1.3}
\]

This matrix is simply Eq. 2.3.1.2 to the power \([m/n]\), followed by swapping the inputs (the smaller must always be on the bottom).

The GCD and multiplication: Multiplication is simply recursive addition, and finding the GCD takes advantage of this fact. For example \(3 \times 2 = 3 + 3 = 2 + 2 + 2\). Since division is the inverse of multiplication, it must be recursive subtraction.

The GCD and long division: When one learns how to divide a smaller number into a larger one, they must learn how to use the GCD. For example, suppose we which to compute \(6 \div 110\). One starts by finding out how many times 6 goes into 11. Since \(6 \times 2 = 12\), which is larger than 11, the answer is 1. We then subtract 6 from 11 to find the remainder 5. This is, of course, the floor function (e.g., \([11/6] = 1, [11/5] = 2\).

Example: Start with the two integers \([873, 582]\). In factored form these are \([\pi^2 \cdot 3^2, \pi^2 \cdot 3 \cdot 2]\). Given the factors, we see that the largest common factor is \(\pi^2 \cdot 3 = 291\) \((\pi^2 = 97)\). When we take the ratio of the two numbers this common factor cancels \(873/582 = \pi^2 \cdot 3 \cdot 3/2 = 1.5\).

Of course if we divide 582 into 873 we will numerically obtain the answer 1.5 \(\in \mathbb{F}\).

Exercise: What does it mean to reach the turning-point when using the Euclidean algorithm? Solution: When \(m/n - [m/n] < n\) we have reached a turning-point. When the remainder is zero (i.e., \(m/n - [m/n] = 0\)), we have reached the GCD.

Exercise: Show that in Matlab/Octave \(\text{rat}(873/582) = 1 + 1/(-2)\) gives the wrong answer \(\text{rats}(837/582)=3/2\). Hint: Factor the two numbers and cancel out the gcd. Solution: Since \(\text{factor}(873) = 3 \cdot 3 \cdot 97\) and \(\text{factor}(582) = 2 \cdot 3 \cdot 97\), the gcd is \(3 \cdot 97\), thus \(3/2 = 1 + 1/2\) is the correct answer. But due to rounding methods, it is not 3/2. As an example, in Matlab/Octave \(\text{rat}(3/2)=2+1/(-2)\). Matlab’s rat () command uses rounding rather than the floor function, which explains the difference. When the rat () function produces negative numbers, rounding is employed.

Exercise: Divide 10 into 99, the floor function (floor(99/10)) must be used, followed by the remainder function (rem(99, 10)). Solution: When we divide a smaller number into a larger one, we must first find the floor followed by the remainder. For example 99/10 = 9 + 9/10 has a floor of 9 and a remainder of 9/10.

Graphical description of the GCD: The Euclidean algorithm is best viewed graphically. In Fig. 2.4 we show what is happening as one approaches the turning-point, at which point the two numbers must be swapped to keep the difference positive, which is addressed by the upper row of Eq. 2.3.1.3.

Below is a simple Matlab/Octave code to find \(l = \gcd(m, n)\) based on the Stillwell (2010) definition of the EA:

```matlab
function k = gcd(m,n)
while m ~= 0
```

%˜/M/gcd0.m
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A=m; B=n;
m=max(A,B); n=min(A,B); %m>n
m=m-n;
endwhile %m=n
k=A;

This program loops until \( m = 0 \).

**Coprimes:** When the GCD of two integers is 1, the only common factor is 1. This is of key importance when trying to find common factors between the two integers. When \( 1 = \gcd(m, n) \) they are said to be *coprime* or “relatively prime,” which is frequently written as \( m \perp n \). By definition, the largest common factor of coprimes is 1. But since 1 is not a prime (\( \pi_1 = 2 \)), they have no common primes. It can be shown (Stillwell, 2010, p. 41-42) that when \( a \perp b \), there exist \( m, n \in \mathbb{Z} \) such that

\[
am + bn = \gcd(a, b) = 1.
\]

This equation may by related to the addition of two fractions having coprime numerators \( (a \perp b) \). For example

\[
\frac{a}{m} + \frac{b}{n} = \frac{an + bm}{mn}.
\]

It is not obvious to me that this is simply \( 1/mn \).

**Exercise:** Show that

\[
\left[ \begin{array}{cc} 0 & 1 \\ 1 & \lfloor \frac{m}{n} \rfloor \end{array} \right] = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[ \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right] \left[ \begin{array}{c} \frac{m}{n} \end{array} \right].
\]

**Solution:** This uses the result of the exercise from a previous page, times the row swap matrix. Consider this result in terms of an Eigenanalysis.

2.3.2 Continued fraction algorithm

As shown in Fig. 2.5, the *continued fraction algorithm* (CFA) starts from a single real decimal number \( x_0 \in \mathbb{R} \), and recursively expands it as a fraction \( x \in \mathbb{F} \) (Graham et al., 1994). Thus the CFA may be used for forming rational approximations to any real number. For example, \( \pi \approx 22/7 \), and excellent approximation, well known to Chinese mathematicians.

The Euclidean algorithm (i.e., GCD), on the other hand, operates on a pair of integers \( m, n \in \mathbb{N} \) and returns their greatest common divisor \( k \in \mathbb{N} \), such that \( m/k, n/k \in \mathbb{F} \) are coprime, thus reducing the ratio to its irreducible form (i.e., \( m/k \perp n/k \)). Note this is done without factoring \( m, n \).

Despite this seemingly irreconcilable difference between the GCD and CFA, the two are closely related, so close that Gauss called the Euclidean algorithm, for finding the GCD, the continued fraction algorithm (CFA) (Stillwell, 2010, P. 48).

At first glance it is not clear why Gauss would be so “confused.” One is forced to assume that Gauss had some deeper insight into this relationship. If so, that insight would be valuable to understand.

Since Eq. 2.3.1.3 may be inverted, the process may be reversed, which is closely related to the *continued fraction algorithm* (CFA) as discussed in Fig. 2.5. This might be the basis behind Gauss’s insight.
Definition of the CFA

1. Start with $n = 0$ and the positive input target $x_0 \in \mathbb{R}^+$. 
   $n = 0, m_0 = 0, x_0 = \pi$
2. Rounding: Let $m_n = \lfloor x_n \rfloor \in \mathbb{N}$ 
   $m_0 = \lfloor \pi \rfloor = 3$
3. The input vector is then: $[m_n, x_n]^T$ 
   $[3, \pi]^T$
4. Remainder: $r_n = x_n - m_n$ 
   $(-0.5 \leq r_n \leq 0.5)$. 
   $r_0 = \pi - 3 \approx 0.1416$
5. Reciprocate:

   $x_{n+1} = \begin{cases} 
   1/r_n, & n \leftarrow n + 1; \text{ Go to step 2 } \quad r_n \neq 0 \\
   0, & \text{ terminate } \quad r_n = 0 
   \end{cases}$ 

   Output: $[m_n, x_{n+1}]^T = [3, 7.06]^T$

Figure 2.5: The CFA of any positive number, $x_0 \in \mathbb{R}^+$, is defined in this figure. Numerical values for $n = 0, x_0 = \pi, m_0 = 0$ are given in blue on the far right. For $n = 1$ the input vector is $[m_1, x_2]^T = [3, 7.0626]^T$. If at any step the remainder is zero, the algorithm terminates (Step 5). Convergence is guaranteed. The recursion may continue to any desired accuracy, and terminates if $r_n = 0$. Alternative rounding schemes are given on p. 188.

Notation: Writing out all the fractions can become tedious. For example, expanding $e = 2.7183 \cdots$ using the Matlab/Octave command `rat(exp(1))` gives the approximation 

$$
\exp(1) = 3 + 1/(−4 + 1/(2 + 1/(5 + 1/(−2 + 1/(−7)))))) - o\left(1.75 \times 10^{-6}\right),
$$

$$
= [3; −4, 2, 5, −2, −7] - o(1.75 \times 10^{-6}).
$$

Here we use a compact bracket notation, $\hat{e}_6 \approx [3; −4, 2, 5, −2, −7]$ where $o()$ indicates the error of the CFA expansion.

Since entries are negative, we deduce that rounding arithmetic is being used by Matlab/Octave (but this is not documented). Note that the leading integer part $m_0$ may be noted by an optional semicolon.\(^\text{19}\)

If the steps are carried further, the values of $m_n \in \mathbb{Z}$ give increasingly more accurate rational approximations. \footnote{Unfortunately Matlab/Octave does not support the bracket notation.}

Example: Let $x_0 \equiv \pi \approx 3.14159 \ldots$. As shown in Fig. 2.6, $a_0 = 3, r_0 = 0.14159, x_1 = 7.065 \approx 1/r_0$, and $a_1 = 7$. If we were to stop here we would have

$$
\hat{\pi}_2 = 3 + \frac{1}{7 + 0.0625} \ldots = 3 + \frac{1}{7} = \frac{22}{7}.
$$

This approximation of $\hat{\pi}_2 = 22/7$ has a relative error of $0.04\%$

$$
\frac{22/7 - \pi}{\pi} \approx 4 \times 10^{-4}.
$$

Example: For a second level of approximation we continue by reciprocating the remainder $1/0.0625 \approx 15.9966$ which rounds to 16 giving a negative remainder of $\approx -1/300$:

$$
\hat{\pi}_3 \approx 3 + 1/(7 + 16) = 3 + 16/(7 \cdot 16 + 1) = 3 + 16/113 = 355/113.
$$

With rounding the remainder is $-0.0034$, resulting in a much more rapid convergence. If floor rounding is used ($15.9966 = 15 - 0.9966$) the remainder is positive and close to 1, resulting in a much less accurate rational approximation for the same number of terms. It follows that there can be a dramatic difference depending on the rounding scheme, which, for clarity, should be specified, not inferred.

\(^\text{19}\)Unfortunately Matlab/Octave does not support the bracket notation.
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Rational approximation examples

\[ \hat{\pi}_2 = \frac{22}{7} = [3; 7] \approx \hat{\pi}_2 + o(1.3 \times 10^{-3}) \]
\[ \hat{\pi}_3 = \frac{355}{113} = [3; 7, 16] \approx \hat{\pi}_3 - o(2.7 \times 10^{-7}) \]
\[ \hat{\pi}_4 = \frac{104348}{33215} = [3; 7, 16, -249] \approx \hat{\pi}_4 + o(3.3 \times 10^{-10}) \]

Figure 2.6: The expansion of \( \pi \) to various orders, using the CFA, along with the order of the error of each rational approximation, with rounding. For example, \( \hat{\pi}_2 = 22/7 \) has an absolute error (\(|22/7 - \pi|\)) of about 0.13%.

Exercise: Find the CFA using the floor function, to 12th order.

Solution: \( \hat{\pi}_{12} = [3; 7, 15, 1, 292, 1, 1, 2, 1, 3, 1] \). Octave/Matlab will give a different answer due to the use of rounding rather than floor.

Example: Matlab/Octave’s \texttt{rat(pi,1e-20)} gives:

\[
3 + 1/(7 + 1/(16 + 1/(-294 + 1/(3 + 1/(-4 + 1/(5 + 1/(-15 + 1/(-3)))))])).
\]

In bracket notation

\[ \hat{\pi}_9 = [3; 7, 16, -294, 3, -4, 5, -15, 03]. \]

Because the sign changes, it is clear that Matlab/Octave use rounding rather than the floor function.

Exercise: Based on several examples, which rounding scheme is the most accurate? Explain why. Solution: Rounding will result in a smaller remainder at each iteration, thus a smaller net error and thus faster convergence. Using the floor truncation will always give positive coefficients, which could could have some applications.

When the CFA is applied and the expansion terminates (\( r_n = 0 \)), the target is rational. When the expansion does not terminate (which is not always easy to determine, as the remainder may be ill-conditioned due to small numerical rounding errors), the number is irrational. Thus the CFA has important theoretical applications regarding irrational numbers. You may explore this using Matlab’s \texttt{rats(pi)} command.

Besides these five basic rounding schemes, there are two other important \( \mathbb{R} \rightarrow \mathbb{N} \) functions (i.e., mappings), which will be needed later: \( \text{mod}(x, y), \text{rem}(x, y) \) with \( x, y \in \mathbb{R} \). The base 10 numbers may be generated from the counting numbers using \( y=\text{mod}(n,10) \).

Exercise:

1. Show how to generate a base-10 real number \( y \in \mathbb{R} \) from the counting numbers \( \mathbb{N} \) using the \( m=\text{mod}(n,10)+k10 \) with \( n, k \in \mathbb{N} \). Solution: Every time \( n \) reaches a multiple of 10, \( m \) is reset to 0 and the next digit to the left is increased by 1, by adding 1 to \( k \), generating the digit pair \( km \). Thus the \text{mod()} function forms the underlying theory behind decimal notation.

2. How would you generate binary numbers (base 2) using the \text{mod}(x, b) function? Solution: Use the same method as in the first example above, but with \( b = 2 \).

3. How would you generate hexadecimal numbers (base 16) using the \text{mod}(x, b) function? Solution: Use the same method as in the first example above, but with \( b = 16 \).

4. Write out the first 19 numbers in hex notation, starting from zero. Solution: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, 10, 11, 12. Recall that \( 10_{16} = 16_{10} \) thus \( 12_{16} = 18_{10} \), resulting in a total of 19 numbers if we include 0.
5. What is $FF_{16}$ in decimal notation? **Solution:** $\text{hex2dec(’ff’) = 255}_{10}$.

**Symmetry:** A continued fraction expansion can have a high degree of “recursive symmetry.” For example, the CFA of

$$R_1 \equiv \frac{1 + \sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1 + \cdots}} = 1.618033988749895 \cdots.$$  

(2.3.2.5)

Here $a_n$ in the CFA is always 1 ($R_1 \equiv [1, 1, 1, \cdots]$), thus the sequence cannot terminate, proving that $\sqrt{5} \not\in \mathbb{Q}$. A related example is $R_2 \equiv \text{rat}(1+\sqrt{2})$, which gives $R_2 = [2, 2, 2, \cdots]$.

When expanding a target irrational number ($x_0 \in \mathbb{R}$), and the CFA is truncated, the resulting rational fraction approximates the irrational target. For the example above, if we truncate at three coefficients ([1; 1, 1]) we obtain

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} = 1 + \frac{1}{2} = 1.5 = \frac{1 + \sqrt{5}}{2} + 0.118 \cdots.$$

Truncation after six steps gives

$$[1, 1, 1, 1, 1, 1] = 13/8 \approx 1.6250 = \frac{1 + \sqrt{5}}{2} + .0070 \cdots.$$

Because all the coefficients are 1, this example converges very slowly. When the coefficients are large (i.e., remainder small), the convergence will be faster. The expansion of $\pi$ is an example of faster convergence.

**In summary:** Every rational number $m/n \in \mathbb{F}$, with $m > n > 1$, may be uniquely expanded as a continued fraction, with coefficients $a_k$ determined using the CFA. When the target number is irrational ($x_0 \in \mathbb{Q}$), the CFA does not terminate; thus, each step produces a more accurate rational approximation, converging in the limit as $n \to \infty$.

Thus the CFA expansion is an algorithm that can, in theory, determine when the target is rational, but with an important caveat: one must determine if the expansion terminates. This may not be obvious. The fraction $1/3 = 0.3333\cdots$ is an example of such a target, where the CFA terminates yet the fraction repeats. It must be that

$$1/3 = 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + \cdots.$$

Here $3*3=9$. As a second example$^{20}$

$$1/7 = 0.142857142857142857 \cdots = 142857 \times 10^{-6} + 142857 \times 10^{-12} + \cdots$$

There are several notations for repeating decimals such as $1/7 = 0.1\overline{42857}$ and $1/7 = 0.1((142857))$. Note that $142857 = 999999/7$. Related identities include $1/11 = 0.090909\cdots$ and $11 \times 0.090909 = 999999$. When the sequence of digits repeats, the sequence is predictable, and it must be rational. But it is impossible to be sure that it repeats because the length of the repeat can be arbitrarily long.

One of the many useful things about the procedure is its generalizations to the expansion of transmission lines, as complex functions of the Laplace complex frequency $s$. As an example, in Assignment DE-3 (problem 2 on §5.4, page 162), a transmission line is developed in terms of the CFA.

**Exercise:** Discuss the relationship between the CFA and the transmission line modeling method on p. 92. **Solution:** The solution is detailed in Appendix D (p. 217).
2.3. APPLICATIONS OF PRIME NUMBERS

2.3.3 Pythagorean triplets (Euclid’s formula)

Euclid’s formula is a method for finding three integer lengths \([a, b, c] \in \mathbb{N}\) that satisfy Eq. 1.1.1.1. It is important to ask: Which set are the lengths \([a, b, c]\) drawn from? There is a huge difference, both practical and theoretical, if they are from the real numbers \(\mathbb{R}\), or the counting numbers \(\mathbb{N}\). Given \(p, q \in \mathbb{N}\) with \(p > q\), the three lengths \([a, b, c]\) of Eq. 1.1.1.1 are given by

\[
a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2. \tag{2.3.3.6}
\]

This result may be directly verified, since

\[
\]

or

\[
p^4 + q^4 + 2p^2q^2 = p^4 + q^4 - 2p^2q^2 + 4p^2q^2.
\]

Thus, Eq. 2.3.3.6 is easily proved once given. Deriving Euclid’s formula (See AE-2, problem #2) is obviously much more difficult, and is similar to the proof of Pell’s equation.

A well-known example is the right triangle depicted in Fig. 2.7, defined by the integer lengths \([3, 4, 5]\) having angles \([30, 60, 90]\) \(\in \mathbb{N}\) [deg] (i.e., \([\pi/6, \pi/3, \pi/2]\) \(\in \mathbb{I}\) [rad]), has one irrational \((\mathbb{I})\) length \(\sqrt{3} \in \mathbb{N}\). Table 2.1: Table of Pythagorean triples computed from Euclid’s formula Eq. 2.3.3.6 for various \([p, q]\). The last three columns are the first, fourth and penultimate values of “Plimpton-322,” along with their corresponding \([p, q]\). In all cases \(c^2 = a^2 + b^2\) and \(p = q + l\), where \(l = \sqrt{c - b} \in \mathbb{N}\).

<table>
<thead>
<tr>
<th>(q)</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>2</th>
<th>3</th>
<th>3</th>
<th>3</th>
<th>5</th>
<th>54</th>
<th>27</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>(p)</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>12</td>
<td>125</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>(a)</td>
<td>3</td>
<td>8</td>
<td>15</td>
<td>5</td>
<td>10</td>
<td>14</td>
<td>15</td>
<td>5</td>
<td>12</td>
<td>21</td>
<td>7</td>
<td>16</td>
</tr>
<tr>
<td>(b)</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>30</td>
<td>36</td>
<td>24</td>
<td>30</td>
</tr>
<tr>
<td>(c)</td>
<td>5</td>
<td>10</td>
<td>17</td>
<td>13</td>
<td>15</td>
<td>20</td>
<td>29</td>
<td>25</td>
<td>34</td>
<td>45</td>
<td>25</td>
<td>34</td>
</tr>
</tbody>
</table>

The set from which the lengths \([a, b, c]\) are drawn was not missed by the early Asians, and documented by the Greeks. Any equation whose solution is based on integers is called a Diophantine equation, named after the Greek mathematician Diophantus of Alexandria (c250 CE) (Fig. 1.1, p. 15).

A stone tablet having the numbers engraved on it, as shown in Fig. 2.8, was discovered in Mesopotamia, from the 19th century [BCE], and cataloged in 1922 by George Plimpton.21. These numbers are \(a\) and \(c\) pairs from PTs \([a, b, c]\). Given this discovery, it is clear that the Pythagoreans were following those who

---

Figure 2.8: “Plimpton-322” is a stone tablet from 1800 [BCE], displaying $a$ and $c$ values of the Pythagorean triplets $[a, b, c]$, with the property $b = \sqrt{c^2 - a^2} \in \mathbb{N}$. Several of the $c$ values are primes, but not the $a$ values. The stone is item 322 (item 3 from 1922) from the collection of George A. Plimpton.

came long before them. Recently a second similar stone, dating between 350 and 50 [BCE] has been reported, that indicates early calculus on the orbit of Jupiter’s moons, the very same moons that in 1687 Rømer observed, to show that the speed of light was finite (p. 20).

It is of interest that PTs play a role on atomic physics, as discussed in Appendix H on page 241.

2.3.4 Pell’s equation

Pell’s equation

$$x_n^2 - Ny_n^2 = (x_n - \sqrt{N}y_n)(x_n + \sqrt{N}y_n) = 1,$$

with non-square $N \in \mathbb{N}$ specified and $x, y \in \mathbb{N}$ unknown, has a venerable history in both physics (p. 27) and mathematics. Given its factored form it is obvious that every solution $x_n, y_n$ has the asymptotic property

$$\frac{x_n}{y_n} \bigg|_{n \to \infty} \to \pm \sqrt{N}.$$

It is believed that Pell’s equation is directly related to the Pythagorean theorem, since they are both simple binomials having integer coefficients (Stillwell, 2010, 48), with Pell’s equation being the hyperbolic version of Eq. 1.1.1.1. For example, with $N = 2$, a solution is $x = 17, y = 12$ (i.e., $17^2 - 2 \cdot 12^2 = 1$).

A $2 \times 2$ matrix recursion algorithm, likely due to the Chinese and used by the Pythagoreans to investigate $\sqrt{N}$, is

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n,$$

where we indicate the index outside the vectors.

Starting with the trivial solution $[x_0, y_0]^T = [1, 0]^T$ (i.e., $x_0^2 - Ny_0^2 = 1$), additional solutions of Pell’s equations are determined, having the property $x_n/y_n \to \sqrt{N} \in \mathbb{F}$, motivated by the Euclid’s formula for Pythagorean triplets (Stillwell, 2010, p. 44).

Note that Eq. 2.3.4.9 is a $2 \times 2$ linear matrix composition method (see p. 81), since the output of one matrix multiply is the input to the next.

Asian solutions: The first solution of Pell’s equation was published by Brahmagupta (c628), who independently discovered the equation (Stillwell, 2010, p. 46). Brahmagupta’s novel solution also used the composition method, but different from Eq. 2.3.4.9. Then in 1150 CE, Bhāskara II independently obtained solutions using Eq. 2.3.4.9 (Stillwell, 2010, p.69). This is the composition method we shall explore here, as summarized in Table B.1 (p. 200).

The best way to see how this recursion results in solutions to Pell’s equation is by example. Initializing the recursion with the trivial solution \( [x_0, y_0]^T = [1, 0]^T \) gives

\[
\begin{bmatrix}
 x_1 \\
y_1
\end{bmatrix} = \begin{bmatrix}
 1 \\
 1 
\end{bmatrix},
\]

\[
\begin{bmatrix}
 x_2 \\
y_2
\end{bmatrix} = \begin{bmatrix}
 2 & 1 \\
 1 & 1
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 1 \\
 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
 x_3 \\
y_3
\end{bmatrix} = \begin{bmatrix}
 2 & 1 \\
 1 & 1
\end{bmatrix} \begin{bmatrix}
 2 \\
 1
\end{bmatrix} = \begin{bmatrix}
 3 \\
 2
\end{bmatrix},
\]

\[
\begin{bmatrix}
 x_4 \\
y_4
\end{bmatrix} = \begin{bmatrix}
 2 & 1 \\
 1 & 1
\end{bmatrix} \begin{bmatrix}
 3 \\
 2
\end{bmatrix} = \begin{bmatrix}
 7 \\
 5
\end{bmatrix},
\]

\[
\begin{bmatrix}
 x_5 \\
y_5
\end{bmatrix} = \begin{bmatrix}
 2 & 1 \\
 1 & 1
\end{bmatrix} \begin{bmatrix}
 7 \\
 5
\end{bmatrix} = \begin{bmatrix}
 17 \\
 12
\end{bmatrix}.
\]

Thus the recursion results in a modified version of Pell’s equation

\[
x_n^2 - 2y_n^2 = (-1)^n,
\]

where only even values of \( n \) are solutions. This sign change had no effect on the Pythagoreans’ goal, since they only cared about the ratio \( y_n/x_n \to \pm \sqrt{2} \).

Modified recursion: The following summarizes the solution (\( \in \mathbb{C} \)) of Pell’s equation for \( N = 2 \) using a slightly modified linear matrix recursion. To fix the \((-1)^n\) problem, multiplying the \( 2 \times 2 \) matrix by \( 1_j = \sqrt{-1} \), which gives

\[
\begin{bmatrix}
 x \\
y
\end{bmatrix} = j \begin{bmatrix}
 1 \\
 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
 x \\
y
\end{bmatrix} = j^2 \begin{bmatrix}
 2 & 1 \\
 1 & 1
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 1 \\
 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
 x \\
y
\end{bmatrix} = j^3 \begin{bmatrix}
 3 & 2 \\
 2 & 1
\end{bmatrix} \begin{bmatrix}
 1 \\
 1
\end{bmatrix} = \begin{bmatrix}
 2 & 1 \\
 1 & 1
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 1 \\
 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
 x \\
y
\end{bmatrix} = j^7 \begin{bmatrix}
 7 & 5 \\
 5 & 3
\end{bmatrix} \begin{bmatrix}
 1 \\
 1
\end{bmatrix} = \begin{bmatrix}
 3 & 2 \\
 2 & 1
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 2 & 1 \\
 1 & 1
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 1 \\
 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
 x \\
y
\end{bmatrix} = j^{12} \begin{bmatrix}
 12 & 9 \\
 9 & 6
\end{bmatrix} \begin{bmatrix}
 1 \\
 1
\end{bmatrix} = \begin{bmatrix}
 5 & 3 \\
 3 & 2
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 2 & 1 \\
 1 & 1
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 1 \\
 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
 x \\
y
\end{bmatrix} = j^{41} \begin{bmatrix}
 41 & 29 \\
 29 & 20
\end{bmatrix} \begin{bmatrix}
 1 \\
 1
\end{bmatrix} = \begin{bmatrix}
 17 & 12 \\
 12 & 9
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 7 & 5 \\
 5 & 3
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 3 & 2 \\
 2 & 1
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 2 & 1 \\
 1 & 1
\end{bmatrix} \begin{bmatrix}
 1 \\
 0
\end{bmatrix} = \begin{bmatrix}
 1 \\
 1
\end{bmatrix}.
\]

Solution to Pell’s equation: By multiplying the matrix by \( 1_j \), all the solutions \( (x_k \in \mathbb{C}) \) to Pell’s equation are determined. The \( 1_j \) factor corrects the alternation in sign, so every iteration yields a solution. For \( N = 2, n = 0 \) (the initial solution) \( [x_0, y_0] = [1, 0] \), \( [x_1, y_1] = j[1, 1] \), and \( [x_2, y_2] = -[3, 2] \). These are easily checked using this recursion.

The solution for \( N = 3 \) is given in Appendix B.2.1 Table. B.1 (page 200) shows that every output of this slightly modified matrix recursion gives solutions to Pell’s equation: \( [1, 0], [1, 1], [4, 2], [10, 6], \ldots, [76, 44], \ldots \).
At each iteration, the ratio \( x_n/y_n \) approaches \( \sqrt{2} \) with increasing accuracy, coupling it to the CFA which may also be used to find approximations to \( \sqrt{N} \). The value of \( 41/29 \approx \sqrt{2} \), with a relative error of \(<0.03\%\). \( \text{CH:START} \) Remove the line "The solution ...". \( \text{CH:END} \)

### 2.3.5 Fibonacci sequence

Another classic problem, also formulated by the Chinese, was the Fibonacci sequence, generated by the relation

\[
f_{n+1} = f_n + f_{n-1}. \tag{2.3.5.11}
\]

Here the next number \( f_{n+1} \in \mathbb{N} \) is the sum of the previous two. If we start from \([0, 1]\), this linear recursion equation leads to the Fibonacci sequence \( f_n = [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots] \). Alternatively, if we define \( y_{n+1} = x_n \), then Eq. 2.3.5.11 may be compactly represented by a \( 2 \times 2 \) companion matrix recursion (see Fibonacci Exercises in AE-1)

\[
\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n, \tag{2.3.5.12}
\]

which has eigenvalues \( (1 \pm \sqrt{5})/2 \).

The correspondence of Eqs. 2.3.5.11 and 2.3.5.12 is easily verified. Starting with \( [x, y]^T_0 = [0, 1]^T \) we obtain for the first few steps

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_0, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_2, \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3, \ldots
\]

From the above \( x_n = [0, 1, 1, 2, 3, 5, \ldots] \) is the Fibonacci sequence, since the next \( x_n \) is the sum of the previous two, and the next \( y_n \) is \( x_n \).

Figure 2.9: This construction is called the Fibonacci spiral. Note how it is constructed out of squares having areas given by the square of the Fibonacci numbers. In this way, the spiral is smooth and the radius increases as the Fibonacci numbers (e.g., \( 8 = 3+5, 13 = 5+8, \text{etc.} \)). (Adapted from https://en.wikipedia.org/wiki/Golden_spiral.)

**Exercise:** Use the Octave/Matlab command `compan(c)` to find the companion matrix of the polynomial coefficients defined by Eq. 2.3.5.11. **Solution:** Using Matlab/Octave: \( f=[1, -1, -1] \); \( C=compan(f) \); returning

\[
C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.3.5.13}
\]

**Exercise:** Find the eigenvalues of matrix \( C \). **Solution:** The characteristic equation is

\[
\det \begin{bmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0
\]

or \( \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - 1/4 - 1 = 0 \), which has roots \( \lambda \pm = (1 \pm \sqrt{5})/2 \approx \{1.618, -0.618\} \).
The mean-Fibonacci sequence: Suppose that the Fibonacci sequence recursion is replaced by the mean of the last two values, namely let
\[ f_{n+1} = \frac{f_n + f_{n-1}}{2}. \] (2.3.5.14)
This seems like a small change. But how does the solution differ? To answer this question it is helpful to look at the corresponding $2 \times 2$ matrix.

Exercise: Find the $2 \times 2$ matrix corresponding to Eq. 2.3.5.14. The $2 \times 2$ matrix may be found using the companion matrix method (p. 66).

Solution: Using Matlab/Octave code:
\[
f = [1, -1/2, -1/2];
C = compan(f);
\]
which returns
\[
C = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}
\] (2.3.5.15)

Exercise: Find the steady state solution for the mean-Fibonacci, starting from $[1, 0]^T$. State the nature of both solutions.

Solution: By inspection one steady-state solution is $[1, 1]^T$, or $f_n = 1^n$. To find the full solution we need to find the two eigenvalues, defined by
\[
\det \begin{bmatrix} 1/2 - \lambda & 1/2 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda/2 - 1/2 = (\lambda - 1/4)^2 - (1/4)^2 - 1/2 = 0
\]
Thus $\lambda_{\pm} = (1 \pm 3)/4 = [1, -0.5]$. Thus the second solution is $(-1/2)^n$, which changes sign at each time step and quickly goes to zero. The full solution is given by $E\Lambda^n E^{-1}[1, 0]^T$ (see Appendix B, p. 197).

Relations to digital signal processing: Today we recognize Eq. 2.3.5.11 as a discrete difference equation, which is a pre-limit (pre Stream 3) recursive form of a differential equation. The $2 \times 2$ matrix form of Eq. 2.3.5.11 is an early precursor to 17th and 18th century developments in linear algebra. Thus the Greeks’ recursive solution for the $\sqrt{2}$ and Bhâskara’s (1030 CE) solution of Pell’s equation are early precursors to discrete-time signal processing, as well as to calculus.

There are strong similarities between Pell’s equation and the Pythagorean theorem. As we shall see in §2, Pell’s equation is related to the geometry of a hyperbola, just as the Pythagorean equation is related to the geometry of a circle. We shall show, as one might assume, that there is a Euclid’s formula for the case of Pell’s equations, since these are all conic sections with closely related conic geometry. As we have seen, the solutions involve $\sqrt{-1}$. The derivation is a trivial extension of that for the Euclid’s formula for Pythagorean triplets. The early solution of Brahmagupta was not related to this simple formula.
Chapter 3

Algebraic Equations: Stream 2

3.1 The physics behind nonlinear Algebra (Euclidean geometry)

Stream 2 is geometry, which led to the merging of Euclid’s geometrical methods and the 9th century development of algebra by al-Khwarizmi (830 CE) (Fig. 1.1, p. 15). This migration of ideas led Descartes and Fermat to develop analytic geometry (Fig. 1.2, p. 18).

The mathematics up to the time of the Greeks, documented and formalized by Euclid, served students of mathematics for more than two thousand years. Algebra and geometry were, at first, independent lines of thought. When merged, the focus returned to the Pythagorean theorem. Algebra generalized the analytic conic section into the complex plane, greatly extending the geometrical approach as taught in Euclid’s Elements. With the introduction of algebra, numbers, rather than lines, could be used to represent a geometrical lengths in the complex plane. Thus the appreciation for geometry grew, given the addition of the rigorous analysis using numbers.

Chronological history post 15th century

16th: Bombelli 1526–1572; Galileo 1564–1642; Kepler 1571–1630; Mersenne, 1588,1648;
17th: Huygens, 1629,1695; Newton 1642–1727; Principia 1687; Bernoulli, Jakob, 1655,1705; Bernoulli, Johann 1667–1748; Pascal,; Fermat, 1607,1665; Descartes, 1596,1648; Bernoulli, Daniel 1667,1748;
18th: Bernoulli, Daniel 1700–1782; Euler 1707–1783; d’Alembert 1717–1783; Lagrange 1736–1833; Laplace 1749–1827; Fourier 1768–1830; Napoleon 1769–1821; Gauss 1777–1855; Cauchy 1789–1857;
20th: Bode, Hendrik Wade 1905–1982

Physics inspires algebraic mathematics: The Chinese used music, art, and navigation to drive mathematics. Unfortunately much of their knowledge has been handed down either as artifacts, such as musical bells and tools, or mathematical relationships documented, but not created, by scholars such as Euclid, Archimedes, Diophantus, and perhaps Brahmagupta. With the invention of algebra by al-Khwarizmi (830CE), mathematics became more powerful, and blossomed. During the 16th and 17th century, it had become clear that differential equations (DEs), such as the wave equation, can characterize a law of nature, at a single point in space and time. This principle was not obvious. A desire to understand motions of objects and planets precipitated many new discoveries. This period, centered around Galileo, Newton and Euler, is illustrated in Fig. 1.2 (p. 18).

As previously described, the law of gravity was first formulated by Galileo using the concept of conservation of energy, which determines how masses are accelerated when friction is not considered
Newton and the speed of sound: 

Once Newton proposed the basic laws of gravity, and explained the elliptical motion of the planets, he proposed the first model of the speed of sound.

In 1630 Mersenne showed that the speed of sound was approximately 1000 [ft/s]. This may be done by timing the difference in the time the flash of an explosion, to the time it is heard. For example, if the explosion is 1 [mi] away, the delay is about 5 seconds. Thus with a simple clock, such as a pendulum, and an explosive, the speed may be accurately measured. If we say the speed of sound is \(c_0\), then the equation for the wavefront would be \(f(x, t) = u(x - c_0 t)\), where function \(u(t) = 0\) for \(t < 0\) and 1 for \(t > 0\). If the wave is traveling in the opposite direction, then the formula would be \(u(x + c_0 t)\).
one also assumes that sounds add in an independent manner (superposition holds) (Postulate P2, p. 106),
then the general solution for the acoustic wave would be

\[ f(x, t) = Au(x - c_o t) + Bu(x + c_o t) \]

where \( A, B \) are the amplitudes of the two waves. This is the solution proposed by d’Alembert in 1747 to the
acoustic wave equation

\[ \frac{\partial^2}{\partial x^2} \varrho(x, t) = \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \varrho(x, t), \tag{3.1.0.3} \]

one of the most important equations of mathematical physics (see Eq. 4.3.0.1, p. 121), 20 years after
Newton’s death.

It had been well established, at least by the time of Galileo, that the wavelength \( \lambda \) and frequency \( f \)
for a pure tone sound wave obeys the relation

\[ f\lambda = c_o. \tag{3.1.0.4} \]

Given what we know today, the general solution to the wave equation may be written in terms of a
sum over the complex exponentials, famously credited to Euler, as

\[ \varrho(x, t) = Ae^{2\pi i (ft - x/\lambda)} + Be^{2\pi i (ft + x/\lambda)}, \tag{3.1.0.5} \]

where \( t \) is time and \( x \) is position and \( ft \) and \( x/\lambda \) are dimensionless. This equation only describes the
steady-state solution, no onsets or dispersion. Thus this solution must be generalized to include these
important effects.

Thus the basics of sound propagation was in Newton’s grasp, and finally published in Principia in
1687. The general solution to Newton’s wave equation \([i.e., p(x, t) = G(t \pm x/c)]\), where \( G \) is any
function, was first published 60 years later by d’Alembert (c1747).

Newton’s value for the speed of sound in air \( c_o \) was incorrect by the thermodynamic constant
\( \sqrt{\eta_o} = \sqrt{1.4} \), a problem that would take 129 years to formulate, by Laplace in 1816 (Rayleigh,
1896, page xx). What was needed was the adiabatic process \((the\ concept\ of\ constant-heat\ energy)\).
For audio frequencies \((0.02-20 \text{ kHz})\), the temperature gradients cannot diffuse the distance of a wave-
length in one cycle \(\text{CH:END}\) (Tisza, 1966; Pierce, 1981; Boyer and Merzbach, 2011), “trapping” the heat
energy in the wave.\(^1\) To repair Newton’s formula for the sound speed it was necessary to define the
dynamic stiffness of air \( \eta_oP_o \), where \( P_o \) \([1 \text{ atm} \text{ or } 10^5 \text{ Pa}] \) is the static stiffness of air. \([1 \text{ Pa}] \) is 1
\([\text{N/m}^2] \). This required replacing Boyle’s Law \( (PV/T = \text{ constant}) \) with the adiabatic expansion law
\( (PV/\eta_o = \text{ constant}) \). But this fix still ignores viscous and thermal losses (Kirchhoff, 1868; Rayleigh,
1896; Mason, 1927; Pierce, 1981).

Today we know that the speed of sound is given by

\[ c_o = \sqrt{\frac{\eta_o P_o}{\rho_o}} = 343, \quad [\text{m/s}] \]

\(\text{CH:START}\) which is a function of the density \( \rho_o = 1.12 \text{ [kg/m}^3] \), \( P_o = 10^5 \text{ [Pa]} \), and the dynamic stiffness \( \eta_oP_o \) of air.\(^2\) The speed of sound stated in other units is 434 \([\text{m/s}] \), 1234.8 \([\text{km/h}] \), 1,125 \([\text{ft/ms}] \),
1125.3 \([\text{ft/s}] \), 4.692 \([\text{km/min}] \), 403 \([\text{m/min}] \), 0.213 \([\text{mi/s}] \), or 767 \([\text{mi/h}] \).

Newton’s success was important because it quantified the physics behind the speed of sound, and
demonstrated that momentum \((mv)\), not mass \( m \), was transported by the wave. His concept was correct,
and his formulation using algebra and calculus represented a milestone in science, assuming no visco-
elastic losses. When including losses, the wave number becomes a complex function of frequency,
leading to Eq. 4.1.1.4 (p. 234).

\(^1\)There were other physical enigmas, such as the observation that sound disappears in a vacuum or that a vacuum
cannot draw water up a column by more than 34 feet.

\(^2\)\( \eta_o = C_p/C_v = 1.4 \) is the ratio of two thermodynamic constants and \( P_o = 10^5 \text{ [Pa]} \) is the barometric pressure of air.
In periodic structures, again the wave number becomes complex due to diffraction, as commonly observed in optics (e.g., diffraction gratings) and acoustics (creeping surface waves). Thus Eq. 3.1.0.4 only holds for the simplest cases, but in general, Eq. 3.1.0.6, the complex analytic (dispersive) function, called the propagation vector \( \kappa(x, s) \) (see below), must be considered.

The corresponding discovery for the formula for the speed of light was made 174 years after *Principia*, by Maxwell (c1861). Maxwell’s formulation also required great ingenuity, as it was necessary to hypothesize an experimentally unmeasured term in his equations, to get the mathematics to correctly predict the speed of light.

It is somewhat amazing that to this day, we have failed to fully understand gravity significantly better than Newton’s theory. This is too harsh given the famous general relativity (c1920) work of Einstein.³

**Case of dispersive wave propagation:** This classic relation \( \lambda f = c \) is deceptively simple, yet confusing, because the wave number⁴ \( k = 2\pi/\lambda \) becomes a complex function of frequency (has both real and imaginary parts) in dispersive media (e.g., acoustic waves in tubes) when losses are considered (Kirchhoff, 1868; Mason, 1928).

A second important example is the case of electron waves in silicon crystals, where the wave number \( k(f) = 2\pi f/c \) is replaced with the complex analytic function of \( s \), called the propagation vector \( \kappa(s) \). In this case the wave becomes the eigen-function of the vector (3D) wave equation

\[
p^{\pm}(x, t) = P_o(s) e^{\pm \kappa(x, s) z},
\]

where \( |\kappa(x, s)| \) is the vector eigenvalue (Brillouin, 1953). In these more general cases \( \kappa(x, s) \) must be a vector complex analytic function of the Laplace frequency \( s = \sigma + \omega j \), and inverted with the Laplace transform (Brillouin, 1960, with help from Sommerfeld). This is because electron “waves” in the dispersive semi-conductor (e.g., silicon) are “causally filtered,” in 3 dimensions, in magnitude, phase and direction \( x \). These 3D dispersion relations are known as *Brillouin zones.*

Silicon is a highly dispersive “wave-filter,” forcing the wavelength to be a functions of both \( s \) and direction. This view is elegantly explained by Brillouin (1953, Chap. 1), in his historic text. While the most famous examples come from quantum mechanics (Condon and Morse, 1929), modern acoustics contains a rich source of related examples (Morse, 1948; Beranek, 1954; Ramo *et al.*, 1965; Fletcher and Rossing, 2008).

### 3.1.1 The first algebra

Prior to the invention of algebra, people worked out problems as sentences, using an obtuse description of the problem (Stillwell, 2010, p. 93). Algebra changed this approach, resulting in a compact language of mathematics, where numbers are represented as abstract symbols (e.g., \( x \) and \( \alpha \)). The problem to be solved could be formulated in terms of sums of powers of smaller terms, the most common being powers of some independent variable (i.e., time or frequency). If we set \( a_n = 1 \)

\[
P_N(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0z^0 = z^n + \sum_{k=0}^{n-1} a_k z^k = \prod_{k=0}^{n} (z - z_k).
\]

is called a **monic polynomial**. The coefficient \( a_n \) cannot be zero, or the polynomial would not be of degree \( n \). The resolution is to force \( a_n = 1 \), since this simplifies the expression and does not change the roots.

The key question is: What values of \( z = z_k \) result in \( P_N(z_k) = 0 \). In other words, what are the roots \( z_k \) of the polynomial? Answering this question consumed thousands of years, with intense efforts by

---

³Gravity waves were first observed experimentally while I was formulating §3 (p. 55).

⁴This term is a misnomer, since the wave number is a complex function of the Laplace frequency \( s = \sigma + \omega j \), thus not a number in the common sense. Much worse \( \kappa(s) = s/c_0 \) must be complex analytic in \( s \), which an even stronger condition. The term wave number is so well established, there is no hope for recovery at this point.
many aspiring mathematicians. In the earliest attempts, it was a competition to evaluate mathematical acumen. Results were held as a secret to the death bed. It would be fair to view this effort as an obsession. Today the roots of any polynomial may be found, to high accuracy, by numerical methods. Finding roots is limited by the numerical limits of the representation, namely by IEEE-754 (p. 33). There are also a number of important theorems.

Of particular interest is composing a circle with a line, by finding the intersection (root). There was no solution to this problem using geometry. The resolution of this problem is addressed in the assignments below.

### 3.1.2 Finding roots of polynomials

The problem of factoring polynomials has a history more than a millennium in the making. While the quadratic \((N = 2)\) was solved by the time of the Babylonians (i.e., the earliest recorded history of mathematics), the cubic solution was finally published by Cardano in 1545. The same year, Cardano’s student solved the quartic \((N = 4)\). In 1826 (281 years later) it was proved that the quintic \((N = 5)\) could not be factored by analytic methods.

As a concrete example we begin with the important but trivial case of the quadratic

\[ P_2(s) = as^2 + bs + c. \] (3.1.2.8)

First note that if \(a = 0\), the quadratic reduces to the monomial \(P_1(s) = bs + c\). Thus we have the necessary condition that \(a \neq 0\). The best way to proceed is to divide \(a\) out and work directly with the monic \(\frac{1}{a}P_2(s) = \frac{1}{a}P_2(s)\). In this way we do not need to worry about the \(a = 0\) exception.

The roots are those values of \(s\) such that \(\frac{1}{a}P_2(s_k) = 0\). One of the first results (recorded by the Babylonians, c2000 BCE) was the factoring of this equation by completing the square. One may isolate \(s\) by rewriting Eq. 3.1.2.8 as

\[ \frac{1}{a}P_2(s) = (s + b/(2a))^2 - (b/(2a))^2 + c/a. \] (3.1.2.9)

The factorization may be verified by expanding the squared term, and canceling \((b/(2a))^2\)

\[ \frac{1}{a}P_2(s) = [s^2 + (b/a)s + (b/(2a))^2] - (b/(2a))^2 + c/a. \]

Setting Eq. 3.1.2.9 to zero and solving for the two roots \(s_{\pm}\) gives the quadratic formula

\[ s_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \bigg|_{a=1} = -b/2 \pm \sqrt{(b/2)^2 - c}. \] (3.1.2.10)

**Role of the discriminant:** This can be further simplified. The term \((b/2)^2 - c > 0\) under the square root is called the discriminant. Nominally in physics and engineering problems, the discriminant is negative, and \(b/2 \ll \sqrt{c}\) may be ignored (the damping is small compared to the resonant frequency), leaving only \(-c\) under the radical. Thus, the most natural way (i.e., corresponding to the most common physical cases) of writing the roots (Eq. 3.1.2.10) is

\[ s_{\pm} \approx -b/2 \pm j\sqrt{|c|} = -\sigma_o \pm s_0. \] (3.1.2.11)

This form separates the real and imaginary parts of the solution in a natural way. The term \(\sigma_o = b/2\) is called the damping, which accounts for losses in a resonant circuit, while the term \(\omega_o = \sqrt{|c|}\), for

\[ s_{\pm} \approx -b/2 \pm j\sqrt{|c|} = -\sigma_o \pm s_0. \] (3.1.2.11)

This is the case for mechanical and electrical circuits having small damping. Physically \(b > 0\) is the damping coefficient and \(\sqrt{c} > 0\) is the resonant frequency. One may then simplify and factor the form as

\[ s^2 + 2bs + c^2 = (s + b + j\sqrt{c})(s + b - j\sqrt{c}). \]
mechanical, acoustical and electrical networks, is called the resonant frequency. The last approximation ignores the (typically) minor correction \( b/2 \) to the resonant frequency, which in engineering applications is almost always ignored. Knowing that there is a correction is highlighted by this formula, making one aware that the small approximation exists (thus can be ignored).

It is not required that \( a, b, c \in \mathbb{R} > 0 \), but for physical problems of interest, this is almost always true (>99.99% of the time).

**Summary:** The quadratic equation and its solution are ubiquitous in physics and engineering. It seems obvious that instead of memorizing the meaningless quadratic formula Eq. 3.1.2.10, one should learn the physically meaningful solution, Eq. 3.1.2.11, obtained via Eq. 3.1.2.9, with \( a = 1 \). Arguably, the factored and normalized form (Eq. 3.1.2.9) is easier to remember, as a method (completing the square), rather than as a formula to be memorized.

Additionally, the real \((b/2)\) and imaginary \( \pm j\sqrt{c} \) parts of the two roots have physical significance as the damping and resonant frequency. Equation 3.1.2.10 has none (it is useless).

No insight is gained by memorizing the quadratic formula. To the contrary, an important concept is gained by learning how to complete the square, which is typically easier than identifying \( a, b, c \) and blindly substituting them into Eq. 3.1.2.10. Thus it’s worth learning the alternate solution (Eq. 3.1.2.11) since it is more common in practice and requires less algebra to interpret the final answer.

**Exercise:** By direct substitution demonstrate that Eq. 3.1.2.10 is the solution of Eq. 3.1.2.8. Hint: Work with \( \hat{P}_2(s) \).

**Solution:** Setting \( a = 1 \) the quadratic formula may be written

\[
\text{CH : START } s \pm \text{ CH : END } = \frac{-b \pm j\sqrt{4c - b^2}}{2}.
\]

Substituting this into \( \hat{P}_2(s) \) gives

\[
\hat{P}_\pm(s_\pm) = s_\pm^2 + bs_\pm + c = -b/2 \pm j\sqrt{b^2 - (b/2)^2}
\]

\[
= \left( \frac{-b \pm \sqrt{b^2 - 4c}}{2} \right)^2 + b \left( \frac{-b \pm \sqrt{b^2 - 4c}}{2} \right) + c
\]

\[
= \frac{1}{4} \left( b^2 \pm 2b\sqrt{b^2 - 4c} + (b^2 - 4c) \right) + \frac{1}{4} \left( -2b^2 \pm 2b\sqrt{b^2 - 4c} \right) + c
\]

\[
= 0.
\]

In third grade I learned the times-table trick for 9:

\[
9 \cdot n = (n - 1) \cdot 10 + (10 - n).
\]

With this simple rule I did not need to depend on my memory for the 9 times tables. For example: \( 9 \cdot 7 = (7 \cdot 1) \cdot 10 + (10 - 7) = 60 + 3 \) and \( 9 \cdot 3 = (3 \cdot 1) \cdot 10 + (9 - 3) = 20 + 7 \). By expanding the above, one can see why it works: \( 9n = n10 - 10 + 10 - n = n(10 - 1) \). Note that the two terms \((n - 1)\) and \((10 - n)\), add to 9.

Learning an algorithm is much more powerful than memorization of the 9 times tables. How one thinks about a problem can have a great impact on ones perception.

**Newton’s method for finding roots of** \( P_N(s) \): Newton is well known for an approximate but efficient method to find the roots of a polynomial. Consider polynomial \( s, P_N(s) \in \mathbb{C} \)

\[
P_N(s) = c_N(s - s_o)^N + c_{N-1}(s - s_o)^{N-1} + \cdots + c_1(s - s_o) + c_0,
\]

\[ (3.1.2.12) \]
where we use Taylor’s formula (p. 72) to determine the coefficients

\[ c_k = \frac{1}{k!} \frac{d^k}{ds^k} P_N(s) \bigg|_{s=s_0}. \tag{3.1.2.13} \]

If our initial guess for the root \( s_1 \) is close to a root \( s_o \) (i.e., \( s_1 - s_o \) is within the radius of convergence), then \( |(s_1 - s_o)^k| \ll |(s_1 - s_o)| \) for \( k \geq 2 \in \mathbb{N} \). Thus we may truncate \( P_N(s_1) \) to its linear term \( c_1 \):

\[
P_N(s_1) \approx (s_1 - s_o) \frac{d}{ds} P_N(s) \bigg|_{s=s_o} + P_N(s_o) = (s_1 - s_o) P'_N(s_o) + P_N(s_o),
\]

where \( P'_N(s_o) \) is shorthand for \( dP_N(s_o)/ds \).

Newton’s approach (approximation) was to define a recursion such that the next guess \( s_{n+1} \) is closer to the root \( s_o \) than the previous guess \( s_n \). Replacing \( s_1 \) by \( s_{n+1} \) and \( s_o \) by \( s_n \) gives

\[
P_N(s_{n+1}) = (s_{n+1} - s_n) P'_N(s_n) + P_N(s_n) \to 0.
\]

Here we assume \( P_N(s_{n+1}) \to 0 \) because \( s_{n+1} \to s_n \) as \( n \to \infty \).

Solving for \( s_{n+1} \):

\[
s_{n+1} = s_n - \frac{P_N(s_n)}{P'_N(s_n)}.	ag{3.1.2.14}
\]

Everything on the right is known, thus \( s_{n+1} \) should converge to the root \( s_o \) as \( n \to \infty \).

In practice, it only takes a few steps to approach the root. In experimental trials (see Fig. 3.2) less than 10 steps gives double-precision floating-point machine accuracy. If any value \( s_n \) is close to a root of \( P'_N \), the recursion fails, giving a large value for \( s_{n+1} \), forcing the method to restart at \( s_{n+1} \), far from the root. In such cases the solution typically converges to a different root. It should not be difficult to detect these large non-convergent steps, by monitoring \( |s_{n+1} - s_n| \), which should be monotonically decreasing.

However, if one assumes that the initial guess \( s_1 \in \mathbb{R} \), and then evaluates the polynomial using real arithmetic, the estimate \( s_{n+1} \in \mathbb{R} \). Thus the iteration will not converge if \( s_o \notin \mathbb{C} \).

Root \( s_o \in \mathbb{C} \) may be found by a recursion, defining sequence \( s_n \to s_o, n \in \mathbb{N} \), such that \( P_N(s_n) \to 0 \) as \( n \to \infty \). As shown in Fig. 3.2, solving for \( s_{n+1} \) using Eq. 3.1.2.14 always gives one of the roots, due to the analytic behavior of the complex logarithmic derivative \( P'_N/P_N \).

With every step \( s_{n+1} \) is closer to the root, converging to the root in the limit. As it comes closer, the linearity assumption becomes more accurate, resulting in a better approximation and thus a faster convergence.

Equation 3.1.2.14 depends on the log-derivative \( d \log P(x)/dx = P'(x)/P(x) \). It follows that even for cases where fractional derivatives of roots are involved (p. 130), Newton’s method should converge, since the log-derivative linearizes the equation.\(^6\)

**Newton’s view:** Newton believed that imaginary roots and numbers have no meaning (p. 117), thus he sought only real roots (Stillwell, 2010, p. 119). In this case Newton’s relation may be explored as a graph, which puts Newton’s method in the realm of analytic geometry.

**Example:** Given a polynomial \( P_2 = 1 - x^2 \), having roots \( \pm 1 \), use Newton’s method to find the roots. \( P_2(x) = -2x \), thus Newton’s iteration becomes

\[
x_{n+1} = x_n + \frac{1 - x_n^2}{2x_n}.
\]

---

\(^6\)This seems like a way to understand fractional, even irrational, roots.
CHAPTER 3. STREAM 2: ALGEBRAIC EQUATIONS

Figure 3.2: Newton’s method applied to the polynomial having real roots \([1, 2, 3, 4]\) (left) and 5 complex roots (right). A random starting point was chosen, and each curve shows the values of \(s_n\) as Newton’s method converges to the root. Different random starting points converge to different roots. The method always results in convergence to a root. Claims to the contrary (Stewart, 2012, p. 347) are a result of forcing the roots to be real. For convergence, one must assume \(s_n \in \mathbb{C}\). For a related discussion see Stillwell (2010, §14.7).

From the Gauss-Lucas theorem, for the case of \(N = 2\), the root of \(P'_2(x)\) is always the average of the roots of \(P_2(x)\).

To start the iteration \((n = 0)\) we need an initial guess for \(x_0\), which is an “initial random guess” of where a root might be. The only place we may not start is at the roots of \(P'_N(x)\).

For \(P_2(x) = 1 - x^2\),

\[
x_1 = x_0 + \frac{1 - x_0^2}{2x_0} = x_0 + \frac{1}{2} (-x_0 + 1/x_0) .
\]

Exercise:

1. Let \(P_2(x) = 1 - x^2\). Choose the expansion point as \(x_0 = 1/2\). Draw a graph describing the first step of the iteration. **Solution:** Start with an \((x, y)\) coordinate system and put points at \(x_0 = (1/2, 0)\) and the vertex of \(P_2(x)\), i.e.: \((0, 1) (P_2(0) = 1)\). Then draw \(1 - x^2\), along with a line from \(x_0\) to \(x_1\). ■

2. From the previous exercise, calculate \(x_1\) and \(x_2\). What number is the algorithm approaching? Is it a root of \(P_2\)?

**Solution:** First we must find \(P'_2(x) = -2x\). Thus the equation we will iterate is

\[
x_{n+1} = x_n + \frac{1 - x_n^2}{2x_n} = \frac{x_n^2 + 1}{2x_n} .
\]

By hand

\[
x_0 = 1/2
\]

\[
x_1 = \frac{(1/2)^2 + 1}{2(1/2)} = \frac{1}{4} + 1 = 5/4 = 1.25
\]

\[
x_2 = \frac{(5/4)^2 + 1}{2(5/4)} = \frac{25/16 + 1}{10/4} = \frac{41}{40} = 1.025.
\]

These estimates rapidly approach the positive real root \(x = 1\). Note that if one starts at the root of \(P'(x) = 0\) (i.e., \(x_0 = 0\)), the first step is indeterminate. ■
3. Write an Octave/Matlab script to check your answer for part (a).  Solution: 

```octave
x=1/2;
for n = 1:3
    x = x+(1-x*x) / (2*x);
end
```

(a) For \( n = 4 \), what is the absolute difference between the root and the estimate, \( |x_r - x_4| \)?  Solution: 4.6E-8 (very small!)

(b) What happens if \( x_0 = -1/2 \)?  Solution: The solution converges to the negative root, \( x = -1 \).

4. Does Newton’s method (Kelley, 2003) work for \( P_2(x) = 1 + x^2 \)? Hint: What are the roots in this case?  Solution: in this case \( P_2'(x) = +2x \) thus the iteration gives 

\[
x_{n+1} = x_n - \frac{1 + x_n^2}{2x_n}.
\]

In this case the roots are \( x_{\pm} = \pm 1 \), namely purely imaginary. Obviously Newton’s method fails because there is no way for the answer to become complex. If like Newton, you didn’t believe in complex numbers, your method would fail to converge to the complex roots (i.e., real in \( \rightarrow \) real out). This is because Octave/Matlab assumes \( x \in \mathbb{R} \) if it is initialized as \( \mathbb{R} \).  

5. What if you let \( x_0 = (1 + j)/2 \) for the case of \( P_2(x) = 1 + x^2 \)?  Solution: By starting with a complex initial value, we fix the Real in = Real out problem.

Basic properties of polynomials

In some sense polynomials such as \( P_N(z) \) are the simplest construction used in algebra, and a summary of their most basic properties is helpful.

1. The degree of a polynomial is \( n \).

2. Polynomials are single valued, namely for every \( z_o \) there is precisely one value for \( P_N(z_o) \).

3. In mathematical physics and engineering is is common to have real coefficients \( a_n \), but complex coefficients are possible.

4. The coefficients of every polynomial are determined by its Taylor series, namely Eq. 3.2.2 (p. 72).

5. If the coefficients are real and positive the \( P_N(x) \) is positive and real if \( x \geq 0 \)

6. The fundamental theorem of algebra states that \( P_N(z) \) has exactly \( n \) roots.

7. The roots of polynomials with positive and real coefficients typically have complex roots, namely if \( P_N(z_k) = 0 \), the \( z_k \in \mathbb{C} \).

8. The region of convergence (RoC) of every polynomial about the expansion point is infinite.

9. The roots of the derivative of a polynomial lie within the convex hull defined by the roots of \( P_N(z) \), as described by the Gauss-Lucas theorem (p. 64).
**Exercise:** Find the logarithmic derivative of $f(x)g(x)$.  

**Solution:** From the definition of the logarithmic derivative and the chain rule for the differentiation of a product:

$$
\frac{d}{dx} \ln f(x)g(x) = \frac{d}{dx} \ln f + \frac{d}{dx} \ln g = \frac{1}{f} \frac{df}{dx} + \frac{1}{g} \frac{dg}{dx}.
$$

**Example:** Assume that polynomial $P_3(s) = (s - a)^2/(s - b)^\pi$. Then

$$
\ln P_3(s) = 2 \ln(s - a) - \pi \ln(s - b)
$$
and

$$
\frac{d}{ds} \ln P_3(s) = \frac{2}{s - a} - \pi \frac{1}{s - b}.
$$

**Reduction by the logarithmic derivative to simple poles:** As shown by the above trivial example, any polynomial, having zeros of arbitrary degree (i.e., $\pi$ in the example), may be reduced to the ratio of two polynomials, by taking the logarithmic derivative, since

$$
L_N(s) = \frac{N(s)}{D(s)} = \frac{d}{ds} \ln P_N(s) = \frac{P'_N(s)}{P_N(s)}.
$$

(3.1.2.15)

Here the starting polynomial is the denominator $D(s) = P_N(s)$ while the numerator $N(s) = P'_N(s)$ is the derivative of $D(s)$. Thus the logarithmic derivative can play a key role in analysis of complex analytic functions, as it reduces higher order poles, even those of irrational degree, to a simple poles (those of degree 1).

The logarithmic derivative $L_N(s)$ has a number of special properties:

1. $L_N(s)$ has simple poles $s_p$ and zeros $s_z$.
2. The poles of $L_N(s)$ are the zeros of $P_N(s)$.
3. The zeros of $L_N(s)$ (i.e., $P'_N(s) = 0$) are the zeros of $P'_N(s)$.
4. $L_N(s)$ is analytic everywhere other than its poles.
5. Since the zeros of $P_N(s)$ are simple (no second-order poles), it is obvious that the zeros of $L_N(s)$ always lie close to the line connecting the two poles. One may easily demonstrate the truth of the statement numerically, and has been quantified by the Gauss-Lucas theorem which specifies the relationship between the roots of a polynomial and those of its derivative. Specifically, the roots of $P'_N$ lie inside the convex hull of the roots of $P_N$.

To understand the meaning of the convex hull, consider the following construction: If stakes are placed at each of the $N$ roots of $P_N(x)$, and a string is then wrapped around the stakes, with all the stakes inside the string, the convex hull is then the closed set inside the string. One can then begin to imagine how the $N - 1$ roots of the derivative must evolve with each set inside the convex hull of the previous set. This concept may be recursed to smaller values of $N$.

6. Newton’s method may be expressed in terms of the reciprocal of the logarithmic derivative, since

$$
s_{k+1} = s_k + \epsilon_o / L_N(s),
$$

where $\epsilon_o$ is called the step size, which is used to control the rate of convergence of the algorithm. If the step size is too large, the root-finding path may jump to a different domain of convergence, thus a different root of $P_N(s)$.

7. Not surprisingly, given all the special proprieties, $L_N(x)$ plays an key role in mathematical physics.
Euler’s product formula: Counting may be written as a linear recursion, simply by adding 1 to the previous value starting from 0. The even numbers may be generated by adding 2, starting from 0. Multiples of $3$ may be similarly generated by adding 3 to the previous value, starting from 0. Such recursions are fundamentally related to prime numbers $\pi_k \in \mathbb{P}$, as first investigated by Euler. This logic is the basis of the sieve (§2.3, p. 39). The basic idea is both simple and important, taking almost everyone by surprise, likely even Euler. It is related on the old idea that the integers may be generated by the geometric series, when viewed as a recursion.

**Example:** Let’s look at counting modulo prime numbers. For example, if $k \in \mathbb{N}$ then

$$k \cdot \text{mod}(k, 2), \quad k \cdot \text{mod}(k, 3), \quad k \cdot \text{mod}(k, 5)$$

are all multiples of the primes $\pi_1 = 2$, $\pi_2 = 3$ and $\pi_3 = 5$.

![Feedback network](image)

To see this define the *step function* $u_n = 0$ for $n < 0$ and $u_n = 1$ for $n \geq 0$ and the *counting number function* $N_n = 0$ for $n < 0$. The counting numbers may be recursively generated from the recursion

$$N_{n+1} = N_{n-M} + u_n,$$

which for $M = 1$ gives $N_n = n$. For $M = 2$ we have $N_n = 0, 2, 4, \cdots$ are the even numbers.

As was first published by Euler in 1737, one may recursively factor out the leading prime term, resulting in Euler’s product formula. Based on the argument given the discussion of the Sieve (p. 39), one may automate the process and create a recursive procedure to identify multiples of the first item on the list, and then remove the multiples of that prime. The lowest number on this list is the next prime. One may then recursively generate all the multiples of this new prime, and remove them from the list. Any numbers that remain are candidates for primes.

The observation that this procedure may be automated with a recursive filter, such as that shown in Fig. 3.3, implies that it may be transformed into the frequency domain, and described in terms of its poles, which are related to the primes. For example, the poles of the filter shown in Fig. 3.3 may be determined by taking the z-transform of the recursion equation, and solving for the roots of the resulting polynomial. The recursion equation is the time-domain equivalent to Riemann’s zeta function $\zeta(s)$, which the frequency domain equivalent representation.

**Exercise:** Show that $N_n = n$ follows from the above recursion. **Solution:** If $n = -1$ we have $N_n = 0$ and $u_n = 0$. For $n = 0$ the recursion gives $N_1 = N_0 + u_0$, thus $N_1 = 0 + 1 = 1$. When $n = 1$ we have $N_2 = N_1 + 1 = 1 + 1 = 2$. For $n = 2$ the recursion gives $N_3 = N_2 + 1 = 3$. Continuing the recursion we find that $N_n = n$. Today we denote such recursions of this form as digital filters. The state diagram for $N_n$ is given in Fig. 3.3.

To start the recursion define $u_n = 0$ for $n < 0$. Thus $u_0 = u_{-1} + 1$. But since $u_{-1} = 0$, $u_0 = 1$. The counting numbers follow from the recursion. A more understandable notation is the convolution of
the step function with itself, namely

\[ n u_n = u_n \ast u_n = \sum_{m=0}^{\infty} u_m u_{m-n} \leftrightarrow \frac{1}{(1 - z)^2} \]

which says that the counting numbers \( \hat{n} \in \mathbb{N} \) are easily generated by convolution, which corresponds to a second order pole at \( s = 0 \), in the Laplace frequency domain.

**Exercise:** Write an Octave/Matlab program that generates removes all the even number, to generate the odd numbers \( N_n = \{1, 0, 3, 0, 5, 0, 7, 0, \ldots\} \).

**Solution:**

```matlab
M=50; N=(0:M-1); u=ones(1,M); u(1)=0; Dem=[1 1]; Num=[1]; n=filter(Num,Dem,u); y2=n.*N; F1=n-y2
which generates: F1 = [0, 1, 0, 3, 0, 5, 0, 7, 0, 9, 0, \ldots].
```

An alternative is to use the \( \text{mod}(n, N) \) function:

\( M=20; \ n=0:M; \ k=\text{mod}(n, 2); \ m=(k==0) \.*n; \)
which generates \( m = [0, 1, 0, 3, 0, 5, 0, 7, 0, \ldots] \).

**Exercise:** Write a program to recursively down-sample \( N_n \) by 2:1.

**Solution:**

```matlab
N=[1 0 3 0 5 0 7 0 9 0 11 0 13 0 15]; M=N(2:2:end);
which gives: M = [1, 3, 5, 7, 9, 11, 13, 15, \ldots].
```

For the next step toward a full sieve (Fig. 2.3, p. 40), generate all the multiples of 3 (the second prime) and subtract these from the list. This will either zero out these number from the list, or create negative items, which may then be removed. Numbers are negative when the number has already been removed because it has a second factor of that number. For example, 6 is already removed because it is a multiple of 2, thus was removed when removing the multiples of prime number 2.

### 3.1.3 Matrix formulation of the polynomial

There is a one-to-one relationship between every constant coefficient differential equation, its characteristic polynomial and the equivalent matrix form of that differential equation, defined by the *companion matrix*. The roots of the monic polynomial are the eigenvalues of the companion matrix \( C_N \) (Horn and Johnson, 1988, p. 147).

**The companion matrix:** The \( N \times N \) companion matrix is defined as

\[
C_N = \begin{bmatrix}
0 & -c_0 \\
1 & 0 & -c_1 \\
0 & 1 & 0 & -c_2 \\
\vdots & \vdots & & \vdots \\
0 & 1 & 0 & \cdots & -c_{N-2} \\
0 & 1 & 0 & 0 & -c_{N-1}
\end{bmatrix}_{N \times N}
\]  

(3.1.3.17)
The constants $c_{N-n}$ are from the monic polynomial of degree $N$

$$P_N(s) = s^N + c_{N-1}s^{N-1} + \cdots + c_2s^2 + c_1s + c_0$$

$$= s^N + \sum_{n=0}^{N-1} c_n s^n,$$

having coefficient vector

$$c_{N-1}^T = [1, c_{N-1}, c_{N-2}, \ldots c_0]^T.$$

Any transformation of a matrix that leaves the eigenvalues invariant (e.g., the transpose) will result in an equivalent definition of $C_n$. For example, the Octave and Matlab companion matrix function $C=compan(A)$ returns the coefficient vector along the top row. (See p. 66).

**Exercise:** Show that the eigenvalues of the 3x3 companion matrix are the same as the roots of $P_3(s)$.

**Solution:** Expanding the determinant of $C_3 - sI_3$ along the right-most column

$$P_3(s) = \begin{vmatrix} -s & 0 & -c_0 \\ 1 & -s & -c_1 \\ 0 & 1 & -(c_2 + s) \end{vmatrix} = c_0 + c_1s + (c_2 + s)s^2 = s^3 + c_2s^2 + c_1s + c_0.$$

Setting this to zero gives the requested result.

**Exercise:** Find the companion matrix for the Fibonacci sequence, defined by the recursion (i.e., difference equation)

$$f_{n+1} = f_n + f_{n-1},$$

initialized with $f_n = 0$ for $n < 0$ and $f_0 = 1$. **Solution:** Taking the “Z transform” gives the polynomial

$$(z^1 - z^0 - z^{-1})F(z) = 0$$

having the coefficient vector $c = [1, -1, -1]$, resulting in the Fibonacci companion matrix

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

The Matlab/Octave companion matrix routine $companion(C)$ uses an alternative definition, which has the same eigenvalues (See p. 52).

**Example:** Matlab/Octave: A polynomial is represented in Matlab/Octave in terms of its coefficient vector. When the polynomial vector for the poles of a differential equation is

$$c_N = [1, c_{N-1}, c_{N-2}, \ldots c_0]^T,$$

the coefficient $c_N = 1$. This normalization guarantees that the leading term is not zero, and the number of roots ($N$) is equal to the degree of the monic polynomial.

### 3.1.4 Working with polynomials in Matlab/Octave

In Matlab/Octave there are seven related functions you must become familiar with:

1. **roots(A)** Vector $A = [a_N, a_{N-1}, \ldots, a_0] \in \mathbb{C}$ are the complex coefficients of polynomial $P_N(z) = \sum_{n=0}^{N} a_n z^n \in \mathbb{C}$, where $N \in \mathbb{N}$ is the degree of the polynomial. It is convenient to force $a_N = 1$, corresponding to dividing the polynomial by this value, when it is not 1, guaranteeing it cannot be zero. Further $R$ is the vector of roots, $[z_1, z_2, \ldots, z_n] \in \mathbb{C}$ such that $polyval(A, z_k) = 0$.

Example: $\text{roots}([1, -1]) = 1$. 

CH:START

CH:END
2. $y = \text{polyval}(A,x)$: This evaluates the polynomial defined by vector $A \in \mathbb{C}^N$ evaluated at $x \in \mathbb{C}$, returning vector $y(x) \in \mathbb{C}$.

Example: \( \text{polyval}([1 \ -1],1)=0, \text{polyval}([1, \ 1],3)=4 \).

3. $P = \text{poly}(R)$: This is the inverse of \( \text{root}() \), returning a vector of polynomial coefficients $P \in \mathbb{C}^N$ of the corresponding characteristic polynomial, starting from either a vector of roots $R$, or a matrix $A$, for example, defined with the roots on the diagonal. The characteristic polynomial is defined as the determinant of \( |A - \lambda I| = 0 \) having roots $R$.

Example: \( \text{poly}([1])=[1, -1] \), \( \text{poly}([1,2])=[1,-3,2] \).

Due to IEEE-754 scaling issues, this can give strange results that are numerically correct, but only within the limits of IEEE-754 accuracy.

4. $R = \text{polyder}(C)$ This routine takes the $N$ coefficients of polynomial $C$ and returns the $N-1$ coefficients of the derivative of the polynomial. This is useful when working with Newton’s method, where each step is proportional to $P_N(x)/P_N'(x)$.

Example: \( \text{polyder}([1,1]) = [1] \)

5. $[K,R] = \text{residue}(N,D)$: Given the ratio of two polynomials $N,D$, residue$(N,D)$ returns vectors $K,R$ such that

$$\frac{N(s)}{D(s)} = \sum_k K_k \frac{1}{s - s_k}, \quad (3.1.4.18)$$

where $s_k \in \mathbb{C}$ are the roots of the denominator $D$ polynomial and $K \in \mathbb{C}$ is a vector of residues, which characterize the roots of the numerator polynomial $N(s)$. The use of residue$(N,D)$ will be discussed on p. 132. This is one of the most valuable time-saving routine I know of.

Example: \( \text{residue}(2, [1 \ 0 \ -1]) = [1 \ -1] \)

6. $C = \text{conv}(A,B)$: Vector $C \in \mathbb{C}^{N+M-1}$ contains the polynomial coefficients of the convolution of the two vector of coefficients of polynomials $A,B \in \mathbb{C}^N$ and $B \in \mathbb{C}^M$.

Example: \( [1, \ 2, \ 1] = \text{conv}([1, \ 1], [1, \ 1]) \).

7. $[C,R] = \text{deconv}(N,D)$: Vectors $C, N, D \in \mathbb{C}$. This operation uses long division of polynomials to find $C(s) = N(s)/D(s)$ with remainder $R(s)$, where $N = \text{conv}(D,C) + R$, namely

$$C = \frac{N}{D} \quad \text{remainder} \ R. \quad (3.1.4.19)$$

Example: Defining the coefficients of two polynomials as $A = [1, a_1, a_2, a_3]$ and $B = [1, b_1, b + 2]$, one may find the coefficients of the product from $C = \text{conv}(A,B)$, and recover $B$ from $C$ with $B = \text{deconv}(C,A)$.

8. $A = \text{compan}(D)$: Vector $D = [1, d_{N-1}, d_{N-2}, \cdots, d_0]^T \in \mathbb{C}$ contains the coefficients of the monic polynomial

$$D(s) = s^N + \sum_{k=1}^{N} d_{N-k}s^k,$$

and $A$ is the companion matrix of vector $D$ (Eq. 3.1.3.17, p. 66). The eigenvalues of $A$ are the roots of monic polynomial $D(s)$.

Example: \( \text{companion}([1 \ -1 \ -1]) = [1 \ 1; \ 1 \ 0] \)
Exercise: Practice the use of Matlab’s/Octave’s related functions, which manipulate roots, polynomials and residues: root(), conv(), deconv(), poly(), polyval(), polyder(), residue(), compan(). Solution: Try Newton’s method for various polynomials. Use \( N=poly(R) \) to provide the coefficients of a polynomial given the roots \( R \). Then use root() to factor the resulting polynomial. Then use Newton’s method and show that the iteration converges to the nearest root.\(^7\)

### 3.2 Eigenanalysis I: eigenvalues of a matrix

At this point we turn a corner in the discussion, to discuss the important topic of eigenanalysis, which starts with the computation of the eigenvalues and their eigenvectors of a matrix. As briefly discussed on p. 36, eigenvectors are mathematical generalizations of resonances, or modes, naturally found in physical systems.

When you pluck the string of a violin or guitar, or hammer a bell or tuning fork, there are natural resonances that occur. These are the eigenmodes of the instrument. The frequency of each mode is related to the eigenvalue, which in physical terms is the frequency of the mode. But this idea goes way beyond simple acoustical instruments. Wave-guides and atoms are resonant systems. The resonances of the hydrogen atom are called the Lyman series, a special case of the Rydberg series and Rydberg atom (Bohr, 1954; Gallagher, 2005).

Thus this topic runs deep in both physics and, eventually, mathematics. In some real sense, eigenanalysis was what the Pythagoreans were seeking to understand. This relationship is rarely spoken about in the open literature, but once you see it, it can never be forgotten, as it colors your entire view of all aspects of modern physics.

#### 3.2.1 Eigenvalues of a matrix

The method for finding eigenvalues is best described by an example.\(^8\) Starting from the matrix Eq. 2.3.5.15 (p. 53), the eigenvalues are defined by the eigenmatrix equation

\[
\begin{bmatrix}
\frac{1}{2} & 1 \\
2 & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
= \lambda
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}.
\]

The unknowns here are the eigenvalue \( \lambda \) and the eigenvector \( e = [e_1, e_2]^T \). First find \( \lambda \) by subtracting the right from the left

\[
\begin{bmatrix}
\frac{1}{2} & 1 \\
2 & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
- \lambda
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
1 - 2\lambda \\
2 - 2\lambda
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} = 0 \quad (3.2.1.1)
\]

The only way that this equation for \( e \) can have a solution is if the matrix is singular. If it is singular, the determinant of the matrix is zero.

**Example:** The determinant of the above \( 2 \times 2 \) example is the product of the diagonal elements, minus the product of the off-diagonal elements, which results in the quadratic equation

\[
-2\lambda(1 - 2\lambda) - 2 = 4\lambda^2 - 2\lambda - 2 = 0.
\]

Completing the square gives

\[
(\lambda - 1/4)^2 - (1/4)^2 - 1/2 = 0, \quad (3.2.1.2)
\]

thus the roots (i.e., eigenvalues) are \( \lambda_{\pm} = \frac{1 \pm \sqrt{3}}{2} = \{1, -1/2\} \).

\(^7\)A Matlab/Octave program that does this may be downloaded from CH:STARThttps://jontalle.web.engr. illinois.edu/uploads/493/M/NewtonJPD.m CH:END

\(^8\)Appendix B (p. 197) is an introduction to the topics of eigenanalysis for \( 2 \times 2 \) matrices.
**Exercise:** Expand Eq. 3.2.1.2 and recover the quadratic equation. **Solution:**

\[(\lambda - 1/4)^2 - (1/4)^2 = \lambda^2 - \lambda/2 + (1/4)^2 = 0.\]

Thus completing the square is equivalent to the original equation. ■

**Exercise:** Find the eigenvalues of matrix Eq. 2.3.4.7. (Hint: see p. 198) **Solution:** This is a minor variation on the previous example. Briefly

\[\text{det} \begin{bmatrix} 1 - \lambda & N \\ 1 & 1 - \lambda \end{bmatrix} = (1 - \lambda)^2 - N = 0.\]

Thus \(\lambda_{\pm} = 1 \pm \sqrt{N}.\) ■

**Exercise:** Starting from \([x_n, y_n]^T = [1, 0]^T\) compute the first 5 values of \([x_n, y_n]^T.\) **Solution:** Here is a Matlab/Octave code for computing \(x_n:\)

\[
x(1:2,1)=\{1;0\};
A=[1 1;2 0]/2;
for k=1:10; x(k+1)=A*x(:,k); end
\]

which gives the rational \((x_n \in \mathbb{Q})\) sequence: 1, 1/2, 3/4, 5/8, 11/2^4, 21/2^5, 43/2^6, 85/2^7, 171/2^8, 341/2^9, 683/2^{10}, \ldots. ■

**Exercise:** Show that the solution to Eq. 2.3.5.14 is bounded, unlike that of the divergent Fibonacci sequence. Explain what is going on. **Solution:** Because the next value is the mean of the last two, the sequence is bounded. To see this one needs to compute the eigenvalues of the matrix Eq. 2.3.5.15 (p. 53). ■

**Eigenanalysis:** The key to the analysis of such equations is called the eigenanalysis, or modal-analysis method. These are also known as resonant modes in the physics literature. Eigenmodes describe the naturally occurring “ringing” found in physical wave-dominated boundary value problems. Each mode’s “eigenvalue” quantifies the mode’s natural frequency. Complex eigenvalues result in damped modes, which decay in time due to energy losses. Common examples include tuning forks, pendulums, bell, and strings of musical instruments, all of which have a characteristic frequency.

Two modes with the same frequency are said to be degenerate. This is a very special condition, with a high degree of symmetry.

Cauchy’s residue theorem (p. 132) is used to find the time-domain response of each frequency-domain complex eigenmode. Thus eigenanalysis and eigenmodes of physics are the same thing (see p. 120), but are described using different notational methods.\(^9\) The “eigen method” is summarized in Appendix B.3, p. 201.

Taking a simple example of a \(2 \times 2\) matrix \(T \in \mathbb{C},\) we start from the definition of the two eigen-equations

\[Te_{\pm} = \lambda_{\pm}e_{\pm},\] (3.2.1.3)

corresponding to two eigenvalues \(\lambda_{\pm} \in \mathbb{C}\) and two \(2 \times 1\) eigenvectors \(e_{\pm} \in \mathbb{C}.\)

**Example:** Assume that \(T\) is the Fibonacci matrix Eq. 2.3.5.12.

The eigenvalues \(\lambda_{\pm}\) may be merged into a \(2 \times 2\) diagonal eigenvalue matrix

\[
\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix},
\]

\(^9\)During the discovery or creation of quantum mechanics, two alternatives were developed: Schrödinger’s differential equation method and Heisenberg’s matrix method. Eventually it was realized the two were equivalent.
while the two eigenvectors $e_+$ and $e_-$ are merged into a $2 \times 2$ eigenvector matrix

$$E = \begin{bmatrix} e_+ & e_- \end{bmatrix} = \begin{bmatrix} e_1^+ & e_1^- \\ e_2^+ & e_2^- \end{bmatrix}, \quad (3.2.1.4)$$

corresponding to the two eigenvalues. Using matrix notation, this may be compactly written as

$$TE = E\Lambda. \quad (3.2.1.5)$$

Note that while $\lambda_\pm$ and $E_\pm$ commute, $E\Lambda \neq \Lambda E$.

From Eq. 3.2.1.5 we may obtain two very important forms:

1. the diagonalization of $T$
   $$\Lambda = E^{-1}TE, \quad (3.2.1.6)$$
   and

2. the eigen-expansion of $T$
   $$T = E\Lambda E^{-1}, \quad (3.2.1.7)$$
   which is used for computing powers of $T$ (i.e., $T^{100} = E^{-1}\Lambda^{100}E$).

Example: If we take

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

then the eigenvalues are given by $(1 - \lambda_\pm)(1 + \lambda_\pm) = -1$, thus $\lambda_\pm = \pm \sqrt{2}$. This method of eigenanalysis is discussed on p. 49 and Appendix B.2 (p. 199).

Exercise: Show that the geometric series formula holds for $2 \times 2$ matrices. Starting with the $2 \times 2$ identity matrix $I_2$ and $a \in \mathbb{C}$, with $|a| < 1$, show that

$$I_2(I_2 - aI_2)^{-1} = I_2 + aI_2 + a^2I_2^2 + a^3I_2^3 + \cdots.$$

Solution: Since $a^kI_2^k = a^kI_2$, we may multiply both sides by $I_2 - aI_2$ to obtain

$$I_2 = I_2 + aI_2 + a^2I_2^2 + a^3I_2^3 + \cdots - aI_2(aI_2 + a^2I_2^2 + a^3I_2^3 + \cdots)$$

$$= |1 + (a + a^2 + a^3 + \cdots) - (a + a^2 + a^3 + \cdots)|I_2$$

$$= I_2$$

This equality requires that the two series converge, which requires that $|a| < 1$. ■

Exercise: Show that when $T$ is not a square matrix, Eq. 3.2.1.3 can be generalized to

$$T_{m,n} = U_{m,m}\Lambda_{m,n}V_{n,n}^\dagger.$$  

This important generalization of eigenanalysis is called a singular value decomposition (SVD).

Summary: The GCD (Euclidean algorithm), Pell’s equation and the Fibonacci sequence may all be written as compositions of $2 \times 2$ matrices. Thus Pell’s equation and the Fibonacci sequence are special cases of the $2 \times 2$ matrix composition

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n.$$
This is an important and common thread of these early mathematical findings. This $2 \times 2$ linearized matrix recursion plays a special role in physics, mathematics and engineering, because one dimensional system equations are solved using the $2 \times 2$ eigenanalysis method. More than several thousand years of mathematical, by trial and error, set the stage for this breakthrough. But it took even longer to be fully appreciated.

The key idea of the $2 \times 2$ matrix solution, widely used in modern engineering, can be traced back to Brahmagupta’s solution of Pell’s equation, for arbitrary $N$. Brahmagupta’s recursion, identical to that of the Pythagoreans’ $N = 2$ case (Eq. 2.3.4.9), eventually led to the concept of linear algebra, defined by the simultaneous solutions of many linear equations. The recursion by the Pythagoreans (6th BCE) predated the creation of algebra by al-Khwārizmī (9th CE century), as seen in Fig. 1.1.

3.2.2 Taylor series

An analytic function is one that

1. may be expanded in a Taylor series

$$P(x) = \sum_{n=0}^{\infty} c_n(x - x_o)^n,$$  (3.2.2.8)

2. converges for $|x - x_o| < 1$, called the RoC, with coefficients $c_n$.

3. The Taylor series coefficients $c_n$ are defined by taking derivatives of $P(x)$ and evaluating them at the expansion point $x_o$, namely

$$c_n = \frac{1}{n!} \frac{d^n}{dx^n} P(x) \bigg|_{x=x_o}.$$  (3.2.2.9)

4. Where as $P(x)$ may be multivalued, the Taylor series is always single-valued.

Properties: The Taylor formula is a prescription for how to uniquely define the coefficients $c_n$. Without the Taylor series formula, we would have no way of determining $c_n$. The proof of the Taylor formula is transparent, simply by taking successive derivatives of Eq. 3.2.2.8, and then evaluating the result at the expansion point. If $P(x)$ is analytic then this procedure will always work. If $P(x)$ fails to have a derivative of any order, then the function is not analytic and Eq. 3.2.2.8 is not valid for $P(x)$. For example, if $P(x)$ has a pole at $x_o$ then it is not analytic at that point.

The Taylor series representation of $P(x)$ has special applications for solving differential equations because:

1. it is single valued,
2. all its derivatives and integrals are uniquely defined,
3. it may be continued into the complex plane by extending $x \in \mathbb{C}$. Typically this involves expanding the series about a different expansion point.

Analytic continuation: A limitation of the Taylor series expansion is that it is not valid outside of its RoC. One method for avoiding this limitation is to move the expansion point. This is called analytic continuation. However, analytic continuation is a non-trivial operation because it (1) requires manipulating an infinite number of derivatives of $P(x)$, (2) at the new expansion point $x_o$, where (3) $P(x - x_o)$ may not have derivatives, due to possible singularities. (4) Thus one needs to know where the singularities of $P(s)$ are in the complex $s$ plane. Due to these many problems, analytic continuation is rarely used, other than as an important theoretical concept.

Example: The trivial case is the geometric series $P(x) = 1/(1 - x)$ about the expansion point $x = 1$. The function $P(x)$ is defined everywhere, except at the singular point $x = 1$, whereas the geometric series is only valid for $|x| < 1$. 
Exercise: Verify that \( P_n = 0 \)

Solution: To obtain \( c_0 \), for \( n = 0 \), there is no derivative (\( d^0 / dx^0 \) indicates no derivative is taken), so we must simply evaluate \( P(x - x_o) = c_0 + c_1(x - x_o) + \cdots \) at \( x = x_o \), leaving \( c_0 \). To find \( c_1 \) we take one derivative which results in \( P'(x) = c_1 + 2c_2(x - x_o) \cdots \). Evaluating this at \( x = x_o \), leaves \( c_1 \). Each time we take a derivative we reduce the degree of the series by 1, leaving the next constant term.

Exercise: Suppose we truncate the Taylor series expansion to \( N \) terms. What is the name of such functions and what are their properties? Solution: When an infinite series is truncated the resulting function is called an \( N \)th degree polynomial

\[
P_N(x) = \sum_{n=0}^{N} c_n (x - x_o)^n
\]

We can find \( c_0 \) by evaluating \( P_N(x) \) at the expansion point \( x_o \), since from the above formula \( P_N(x_o) = c_0 \). From the Taylor formula \( c_1 = P_N'(x_o) \).

Exercise: How many roots do \( P_N(x) \) and \( P_N'(x) \) have? Solution: According to the fundamental theorem of algebra \( P_N(x) \) has \( N \) roots. \( P_N'(x) \) has \( N - 1 \) roots. The Gauss-Lucas theorem states that the \( N - 1 \) roots of \( P_N'(x) \) lie inside the convex hull (p. 64) of the \( N \) roots of \( P_N(x) \).

Exercise: Would it be possible to find the inverse Gauss-Lucas theorem, that states where the roots of the integral of a polynomial might be? Solution: With each integral there is a new degree of freedom that must be accommodated for. Thus this problem is much more difficult. But since there is only 1 extra degree of freedom, it does not seem impossible. To solve this problem a constraint will be needed.

Role of the Taylor series: The Taylor series plays a key role in the mathematics of differential equations and their solution, as the coefficients of the series uniquely determine the analytic series representation via its derivatives. The implications and limitations of the power series representation are very specific: if the series fails to converge (i.e., outside the RoC), it is essentially meaningless.

A very important fact about the RoC: It is only relevant to the series, not the function being expanded. Typically the function has a pole at the radius of the RoC, where the series fails to converge. However the function being expanded is valid everywhere, other than at the pole. It seems that this point has been poorly explained in many texts. Besides the RoC is the region of divergence RoD, which is the RoC’s complement.

The Taylor series does not need to be infinite to converge to the function it represents, since it obviously works for any polynomial \( P_N(x) \) of degree \( N \). But in the finite case (\( N < \infty \)), the RoC is infinite, and the series is the function \( P_N(x) \) exactly, everywhere. Of course \( P_N(x) \) is called a polynomial of degree \( N \). When \( N \to \infty \), the Taylor series is only valid within the RoC, and it is (typically) the representation of the reciprocal of a polynomial.

These properties are both the curse and the blessing of the analytic function. On the positive side, analytic functions are the ideal starting point for solving differential equations, which is exactly how they were used by Newton and others. Analytic functions are “smooth” since they are infinitely differentiable, with coefficients given by Eq. 3.2.2.9. They are single valued, so there can be no ambiguity in their interpretation.

Two well-known analytic functions are the geometric series \((|x| < 1)\)

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n
\]

(3.2.2.10)
and exponential series ($|x| < \infty$)

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \tag{3.2.2.11}$$

**Exercise:** Relate the Taylor series expressions for Eq. 3.2.2.10 to the following functions:

1. $$F_1(x) = \int_0^x \frac{1}{1 - x} \, dx \quad \tag{3.2.2.12}$$
   **Solution:**
   $$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

2. $$F_2(x) = \frac{d}{dx} \frac{1}{1 - x} \quad \tag{3.2.2.13}$$
   **Solution:**
   $$1 + 2x + 3x^2 + \ldots$$

3. $$F_3(x) = \ln \frac{1}{1 - x} \quad \tag{3.2.2.14}$$
   **Solution:**
   $$1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \ldots$$

4. $$F_4(x) = \frac{d}{dx} \ln \frac{1}{1 - x} \quad \tag{3.2.2.15}$$
   **Solution:**
   $$1 + x + x^2 + x^3 + \ldots$$

**Exercise:** Using symbolic manipulation (Matlab, Octave, Mathematica), expand the given function $F(s)$ in a Taylor series, and find the recurrence relation between the Taylor coefficients $c_n$, $c_{n-1}$, $c_{n-2}$. Assume $a \in \mathbb{C}$ and $T \in \mathbb{R}$.

1. $$F(s) = e^{as}$$
   **Solution:** A Google search on “octave syms taylor” is useful to answer this question. The Matlab/Octave code is to expand this in a taylor series is
   ```octave
   syms s
   taylor(exp(s), s, 0, 'order', 10)
   ```

**Exercise:** Find the coefficients of the following functions by the method of Eq. 3.2.2.9, and give the RoC.

1. $w(x) = \frac{1}{1-x^2}$. **Solution:** From a straightforward expansion we know the coefficients are
   $$\frac{1}{1-x^2} = 1 + x + (x^3)^2 + (x^3)^3 + \ldots = 1 + x + x^2 + -x^3 + \ldots$$
   Working this out using Eq. 3.2.2.9 is more work:
   $$c_0 = \frac{1}{a} w(0) = 1; c_1 = \frac{1}{a} \frac{dw}{dx} \bigg|_{x=0} = -\frac{-1}{(1-x^2)^2} \bigg|_{x=0} = 1; c_2 = \frac{1}{a} \frac{d^2 w}{dx^2} \bigg|_{x=0} = \frac{1}{2} \frac{2}{(1-x^2)^3} \bigg|_{x=0} = -1;$$
   $$c_3 = \frac{1}{a} \frac{d^3 w}{dx^3} \bigg|_{x=0} = \frac{-1}{(1-x^2)^4} \bigg|_{x=0} = -j.$$
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\[ c_0 = 1; c_1 = \frac{d}{dx} \sum (jx)^n \bigg|_0 = j; c_2 = \frac{1}{2!} \frac{d^2}{dx^2} \sum (jx)^n \bigg|_0 = 2(j)^2; \]
\[ c_3 = \frac{1}{3!} \frac{d^3}{dx^3} \sum (jx)^n \bigg|_0 = (j)^3 = -j; \]
\[ \cdots; \]
\[ c_n = \frac{1}{n!} \frac{d^n}{dx^n} \sum (jx)^n \bigg|_0 = j^n. \]

The RoC is \( |x| < 1. \)

2. \( w(x) = e^{xj}. \) Solution: \( c_n = \frac{1}{n!} j^n. \) The RoC is \( |x| < \infty. \) Functions with an \( \infty \) RoC are called entire. Thus \( c_n = jc_{n-1}/n. \)

Brune impedances: A third special family of functions is formed from ratios of two polynomials \( Z(s) = N(s)/D(s), \) commonly used to define an impedance \( Z(s), \) denoted the Brune impedance. Impedance functions are a very special class of complex analytic functions because they must have a non-negative real part

\[ \Re Z(s) = \frac{\Re N(s)}{\Re D(s)} \geq 0, \]

so as to obey conservation of energy. A physical Brune impedance cannot have a negative resistance (the real part); otherwise it would act like a power source, violating conservation of energy. Most impedances used in engineering applications are in the class of Brune impedances, defined by the ratio of two polynomials, of degrees \( m \) and \( n \)

\[ Z_{\text{Brune}}(s) = \frac{P_M(s)}{P_N(s)} = \frac{s^M + a_1 s^{M-1} \cdots a_0}{s^N + b_1 s^{N-1} \cdots b_0}, \quad (3.2.2.16) \]

where \( M = N \pm 1 \) (i.e., \( N = M \pm 1 \)). This fraction of polynomials is sometimes known as a “Padé approximation,” with poles and zeros, defined as the complex roots of the two polynomials. The key property of the Brune impedance is that the real part of the impedance is non-negative (positive or zero) in the right \( s \) half-plane

\[ \Re Z(s) = \Re [R(\sigma, \omega) + jX(\sigma, \omega)] = R(\sigma, \omega) \geq 0 \quad \text{for} \quad \Re s = \sigma \geq 0. \quad (3.2.2.17) \]

Since \( s = \sigma + \omega j, \) the complex frequency (s) right half-plane (RHP) corresponds to \( \Re s = \sigma \geq 0). \) This condition defines the class of positive-real functions, also known as the Brune condition, which is frequently written in the abbreviated form

\[ \Re Z(\Re s \geq 0) \geq 0. \quad (3.2.2.18) \]

As a result of this positive-real (PR) constraint, the subset of Brune impedances (those given by Eq. 3.2.2.16 satisfying Eq. 3.2.2.17) must be complex analytic in the entire right \( s \) half-plane. This is a powerful constraint that places strict limitations on the locations of both the poles and the zeros of every positive-real Brune impedance.

Exercise: Show that \( Z(s) = 1/\sqrt{s} \) is positive-real, but not a Brune impedance. Solution: Since it may not be written as the ratio of two polynomials, it is not in the Brune impedance class. By writing \( Z(s) = |Z(s)|e^{i\phi} \) in polar coordinates, since \(-\pi/4 \leq \phi \leq \pi/4 \) when \( |s| < \pi/2, \) \( Z(s) \) satisfies the Brune condition, thus is positive-real.
Determining the region of convergence (RoC): Determining the RoC for a given analytic function is quite important, and may not always be obvious. In general the RoC is a circle whose radius extends from the expansion point out to the nearest pole. Thus when the expansion point is moved, the RoC changes, since the location of the pole is fixed.

Example: For the geometric series (Eq. 3.2.2.10), the expansion point is \( x_o = 0 \), and the RoC is \(|x| < 1\), since \( 1/(1 - x) \) has a pole at \( x = 1 \). We may move the expansion point by a linear transformation, for example, by replacing \( x \) with \( z + 3 \). Then the series becomes \( 1/((z + 3) - 1) = 1/(z + 2) \), so the RoC becomes 3 because in the \( z \) plane the pole has moved to \(-2\).

Example: A second important example is the function \( 1/(x^2 + 1) \), which has the same RoC as the geometric series, since it may be expressed in terms of its residue expansion (aka, partial fraction expansion)

\[
\frac{1}{x^2 + 1} = \frac{1}{(x + 1j)(x - 1j)} = \frac{1}{2j} \left( \frac{1}{x - 1j} - \frac{1}{x + 1j} \right).
\]

Each term has an RoC of \(|x| < |1j| = 1\). The amplitude of each pole is called the residue, defined in Eq. 4.4.1.4, p. 132. The residue for the pole at \( 1j \) is \( 1/2j \).

Exercise: Verify the above expression is correct, and show that the residues are \( \pm 1/2j \). Solution: Cross-multiply and cancel, leaving 1, as required. The RoC is the coefficient on the pole. Thus the residue of the pole at \( xj \) is \( j/2 \).

Exercise: Find the residue of \( \frac{d}{dz} z^n \). Solution: Taking the derivative gives \( \pi z^{n-1} \) which has a pole at \( z = 0 \). Applying the formula for the residue (Eq. 4.4.1.4, p. 132) we find

\[
e^{-1} = \pi \lim_{z \to 0} zz^{n-1} = \pi \lim_{z \to 0} z^n = 0.
\]

Thus the residue is zero.

3.2.3 Analytic functions

Any function that has a Taylor series expansion is called an analytic function. Within the RoC, the series expansion defines a single-valued function. Polynomials \( 1/(1 - x) \) and \( e^x \) are examples of analytic functions that are real functions of their real argument \( x \).

Every analytic function has a corresponding differential equation, which is determined by the coefficients \( a_k \) of the analytic power series. An example is the exponential, which has the property that it is the eigen-function of the derivative operation

\[
\frac{d}{dx} e^{ax} = ae^{ax},
\]

which may be verified using Eq. 3.2.2.11. This relationship is a common definition of the exponential function, which is very special because it is the eigen-function of the derivative.

The complex analytic power series (i.e., complex analytic functions) may also be integrated, term by term, since

\[
\int^x f(x)dx = \sum \frac{a_k}{k + 1} x^{k+1}.
\]

Newton took full advantage of this property of the analytic function and used the analytic series (Taylor series) to solve analytic problems, especially for working out integrals, allowing him to solve differential equations. To fully understand the theory of differential equations, one must master single-valued analytic functions and their analytic power series.
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**Single- vs. multi-valued functions**

Polynomials and their $\infty$-degree extensions (analytic functions) are single valued: for each $x$ there is a single value for $P_N(x)$. The roles of the domain and codomain may be swapped to obtain an inverse function, with properties that can be very different from those of the function. For example, $y(x) = x^2 + 1$ has the inverse $x = \pm \sqrt{y - 1}$, which is double valued, and complex when $y < 1$. Periodic functions such as $y(x) = \sin(x)$ are even more “exotic” since $x(y) = \arcsin(x) = \sin^{-1}(x)$ has an infinite number of $x(y)$ values for each $y$. This problem was first addressed in Riemann’s 1851 PhD thesis, written while he was working with Gauss.

**Exercise:** Let $y(x) = \sin(x)$. Then $dy/dx = \cos(x)$. Show that $dx/dy = \pm 1/\sqrt{1-y^2}$. **Solution:** Since $\sin^2 x + \cos^2 x = 1$, it follows that $y^2(x) + (dy/dx)^2 = 1$. Thus $dy/dx = \pm 1/\sqrt{1-y^2}$. Taking the reciprocal gives the result result.

To fully understand, Google implicit function theorem (D’Angelo, 2017, p. 104).

**Exercise:** Evaluate the integral

$$I(y) = \int^y \frac{dy}{\sqrt{1-y^2}}.$$

**Solution:** From the previous exercise we know that

$$x(y) = \int^x dx = \int^y \frac{dy}{\sqrt{1-y^2}}.$$

But since $y(x) = \sin(x)$ it follows that $x(y) = \sin^{-1}y = \arcsin(y)$.

**Exercise:** Find the Taylor series coefficients of $y = \sin(x)$ and $x = \sin^{-1}(y)$. Hint: Use symbolic Octave. Note $\sin^{-1}(y) = \arcsin(y)$.

**Solution:**

```matlab
syms s; taylor(sin(s), ’order’, 10);
```

$$\sin(s) = s - s^3/3! + s^5/5! - s^7/7! + \cdots +$$

and

```matlab
syms s; taylor(asin(s), ’order’, 15);
```

$$\arcsin(s) = s + 1/6 s^3 + 3/40 s^5 + 5/112 s^7 + 35/1152 s^9 + 63/2816 s^{11} + 231/13312 s^{13} + \cdots +$$

$$= s + 1/3 \cdot 2^1 s^3 + 3/5 \cdot 2^3 s^5 + 5/7 \cdot 2^5 s^7 + 7 \cdot 5 / 9 \cdot 2^7 s^9 + 7 \cdot 3^2 / 11 \cdot 2^9 s^{11} + 3 \cdot 7 \cdot 11 / 13 \cdot 2^{10} s^{13} + \cdots +$$

Note that every complex analytic function may be expanded in a Taylor series, within its RoC. It follows that the inverse is also complex analytic, as demonstrated in this case using symbolic algebra.

**Exercise:** What is the necessary condition, that if $dy/dx = F(x)$, then $dx/dy = 1/F(x)$. **Solution:** This will be true when $df(x)/dx = F(x)$ is complex analytic because the FTCC defines the antiderivative. In this case $dy/dx = (dx/dy)^{-1}$ (except at singular points, where its not analytic).

3.2.4 Complex analytic functions

When the argument of an analytic function $F(x)$ is complex, that is, $x \in \mathbb{R}$ is replaced by $s = \sigma + \omega \gamma \in \mathbb{C}$. Recall that $\mathbb{R} \subset \mathbb{C}$. Thus

$$F(s) = \sum_{n=0}^{\infty} c_n (s - s_o)^n, \quad (3.2.4.20)$$
with \( c_n \in \mathbb{C} \), that function is said to be a complex analytic.

An important example is where the exponential becomes complex, since
\[
e^{st} = e^{(\sigma + \omega) t} = e^{\sigma t} e^{\omega t} = e^{\sigma t} \left[ \cos(\omega t) + j \sin(\omega t) \right].
\] (3.2.4.21)

Taking the real part gives
\[
\Re\{e^{st}\} = e^{\sigma t} e^{\omega t} + e^{-\omega t} = e^{\sigma t} \cos(\omega t),
\]
and \( \Im\{e^{st}\} = e^{\sigma t} \sin(\omega t) \). Once the argument is allowed to be complex, it becomes obvious that the exponential and circular functions are fundamentally related. This exposes the family of entire circular functions [i.e., \( e^s \), \( \sin(s) \), \( \cos(s) \), \( \tan(s) \), \( \cosh(s) \), \( \sinh(s) \)] and their inverses [\( \ln(s) \), \( \arcsin(s) \), \( \arccos(s) \), \( \arctan(s) \), \( \cosh^{-1}(s) \), \( \sinh^{-1}(s) \)], first fully elucidated by Euler (c1750) (Stillwell, 2010, p. 315).

Note that because \( \sin(\omega t) \) is periodic, its inverse must be multi-valued. What was needed is some systematic way to account for this multi-valued property. This extension to multi-valued functions is called a branch cut, invented by Riemann in his 1851 PhD Thesis, supervised by Gauss, in the final years of Gauss’s life.

The Taylor series of a complex analytic function: However, there is a fundamental problem: we cannot formally define the Taylor series for the coefficients \( c_k \) until we have defined the derivative with respect to the complex variable \( dF(s)/ds \), with \( s \in \mathbb{C} \). Thus simply substituting \( s \) for \( x \) in an analytic function leaves a major hole in one’s understanding of the complex analytic function.

Impact of complex analytic mathematics on physics: It seems likely, if not obvious, that the success of Newton was his ability to describe physics by the use of mathematics. He was inventing new mathematics at the same time he was explaining new physics. The same might be said for Galileo. It seems likely that Newton was extending the successful techniques and results of Galileo’s work on gravity (Galileo, 1638). Galileo died on Jan 8, 1642, and Newton was born Jan 4, 1643, just short of one year later. Obviously Newton was well aware of Galileo’s great success, and naturally would have been influenced by him (see p. 19).

The application of complex analytic functions to physics was dramatic, as may be seen in the six volumes on physics by Arnold Sommerfeld (1868-1951), and from the productivity of his many (36) students (e.g., Debye, Lenz, Ewald, Pauli, Guillemin, Bethe, Heisenberg, Morse and Seebach, to name a few), notable coworkers (i.e., Leon Brillouin) and others (i.e., John Bardeen), upon whom Sommerfeld had a strong influence. Sommerfeld is famous for training many students who were awarded the Nobel Prize in Physics, yet he never won a Nobel (the prize is not awarded in mathematics). Sommerfeld brought mathematical physics (the merging of physical and experimental principles via mathematics) to a new level with the use of complex integration of analytic functions to solve otherwise difficult problems, thus following the lead of Newton who used real integration of Taylor series to solve differential equations (Brillouin, 1960, Ch. 3 by Sommerfeld, A.).
3.3 Root classification by convolution

Following the exploration of algebraic relationships by Fermat and Descartes, the first theorem was being formulated by d’Alembert. The idea behind this theorem is that every polynomial of degree \( N \) (Eq. 3.1.1.7) has at least one root. This may be written as the product of the root and a second polynomial having constant coefficients, of order \( N \) and the term order \( N - 1 \). By the recursive application of this concept, it is clear that every polynomial of degree \( N \) has \( N \) roots. Today this result is known as the fundamental theorem of algebra:

> Every polynomial equation \( P(z) = 0 \) has a solution in the complex numbers. As Descartes observed, a solution \( z = a \) implies that \( P(z) \) has a factor \( z - a \). The quotient

\[
Q(z) = \frac{P(z)}{z - a} = \frac{P(z)}{a} \left[ 1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \cdots \right]
\]

is then a polynomial of one lower degree. … We can go on to factorize \( P(z) \) into \( n \) linear factors.


The ultimate expression of this theorem is given by Eq. 3.1.1.7 (p. 58), which indirectly states that an \( n \)th degree polynomial has \( n \) roots. We shall use the term degree when speaking of polynomials and the term order when speaking of differential equations. A general rule is that order applies to the time domain and degree to the frequency domain, since the Laplace transform of a differential equation, having constant coefficients, of order \( N \), is a polynomial of degree \( N \) in Laplace frequency \( s \).

Today this theorem is so widely accepted we fail to appreciate it. Certainly about the time you learned the quadratic formula, you were prepared to understand the concept of polynomials having roots. The simple quadratic case may be extended to a higher degree polynomial. The Octave/Matlab command \texttt{roots([1,a_2,a_1,a_0])} provides the roots \([s_1,s_2,s_3]\) of the cubic equation, defined by the coefficient vector \([1,a_2,a_1,a_0]\). The command \texttt{poly([s_1,s_2,s_3])} returns the coefficient vector. I don’t know the largest degree that can be accurately factored by Matlab/Octave, but I’m sure its well over \( N = 10^3 \). Today, finding the roots numerically is a solved problem.

**Factorization versus convolution:** The best way to gain insight into the polynomial factorization problem is through the inverse operation, multiplication of monomials. Given the roots \( x_k \), there is a simple algorithm for computing the coefficients \( a_k \) of \( P_N(x) \) for any \( n \), no matter how large. This method is called convolution. Convolution is said to be a trap-door function since it is easy, while the inverse, factoring (deconvolution), is hard, and analytically intractable for degree \( N \geq 5 \) (Stillwell, 2010, p. 102).

### 3.3.1 Convolution of monomials

As outlined by Eq. 3.1.1.7, a polynomial has two equivalent descriptions, first as a series with coefficients \( a_n \) and second in terms of its roots \( x_r \). The question is: What is the relationship between the coefficients and the roots? The simple answer is that they are related by convolution.

Let us start with the quadratic

\[
(x + a)(x + b) = x^2 + (a + b)x + ab,
\]

where in vector notation \([-a, -b]\) are the roots and \([1, a + b, ab]\) are the coefficients.

To see how the result generalizes, we may work out the coefficients for the cubic \((N = 3)\). Multiplying the following three factors gives

\[
(x - 1)(x - 2)(x - 3) = (x^2 - 3x + 2)(x - 3) = x(x^2 - 3x + 2) - 3(x^2 - 3x + 2) = x^3 - 6x^2 + 11x - 6.
\]

(3.3.1.3)
When the roots are [1, 2, 3] the coefficients of the polynomial are [1, −6, 11, −6]. To verify, substitute the roots into the polynomial, and show that they give zero. For example, \( r_1 = 1 \) is a root since 
\[ P_5(1) = 1 - 6 + 11 - 6 = 0. \]

As the degree increases, the algebra becomes more difficult. Imagine trying to work out the coefficients for \( N = 100 \). What is needed is a simple way of finding the coefficients from the roots. Fortunately, convolution keeps track of the book-keeping, formalizing the procedure, along with Newton’s deconvolution method for finding roots of polynomials (p. 61).

**Convolution of two vectors:** To get the coefficients by convolution, write the roots as two vectors [1, \( a \)] and [1, \( b \)]. To find the coefficients we must convolve the vectors, indicated by \([1, a] \ast [1, b] \), where \( \ast \) denotes convolution. Convolution is a recursive operation. The convolution of \([1, a] \ast [1, b] \) is done as follows: reverse one of the two monomials, padding unused elements with zeros. Next slide one monomial against the other, forming the local scalar-product, and the result placed in the output vector. Outside the range shown, all the elements are zero. In summary,
\[
[1, −1] \ast [1, −2] = [1, −1 − 2, 2] = [1, −3, 2].
\]

In general
\[
[a, b] \ast [c, d] = [ac, bc + ad, bd],
\]

Convolving a third term [1, −3] with [1, −3, 2] gives (Eq. 3.3.1.3)
\[
[1, −3] \ast [1, −3, 2] = [1, −3 − 3, 9 + 2, −6] = [1, −6, 11, −6],
\]
which is identical to the cubic example, found by the algebraic method.

By convolving one monomial factor at a time, the overlap is always two elements, thus it is never necessary to compute more than two multiplies and an add for each output coefficient. This greatly simplifies the operations (i.e., they are easily done in your head). Thus the final result is more likely to be correct. Comparing this to the algebraic method, convolution has the clear advantage.

**Exercise:** What are the three nonlinear equations that one would need to solve to find the roots of a cubic? **Solution:** From our formula for the convolution of three monomials we may find the nonlinear “deconvolution” relations between the roots \([−a, −b, −c] \) and the cubic’s coefficients \([1, \alpha, \beta, \gamma] \)

\[
(x + a) \ast (x + b) \ast (x + c) = (x + c) \ast (x^2 + (a + b)x + ab)
\]
\[
= x \cdot (x^2 + (a + b)x + ab) + c \cdot (x^2 + (a + b)x + ab)
\]
\[
= x^3 + (a + b + c)x^2 + (ab + ac + cb)x + abc
\]
\[
= [1, a + b + c, ab + ac + cb, abc].
\]

\[\text{By working with the negative roots we may avoid an unnecessary and messy alternating sign problem.}\]
It follows that the nonlinear equations must be

\[
\begin{align*}
\alpha &= a + b + c \\
\beta &= ab + ac + bc \\
\gamma &= abc.
\end{align*}
\]

These may be solved by the classic cubic solution, which therefore is a deconvolution problem, also known as long division of polynomials. It follows that the following long-division of polynomials must be true:

\[
\frac{x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc}{x + a} = x^2 + (b + c)x + bc
\]

The product of monomial \( P_1(x) \) with a polynomial \( P_N(x) \) gives \( P_{N+1}(x) \): This statement is another way of stating the fundamental theorem of algebra. Each time we convolve a monomial with a polynomial of degree \( N \), we obtain a polynomial of degree \( N + 1 \). The convolution of two monomials results in a quadratic (degree 2 polynomial). The convolution of three monomials gives a cubic (degree 3). In general, the degree \( k \) of the product of two polynomials of degree \( n, m \) is the sum of the degrees \( (k = n + m) \). For example, if the degrees are each 5 \((n = m = 5)\), then the resulting degree is 10.

While we all know this theorem from high school algebra class, it is important to explicitly identify the fundamental theorem of algebra.

Note that the degree of a polynomial is one less than the length of the vector of coefficients. Since the leading term of the polynomial cannot be zero, else the polynomial would not have degree \( N \), when looking for roots, the coefficient can (and should always) be normalized to 1.

In summary, the product of two polynomials of degree \( m, n \) having \( m \) and \( n \) roots gives a polynomial of degree \( m + n \). This is an analysis process of merging polynomials, by coefficient convolution. Multiplying polynomials is a merging process, into a single polynomial.

Composition of polynomials: Convolution is not the only important operation between two polynomials. Another is composition, which may be defined for two functions \( f(z), g(z) \). Then the composition \( c(z) = f(z) \circ g(z) = f(g(z)) \). As a specific example, suppose \( f(z) = 1 + z + z^2 \) and \( g(z) = e^{2z} \).

With these definitions

\[
f(z) \circ g(z) = 1 + e^{2z} + (e^{2z})^2 = 1 + e^{2z} + e^{4z}.
\]

Note that \( f(z) \circ g(z) \neq g(z) \circ f(z) \).

Exercise: Find \( g(z) \circ f(z) \). Solution: \( e^{2f(z)} = e^{2(1 + z + z^2)} = e^2 e^{(1 + z + z^2)} = e^3 e^{z^2} \).■

3.3.2 Residue expansions of rational functions

As discussed on p. 67, there are at least seven important Matlab/Octave routines that are closely related: \( \text{conv()} \), \( \text{deconv()} \), \( \text{poly()} \), \( \text{polyder()} \), \( \text{polyval()} \), \( \text{residue()} \), \( \text{root()} \). Several of these are complements of each other, or do a similar operation in a slightly different way. Routines \( \text{conv()} \), \( \text{poly()} \) build polynomials from the roots, while \( \text{root()} \) solves for the roots given the polynomial coefficients. The operation \( \text{residue()} \) converts the ratio of two polynomials and expands it in a partial fraction expansion, with poles and residues.

When lines and planes are defined, the equations are said to be linear in the independent variables. In keeping with this definition of linear, we say that the equations are non-linear when the equations have degree greater than 1 in the independent variables. The term bilinear has a special meaning, in that both the domain and codomain are linearly related by lines (or planes). As an example, impedance is
defined in frequency as the ratio of the voltage over the current, but it frequently has a representation as the ratio of two polynomials $N(s)$ and $D(s)$

$$Z(s) = \frac{N(s)}{D(s)} = sL_o + R_o + \sum_{k=0}^{K} \frac{K_k}{s - s_k}. \quad (3.3.2.4)$$

Here $Z(s)$ is the impedance and $V$ and $I$ are the voltage and current at radian frequency $\omega$.\(^{11}\)

Such an impedance is typically specified as a rational or bilinear function, namely the ratio of two polynomials, $P_N(s) = N(s) = [a_N, a_{n-1}, \ldots, a_0]$ and $P_K(s) = D(s) = [b_K, b_{K-1}, \ldots, b_0]$ of degrees $N, K \in \mathbb{N}$, as functions of complex Laplace frequency $s = \sigma + j\omega$, having simple roots. Most impedances are rational functions since they may be written as $D(s)V = N(s)I$. Since $D(s)$ and $N(s)$ are both polynomials in $s$, rational functions are also called bilinear transformation or in the mathematical literature Möbius transformation, which comes from a corresponding scalar differential equation, of the form

$$\sum_{k=0}^{K} b_k \frac{d^k}{dt^k} i(t) = \sum_{n=0}^{N} a_n \frac{d^n}{dt^n} v(t) \iff I(\omega) = \left[ b_k s^k \right] \sum_{k=0}^{K} b_k s^k = V(\omega) \sum_{n=0}^{N} a_n s^n. \quad (3.3.2.5)$$

This construction is also known as the ABCD method in the engineering literature (Eq. 3.5.0.1, p. 92). This equation, as well as 3.3.2.4, follows from the Laplace transform (see p. 103) of the differential equation (on left), by forming the impedance $Z(s) = V/I = A(s)/B(s)$. This form of the differential equation follows from Kirchhoff’s voltage and current laws (KCL, KVL) or from Newton’s laws (for the case of mechanics).

**The physical properties of an impedance:** Based on d’Alembert’s observation that the solution to the wave equation is the sum of forward and backward traveling waves, the impedance may be rewritten in terms of forward and backward traveling waves (see p. 120)

$$Z(s) = \frac{V}{I} = \frac{V^+ + V^-}{I^+ - I^-} = r_o \frac{1 + \Gamma(s)}{1 - \Gamma(s)}, \quad (3.3.2.6)$$

where $r_o = P^+/I^+$ is called the characteristic impedance of the transmission line (e.g., wire) connected to the load impedance $Z(s)$, and $\Gamma(s) = V^-/V^+ = I^-/I^+$ is the reflection coefficient corresponding to $Z(s)$. Any impedance of this type is called a Brune impedance due to its special properties (Brune, 1931a; Van Valkenburg, 1964a). Like $Z(s)$, $\Gamma(s)$ is causal and complex analytic. The impedance and the reflectance function $\Gamma(s)$ must both be complex analytic, since they are related to the bilinear transformation, which assures the mutual complex analytic properties.

Due to the bilinear transformation, the physical properties of $Z(s)$ and $\Gamma(s)$ are very different. Specifically, the real part of the load impedance will be non-negative ($\Re\{Z(\omega_j)\} \geq 0$), if and only if $|\Gamma(s)| \leq 1$. In the time domain, the impedance $z(t) \leftrightarrow Z(s)$ must have a value of $r_o$ at $t = 0$. Correspondingly, the time domain reflectance $\gamma(t) \leftrightarrow \Gamma(s)$ must be zero at $t = 0$.

This is the basis of conservation of energy, which may be traced back to the properties of the reflectance $\Gamma(s)$.

**Exercise:** Show that if $\Re\{Z(s)\} \geq 0$ then $|\Gamma(s)| \leq 1$. **Solution:** Taking the real part of Eq. 3.3.2.6, which must be $\geq 0$, we find

$$\Re\{Z(s)\} = \frac{r_o}{2} \left[ \frac{1 + \Gamma(s)}{1 - \Gamma(s)} + \frac{1 + \Gamma^*(s)}{1 - \Gamma^*(s)} \right] = r_o \frac{1 - |\Gamma(s)|^2}{2 |1 + \Gamma(s)|^2} \geq 0.$$  

Thus $|\Gamma| \leq 1$. ■

\(^{11}\)Note that the relationship between the impedance and the residues $K_k$ is a linear one, best solved by setting up a linear system of equations in the unknown residues.
3.4 Introduction to Analytic Geometry

Analytic geometry came about with the merging of Euclid’s geometry with algebra. The combination of Euclid’s (323 BCE) geometry and al-Khwarizmi’s (830 CE) algebra resulted in a totally new and powerful tool, analytic geometry, independently worked out by Descartes and Fermat (Stillwell, 2010). The addition of matrix algebra during the 18th century enabled analysis in more than three dimensions, which today is one of the most powerful tools used in artificial intelligence, data science and machine learning. The utility and importance of these new tools cannot be overstated. The timeline for this period is provided in Fig. 1.2.

There are many important relationships between Euclidean geometry and 16th century algebra. An attempt at a detailed comparison is summarized in Table ???. Important similarities include vectors, their Pythagorean lengths

\[ c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \]  

(3.4.0.1)

\[ a = x_2 - x_1 \] and \[ b = y_2 - y_1 \], and the angles. Euclid’s geometry had length and angles, but no concept of coordinates, thus of vectors. One of the main innovations of analytic geometry is that one may compute with real, and soon after, complex numbers.

There are several new concepts that come with the development of analytic geometry:

1. Composition of functions: If \( y = f(x) \) and \( z = g(y) \) then the composition of functions \( f \) and \( g \) is denoted \( z(x) = g \circ f(x) = g(f(x)) \).

2. Elimination: Given two functions \( f(x, y) \) and \( g(x, y) \), elimination removes either \( x \) or \( y \). This procedure, known to the Chinese, is called Gaussian elimination.

3. Intersection: While one may speak of the intersection of two lines to define a point, or two planes to define a line. This is a special case of elimination when the functions \( f(x, y) \), \( g(x, y) \) are linear in their arguments. The term intersection is also an important but very different concept in set theory.

4. Vectors: Analytic geometry provides the concept of a vector (see Appendix A), as a line with length and orientation (i.e., direction). Analytic geometry defines vectors in any number of dimensions, as ordered sets of points.

5. Analytic geometry extends the ideas of Euclidean geometry with the introduction of the scalar (dot) product of two vectors \( f \cdot g \), and the vector (cross) product \( f \times g \) (see Fig. 3.4).

What algebra also added to geometry was the ability to compute with complex numbers. For example, the length of a line (Eq. 3.4.0.1) was measured in Geometry with a compass: numbers played no role. Once algebra was available, the line’s Euclidean length could be computed numerically, directly from the coordinates of the two ends, defined by the 3-vector

\[ e = x\hat{x} + y\hat{y} + z\hat{z} = [x, y, z]^T, \]

which represents a point at \((x, y, z) \in \mathbb{R}^3 \subset \mathbb{C}^3\) in three dimensions, having direction, from the origin \((0, 0, 0)\) to \((x, y, z)\). An alternative matrix notation is \(e = [x, y, z]^T\), a column vector of three numbers. These two notations are different ways of representing vector \(e\).

By defining the vector, analytic geometry allows Euclidean geometry to become quantitative, beyond the physical drawing of an object (e.g., a sphere, triangle or line). With analytic geometry we have the Euclidean concept of a vector, a line having a magnitude (length) and direction, but analytic, defined in terms of physical coordinates (i.e., numbers). The difference between two vectors defines a third vector, a concept already present in Euclidean geometry. For the first time, complex numbers were allowed into geometry (but rarely used before Cauchy and Riemann).
Scalar product of two vectors: When using algebra, many concepts, obvious with Euclid’s geometry, may be made precise. There are many examples of how algebra extends Euclidean geometry, the most basic being the scalar product (aka, dot product) between vectors $\mathbf{x}, \kappa \in \mathbb{C}^3$

$$\mathbf{x} \cdot \kappa = (x\hat{x} + y\hat{y} + z\hat{z}) \cdot (\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}), \quad \kappa \in \mathbb{C}$$

Scalar products play an important role in vector algebra and calculus (see Appendix A.3, p. 190).

In matrix notation the scalar product is written as (p. 81)

$$\mathbf{x} \cdot \kappa = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = x\alpha + y\beta + z\gamma. \quad (3.4.0.2)$$

If $\kappa(s) \in \mathbb{C}^3$ is a function of complex function of frequency $s$, then the scalar product is complex function of $s$.

Norm (length) of a vector: The norm of a vector (Appendix A.3, p. 190)

$$||\mathbf{e}|| \equiv +\sqrt{\mathbf{e} \cdot \mathbf{e}} \geq 0$$

is defined as the positive square root of the scalar product of the vector with itself. This is a generalization of the length, in any number of dimensions, forcing the sign of the square-root to be non-negative. The length is a concept of Euclidean geometry, and it must always be positive and real. A complex (or negative) length is not physically meaningful. More generally, the Euclidean length of a line is given as the norm of the difference between two real vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}$

$$||\mathbf{e}_1 - \mathbf{e}_2||^2 = (\mathbf{e}_1 - \mathbf{e}_2) \cdot (\mathbf{e}_1 - \mathbf{e}_2)$$

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \geq 0. \quad (3.4.0.3)$$

From this formula we see that the norm of the difference of two vectors is simply a compact expression for the Euclidean length. A zero-length vector, such as is a point, is the result of the fact that

$$|x - x|^2 = (x - x) \cdot (x - x) = 0.$$
Integral definition of a scalar product: Up to this point, following Euclid, we have only considered a vector to be a set of elements \( \{ x_i \} \in \mathbb{R} \), index over \( n \in \mathbb{N} \), as defining a linear vector space with scalar product \( x \cdot y \), with the scalar product defining the norm or length of the vector \( ||x|| = \sqrt{x \cdot x} \). Given the scalar product, the norm naturally follows.

At this point an obvious question presents itself: Can we extend our definition of vectors to differentiable functions (i.e., \( f(t) \) and \( g(t) \)), indexed over \( t \in \mathbb{R} \), with coefficients labeled by \( t \in \mathbb{R} \), rather than by \( n \in \mathbb{N} \)? Clearly, if the functions are analytic, there is no obvious reason why this should be a problem, since analytic functions may be represented by a convergent series having Taylor coefficients, thus are integrable term by term.

Specifically, under certain conditions, the function \( f(t) \) may be thought of as a vector, defining a normed vector space. This intuitive and somewhat obvious idea is powerful. In this case the scalar product can be defined in terms of the integral

\[
 f(t) \cdot g(t) = \int f(t)g(t)dt = ||f(t)|| ||g(t)|| \cos \theta.
\]

This definition of the vector scalar product allows for a significant but straightforward generalization of our vector space, which will turn out to be both useful and an important extension of the concept of a normed vector space. In this space we can define the derivative of a norm with respect to \( t \), which is not possible for the case of the discrete case, indexed over \( n \). The distinction introduces the concept of analytic continuity in the index \( t \), which does not exist for the discrete index \( n \in \mathbb{N} \).

CH:START

Pythagorean theorem and the Schwarz inequality: Regarding Fig. 3.4, suppose we compute the difference between vector \( A \in \mathbb{R} \) and \( \alpha B \in \mathbb{R} \) as \( L = ||A - \alpha B|| \in \mathbb{R} \), where \( \alpha \in \mathbb{R} \) is a scalar that modifies the length of \( B \). We seek the value of \( \alpha \), which we denote as \( \alpha^* \), that minimizes the length of \( L \). From simple geometrical considerations, \( L(\alpha) \) will be minimum when the difference vector is perpendicular to \( B \), as shown in the figure by the dashed line from the tip of \( A \perp B \).

To show this algebraically we write out the expression for \( L(\alpha) \) and take the derivative with respect to \( \alpha \), and set it to zero, which gives the formula for \( \alpha^* \). The argument does not change, but the algebra greatly simplifies, if we normalize \( A, B \) to be unit vectors \( a = A/||A|| \) and \( b = B/||B|| \), which each have norm \( 1 \)

\[
 L^2 = (a - \alpha b) \cdot (a - \alpha b) = 1 - 2\alpha a \cdot b + \alpha^2.
\]

Thus the length is shortest \( L = L_s \), as shown in Fig. 3.4) when

\[
 \frac{d}{d\alpha} L^2 = -2a \cdot b + 2\alpha^* = 0.
\]

Solving for \( \alpha^* \in \mathbb{R} \) we find \( \alpha^* = a \cdot b \). Since \( L_s > 0 (a \neq b) \), Eq. 3.4.0.4 becomes

\[
 1 - 2|a \cdot b|^2 + |a \cdot b|^2 = 1 - |a \cdot b|^2 > 0.
\]

In conclusion \( \cos \theta \equiv |a \cdot b| < 1 \). In terms of \( A, B \) this is CH:START \( |A \cdot B| < ||A|| ||B|| \cos \theta \), as shown adjacent to \( B \) in Fig. 3.4. CH:END Thus the scalar product between two vectors is their direction cosine. Furthermore, since this forms a right triangle, the Pythagorean theorem must hold. The triangle inequality says that the lengths of the two sides must be greater than the hypotenuse. Note that \( \Theta \in \mathbb{R} \neq \mathbb{C} \). This derivation is an abbreviated version of a related discussion on p. 89. Equality cannot be obtained because the Fourier space forms an open set which gives rise to Gibbs ringing.
Vector cross ($\times$) and wedge ($\wedge$) product of two vectors: The vector product (aka, cross-product) $A \times B$ and exterior product (aka, wedge-product) $A \wedge B$ are the second and third types of vector products. As shown in Fig. 3.4,

$$C = A \times B = (a_1\hat{x} + a_2\hat{y} + a_3\hat{z}) \times (b_1\hat{x} + b_2\hat{y} + b_3\hat{z}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

is $\perp$ to the plane defined by $A$ and $B$. The cross–product is strictly limited to two input vectors $A$ and $B \in \mathbb{R}^2$, taken from three real dimensions (i.e., $\mathbb{R}^3$).

The exterior (wedge) product generalizes the cross–product, since it may be defined in terms of any two vectors $A, B \in \mathbb{C}^2$ taken from $n$ dimensions ($\mathbb{C}^n$), with output in $\mathbb{C}^1$. Thus the cross–product is composed of three wedge–products.

Example: If we define $A = 3j\hat{x} - 2\hat{y} + 0\hat{z}$ and $B = 1\hat{x} + 1\hat{y} + 0\hat{z}$ then the cross–product is

1. $A \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3j & -2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = (3j + 2)\hat{z}$.

Since $a_1 \in \mathbb{C}$, this example violates the common assumption that $A \in \mathbb{R}^3$.

2. The wedge–product $A \wedge B$ takes two vectors and returns a scalar, which is the magnitude of a vector $\perp$ to the plane defined by the two input vectors (see Fig. 3.4). It is defined as

$$A \wedge B = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 3j & 1 \\ -2 & 1 \end{vmatrix}$$

$$= (3j\hat{x} - 2\hat{y}) \wedge (\hat{x} + \hat{y})$$

$$= 3 \cdot 0 \hat{x} \wedge \hat{x} - 0 \hat{y} \wedge \hat{x} + 3j\hat{x} \wedge \hat{y} - 2 \hat{y} \wedge \hat{y}$$

$$= (3j + 2)||\hat{x} \wedge \hat{y}||$$

$$= (3j + 2)||\hat{z}|| = 3j + 2.$$

This defines a compact and useful algebra (Hestenes, 2003). From the above example we see that the absolute value of the wedge–product $|a \wedge b| = |a \times b|$, namely

$$|(a_2\hat{y} + a_3\hat{z}) \wedge (b_2\hat{y} + b_3\hat{z})| = |a \times b|.$$

The wedge–product is especially useful because it is zero when the two vectors are co-linear, namely $\hat{x} \wedge \hat{x} = 0$ and $\hat{x} \wedge \hat{y} = 1$, where $\hat{x}, \hat{y}$ are unit vectors. Since

$$\hat{a} \cdot \hat{b} = ||\hat{a}|| ||\hat{b}|| \cos \theta \quad \text{and} \quad \hat{a} \wedge \hat{b} = ||\hat{a}|| ||\hat{b}|| \sin \theta,$$

it follows that

$$\hat{a} \cdot \hat{b} + j\hat{a} \wedge \hat{b} = ||\hat{a}|| ||\hat{b}|| e^{j\theta},$$

which may be viewed as the complex scalar product, with the right hand side the polar form.

Scalar triple product: The triple of a third vector $C$ with the vector product $A \times B \in \mathbb{R}$ is

$$C \cdot (A \times B) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \in \mathbb{R}^3,$$

which equals the volume of a parallelepiped.
3.4. INTRODUCTION TO ANALYTIC GEOMETRY

3.4.1 Generalized vector product

As shown in Fig. 3.4, any two vectors \( \mathbf{A}, \mathbf{B} \in \{\hat{x}, \hat{y}\} \), define a plane. There are two types of vector products:\(^{12}\) the 1) scalar product

\[
\mathbf{A} \cdot \mathbf{B} = ||\mathbf{A}|| ||\mathbf{B}|| \cos \theta \in \mathbb{R},
\]

and 2) vector wedge–product

\[
\mathbf{A} \wedge \mathbf{B} = ||\mathbf{A}|| ||\mathbf{B}|| \sin \theta \in \mathbb{R},
\]

each a real scalar.\(^{13}\) As show in the figure, these two products form a right triangle, thus may be naturally merged, defining the complex analytic generalize vector product

\[
\mathbf{A} \Join \mathbf{B} = \mathbf{A} \cdot \mathbf{B} + \mathbf{j}\mathbf{A} \wedge \mathbf{B} = ||\mathbf{A}|| ||\mathbf{B}|| e^{\mathbf{j}\theta} \in \mathbb{C}. \quad (3.4.1.5)
\]

An interesting example based on the Poynting vector comes from Maxwell’s equations

\[
\frac{1}{\mu_0}(\mathbf{E} \cdot \mathbf{B} + \mathbf{j}\mathbf{E} \wedge \mathbf{B}) \quad \text{W/m}^2.
\]

Impact of Analytic Geometry:

The most obvious impact of analytic geometry was its detailed analysis of the conic sections, using algebra rather than drawings via a compass and ruler. An important example is the composition of the line and circle, a venerable construction, presumably going back to before Diophantus (250CE). Once algebra was invented, the composition could be done using formulas. With this analysis came complex numbers.

The first two mathematicians to appreciate this mixture of Euclid’s geometry and the new algebra were Fermat and Descartes; soon Newton contributed to this effort by the addition of physics (e.g., calculations in acoustics, orbits of the planets, and the theory of gravity and light, significant concepts for 1687.

Given these new methods, many new solutions to problems emerged. The complex roots of polynomials continued to appear, without any obvious physical meaning. Complex numbers seem to have been viewed as more of an inconvenience than a problem. Newton’s solution to this dilemma was to simply ignore the “imaginary” cases (Stillwell, 2010, p. 115-119).

3.4.2 Development of Analytic Geometry

Intersection and Gaussian elimination:

The first “algebra” (al-jabr) is credited to al-Khwarizmi (830 CE). Its invention advanced the theory of polynomial equations in one variable, Taylor series, and composition versus intersections of curves. The solution of the quadratic equation had been worked out thousands of years earlier, but with algebra a general solution could be defined. The Chinese had found the way to solve several equations in several unknowns, for example, finding the values of the intersection of two circles. With the invention of algebra by al-Khwarizmi, a powerful tool became available to solve more difficult problems.

Composition and Elimination

In algebra there are two contrasting operations on functions: composition and elimination.

\(^{12}\)https://en.wikipedia.org/wiki/Bivector

\(^{13}\)In some texts the wedge–product is called the vector exterior–product.
Composition: Composition is the merging of functions, by feeding one into the other. If the two functions are \( f, g \) then their composition is indicated by \( f \circ g \), meaning the function \( y = f(x) \) is substituted into the function \( z = g(y) \), giving \( z = g(f(x)) \).

Composition is not limited to linear equations, even though that is where it is most frequently applied. To compose two functions, one must substitute one equation into the other. That requires solving for that substitution variable, which is not always possible in the case of nonlinear equations. However, many tricks are available that may work around this restriction. For example, if one equation is in \( x^2 \) and the other in \( x^3 \) or \( \sqrt{x} \), it may be possible to multiply the first by \( x \) or square the second. The point is that one of the variables must be isolated so that when it is substituted into the other equations, the variable is removed from the mix.

Example: Let \( y = f(x) = x^2 - 2 \) and \( z = g(y) = y + 1 \). Then
\[
g \circ f = g(f(x)) = (x^2 - 2) + 1 = x^2 - 1.
\]
In general composition does not commute (i.e., \( f \circ g \neq g \circ f \)), as is easily demonstrated. Swapping the order of composition for our example gives
\[
f \circ g = f(g(y)) = z^2 - 2 = (y + 1)^2 - 2 = y^2 + 2y - 1.
\]

Intersection: Complementary to composition is intersection (i.e., decomposition). For example, the intersection of two lines is defined as the point where they meet. This is not to be confused with finding roots. A polynomial of degree \( N \) has \( N \) roots, but the points where two polynomials intersect has nothing to do with the roots of the polynomials. The intersection is a function (equation) of lower degree, implemented by Gaussian elimination.

A system of linear equations \( Ax = y \) has many interpretations, and one should not be biased by the notation. As engineers we are trained to view \( x \) as the input and \( y \) as the output, in which case then \( y = Ax \) seems natural, much like the functional relation \( y = f(x) \). But what does the linear relation \( x = Ay \) mean, when \( x \) is the input? The obvious answer is that \( y = A^{-1}x \).

But when working with systems of equations, there are many uses of equations, and we need to become more flexible in our interpretation. For example \( y = A^2x \) has a useful meaning, and in fact we saw this type of relationship when working with Pell’s equation (p. 50) and the Fibonacci sequence (p. 52). As another example consider
\[
\begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix} = \begin{bmatrix}
  a_{1x} & a_{1y} \\
  a_{2x} & a_{2y}
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix},
\]
which is reminiscent of a three-dimensional surface \( z = f(x, y) \). We shall find that such generalizations are much more than a curiosity.

Intersection of two lines Unless they are parallel, two lines meet at a point. In terms of linear algebra this may be written as two linear equations\(^\text{14}\) (on the left), along with the intersection point \([x_1, x_2]^T\), given by the inverse of the \( 2 \times 2 \) set of equations (on the right)
\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix} \quad \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
  d & -b \\
  -c & a
\end{bmatrix} \begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix}.
\]

By substituting the expression for the intersection point \([x_1, x_2]^T\) into the original equation, we see that it satisfies the equations. Thus the equation on the right is the solution to the equation on the left.

\(^\text{14}\)When writing the equation \( Ax = y \) in matrix format, the two equations are \( ax_1 + bx_2 = y_1 \) and \( cx_1 + dx_2 = y_2 \) with unknowns \((x_1, x_2)\), whereas in the original equations \( ay_1 + by_2 = c \) and \( dy_1 + ey_2 = f \), they were \( y, x \). Thus in matrix format, the names are changed. The first time you see this scrambling of variables, it can be confusing.
3.4. INTRODUCTION TO ANALYTIC GEOMETRY

Note the structure of the inverse: 1) The diagonal values \((a, d)\) are swapped, 2) the off-diagonal values \((b, c)\) are negated and 3) the \(2 \times 2\) matrix is divided by the determinant \(\Delta = ad - bc\). If \(\Delta = 0\), there is no solution. When the determinant is zero \((\Delta = 0)\), the slopes of the two lines are equal, thus the lines are parallel. Only if the slopes differ can there be a unique solution.

**Exercise:** Show that the equation on the right is the solution of the equation on the left. **Solution:** By a direct substitution (composition) of the right equation into the left equation

\[
\begin{bmatrix}
a & b \\
c & d \\
\end{bmatrix} \cdot \frac{1}{\Delta} \begin{bmatrix}
d & -b \\
-c & a \\
\end{bmatrix} \begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
ad - bc & -ab + ab \\
-dc & -cb + ad \\
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
\Delta & 0 \\
0 & \Delta \\
\end{bmatrix},
\]

which gives the identity matrix. ■

Algebra will give the solution when geometry cannot. When the two curves fail to intersect on the real plane, the solution still exists, but is complex valued. In such cases, geometry, which only considers the real solutions, fails. For example, when the coefficients \([a, b, c, d]\) are complex, the solution exists, but the determinant can be complex. Thus algebra is much more general than geometry. Geometry fails when the solution has a complex intersection.

3.4.3 Applications of scalar products

Another important example of algebraic expressions in mathematics is Hilbert’s generalization of the Pythagorean theorem (Eq. 1.1.1.1), known as the Schwarz inequality, is shown in Fig. 3.5. What is special about this generalization is that it proves that when the vertex is \(90^\circ\), the Euclidean length of the leg is minimum.

Vectors may be generalized to have \(\infty\) dimensions: \(\vec{U}, \vec{V} = [v_1, v_2, \cdots, v_\infty]\). The Euclidean inner product (i.e., scalar product) between two such vectors generalizes the finite dimensional case

\[
\vec{U} \cdot \vec{V} = \sum_{k=1}^{\infty} u_k v_k.
\]

As with the finite case, the norm \(||\vec{U}|| = \sqrt{\vec{U} \cdot \vec{U}} = \sqrt{\sum u_k^2}|| is the scalar product of the vector with itself, defining the length of the infinite component vector. Obviously there is an issue of convergence, if the norm for the vector is to have a finite length.

It is a somewhat arbitrary requirement that \(a, b, c \in \mathbb{R}\) for the Pythagorean theorem (Eq. 1.1.1.1). This seems natural enough since the sides are lengths. But, what if they are taken from the complex numbers, as for the lossy vector wave equation, or the lengths of vectors in \(\mathbb{C}^n\)? Then the equation generalizes to

\[
c \cdot c = ||c||^2 = \sum_{k=1}^{n} |c_k|^2,
\]

where \(||c||^2 = (c, c)\) is the inner (dot) product of a vector \(c \in \mathbb{C}\) with itself. As before, \(||c|| = \sqrt{||c||^2}\) is the norm of vector \(c\), akin to a length.

**Schwarz inequality** The Schwarz inequality\(^{15}\) says that the magnitude of the inner product of two vectors is less than or equal to the product of their lengths

\[
|U \cdot V| \leq ||U|| ||V||.
\]

\(^{15}\)A simplified derivation is provided on p. 85.
CHAPTER 3. STREAM 2: ALGEBRAIC EQUATIONS

This may be simplified by normalizing the vectors to have unit length \( \vec{U} = U/||U||, \ vec{V} = V/||V|| \), in which case \(-1 < \vec{U} \cdot \vec{V} \leq 1 \). Another simplification is to define the scalar product in terms of the direction cosine

\[
\cos \theta = |\vec{U} \cdot \vec{V}| \leq 1.
\]

A proof of the Schwarz inequality is as follows: From these definitions we may define the minimum difference between the two vectors as the perpendicular from the end of the first to the intersection with the second. As shown in Fig. 3.5, \( \vec{U} \perp \vec{V} \) may be found by minimizing the length of the vector difference:

\[
\min_{\alpha} ||V - \alpha U||^2 = ||V||^2 + 2\alpha V \cdot U + \alpha^2 ||U||^2 > 0
\]

\[
0 = \partial_{\alpha} (V - \alpha U) \cdot (V - \alpha U)
\]

\[
= V \cdot U - \alpha^* ||U||^2
\]

\[
\therefore \alpha^* = V \cdot U/||U||^2.
\]

The Schwarz inequality follows:

\[
I_{\min} = ||V - \alpha^* U||^2 = ||V||^2 - \frac{|U \cdot V|^2}{||U||^2} > 0
\]

\[
0 \leq |U \cdot V| \leq ||U|| \ ||V||.
\]

An important example of such a vector space includes the definition of the Fourier transform, where we may set

\[
U(\omega) = e^{-i\omega_0 t}, \quad V(\omega) = e^{i\omega t}, \quad U \cdot V = \int_\omega e^{i\omega t} e^{-i\omega_0 t} \frac{d\omega}{2\pi} = \delta(\omega - \omega_0).
\]

It seems that the Fourier transform is a result that follows from a minimization, unlike the Laplace transform that follows from a causal system. This explains the important differences between the two, in terms of their properties (unlike the LT, the FT is not complex analytic). Recall that

\[
U \cdot V + jU \wedge V = ||U|| \ ||V|| e^{j\theta}.
\]

We further explore this topic on p. 99.

3.4.4 Gaussian Elimination

The method for finding the intersection of equations is based on the recursive elimination of all the variables but one. This method, known as Gaussian elimination (Appendix 5, p. 194), works across a broad range of cases, but may be defined as a systematic algorithm when the equations are linear in the variables (Strang et al., 1993). Rarely do we even attempt to solve problems in several variables of degree greater than 1. But Gaussian eliminations may still work in such cases (Stillwell, 2010, p. 90).
In Appendix A.3.5 (p. 196) the inverse of a $2 \times 2$ linear system of equations is derived. Even for a $2 \times 2$ case, the general solution requires a great deal of algebra. Working out a numeric example of Gaussian elimination is more instructive. For example, suppose we wish to find the intersection of the two equations

$$
\begin{align*}
x - y &= 3 \\
2x + y &= 2.
\end{align*}
$$

This $2 \times 2$ system of equations is so simple that you may immediately visualize the solution: By adding the two equations, $y$ is eliminated, leaving $3x = 5$. But doing it this way takes advantage of the specific example, and we need a method for larger systems of equations. We need a generalized (algorithmic) approach. This general approach is called Gaussian elimination.

Start by writing the equations in matrix form (note this is not of the form $Ax = y$)

$$
\begin{pmatrix}
1 & -1 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
3 \\
2
\end{pmatrix}.
$$

Next, eliminate the lower left term ($2x$) using a scaled version of the upper left term ($x$). Specifically, multiply the first equation by $-2$, add it to the second equation, replacing the second equation with the result. This gives

$$
\begin{pmatrix}
1 & -1 \\
0 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
3 \\
2 - 3 \cdot 2
\end{pmatrix}
=
\begin{pmatrix}
3 \\
-4
\end{pmatrix}.
$$

Note that the top equation did not change. Once the matrix is “upper triangular” (zero below the diagonal) you have the solution. Starting from the bottom equation, $y = -4/3$. Then the upper equation gives $x - (-4/3) = 3$, or $x = 3 - 4/3 = 5/3$.

In principle, Gaussian elimination is easy, but if you make a calculation mistake along the way, it is very difficult to find your error. The method requires a lot of mental labor, with a high probability of making a mistake. Thus you do not want to apply this method every time. For example, suppose the elements are complex numbers, or polynomials in some other variable such as frequency. Once the coefficients become more complicated, the seemingly trivial problem becomes corrosive. There is a much better way, that is easily verified, which puts all the numerics at the end, in a single step.

The above operations may be automated by finding a carefully chosen upper-diagonalize matrix $G$. For example, define the “Gaussian matrix” that zeros the element 2 in the matrix of Eq. 3.4.4.9.

More generally let

$$
G = \begin{pmatrix}
1 & 0 \\
a & 1
\end{pmatrix}
$$

Multiplying Eq. 3.4.4.9 by $G$, we find

$$
\begin{pmatrix}
1 & 0 \\
a & 1
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
1 & -1 \\
2 + \alpha & 1 - \alpha
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
3 \\
3a + 2
\end{pmatrix},
$$

thus we obtain Eq. 3.4.4.10 if we let $a = -2$ (we choose $a$ to force the lower-left to be zero). At this point we can either back-substitute and obtain the solution, as we did above, or find a matrix $L$ that finishes the job, by removing elements above the diagonal. The key is that the determinant of this matrix is 1.

Exercise: Using $G$ and $A$ from the discussion above, show that $det(G) = det(GA) = 3$.

Solution: A common notation is to denote $det(A) = |A|$. The two sides of the identity are

$$
|A| = det \begin{pmatrix}
1 & -1 \\
2 & 1
\end{pmatrix} = 1 + 2 = 3, \quad |GA| = det \begin{pmatrix}
1 & -1 \\
0 & 3
\end{pmatrix} = 3,
$$

and $|G| = 1$. Thus $|GA| = |G||A| = 3$. ☐
Matrix inverse: In Appendix A.3.5, p. 196, the inverse of a general $2 \times 2$ matrix takes three steps: 1) swap the diagonal elements, 2) reverse the signs of the off-diagonal elements and 3) divide by the determinant $\Delta = ad - bc$. Specifically

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$  \hfill (3.4.4.13)

There are very few things that you must memorize, but the inverse of a $2 \times 2$ is one of them. It needs to be in your mental tool-kit, like the completion of squares (p. 59).

While it is difficult to compute the inverse matrix from scratch (Appendix 5), it takes only a few seconds (four dot products) to verify it (steps 1 and 2)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad - bc}{\Delta} \begin{bmatrix} ab - ab + ab & \vdots \\ cd - cd & -bc + ad \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}.$$  \hfill (3.4.4.14)

Thus dividing by the determinant gives the $2 \times 2$ identity matrix. A good strategy (don’t trust your memory) is to write down the inverse as best you recall, and then verify.

Using the $2 \times 2$ matrix inverse on our example (Eq. 3.4.4.9), we find

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1+2} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{5}{3} \begin{bmatrix} 5/3 \\ -4/3 \end{bmatrix}.$$  \hfill (3.4.4.15)

If you use this method, you will rarely (never) make a mistake, and the solution is easily verified. Either you can check the numbers in the inverse, as was done in Eq. 3.4.4.14, or you can substitute the solution back into the original equation.

Augmented matrix: There is one minor notational improvement. Rather than writing the matrix equation as $Ax = y$ we place the $y$ vector next to the elements of $A$, to remove the equal sign, which is cumbersome. In this case we write $GA_{aug}$

$$GA_{aug} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -4 \end{bmatrix}.$$

3.5 Transmission (ABCD) matrix composition method

Matrix composition: Matrix multiplication represents a composition of $2 \times 2$ matrices, because the input to the second matrix is the output of the first (this follows from the definition of composition: $f(x) \circ g(x) = f(g(x))$). Thus the ABCD matrix is also known as the transmission matrix method, or occasionally the chain matrix. The general expression for the transmission matrix $T(s)$ is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.$$  \hfill (3.5.0.1)

The four coefficients $A(s), B(s), C(s), D(s)$ are all complex functions of the Laplace frequency $s = \sigma + j\omega$ (p. 104). Typically they are polynomials in $s$, $C(s) = s^2 + 1$, for example. A sum and parallel combination of inductors, capacitors and resistors always results in an impedance given by the ratio of two polynomials (Brune impedance). Thus such methods are called lumped-element networks. A symbolic eigen analysis of $2 \times 2$ matrices may be found in Appendix B.3 (p. 201).

It is a standard convention to always define the current into the node, but since the input current (on the left) is the same as the output current on the right ($I_2$), hence the need for the negative sign on $I_2$ to match the sign convention of current into every node. When using this construction, all the signs will match.

We have already used $2 \times 2$ matrix composition for: (1) representing complex numbers (p. 31), (2) computing the $gcd(m, n)$ of $m, n \in \mathbb{N}$ (p. 42), (3) computing Pell’s equation (p. 50), and (4) the Fibonacci sequence (p. 52). It would appear that $2 \times 2$ matrices have high utility.
Definitions of $A, B, C, D$: By writing out the equations corresponding to Eq. 3.5.0.1, it is trivial to show that:

\[
A(s) = \frac{V_1}{V_2} \bigg|_{I_2=0}, \quad B(s) = -\frac{V_1}{I_2} \bigg|_{V_2=0}, \quad C(s) = \frac{I_1}{V_2} \bigg|_{I_2=0}, \quad D(s) = -\frac{I_1}{I_2} \bigg|_{V_2=0}. \tag{3.5.0.2}
\]

Each equation has a physical interpretation, and a corresponding name. Functions $A$ and $C$ are said to be blocked, because the output current $I_2$ is zero. Functions $B$ and $D$ are said to be short-circuited, because the output voltage $V_2$ is zero.

For example, in a mechanical system blocked would correspond to an output isometric (no length change) velocity of zero. In mechanics the isometric force is defined as the maximum applied force, conditioned on zero velocity (aka, the blocked force). Thus the short-circuited force (aka $B$) would correspond to zero force, which is nonsense. Thus these engineering-centric terms do not gracefully generalize, so better terminology is needed. Much of this was sorted out by Thévenin c1883 (Van Valkenburg, 1964a; Johnson, 2003) and (Kennelly, 1893).

$A, D$ are called voltage (force) and current (velocity) transfer functions, since they are ratios of voltages and currents, whereas $B, C$ are known as the transfer impedance and transfer admittance. For example, the unloaded (blocked) ($I_2 = 0$) output voltage $V_2 = I_1/C$ corresponds to the isometric force in mechanics. In this way each term expresses an output (port 2) in terms of an input (port 1), for a given load condition.

Example: Figure 3.6 Two examples of networks that may be analyzed using the ABCD transmission method.

Exercise: Explain why $C$ is given as above. Solution: Writing out the lower equation gives $I_1 = CV_2 - DI_2$. Setting $I_2 = 0$, we obtain the equation for $C = I_1/V_2|_{I_2=0}$.

Exercise: Can $C = 0$? Solution: Yes, if $I_2 = 0$ and $I_1 = I_2, C = 0$. For $C \neq 0$ there needs to be a finite “shunt” impedance across $V_1$, so that $I_1 \neq I_2 = 0$.

3.5.1 Thévenin parameters of a source

An important concept in circuit theory is that of the Thévenin parameters, the open-circuit voltage and the short-circuit current, the ratio of which defines the Thévenin impedance (Johnson, 2003). The open-circuit voltage is defined as the voltage $V_2$ when the load current $I_2 = 0$, which was shown in the previous exercise to be $V_2/I_1|_{I_2=0} = 1/C$. 

Thévenin Voltage: From Eq. 3.5.0.1 there are two definitions for the Thévenin voltage $V_{\text{Thév}}$, depending on the source on the left:

$$\frac{V_{\text{Thév}}}{I_1} \bigg|_{I_2=0} = \frac{1}{C} \quad \text{and} \quad \frac{V_{\text{Thév}}}{I_1} \bigg|_{I_2=0} = \frac{1}{A}. \quad (3.5.1.3)$$

A more general expression is necessary when the source impedance is finite (neither voltage or current).

Thévenin impedance The Thévenin impedance is the impedance looking into port 2 with $V_1 = 0$, thus

$$Z_{\text{Thév}} = \frac{V_2}{I_2} \bigg|_{V_1=0}. \quad (3.5.1.4)$$

From the upper equation of Eq. 3.5.0.1, with $V_1 = 0$, we obtain $AV_2 = BI_2$, thus

$$Z_{\text{Thév}} = \frac{B}{A}. \quad (3.5.1.5)$$

### 3.5.2 The impedance matrix

With a bit of algebra, one may find the impedance matrix in terms of $A$, $B$, $C$, $D$ (Van Valkenburg, 1964a, p. 310)

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A & \Delta_T \\ 1 & D \end{bmatrix} \begin{bmatrix} I_1 \\ -I_2 \end{bmatrix}. \quad (3.5.2.6)$$

The determinate of the transmission matrix is $\Delta_T = \pm 1$, and if $C = 0$, the impedance matrix does not exist (See (P6) on p. 107 for a discussion).

Definitions of $z_{11}(s), z_{12}(s), z_{21}(s), z_{22}(s)$: The definitions of the matrix elements are easily read off of the equation, as

$$z_{11} \equiv \frac{V_1}{I_1} \bigg|_{I_2=0}, \quad z_{12} \equiv -\frac{V_1}{I_2} \bigg|_{I_1=0}, \quad z_{21} \equiv \frac{V_2}{I_1} \bigg|_{I_2=0}, \quad z_{22} \equiv -\frac{V_2}{I_2} \bigg|_{I_1=0}. \quad (3.5.2.7)$$

These definitions follow trivially from Eq. 3.5.2.6 and each element has a physical interpretation. For example, the unloaded ($I_2 = 0$, aka blocked or isometric) input impedance is $z_{11}(s) = A(s)/C(s)$ while the unloaded transfer impedance is $z_{22}(s) = 1/C(s)$. For reciprocal systems (P6, p. 107) $z_{12} = z_{21}$ since $\Delta_T = 1$. For anti-reciprocal systems, such as dynamic (aka, magnetic) loudspeakers and microphones (Kim and Allen, 2013)), $\Delta_T = -1$, thus $z_{21} = -z_{12} = 1/C$. Finally $z_{22}$ is the impedance looking into port 2 with port 1 open/blocked ($I_1 = 0$). The problem with these basic definitions is that their physical interpretation is unclear. This problem may be solved by referring to Fig. 3.7 which is easier to understand than that of Eq. 3.5.2.6, as it allows one to quickly visualize the many relationships. Specifically the circuit of Fig. 3.7 is given by the $T(s)$ matrix

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 + zayb & z_c(1 + zayb) + za \\ yb & 1 + ybz_c \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \quad (3.5.2.8)$$

Note it is trivial to invert the $T(s)$ matrix because $\Delta_T = \pm 1$.

From the circuit elements defined in Fig. 3.7 (i.e., $za, z_c, y_b$) one may easily compute the impedance matrix element of Eq. 3.5.2.6 (i.e., $z_{11}, z_{12}, z_{21}, z_{22}$). For example, the impedance matrix element $z_{11}$, in terms of $za$ and $yb$, is easily read off of Fig. 3.7 as the sum of the series and shunt impedances

$$z_{11}(s) \big|_{I_2=0} = za + 1/yb = \frac{A}{C}.$$
3.5. TRANSMISSION (ABCD) MATRIX COMPOSITION METHOD

Figure 3.7: Equivalent circuit for a transmission matrix, which allows one to better visualize the matrix elements in terms of complex impedances \( z_a(s) \), \( z_c(s) \), \( y_b(s) \), as defined this figure.

Given the impedance matrix we may then compute transmission matrix \( T(s) \). Namely, from Eq. 3.5.2.6,

\[
\frac{1}{C(s)} = z_{21}, \quad A(s) = \frac{z_{11}}{C(s)}.
\]

The theory is best modeled using the transmission matrix (Eq. 3.5.0.1), while experimental data are best modeled using the impedance matrix formulation (Eq. 3.5.2.6).

Rayleigh reciprocity: Figure 3.7 is particularly helpful in understanding the Rayleigh reciprocity postulate \( P6 \) (Van Valkenburg, 1964a, §14), CH:START

\[
\frac{V_2}{I_1} \bigg|_{I_2=0} = \frac{V_1}{I_2} \bigg|_{I_1=0}.
\]

This says that the output voltage over the input current is symmetric, which is obvious from Fig. 3.7.

The Thévenin voltage \((V_{Thev})\), impedance \((Z_{Thev})\) and reciprocity are naturally explained in terms of Fig. 3.7. This is the normal case of magnetic circuits, such as loudspeakers, where \( y_b \) is represented by a gyrator (Kim and Allen, 2013), or electron spin in quantum mechanics.

3.5.3 Network power relations

Impedance is a very general concept, closely tied to the definition of power \( P(t) \) (and energy). Power is defined as the product of the effort (force) and the flow (current). As described in Fig. 3.1, these concepts are very general, applying to mechanics, electrical circuits, acoustics, thermal circuits, or any other case where conservation of energy applies. Two basic variables are defined, generalized force and generalized flow, also called conjugate variables. The product of the conjugate variables is the power, and the ratio is the impedance. For example, for the case of voltage and current, CH:START

\[
P(t) \equiv v(t)i(t), \quad v(t) = z(t) \ast i(t), \quad i(t) = y(t) \ast v(t)
\]

where \( \ast \) defines convolution (§4.5.4 page 137)

\[
v(t) = z(t) \ast i(t) \equiv \int_{t=0}^{\infty} z(\tau)i(t-\tau)d\tau \leftrightarrow Z(\omega)I(\omega).
\]

Power vs. power series, linear vs. nonlinear Another place where equations of second degree appear in physical applications is in energy and power calculations. The electrical power is given by the product of the voltage \( v(t) \) and current \( i(t) \) (or in mechanics as the force times the velocity). For example if we define \( P = v(t)i(t) \) to be the power \( P \) [watts], then the total energy [joules] at time \( t \) is (Van Valkenburg, 1964a, §14)

\[
E(t) = \int_0^t v(t)i(t)dt.
\]
From this observe that the power is the rate of change of the total energy

\[ P(t) = \frac{d}{dt} E(t), \]

reminiscent of the fundamental theorem of calculus [Eq. 4.2.0.2, (p. 118)].

### 3.5.4 Ohm’s law and impedance

The ratio of voltage over the current is called the impedance which has units of [ohms]. For example given a resistor of \( R = 10 \) [ohms],

\[ v(t) = R \, i(t). \]

Namely, 1 [amp] flowing through the resistor would give 10 [volts] across it. Merging the linear relation due to Ohm’s law with the definition of power shows that the instantaneous power in a resistor is quadratic in voltage and current

\[ P(t) = \frac{v(t)^2}{R} = i(t)^2 R. \]  
(3.5.4.9)

Note that Ohm’s law is linear in its relation between voltage and current whereas the power and energy are **nonlinear**.

Ohm’s law generalizes in a very important way, allowing the impedance (e.g., resistance) to be a linear complex analytic function of complex frequency \( s = \sigma + \omega \mathbf{1} \) (Kennelly, 1893; Brune, 1931a). Impedance is a fundamental concept in many fields of engineering. For example:\(^{16}\) Newton’s second law \( F = ma \) obeys Ohm’s law, with mechanical impedance \( Z(s) = sm \). Hooke’s law \( F = kx \) for a spring is described by a mechanical impedance \( Z(s) = k/s \). In mechanics a “resistor” is called a dash-pot and its impedance is a positive and real constant.

**Kirchhoff’s laws KCL, KVL:** The laws of electricity and mechanics may be written down using Kirchhoff’s current and voltage laws (KCL, KVL), which lead to linear systems of equations in the currents and voltages (velocities and forces) of the system under study, with complex coefficients having positive-real parts.

Points of major confusion are a number of terms that are misused, and overused, in the fields of mathematics, physics and engineering. Some of the most obviously abused terms are **linear/nonlinear**, **energy**, **power**, **power series**. These have multiple meanings, which can, and are, fundamentally in conflict.

**Transfer functions (transfer matrix):** The only method that seems to work, to sort this out, is to cite the relevant physical application, in specific contexts. The most common standard reference is a physical system that has an input \( x(t) \) and an output \( y(t) \). If the system is linear, then it may be represented by its impulse response \( h(t) \). In such cases the system equation is

\[ y(t) = h(t) \ast x(t) \leftrightarrow Y(\omega) = H(s)|_{\omega=0} X(\omega); \]

namely, the convolution of the input with the impulse response gives the output. From Fourier analysis this relation may be written in the real frequency domain as a product of the Laplace transform of the impulse response, evaluated on the \( \omega \mathbf{1} \) axis and the Fourier transform of the input \( X(\omega) \leftrightarrow x(t) \) and output \( Y(\omega) \leftrightarrow y(t) \).

If the system is nonlinear, then the output is not given by a convolution, and the Fourier and Laplace transforms have no obvious meaning.

\(^{16}\) In acoustics the pressure is a potential, like voltage. The force per unit area is given by \( f = -\nabla p \), thus \( F = -\int \nabla p \, dS \). Velocity is analogous to a current. In terms of the velocity potential, the velocity per unit area is \( v = -\nabla \phi \).
The question that must be addressed is why the power is nonlinear, whereas a power series of $H(s)$ is linear: Both have powers of the underlying variables. This is confusing and rarely, if ever, addressed. The quick answer is that powers of the Laplace frequency $s$ correspond to derivatives, which are linear operations, whereas the product of the voltage $v(t)$ and current $i(t)$ is nonlinear. The important and interesting question will be addressed on p. 106, in terms of the system postulates of physical systems.

Table 3.1: The generalized impedance is defined as the ratio of a force over a flow, a concept that also holds in mechanics and acoustics. In mechanics, the ‘force’ is equal to the mechanical force on an element (e.g. a mass, dash-pot, or spring), and the ‘flow’ is the velocity. In acoustics, the ‘force’ is pressure, and the ‘flow’ is the volume velocity or particle velocity of air molecules.

<table>
<thead>
<tr>
<th>Case</th>
<th>Force</th>
<th>Flow</th>
<th>Impedance</th>
<th>units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrical</td>
<td>voltage (V)</td>
<td>current (I)</td>
<td>$Z = V/I$</td>
<td>[Ω]</td>
</tr>
<tr>
<td>Mechanics</td>
<td>force (F)</td>
<td>velocity (U)</td>
<td>$Z = F/U$</td>
<td>mechanical [Ω]</td>
</tr>
<tr>
<td>Acoustics</td>
<td>pressure (P)</td>
<td>particle velocity (V)</td>
<td>$Z = P/V$</td>
<td>specific [Ω]</td>
</tr>
<tr>
<td>Acoustics</td>
<td>mean pressure (P)</td>
<td>volume velocity (V')</td>
<td>$Z = P/V'$</td>
<td>acoustic [Ω]</td>
</tr>
<tr>
<td>Thermal</td>
<td>temperature (T)</td>
<td>heat-flux (J)</td>
<td>$Z = T/J$</td>
<td>thermal [Ω]</td>
</tr>
</tbody>
</table>

**Ohm’s law:** In general, impedance is defined as the ratio of a force over a flow. For electrical circuits (Table 3.1), the voltage is the ‘force’ and the current is the ‘flow.’ Ohm’s law states that the voltage across and the current through a circuit element are linearly related by the impedance of that element (which is typically a complex function of the complex Laplace frequency $s = \sigma + \omega j$). For resistors, the voltage over the current is called the resistance, and is a constant (e.g. the simplest case, $V/I = R \in \mathbb{R}$). For inductors and capacitors, the impedance depends on the Laplace frequency $s$ (e.g. $V/I = Z(s) \in \mathbb{C}$).

As discussed in Table 3.1, the impedance concept also holds for mechanics and acoustics. In mechanics, the ‘force’ is equal to the mechanical force on an element (e.g. a mass, dash-pot, or spring), and the ‘flow’ is the velocity. In acoustics, the ‘force density’ is pressure, and the ‘flow’ is the volume velocity or particle velocity of air molecules. In this section we shall derive the method of the linear composition of systems, also known as the ABCD transmission matrix method, or in the mathematical literature, the Möbius (bilinear) transformation. Using the method of matrix composition, a linear system of $2 \times 2$ matrices can represent a significant family of networks. By the application of Ohm’s law to the circuit shown in Fig. 3.8, we can model a cascade of such cells, which characterize transmission lines (Campbell, 1903).

Example of the use of the ABCD matrix composition: Figure 3.8 characterizes a network composed of a series inductor (mass) having an impedance $Z_l = sL$, and a shunt capacitor (compliance) having an admittance $Y_c = sC \in \mathbb{C}$. As determined by Ohm’s law, each equation describes a linear relation between the current and the voltage. For the inductive impedance, applying Ohm’s law gives

$$Z_l(s) = (V_1 - V_2)/I_1,$$
where \( Z_l(s) = Ls \in \mathbb{C} \) is the complex impedance of the inductor. For the capacitive impedance, applying Ohm’s law gives
\[
Y_c(s) = (I_1 + I_2)/V_2,
\]
where \( Y_c = sC \in \mathbb{C} \) is the complex admittance of the capacitor.

Each of these linear impedance relations may be written in a \( 2 \times 2 \) matrix format. The series inductor \((C = 0)\) equation gives
\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix},
\]
while the shunt capacitor \((L = 0)\) equation yields
\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}.
\]

When the second matrix equation for the shunt admittance (Eq. 3.5.4.11) is substituted into the series impedance equation (Eq. 3.5.4.10), we find the ABCD matrix composition \( (I_{12} = T_1 \circ T_2) \) for the cell is simply the product of two matrices
\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 + Z_l Y_c & Z_l \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{3.5.4.12}
\]

Note that the determinant of the matrix \( \Delta = AD - BC = 1 \). This is not an accident since the determinant of the two matrices are each 1, thus the determinant of their product is 1. Every cascade of series and shunt elements will always have \( \Delta = 1 \).

For the case of Fig. 3.8, Eq. 3.5.4.12 has \( A(s) = 1 + s^2LC, \quad B(s) = sL, \quad C(s) = sC \) and \( D = 1 \). This equation characterizes the four possible relations of the cell’s input and output voltage and current. For example, the ratio of the output to input voltage, with the output unloaded, is
\[
\begin{align*}
\frac{V_2}{V_1} \bigg|_{I_2 = 0} &= \frac{1}{\Delta} = \frac{1}{1 + Z_l Y_c} = \frac{1}{1 + s^2LC}.
\end{align*}
\]
This is known as the voltage divider relation. To derive the current divider relation, use the lower equation with \( V_2 = 0 \)
\[
\frac{-I_2}{I_1} \bigg|_{V_2 = 0} = 1.
\]

**Exercise:** What happens if the roles of \( Z \) and \( Y \) are reversed? **Solution:**
\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1 + Z_l Y_c & Z_l \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{3.5.4.13}
\]
This simply is the same network reversed in direction. ■

**Exercise:** What happens if the series element is a capacitor and the shunt an inductor? **Solution:**
\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 1/Y_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/Z_l & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1 + 1/Z_l Y_c & 1/Y_c \\ 1/Z_l & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{3.5.4.14}
\]
This circuit is a high-pass filter rather than a low-pass. ■
Properties of the transmission matrix: The transmission matrix is always constructed from the product of elemental matrices of the form

\[
\begin{bmatrix}
  1 & Z(s) \\
  0 & 1
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
  1 & 0 \\
  Y(s) & 1
\end{bmatrix}.
\]

Thus for the case of reciprocal systems (P6, p. 107),

\[\Delta_T = \det \begin{bmatrix}
  A(s) & B(s) \\
  C(s) & D(s)
\end{bmatrix} = 1,\]

since the determinant of the product of each elemental matrix is 1, the determinant their product is 1. An anti-reciprocal system may be synthesized by the use of a gyrator, and for such cases \(\Delta_T = -1\).

The eigenvalue and vector equations for a T matrix are summarized in Appendix B (p. 197) and discussed in Appendix B.3 (p. 201). The basic postulates of network theory also apply to the matrix elements \(A(s), B(s), C(s), D(s)\), which place restrictions on their functional relationships. For example property P1 (p. 106) places limits on the poles and/or zeros of each function since the time response must be causal.

### 3.6 Signals: Fourier transforms

Two fundamental tools in engineering mathematics are the Fourier and the Laplace transforms, which deal with time–frequency analysis (Papoulis, 1962).

**Notation:** The Fourier transform \((\mathcal{F} T)\) takes a time domain signal \(f(t) \in \mathbb{R}\) and transforms it to the frequency domain by taking the scalar product (aka dot product) of \(f(t)\) with the complex time vector \(e^{-j\omega t}\):

\[f(t) \leftrightarrow \mathcal{F}(\omega) = f(t) \cdot e^{-j\omega t},\]

where \(F(\omega)\) and \(e^{-j\omega t} \in \mathbb{C}\) and \(\omega, t \in \mathbb{R}\). The scalar product between two vectors results in a scalar (number), as discussed in Appendix A.3 (p. 190).

**Definition of the Fourier transform:** The forward transform takes \(f(t)\) to \(F(\omega)\) while the inverse transform takes \(F(\omega)\) to \(\tilde{f}(t)\). The tilde symbol indicates that, in general, the recovered inverse transform signal can be slightly different from \(f(t)\), with examples in Table 3.2.

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad \tilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \quad (3.6.0.1)
\]

It is accepted in the engineering and physics literature to use the case of the variable to indicate the type of argument. A time domain function is \(f(t)\), where \(t\) has units of seconds [s], is in lower case. Its Fourier transform is in upper case \(F(\omega)\) and is a function of frequency, having units of either hertz [Hz] or radians \([2\pi \text{ Hz}]\). This case convention helps the reader parse the variable under consideration. This is a helpful notation, but not in agreement with the notation used in mathematics, where units are rarely cited.

**Types of Fourier Transforms:** As summarized in Fig. 3.3 each \(\mathcal{F} T\) type is determined by its time and frequency symmetries. A time function \(f(t)\) may be either continuous in time, with \(-\infty < t < \infty\), discrete in time \(f_n = f(t_n)\), with \(t_k = kT_n\), where \(T_n\) is called the Nyquist sample period, or periodic in time \(f((t))_{T_p} = f(t + kT_p)\), where \(T_p\) is called the period. Here \(k, n \in \mathbb{Z}\) and \(T_n, T_p \in \mathbb{R}\).
Table 3.2: Basic (Level I) Fourier transforms. Note \( a > 0 \in \mathbb{R} \) has units \( \text{[rad/s]} \). To flag this necessary condition, we use \( |a| \) to assure this condition will be met. The other constant \( T_o \in \mathbb{R} \text{[s]} \) has no restrictions, other than being real. Complex constants may not appear as the argument to a delta function, since complex numbers do not have the order property.

<table>
<thead>
<tr>
<th>( f(t) \leftrightarrow F(\omega) )</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{\delta}(t) \leftrightarrow 1(\omega) \equiv 1 \forall \omega )</td>
<td>Dirac</td>
</tr>
<tr>
<td>( 1(t) \equiv 1 \forall t \leftrightarrow 2\pi \tilde{\delta}(\omega) )</td>
<td>Dirac</td>
</tr>
<tr>
<td>( \text{sgn}(t) = \frac{t}{</td>
<td>t</td>
</tr>
<tr>
<td>( \tilde{\delta}(t - T_o) \leftrightarrow e^{-j\omega T_o} )</td>
<td>delay</td>
</tr>
<tr>
<td>( \tilde{\delta}(t - T_o) \ast f(t) \leftrightarrow F(\omega)e^{-j\omega T_o} )</td>
<td>delay</td>
</tr>
<tr>
<td>( \tilde{u}(t)e^{-</td>
<td>a</td>
</tr>
<tr>
<td>( \text{rec}(t) = \frac{1}{T_o} \left[ \tilde{u}(t) - \tilde{u}(t - T_o) \right] \leftrightarrow \frac{1}{T_o} \left( 1 - e^{-j\omega T_o} \right) )</td>
<td>pulse</td>
</tr>
<tr>
<td>( \tilde{u}(t) \ast \tilde{u}(t) \leftrightarrow \tilde{\delta}^2(\omega) )</td>
<td>Not defined</td>
</tr>
</tbody>
</table>

Table 3.3: Abbreviations: \( \mathcal{F} \mathcal{T} \): Fourier Transform; FS: Fourier Series; DTFT: Discrete time Fourier transform; DFT: Discrete Fourier transform (the FFT is a “fast” DFT):

<table>
<thead>
<tr>
<th>FREQUENCY \ TIME</th>
<th>continuous ( \omega )</th>
<th>discrete ( \omega_k )</th>
<th>periodic ((\omega)) (\Omega_o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{F} \mathcal{T} )</td>
<td>–</td>
<td>DFT (FFT)</td>
<td>FS</td>
</tr>
<tr>
<td>( \text{DTFT} )</td>
<td>–</td>
<td>DTFT</td>
<td>DFT (FFT)</td>
</tr>
</tbody>
</table>

A general rule is that if a function is discrete in one domain (time or frequency) it is periodic in the other domain (frequency or time). For example, the discrete time function \( f_n \) must have a periodic frequency response, namely \( f_n \leftrightarrow F((\omega))_{T_p} \). This is the case of discrete-time Fourier transform (DTFT). Alternatively when the time function is periodic, the frequencies must be discrete, namely \( f((t))_{T_p} \leftrightarrow F(\omega_k) \). This is the case of the Fourier series (FS). When both time and frequencies are discrete, both the time and frequencies must be periodic. This is the case of the discrete Fourier transform (DFT). These three cases are summarized in Fig. 3.4.

1. Both time \( t \) and frequency \( \omega \) are real.
2. For the forward transform (time to frequency), the sign of the exponential is negative.
3. The limits on the integrals in both the forward and reverse FTs are \([-\infty, \infty]\).
4. When taking the inverse Fourier transform, the scale factor of \( 1/2\pi \) is required to cancel the \( 2\pi \) in the frequency differential \( d\omega = 2\pi df \).
5. The Fourier step function is defined by the use of superposition of 1 and $\text{sgn}(t) = t/|t|$ as

$$\tilde{u}(t) \equiv \frac{1 + \text{sgn}(t)}{2} = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}.$$ 

Taking the FT of a delayed step function

$$\tilde{u}(t - T_o) \leftrightarrow \frac{1}{2} \int_{-\infty}^{\infty} \left[1 - \text{sgn}(t - T_o)\right] e^{-j\omega t} dt = \pi \delta(\omega) + \frac{e^{-j\omega T_o}}{j\omega}.$$ 

Thus the FT of the step function has the term $\pi \delta(\omega)$ due to the 1 in the definition of the Fourier step. This term introduces a serious flaw with the FT of the step function: While it appears to be causal, it is not. Compare this to the convolution $u(t) \ast u(t)$ in Table C.3 on page 208.

6. The convolution $\tilde{u}(t) \ast \tilde{u}(t)$ is not defined because both $1 \ast 1$ and $\tilde{\delta}^2(\omega)$ are not defined.

7. The inverse $\mathcal{F}^{-1}$ has convergence issues whenever there is a discontinuity in the time response. This we indicate with a hat over the reconstructed time response. The error between the target time function and the reconstructed is zero in the root-mean sense, but not point-wise.

Specifically, at the discontinuity point for the Fourier step function ($t = 0$), $\tilde{u}(t) \neq u(t)$, yet $\int |\tilde{u}(t) - u(t)|^2 dt = 0$. At the point of the discontinuity the reconstructed function displays Gibbs ringing (it oscillates around the step, hence does not converge at the jump). The $\mathcal{L}^{-1}$ does not exhibit Gibbs ringing, thus is exact.

8. The FT is not always analytic in $\omega$, as in this example of the step function. The step function cannot be expanded in a Taylor series about $\omega = 0$ because $\tilde{\delta}(\omega)$ is not analytic in $\omega$.

9. The Fourier $\delta$ function is denoted $\tilde{\delta}(t)$, to differentiate it from the Laplace delta function $\delta(t)$. They differ because the step functions differ, due to the convergence problem.

10. One may define

$$\tilde{u}(t) = \int_{-\infty}^{t} \tilde{\delta}(t) dt,$$

and define the somewhat questionable notation

$$\tilde{\delta}(t) = \frac{d}{dt} \tilde{u}(t),$$

since the Fourier step function is not analytic.

11. The rec($t$) function is defined as

$$\text{rec}(t) = \frac{\tilde{u}(t) - \tilde{u}(t - T_o)}{T_o} = \begin{cases} 0 & t < 0 \\ 1/T_o & 0 < t < T_o \\ 0 & t > T_o \end{cases}.$$ 

It follows that $\tilde{\delta}(t) = \lim_{T_o \to 0}$. Like $\tilde{\delta}(t)$, the rec($t$) has unit area.

12. When a function is periodic in one domain ($t, f$) it must be discrete in the other (Fig. 3.4).
Table 3.4: The general rule is that if a function is discrete in one domain (time or frequency) it is periodic in the other.
Abbreviations: FT: Fourier Transform; FS: Fourier Series; DTFT: Discrete time Fourier transform; DFT: Discrete Fourier transform (the FFT is a “fast” DFT);

<table>
<thead>
<tr>
<th>FREQUENCY \ TIME</th>
<th>continuous $t$</th>
<th>discrete $t_k$</th>
<th>periodic ((($t$)$_T$$_o$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>continuous $\omega$</td>
<td>FT</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>discrete $\omega_k$</td>
<td>–</td>
<td>DFT (FFT)</td>
<td>FS</td>
</tr>
<tr>
<td>periodic ((($\omega$)$_\Omega$)</td>
<td>–</td>
<td>DTFT</td>
<td>DFT (FFT)</td>
</tr>
</tbody>
</table>

Table 3.5: Summary of key properties of FTs.

<table>
<thead>
<tr>
<th>FT Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d}{dt} v(t) \leftrightarrow j\omega V(\omega)$</td>
</tr>
<tr>
<td>$f(t) \star g(t) \leftrightarrow F(\omega) G(\omega)$</td>
</tr>
<tr>
<td>$f(t) g(t) \leftrightarrow \frac{1}{2\pi} F(\omega) \star G(\omega)$</td>
</tr>
<tr>
<td>$f(at) \leftrightarrow \frac{1}{a} F \left(\frac{\omega}{a}\right)$</td>
</tr>
</tbody>
</table>

**Periodic signals:** Besides these two basic types of time–frequency transforms, there are several variants that depend on the symmetry in time and frequency. For example, when the time signal is sampled (discrete in time), the frequency response becomes periodic, leading to the *discrete-time Fourier transform* (DTFT). When a time response is periodic, the frequency response is sampled (discrete in frequency), leading to the *Fourier series*. These two symmetries may be simply characterized as *periodic in time $\Rightarrow$ discrete in frequency*, and *periodic in frequency $\Rightarrow$ discrete in time*. When a function is discrete both in time and frequency, it is necessarily periodic in time and frequency, leading to the *discrete Fourier transform* (DFT). The DFT is typically computed with an algorithm called the *fast Fourier transform* (FFT), which can dramatically speed up the calculation when the data is a power of 2 in length.

**Causal-periodic signals:** An special symmetry occurs given functions that are causal and periodic in frequency. The best example is the *z transform*, which are causal (one-sided in time) discrete-time signals. The harmonic series (Eq. 3.2.2.10, p. 73) is the $z$-transform of the discrete-time step function and is thus, due to symmetry, analytic within the RoC in the complex frequency ($z$) domain.

The double brackets on $f((t))_{T_o}$ indicate that $f(t)$ is periodic in $t$ with period $T_o$, i.e., $f(t) = f(t + kT_o)$ for all $k \in \mathbb{N}$. Averaging over one period and dividing by the $T_o$ gives the average value.

**Exercise:** Consider the *Fourier series* scalar (dot) product (Eq. 3.4.0.2, p. 84) between “vectors” $f((t))_{T_o}$ and $e^{-j\omega_k t}$

$$F(\omega_k) = f((t))_{T_o} \cdot e^{-j\omega_k t} = \frac{1}{T_o} \int_0^{T_o} f(t) e^{-j\omega_k t} dt,$$

where $\omega_k = 2\pi/T_o$, $f(t)$ has period $T_o$, i.e., $f(t) = f(t + nT_o) = e^{j\omega_0 t}$ with $n \in \mathbb{N}$, and $\omega_k = k\omega_0$. 

Table 3.6: As summarized in this table of scalar products (dot products), each of the various types of Fourier transforms differ in their support in time and frequency. The transform types are Fourier Transform, Continuous Fourier Series, Discrete time Fourier transform, Discrete time Fourier transform and Fast Fourier transform (fast version of the DFT).

<table>
<thead>
<tr>
<th>Domain</th>
<th>scalar product</th>
<th>form</th>
<th>ON</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}T$</td>
<td>$-\infty &lt; t \in \mathbb{R} &lt; \infty$</td>
<td>$x(t) \cdot y(t)$</td>
<td>$\int_{-\infty}^{\infty} x(t) y(t) dt$</td>
</tr>
<tr>
<td>FS</td>
<td>$0 \leq t \in \mathbb{R} \leq T$</td>
<td>$x((t)) \cdot y((t))$</td>
<td>$\frac{1}{T} \int_{t=0}^{T} x(t) y(t) dt$</td>
</tr>
<tr>
<td>DTFT</td>
<td>$-\infty &lt; t_n &lt; \infty$</td>
<td>$x_n \cdot y_n$</td>
<td>$\sum_{n=-\infty}^{\infty} x_n y_n$</td>
</tr>
<tr>
<td>DFT/FFT</td>
<td>$0 \leq t_n = nT \leq (N-1)T$</td>
<td>$x_n y_n$</td>
<td>$\sum_{n=0}^{N-1} x_n y_n$</td>
</tr>
<tr>
<td>FS</td>
<td>$0 \leq f_k = k \pi/N \leq \frac{(N-1)\pi}{N}$</td>
<td>$X_k Y_k$</td>
<td>$\frac{1}{N} \sum_{n=0}^{N-1} X_k Y_k$</td>
</tr>
</tbody>
</table>

What is the value of the Fourier series scalar product? **Solution:** Evaluating the scalar product we find

$$e^{j2\pi n t} \cdot e^{-j2\pi k t} = \frac{1}{T_o} \int_{0}^{T_o} e^{j2\pi n t} e^{-j2\pi k t} dt = \frac{1}{T_o} \int_{0}^{T_o} e^{2\pi j(n-k)t/t_o} dt = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}.$$ 

The two signals (vectors) are orthogonal. 

**Exercise:** Consider the discrete time $\mathcal{F}T$ (DTFT) as a scalar (dot) product (Eq. 3.4.0.2), between “vectors” $f_n = f(t)|_{t_n}$ and $e^{-j\omega t_n}$ where $t_n = nT_s$ and $T_s = 1/2F_{max}$ is the sample period. **Solution:** The scalar product over $n \in \mathbb{Z}$ is

$$F((\omega))_{2\pi} = f_n \cdot e^{-j\omega t_n} = \sum_{n=-\infty}^\infty f_n e^{-j\omega t_n},$$

where $\omega_0 = 2\pi/T_o$ and $\omega_k = k\omega_0$ is periodic (i.e., $F(\omega) = F(\omega + k\omega_0)$).

### 3.7 Systems: Laplace transforms

The Laplace transform takes real causal signals $f(t)u(t) \in \mathbb{R}$, as a function of real time $t \in \mathbb{R}$, that are strictly zero for negative time ($f(t) = 0$ for $t < 0$), and transforms them into complex analytic functions.
\(F(s) \in \mathbb{C}\) of complex frequency \(s = \sigma + \omega j\). As for the case of Fourier transform, we use the same notation: \(f(t) \leftrightarrow F(s)\).

When a signal is zero for negative time \(f(t < 0) = 0\), it is said to be causal, and the resulting transform \(F(s)\) must be complex analytic over significant regions of the \(s\) plane. For a function of time to be causal, time must be real \((t \in \Re)\), since if it were complex, it would lose the order property (thus it could not be causal). It is helpful to emphasize the causal nature of \(f(t)u(t)\) to force causality, with the Heaviside step function \(u(t)\).

Any restriction on a function (e.g., real, causal, periodic, positive real part, etc.) is called a symmetry property. There are many forms of symmetry (p. 106). The concept of symmetry is very general and widely used in both mathematics and physics, where it is more generally known as group theory. As shown in Fig. 3.7 the two most common \(\mathcal{F}\) symmetries are continuous and discrete time signal. One-sided periodic transforms also exist, such as the system shown in Fig. 3.3 (p. 65).

**Definition of the Laplace transform:** The forward and inverse Laplace transforms (see box equations). Here \(s = \sigma + j\omega \in \mathbb{C} [2\pi\text{Hz}]\) is the complex Laplace frequency in radians and \(t \in \mathbb{R} [s]\) is the time in seconds. Tables of the more common transforms are provided in Appendix C. Table C.3 (p. 208). Properties of more advanced \(\mathcal{LT}\)s are given in Table C.4.

Forward and inverse Laplace transforms:

\[
F(s) = \int_{0}^{\infty} f(t) e^{-st} \, dt \quad \quad \quad f(t) = \frac{1}{2\pi j} \int_{\sigma-o-j\infty}^{\sigma-o+j\infty} F(s) e^{st} \, ds \quad (3.7.0.1)
\]

When dealing with engineering problems it is convenient to separate the signals we use from the systems that process them. We do this by treating signals, such as a music signal, differently from a system, such as a filter. In general signals may start and end at any time. The concept of causality has no mathematical meaning in signal space. Systems, on the other hand, obey very rigid rules (to assure that they remain physical). These physical restrictions are described in terms of the system postulates, which are discussed on p. 106. There is a question as to why postulates are needed, and which ones are the best choices. These questions are discussed in his lectures Feynman (1968, 1970a). The original video is also available online in many places, e.g., via youtube.\(^1\)\(^2\)\(^3\) There may be no definitive answers to these questions, but having a set of postulates is a useful way of thinking about physics.

**Table 3.7:** Laplace transforms \(\mathcal{LT}\)s are complementary to the class of Fourier transforms \(\mathcal{FT}\) due to the fact that the time function must be a causal function. All \(\mathcal{LT}\)s are complex analytic in the complex frequency \(s = \sigma + \omega j\) domain. As an example, a causal function that is continuous but one-sided in time is the step function \(u(t)\), which has the \(\mathcal{LT}\) \(u(t) \leftrightarrow 1/s\). When a function is discrete in time, but one sided, it has a \(\mathcal{Z}\) transform. The discrete-time step function is \(u_n = u[n] \leftrightarrow 1/(1 - z^{-n})\). Abbreviations: \(\mathcal{LT}\): Laplace Transform; \(\mathcal{Z}\)-transform: \(z\) transforms Series;

<table>
<thead>
<tr>
<th>FREQUENCY \ TIME</th>
<th>continuous (t)</th>
<th>discrete (t[k])</th>
<th>causal-periodic ((t)T_o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>continuous (\omega)</td>
<td>(\mathcal{LT})</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>discrete (\omega[k])</td>
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<tr>
<td>periodic (z^j)</td>
<td>–</td>
<td>(z)-Transform</td>
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\(^1\)https://www.youtube.com/watch?v=JXAfEBbaz_4
\(^2\)https://www.youtube.com/watch?v=YaUlqXRFMMY
\(^3\)https://www.youtube.com/watch?v=xnzB_IHGayg
where the path represented by ‘$\subset \infty$’ is a semicircle of infinite radius. For a causal, ‘stable’ (e.g., doesn’t “blow up” in time) signal, all of the poles of $F(s)$ must be inside of the Laplace contour, in the left half $s$-plane.

![Figure 3.9: Three-element mechanical resonant circuit consisting of a spring, mass and dash-pot (e.g., viscous fluid).](image)

**Example:** Hooke’s law for a spring states that the force $f(t)$ is proportional to the displacement $x(t)$, i.e., $f(t) = Kx(t)$. The formula for a dash-pot is $f(t) = Rv(t)$, and Newton’s famous formula for mass is $f(t) = d[Mv(t)]/dt$, which for a constant $M$ is $f(t) = Mdv/dt$.

The equation of motion for the mechanical oscillator in Fig. 3.9 is given by Newton’s second law; the sum of the forces must balance to zero

$$M\frac{d^2x(t)}{dt^2} + Rd\frac{dv(t)}{dt} + Kx(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s). \quad (3.7.0.2)$$

These three constants, the mass $M$, resistance $R$ and stiffness $K \in \mathbb{R} \geq 0$ are real and non-negative. The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$ and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

Newton’s second law (c1650) is the mechanical equivalent of Kirchhoff’s (c1850) voltage law (KVL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., $F = qE$). The current may be thought of as the “flow” of charge.

Equation DE-3.2 may be re-expressed in the frequency domain in terms of an impedance (i.e., Ohms Law), defined as the ratio of the force $F(s)$ to velocity $V(s) = sX(s)$, and the sum of three impedances:

$$Z(s) = \frac{F(s)}{V(s)} = \frac{Ms^2 + Rs + K}{s} = Ms + R + \frac{K}{s}. \quad (3.7.0.3)$$

**Example:** The divergent series

$$e^t u(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \leftrightarrow \frac{1}{s-1}$$

is a valid description of $e^t u(t)$, with an unstable pole at $s = 1$. For values of $|x - x_0| < 1 (x \in \mathbb{R})$, the analytic function $P(x)$ is said to have a region of convergence (RoC). For cases where the argument is complex ($s \in \mathbb{C}$), this is called the radius of convergence (RoC). We might call the region $|s - s_0| > 1$ the region of divergence (RoD), and $|s - s_0| = 0$, the singular circle. Typically the underlying function $P(s)$, defined by the series, has a pole on the singular circle.

There seems to be a conflict with the time response $f(t) = e^\sigma t u(t)$, which has a divergent series (unstable pole). I’m not sure how to explain this conflict, other than to point out that $t \in \mathbb{R}$; thus, the series expansion of the diverging exponential is real-analytic, not complex analytic. First $f(t)$ has a Laplace transform with a pole at $s = 1$, in agreement with its unstable nature. Second, every analytic function must be single valued. This follows from the fact that each term in Eq. 3.2.2.8 is single valued. Third, analytic functions are “smooth” since they may be differentiated an infinite number of times and the series still converges.

The key idea that every impedance must be complex analytic and $\geq 0$ for $\sigma > 0$ was first proposed by Otto Brune in his PhD at MIT, supervised by Ernst A. Guillemin, an MIT electrical engineering professor who played an important role in the development of circuit theory and was a student of Arnold
Sommerfeld. Other MIT advisers were Norbert Wiener and Vannevar Bush. Brune’s primary, but non-MIT, advisor was W. Cauer, who was trained in 19th century German mathematics, perhaps under Sommerfeld (Brune, 1931b).

**Summary:** While the definitions of the FT ($F_T$) and LT ($L_T$) transforms may appear similar, they are not. The key difference is that the time response of the Laplace transform is causal, leading to a complex analytic frequency response. The frequency response of the Fourier transform is complex, but not complex analytic, since the frequency $\omega$ is real. Fourier transforms do not have poles.

The concept of symmetry is helpful in understanding the many different types of time-frequency transforms. Two fundamental types of symmetry are causality (P1) and periodicity.

The Fourier transform $F_T$ characterizes the steady-state response while the Laplace transform $L_T$ characterizes both the transient and steady-state response. Given a causal system force response Eq. 3.7.0.3, $F(s) \leftrightarrow f(t)$ with input velocity $V(s) \leftrightarrow v(t)$, the response is

$$f(t) = z(t) \star v(t) \leftrightarrow Z(\omega) = F(s)\Bigg|_{s = j\omega} V(\omega),$$

which says that the force is given as the convolution of the mechanical impedance $z(t)$ with the input velocity $v(t)$.

### 3.7.1 System postulates

Solutions of differential equations, such as the wave equation, are conveniently described in terms of mathematical properties, which we present here in terms of 11 system postulates (see Appendix E, p. 221 for greater detail):

(P1) **Causality** (non-causal/acausal): Causal systems respond when acted upon. All physical systems obey causality. An example of a causal system is an integrator, which has a response of a step function. Filters are also examples of causal systems. Signals represent acausal responses. They do not have a clear beginning or end, such as the sound of the wind or traffic noise. A causal linear system is typically complex analytic and is naturally represented in the complex $s$ plane via Laplace transforms. A nonlinear system may be causal, but not complex analytic.

(P2) **Linearity** (nonlinear): Linear systems obey superposition. Let two signals $x(t)$ and $y(t)$ are the inputs to a linear system, producing outputs $x'(t)$ and $y'(t)$. When the inputs are presented together as $ax(t) + by(t)$ with weights $a, b \in \mathbb{C}$, the output will be $ax'(t) + by'(t)$. If either $a$ or $b$ is zero, the corresponding signal is removed from the output.

Nonlinear systems mix the two inputs, thereby producing signals not present in the input. For example, if the inputs to a nonlinear system are two sine waves, the output will contain distortion components, having frequencies not present at the input. An example of a nonlinear system is one that multiplies the two inputs. A second is a diode, which rectifies a signal, letting current flow only in one direction. Most physical systems have some degree of nonlinear response, but this is not always desired. Other systems are designed to be nonlinear, such as the diode example.

(P3) **Passive** (active): An active system has a power source, such as a battery, while a passive system has no power source. While you may consider a transistor amplifier to be active, it is only so when connected to a power source. Brune impedances satisfy the positive-real condition (Eq. 3.2.2.18, p. 75).

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20It must be noted that University of Illinois Prof. ‘Mac’ Van VanValkenburg was arguably more influential in circuit theory during the same period. Mac’s books are certainly more accessible, but perhaps less widely cited.
(P4) Real (complex) time response: Typically systems are “real in, real out.” They do not naturally have complex responses (real and imaginary parts). While a Fourier transform takes real inputs and produces complex outputs, this is not an example of a complex time response. P4 is a characterization of the input signal, not its Fourier transform.

(P5) Time-invariant (time varying): For a system to be a time varying system the output must depend on when the input signal starts or stops. If the output, relative to the input, is independent of the starting time, then the system is said to be time-invariant.

(P6) Reciprocal (non- or anti-reciprocal): In many ways this is the most difficult property to understand. It is best characterized by the ABCD matrix (p. 92). If $\Delta_T = 1$, the system is said to be reciprocal. If $\Delta_T = -1$, it is said to be anti-reciprocal. The impedence matrix is reciprocal when $z_{12} = z_{21}$ and anti-reciprocal when $z_{12} = -z_{21}$. Dynamic loudspeakers are anti-reciprocal and must be modeled by a gyrator, which may be thought of as transformers which swaps the force and flow variables. For example, the input impedance of a gyrator terminated by an inductor, is a capacitor. This property is best explained by Fig. 3.7 (p. 95). For an extended discussion on reciprocity, see p. 224.

(P7) Reversibility (non-reversible): If swapping the input and output of a system leaves the system invariant, it is said to be reversible. When $A = D$, the system is reversible. Note the distinction between reversible and reciprocal.

(P8) Space-invariant (space-variant): If a system operates independently as a function of where it physically is in space, then it is space-invariant. When the parameters that characterize the system depend on position, it is space-variant.

(P9) Deterministic (random): Given the wave equation, along with the boundary conditions, the system’s solution may be deterministic, or not, depending on its extent. Consider a radar or sonar wave propagating out into uncharted territory. When the wave hits an object, the reflection can return waves that are not predicted, due to unknown objects. This is an example where the boundary condition is not known in advance.

(P10) Quasistatic ($ka < 1$) Quasistatics follows the Nyquist sampling theorem, for systems that have dimensions that are small compared to the local wavelength (Nyquist, 1924). This assumption fails when the frequency is raised (the wavelength becomes short). Thus this is also known as the long-wavelength approximation. Quasistatics is typically stated as $ka < 1$, where $k = 2\pi/\lambda = \omega/c_o$ and $a$ is the smallest dimension of the system. See p. 178 for a detailed discussion of the role of quasi-statics in acoustic horn wave propagation.

Postulate (P10) is closely related to the Feynman lecture The “underlying unity” of nature where Feynman asks (Feynman, 1970b, Ch. 12-7): “Why do we need to treat the fields as smooth?” His answer is related to the wavelength of the probing signal relative to the dimensions of the object being probed. This raises the fundamental question: Are Maxwell’s equations a band-limited approximation to reality? Today we have no definite answer to this question.

The following quote seems relevant:

The Lorentz force formula and Maxwell’s equations are two distinct physical laws, yet the two methods yield the same results.

Why the two results coincide was not known. In other words, the flux rule consists of two physically different laws in classical theories. Interestingly, this problem was also a motivation behind the development of the theory of relativity by Albert Einstein. In

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1905, Einstein wrote in the opening paragraph of his first paper on relativity theory, “It is known that Maxwell’s electrodynamics – as usually understood at the present time – when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena.” But Einstein’s argument moved away from this problem and formulated special theory of relativity, thus the problem was not solved.

Richard Feynman once described this situation in his famous lecture (The Feynman Lectures on Physics, Vol. II, 1964), “we know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. · · · We have to understand the “rule” as the combined effects of two quite separate phenomena.”

(P11) Periodic ↔ discrete. When a function is discreet in one domain (e.g., time or frequency), it is periodic in the other (frequency or time).

**Summary discussion of the 11 system postulates:** Each postulate has at least two categories. For example, (P1) is either causal, non-causal or acausal while (P2) is either linear or non-linear. (P6) and (P9) only apply to 2-port algebraic networks (those having an input and an output). The others apply to both 2– and 1–port networks (e.g., an impedance is a 1-port). An important example of a 2-port is the anti-reciprocal transmission matrix of a dynamic (EM) loudspeaker, p. 221).

Related forms of these postulates may be found in the network theory literature (Van Valkenburg, 1964a,b; Ramo et al., 1965). Postulates (P1-P6) were introduced by Carlin and Giordano (1964) and (P7-P9) were added by Kim et al. (2016). While linearity (P2), passivity (P3), realness (P4) and time-invariant (P5) are independent, causality (P1) is a consequence of linearity (P2) and passivity (P3) (Carlin and Giordano, 1964, p. 5).

### 3.7.2 Probability

Many things in life follow rules we don’t understand, thus are unpredictable, yet have structure due to some underlying poorly understood physics (e.g., quantum mechanics). Unlike mathematicians, engineers are taught to deal with uncertainty, in terms of random processes, using probability theory. For many this starts out as a large set of boring incomprehensible definitions, but once you begin to understand it, becomes interesting mathematics. It needs to be in your skin. If you don’t have an intuition for it, either keep working on it, or else find another job. Don’t memorize a bunch of formulae, because that won’t work over the long run.

Some view probability as combinatorics. This is wrong. It is much more than that. From my auditory view of speech in noise, probability is about the signal processing of noise and signals (i.e., not combinatorics). The goal of probability is to find correlations in observations, such as the relative frequency of observations in sequential observations of events. Hamming (2004) contains an insightful discussion on probability.

**Definitions:**

1. An **event** is the term to describe an unpredictable outcome (Papoulis and Pillai, 2002).

    **Example:** Measuring the temperature $T(x, t) \in \mathbb{R}$ with $x \in \mathbb{R}^3$ at time $t$ [s] is an event.

    **Example:** Measuring the temperature every hour gives 24 events per day [degrees/hr].

    **Example:** The single toss of a coin, resulting in $\{H, T\}$, is an event.

    **Exercise:** What are the units of a temperature event? **Solution:** While it would seem to be degrees, that unit is not the data that is being observed. Rather it is the relative frequency of
temperatures that is important. For example, how many times was the even between 20 and 21° or between 22 and 27°. That events have no units is not obvious.

2. A trial is \(N\) events.

3. An experiment \(\{M, N\}\) is defined as \(M\) trials of \(N\) events.

4. Number of events: One must always keep track of the number of events so that one can compute the mean (i.e., average) and the uncertainty of an observable outcome.

5. The mean of many trials is the average.

6. A random variable \(X\) is defined as the outcome from an experiment. A random variable rarely has stated units. 

Example: Flipping a coin \(N = 8\) times defines the number of trials.

Exercise: Give the units of coin flips?

\[X \equiv \{H, H, H, T, H, T, T, T\}.

Solution: The random variable \(\{H, T\}\) does not have units, it has random outcomes.

Exercise: How do you give mean to something that doesn’t have units? Solution: One must get creative. Let \(H = 1\) and \(T = -1\) so that the mean can be zero.

Exercise: What is the mean and standard deviation of the coin toss? Solution: To compute the mean (or standard deviation) assign numbers to \(H\) and \(T\). For example let \(H = 1\) and \(T = 0\). Then use the usual formula to compute the numerical values.

7. The Expected value is the mean of \(N\) events.

Exercise: What is the difference between the mean, expected value and the average? Solution: These all mean the same thing. Having several words that mean the same thing is one of the many things that makes probability theory so arbitrary. It is sloppy to have unnecessary terminology.

Exercise: How to assign a numerical mean to random outcomes \(\{H, T\}\). Solution: If we let \(H = 1\) and \(T = 0\) then the mean is

\[\mu = (1 + 1 + 1 + 0 + 1 + 0 + 1 + 0)/8 = 5/8.

8. Note: It is critically important to keep track of the number of events \((N = 8\) in the above example). In some sense \(N\) is more important than the actual measured sequence. It is helpful to think of \(N\) as the independent variable, and \(X\) as the dependent variable. That is, think of \(X(N)\) not \(N(X)\).

Example: To form a trial by flipping a coin \(N = 10\) times. We form an experiment by \(M\) repeated trials \((M = 1000)\).
3.8 Complex analytic mappings (domain-coloring)

One of the most difficult aspects of complex functions of a complex variable is visualizing the mappings from the $z = x + yj$ to $w(z) = u + vj$ planes. For example, $w(z) = \sin(x)$ is trivial

$$\sin(yj) = \frac{e^{-y} - e^{y}}{2j} = -j \sinh(y)$$

is pure imaginary. However, the more general case

$$w(z) = \sin(z) \in \mathbb{C}$$

when $z = x + yj$ is real (i.e., $y = 0$), because $\sin(x)$ is real. Likewise for the case where $x = 0$, is not easily visualized. And when $u(x, y)$ and $v(x, y)$ are less well known functions, $w(z)$ can be even more difficult to visualize. For example, if $w(z) = J_0(z)$ then $u(x, y), v(x, y)$ are the real and imaginary parts of the Bessel function.

A software solution: Fortunately with computer software today, this problem can be solved by adding color to the chart. An Octave/Matlab script\(^\text{22}\) \texttt{viz.m} has been devised to make the charts shown in Fig. 3.10. Such charts are known as domain-coloring. CH:END

Rather than plotting $u(x, y)$ and $v(x, y)$ separately, domain-coloring allows us to display the entire function on one color chart (i.e., colorized plot). For this visualization we see the complex polar form of $w(s) = |w|e^{j\omega}$, rather than the $2 \times 2$ (four-dimensional) Cartesian graph $w(x + yj) = u(x, y) + v(x, y)j$. On the left is the reference condition, the identity mapping ($w = s$), and on the right the origin has been shifted to the right and up by $\sqrt{2}$.

Mathematicians typically use the abstract (i.e., non-physical) notation $w(z)$, where $w = u + vi$ and $z = x + yj$. Engineers typically work in terms of a physical complex impedance $Z(s) = R(s) + jX(s)$, having resistance $R(s)$ and reactance $X(s)$ [ohms], as a function of the complex Laplace radian frequency $s = \sigma + \omega j$ [rad], as used, for example, with the Laplace transform (p. 103).

Fig. 3.10: Left: Domain-colorized map showing the complex mapping from the $s = \sigma + \omega j$ plane to the $w(s) = u(\sigma, \omega) + v(\sigma, \omega)j$ plane. This mapping may be visualized by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase) of the mapping. Right: This shows the $w(s) = s - \sqrt{3}$ plane, shifted to the right and up by $\sqrt{2}/2 = 0.707$. The white and black lines are the iso-real and iso-imaginary contours of the mapping.

Visualizing complex functions: CH:STARTThe mapping from $s = \sigma + \omega j$ to $w(s) = u(\sigma, \omega j) + v(\sigma, \omega j)$ is difficult to visualize because for each point in the domain $s = \sigma + \omega j$, we would like to

\(^{22}\)\url{https://jontalle.web.engr.illinois.edu/uploads/298/viz.zip}

\(^{23}\)Colors vary depending on both the display medium and the eye.
3.8. COMPLEX ANALYTIC MAPPINGS (DOMAIN-COLORING)

represent both the magnitude and phase (or real and imaginary parts) of \( w(s) \). A good way to visualize these mappings is to use color (hue) to represent the phase and intensity (dark to light) to represent the magnitude.

**Example:** Figure 3.11 shows a colorized plot of \( w(z) = \sin(\pi(s-j)/2) \) resulting from the Matlab/Octave command `zviz sin(pi*(s-j)/2)`. The abscissa (horizontal axis) is the real \( \sigma \) axis and the ordinate (vertical axis) is the complex \( j\omega \) axis. The graph is offset along the ordinate axis by \( j1 \), since the argument \( s-j \) causes a shift of the sine function by 1 in the positive imaginary direction. The visible zeros of \( w(s) \) appear as dark regions at \((-2,1), (0,1), (2,1)\). As a function of \( \sigma, w(\sigma+1j) \) oscillates between red (phase is zero degrees), meaning the function is positive and real, and sea-green (phase is 180°), meaning the function is negative and real.

**To use the program use the syntax zviz <function of s>** (for example, type `zviz s.^2`). Note the period between \( s \) and \( -2 \). This will render a domain-coloring (aka colorized) version of the program. A good example for testing is `zviz z-sqrt(j)`, which will show a dark spot (zero) at \((1+1j)/\sqrt{2} = 0.707(1+1j)\).

Along the vertical axis, the displayed function is either a \( \cosh(y) \) or \( \sinh(y) \), depending on the value of \( x \). The intensity becomes lighter as |\( w \)| increases.

**What is being plotted?** The axes are either \( s = \sigma \) and \( \omega \), or \( z = x \) and \( y \). Superimposed on the \( s \) axis is the function \( w(s) = u(\sigma,\omega) + v(\sigma,\omega)j \), represented in polar coordinates by the intensity and color of \( w(s) \). The density (dark vs. light) displays the magnitude |\( w(s) \)| while the color (hue) displays the angle (\( \angle w(s) \)) as a function of \( s \) while the intensity becomes darker as |\( w \)| decreases, and lighter as |\( w(s) \)| increases. The angle \( \angle(w) \) to color mapping is defined by Fig. 3.10. For example red is 0°, green is 90°, purple is -90°, and blue-green is 180°.

![Plot of \( w = \sin(0.5\pi(s-j)) \)](image)

Figure 3.11: Plot of \( w = \sin(0.5\pi(s-j)) \).

**Example:** Additional examples are given in Fig. 3.12 using the notation \( w(s) = u(\sigma,\omega) + v(\sigma,\omega)j \) showing the two complex mappings \( w = e^z \) (left) and its inverse \( s = \ln(w) \). The exponential is relatively easy to understand because \( w(s) = |e^{\phi}e^{\omega j}| = e^\phi \).

The red region is where \( \omega \approx 0 \) in which case \( w \approx e^\sigma \). As \( \sigma \) becomes large and negative, \( w \to 0 \), thus the entire field becomes dark on the left. The field is becoming light on the right where \( w = e^\sigma \to \infty \).

If we let \( \sigma = 0 \) and look along the \( \omega \) axis, we see that the function is changing phase, sea-green (90°) at the top and violet (-90°) at the bottom.

**In the right panel note the zero for \( s(z) = \ln(z) = \ln|z| + \omega j \) at \( z = 1 \). The root of the log function is \( \log(z_r) = 0, w_r = 1, \angle z = \phi = 0, \) since \( \log(1) = 0 \). More generally, the log of \( z = |z|e^{\omega j} \) is \( s(z) = \ln|z| + \phi j \). Thus \( s(w) \) can be zero only when the angle of \( w \) is zero.

**The ln(z) function has a branch cut along the \( \phi(z) = \angle z = 180^0 \) axis. As one crosses over the cut, the phase goes above 180°, and the plane changes to the next sheet of the log function.**

The only sheet with a zero is the principle value, as shown. For all others, the log function is
Figure 3.12: This domain-color map allows one to visualize complex mappings by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase). CH:START The white and black lines are the iso-real and iso-imaginary contours of the complex mapping. CH:END Left: This figure shows the domain-color map for the complex mapping from the \( s = \sigma + \omega j \) plane to the \( w(s) = u + v j = e^{\sigma + \omega j} \) plane, which goes to zero as \( \sigma \to -\infty \), causing the domain-color map to become dark for \( \sigma < -2 \). The white and black lines are always perpendicular because \( e^s \) is complex analytic everywhere. CH:END Right: This shows the principal value of the inverse function \( s(z) = \log(z) \), which has a zero (dark) at \( x = 1 \), since \( \log(1) = 0 \) and the imaginary part is zero. Note the branch cut, where the color is discontinuous, from \( x = [0, -\infty) \). On branches other than the one shown, there are no zeros since the phase \( (\angle s = 2\pi n \in \mathbb{Z}) \) is not zero. \( n \) is called the branch index. See §4.3.3 (p. 126) for a discussion of branch cuts and multi-valued functions. CH:END

either increasing or decreasing monotonically, and there is no zero, as seen for sheet 0 (the one showing in Fig. 3.12).

3.8.1 Riemann sphere: 3\textsuperscript{d} extension of chord and tangent method

Once algebra was formulated by c830 CE, mathematicians were able to expand beyond the limits set by geometry on the real plane, and the verbose descriptions of each problem in prose (Stillwell, 2010, p. 93). The geometry of Euclid’s *Elements* had paved the way, but after 2000 years, the addition of the language of algebra changed everything. The analytic function was a key development, heavily used by both Newton and Euler. Also the investigations of Cauchy made important headway with his work on complex variables. Of special note was integration and differentiation in the complex plane of complex analytic functions, which is the topic of stream 3.

It was Riemann, working with Gauss in the final years of Gauss’s life, who made the breakthrough, with the concept of the extended complex plane. This concept was based on the composition of a line with the sphere, similar to the derivation of Euclid’s formula for Pythagorean triplets (p. 49). While the importance of the extended complex plane was unforeseen, it changed analytic mathematics forever, along with the physics it supported. It unified and thus simplified many important integrals, to the extreme. The basic idea is captured by the fundamental theorem of complex integral calculus (Table 4.1, p. 117).

The idea is outlined in Fig. 3.13. On the left is a circle and a line. The difference between this case and the derivation of the Pythagorean triplets is that the line starts at the north pole, and ends on the real \( x \in \mathbb{R} \) axis at point \( x \). At point \( x' \), the line cuts through the circle. Thus the mapping from \( x \) to \( x' \) takes every point on \( \mathbb{R} \) to a point on the circle. For example, the point \( x = 0 \) maps to the south pole (not indicated). To express \( x' \) in terms of \( x \) one must compose the line and the circle, similar to the

\[ w = \exp(s) \]

\[ \sigma \]

\[ j\omega \]

\[ 2 \]

\[ 1 \]

\[ 0 \]

\[ -1 \]

\[ -2 \]

\[ s = \log(z) \]

\[ x \]

\[ jy \]

\[ 2 \]

\[ 1 \]

\[ 0 \]

\[ -1 \]

\[ -2 \]

\[ \text{Figure 3.12:} \] This domain-color map allows one to visualize complex mappings by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase). CH:START The white and black lines are the iso-real and iso-imaginary contours of the complex mapping. CH:END Left: This figure shows the domain-color map for the complex mapping from the \( s = \sigma + \omega j \) plane to the \( w(s) = u + v j = e^{\sigma + \omega j} \) plane, which goes to zero as \( \sigma \to -\infty \), causing the domain-color map to become dark for \( \sigma < -2 \). The white and black lines are always perpendicular because \( e^s \) is complex analytic everywhere. CH:END Right: This shows the principal value of the inverse function \( s(z) = \log(z) \), which has a zero (dark) at \( x = 1 \), since \( \log(1) = 0 \) and the imaginary part is zero. Note the branch cut, where the color is discontinuous, from \( x = [0, -\infty) \). On branches other than the one shown, there are no zeros since the phase \( (\angle s = 2\pi n \in \mathbb{Z}) \) is not zero. \( n \) is called the branch index. See §4.3.3 (p. 126) for a discussion of branch cuts and multi-valued functions. CH:END

either increasing or decreasing monotonically, and there is no zero, as seen for sheet 0 (the one showing in Fig. 3.12).
3.8. COMPLEX ANALYTIC MAPPINGS (DOMAIN-COLORING)

Figure 3.13: The left panel shows how the real line may be composed with the circle. Each real $x$ value maps to a corresponding point $x'$ on the unit circle. The point $x \to \infty$ maps to the north pole $N$. This simple idea may be extended with the composition of the complex plane with the unit sphere, thus mapping the plane onto the sphere. As with the circle, the point on the complex plane $z \to \infty$ maps onto the north pole $N$. This construction is important because while the plane is open (does not include $z \to \infty$), the sphere is analytic at the north pole. Thus the sphere defines the closed extended plane. Figure adapted from Stillwell (2010, pp. 299-300).

composition used in the derivation of Euclid’s formula (p. 49). The points on the circle, indicated here by $x'$, require a traditional polar coordinate system, having a unit radius and an angle defined between the radius and a vertical line passing through the north pole. When $x \to \infty$ the point $x' \to N$, known as the point at infinity. But this idea goes much further, as shown on the right half of Fig. 3.13.

Here the real tangent line is replaced by a tangent complex plane $z \in \mathbb{C}$, and the complex puncture point $z' \in \mathbb{C}$, in this case on the complex sphere, called the extended complex plane. This is a natural extension of the chord/tangent method on the left, but with significant consequences. The main difference between the complex plane $z$ and the extended complex plane, other than the coordinate system, is what happens at the north pole. The point at $|z| = \infty$ is not defined on the plane, whereas on the sphere, the point at the north pole is simply another point, like every other point on the sphere.

**Open vs. closed sets:** Mathematically the plane is said to be an open set, since the limit $z \to \infty$ is not defined, whereas on the sphere, the point $z'$ is a member of a closed set, since the north pole is defined. The distinction between an open and closed set is important, because the closed set allows the function to be complex analytic at the north pole, which it cannot be on the plane (since the point at infinity is not defined).

The $z$ plane may be replaced with another plane, say the $w = F(z) \in \mathbb{C}$ plane, where $w$ is some function $F$ of $z \in \mathbb{C}$. For the moment we shall limit ourselves to complex analytic functions of $z$, namely $w = F(z) = u(x, y) + v(x, y) \mathbf{i} = \sum_{n=0}^{\infty} c_n z^n$.

In summary, given a point $z = x + y\mathbf{i}$ on the open complex plane, we map it to $w = F(z) \in \mathbb{C}$, the complex $w = u + v\mathbf{i}$ plane, and from there to the closed extended complex plane $w'(z)$. The point of doing this is that it allows us to allow the function $w'(z)$ to be analytic at the north pole, meaning it can have a convergent Taylor series at the point at infinity $z \to \infty$. Since we have not yet defined $dw(z)/dz$, the concept of a complex Taylor series remains undefined.

### 3.8.2 Bilinear transformation

In mathematics the bilinear transformation has special importance because it is linear in its action on both the input and output variables. Since we are engineers we shall stick with the engineering terminology. But if you wish to read about this on the internet, be sure to also search for the mathematical term, Möbius transformation.

When a point on the complex plane $z = x + y\mathbf{j}$ is composed with the bilinear transform $(a, b, c, d \in \mathbb{C})$, the result is $w(z) = u(x, y) + v(x, y)\mathbf{j}$ (this is related to the Möbius transform, p. 31)

$$w = \frac{az + b}{cz + d}. \tag{3.8.2.1}$$

The transformation from $z \to w$ is a cascade of four independent compositions:
1. translation \((w = z + b): a = 1, b \in \mathbb{C}, c = 0, d = 1)\),

2. scaling \((w = |a|z): a \in \mathbb{R}, b = 0, c = 0, d = 1)\),

3. rotation \((w = \frac{a}{|a|}z): a \in \mathbb{C}, b = 0, c = 0, d = |a|)\) and

4. inversion \((w = \frac{1}{z}: a = 0, b = 1, c = 1, d = 0)\).

Each of these transformations is a special case of Eq. 3.8.2.1, with the inversion the most complicated. I highly recommend a video showing the effect of the bilinear (Möbius) transformation on the plane (Arnold, D. and Rogness, J., 2019).25

The bilinear transformation is the most general way to move the expansion point in a complex analytic expansion. For example, starting from the harmonic series, the bilinear transform gives

\[
\frac{1}{1 - w} = \frac{1}{1 - \frac{az + b}{cz + d}} = \frac{cz + d}{(c - a)z + (d - b)} = \frac{1}{1 - \frac{2}{c} \cdot \frac{z + \frac{d}{c}}{\frac{a - b}{c - a}}}
\]

The RoC is transformed from \(|w| < 1\) to \(|(az - b)/(cz - d)| < 1\). An interesting application might be in moving the expansion point until it is on top of the nearest pole, so that the RoC goes to zero. This might be a useful way of finding a pole, for example.

When the extended plane (Riemann sphere) is analytic at \(z = \infty\), one may take the derivatives there, defining the Taylor series with the expansion point at \(\infty\). When the bilinear transformation rotates the Riemann sphere, the point at infinity is translated to a finite point on the complex plane, revealing the analytic nature at infinity. A second way to transform the point at infinity is by the bilinear transformation \(\zeta = 1/z\), mapping a zero (or pole) at \(z = \infty\) to a pole (or zero) at \(\zeta = 0\). Thus this construction of the Riemann sphere and the Möbius (bilinear) transformation allows us to understand the point at infinity, and treat it like any other point. If you felt that you never understood the meaning of the point at \(\infty\) (likely), this should help.

25 https://www.youtube.com/watch?v=0z1fIsUNh04
Stream 3 is $\infty$, a concept which typically means unbounded (immeasurably large), but in the case of calculus, $\infty$ means infinitesimal (immeasurably small), since taking a limit requires small numbers. Taking a limit means you may never reach the target, a concept that the Greeks called Zeno’s paradox (Stillwell, 2010, p. 76).

When speaking of the class of ordinary (versus vector) differential equations, the term scalar is preferable, since the term “ordinary” is vague, if not meaningless. There is a special subset of fundamental theorems for scalar calculus, all of which are about integration, as summarized in Table 4.1 in §4.2, starting with Leibniz’s theorem.

Following the integral theorems on scalar calculus are those on vector calculus, without which there can be no understanding of Maxwell’s equations. Of these, the fundamental theorem of vector calculus (also known as, Helmholtz decomposition), Gauss’s law and Stokes’s theorem form three cornerstones of modern vector field analysis. These theorems allow one to connect the differential (point) and macroscopic (integral) relationships. For example, Maxwell’s equations may be written as either vector differential equations, as shown by Heaviside (along with Gibbs and Hertz), or in integral form. It is helpful to place these two forms side-by-side, to fully appreciate their significance. To understand the differential (microscopic) view, one must understand the integral (macroscopic) view. These are presented in Fig. 5.5 (p. 165) and Fig. 5.6 (p. 166).

4.1 The beginning of modern mathematics

As outlined in Fig. 1.2 (p. 18), mathematics as we know it today, began in the 16th to 18th centuries, arguably starting with Galileo, Descartes, Fermat, Newton, the Bernoulli family, and most importantly, Euler. Galileo was formidable, due to his fame, fortune, and his “successful” stance against the powerful Catholic establishment. His creativity in scientific circles was certainly well known due to his many skills and accomplishments. Descartes and Fermat were at the forefront of merging algebra and geometry. While Fermat kept meticulous notebooks, he did not publish, and tended to be secretive. Thus Descartes’s contributions were more widely acknowledged, but not necessarily deeper.

Regarding the development of calculus, much was yet to be developed by Newton and Leibniz, using term-by-term integration of functions based on Taylor series representation. This was a powerful technique, but as stated earlier, incomplete because the Taylor series can only represent single-valued functions, within the RoC. But more importantly, Newton (and others) failed to recognize (i.e., rejected) the powerful generalization to complex analytic functions. The first major breakthrough was Newton’s publication of *Principia* (1687), and the second was Riemann (1851), advised by Gauss but possibly more influenced by Cauchy.
Following Newton’s lead, the secretive and introverted behavior of the typical mathematician dramatically changed with the Bernoulli family (Fig. 3.1, p. 56). The oldest brother Jacob taught his much younger brother Johann, who then taught his son Daniel. But Johann’s star pupil was Euler. Euler first mastered all the tools and then published, with a prolificacy previously unknown.

**Euler and the circular functions:** The first major task was to understand the family of analytic circular functions, \(e^x, \sin(x), \cos(x),\) and \(\log(x)\), a task begun by the Bernoulli family, but mastered by Euler. Euler sought relations between these many functions, some of which may not be thought of as being related, such as the log and sin functions. The connection that may “easily” be made is through their complex Taylor series representation (Eq. 3.2.2.9, p. 72). By the manipulation of the analytic series representations, the relationship between \(e^x,\) and the \(\sin(x)\) and \(\cos(x),\) was precisely captured with the relation

\[
e^{j\omega} = \cos(\omega) + j\sin(\omega),
\]

and its analytic inverse (Greenberg, 1988, p. 1135)

\[
\tan^{-1}(z) = \frac{1}{2j} \ln \left( \frac{1-jz}{1+jz} \right) = \frac{j}{2} \ln \left( \frac{1-zj}{1+zj} \right).
\]

**Exercise:** Starting from Eq. 4.1.0.1, derive Eq. 4.1.0.2. **Solution:** Let \(z(\omega) = \tan \omega,\) then

\[
z(\omega) = \frac{\sin \omega}{\cos \omega} = \tan(\omega) = -j \frac{e^{j\omega} - e^{-j\omega}}{e^{j\omega} + e^{-j\omega}} = -j \frac{e^{2j\omega} - 1}{e^{2j\omega} + 1}.
\]

Solving for \(e^{-2j\omega}\)

\[
e^{-2j\omega} = \frac{1 - zj}{1 + zj}.
\]

Taking \(\ln()\) of both sides and using the definition of \(z(\omega)\) gives Eq. 4.1.0.2

\[
\omega = \tan^{-1}(z) = \frac{j}{2} \ln \frac{1 - zj}{1 + zj}.
\]

These equations are the basis of transmission lines (TL) and the Smith chart. Here \(z(\omega)\) is the TL’s input impedance and Eq. 4.1.0.4 is the reflectance. ■

![Figure 4.1: Colorized plots of \(\omega(z) = \tan^{-1}(z)\) and \(\omega(z) = \frac{j}{2} \ln(1 - iz)/(1 + iz),\) verifying they are the same complex analytic function.](image)

While many high school students memorize Euler’s relation, it seems unlikely they appreciate the utility of complex analytic functions (Eq. 3.2.4.21, p. 78).
4.2. FUNDAMENTAL THEOREMS OF SCALAR CALCULUS

History of complex analytic functions: Newton (c1650) famously ignored imaginary numbers, and called them imaginary in a disparaging (pejorative) way. Given Newton’s prominence, his view certainly must have keenly attenuated interest in complex algebra, even though it had been previously described by Bombelli in 1526, likely based on his serendipitous finding of Diophantus’s book Arithmetic in the Vatican library.

Euler derived his relationships using real power series (i.e., real analytic functions). While Euler was fluent with \( j = \sqrt{-1} \), he did not consider functions to be complex analytic. That concept was first explored by Cauchy almost a century later. The missing link to the concept of complex analytic is the definition of the derivative with respect to the complex argument

\[
F'(s) = \frac{dF(s)}{ds},
\]  

(4.1.0.5)

where \( s = \sigma + \omega j \), without which the complex analytic Taylor coefficients may not be defined.

Euler did not appreciate the role of complex analytic functions because these were first fully appreciated well after his death (1785) by Augustin-Louis Cauchy (1789–1857), and further extended by Riemann in 1851 (p. 94).

Table 4.1: Summary of the fundamental theorems of integral calculus, each of which deals with integration. There are at least two main theorems related to scalar calculus, and three more for vector calculus.

<table>
<thead>
<tr>
<th>Name</th>
<th>Mapping</th>
<th>p.</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leibniz (FTC)</td>
<td>( \mathbb{R}^1 \rightarrow \mathbb{R}^0 )</td>
<td>117</td>
<td>Area under a real curve</td>
</tr>
<tr>
<td>Cauchy (FTCC)</td>
<td>( \mathbb{C}^1 \rightarrow \mathbb{R}^0 )</td>
<td>117</td>
<td>Area under a complex curve</td>
</tr>
<tr>
<td>Cauchy’s theorem</td>
<td>( \mathbb{C}^1 \rightarrow \mathbb{C}^0 )</td>
<td>132</td>
<td>Close integral over analytic region is zero</td>
</tr>
<tr>
<td>Cauchy’s integral formula</td>
<td>( \mathbb{C}^1 \rightarrow \mathbb{C}^0 )</td>
<td>132</td>
<td>Fundamental theorem of complex integral calculus</td>
</tr>
<tr>
<td>residue theorem</td>
<td>( \mathbb{C}^1 \rightarrow \mathbb{C}^0 )</td>
<td>132</td>
<td>Residue integration</td>
</tr>
<tr>
<td>Helmholtz’s theorem</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.2 Fundamental theorems of scalar calculus

History of the fundamental theorem of scalar calculus: In some sense, the story of calculus begins with the fundamental theorem of calculus (FTC), also known generically as Leibniz’s formula, as shown in Table 4.1. The simplest integral is the length of a line \( L = \int_0^t dx \). If we label a point on a line as \( x = 0 \) and wish to measure the distance to any other point \( x \), we form the line integral between the two points. If the line is straight, this integral is simply the Euclidean length given by the difference between the two ends (Eq. 3.4.0.3, p. 84).

If \( F(\chi) \in \mathbb{R} \) describes a height above the line \( \chi \in \mathbb{R} \), then \( f(x) \)

\[
f(x) - f(0) = \int_{x=0}^{x} F(\chi) d\chi
\]  

(4.2.0.1)

may be viewed as the anti-derivative of \( F(\chi) \). Here \( \chi \) is a dummy variable of integration. Thus the area under \( F(\chi) \) only depends on the difference of the area evaluated at the end points. It makes intuitive sense to view \( f(x) \) as the anti-derivative of \( F(\chi) \).

This property of the area as an integral over an interval, only depending on the end points, has important consequences in physics in terms of conservation of energy, allowing for important generalizations.
For example, as long as \( \chi \in \mathbb{R} \), one may let \( F(\chi) \in \mathbb{C} \) with no loss of generality, due to the linear property (P1, p. 106) of the integral.

If \( f(x) \) is analytic (Eq. 3.2.2.8, p. 72), then

\[
F(x) = \frac{d}{dx} f(x)
\]  

(4.2.0.2)

is an exact real differential. It follows that \( F(x) \) is analytic. This is known as the *fundamental theorem of (real) calculus* (FTC). Thus Eq. 4.2.0.2 may be viewed as an *exact real differential*. This is easily shown by evaluating

\[
\frac{d}{dx} f(x) = \lim_{\delta \to 0} \frac{f(x + \delta) - f(x)}{\delta} = F(x),
\]

starting from the *anti-derivative* Eq. 4.2.0.1. If \( f(x) \) is not analytic then the limit may not exist, so this is a necessary condition.

There are many important variations on this very basic theorem (i.e., p. 115). For example, the limits could depend on time. Also when taking Fourier transforms, the integrand depends on both time \( t \in \mathbb{R} \) and frequency \( \omega \in \mathbb{R} \) via a complex exponential “kernel” function \( e^{\pm \omega t} \in \mathbb{C} \), which is analytic in both \( t \) and \( \omega \).

4.2.1 The fundamental theorems of complex calculus

The *fundamental theorem of complex calculus* (FTCC) states (Greenberg, 1988, p. 1197) that for any complex analytic function \( F(s) \in \mathbb{C} \) with \( s = \sigma + \omega j \in \mathbb{C} \),

\[
f(s) - f(s_0) = \int_{s_0}^{s} F(\zeta) d\zeta.
\]

(4.2.1.3)

Equations 4.2.0.1 and 4.2.1.3 differ because the path of the integral is complex. Thus the line integral is over \( s \in \mathbb{C} \) rather than a real integral over \( \chi \in \mathbb{R} \). The *fundamental theorem of complex calculus* (FTCC) states that the integral only depends on the end points, since

\[
F(s) = \frac{d}{ds} f(s).
\]

(4.2.1.4)

Comparing exact differentials Eq. 4.1.0.5 (FTCC) and Eq. 4.2.0.2 (FTC), we see that \( f(s) \in \mathbb{C} \) must be *complex analytic*, and have a Taylor series in powers in \( s \in \mathbb{C} \). It follows that \( F(s) \) is also complex analytic.

**Complex analytic functions:** The definition of a *complex analytic function* \( F(s) \) of \( s \in \mathbb{C} \) is that the function may be expanded in a Taylor series (Eq. 3.2.4.20, p. 77) about an *expansion point* \( s_0 \in \mathbb{C} \). This definition follows the same logic as the FTC. Thus we need a definition for the coefficients \( c_n \in \mathbb{C} \), which most naturally follow from Taylor’s formula

\[
c_n = \frac{1}{n!} \left. \frac{d^n}{ds^n} F(s) \right|_{s=s_0}.
\]

(4.2.1.5)

The requirement that \( F(s) \) have a Taylor series naturally follows by taking derivatives with respect to \( s \) at \( s_0 \). The problem is that both integration and differentiation of functions of complex Laplace frequency \( s = \sigma + \omega j \) have not yet been defined.

Thus the question is: What does it mean to take the derivative of a function \( F(s) \in \mathbb{C} \), \( s = \sigma + \omega j \in \mathbb{C} \), with respect to \( s \), where \( s \) defines a plane rather than a real line? We learned how to form the derivative on the real line. Can the same derivative concept be extended to the complex plane?
The answer is affirmative. The question may be resolved by applying the rules of the real derivative when defining the derivative in the complex plane. However, for the complex case, there is an issue regarding direction. Given any analytic function \( F(s) \), is the partial derivative with respect to \( \sigma \) different from the partial derivative with respect to \( \omega \)? For complex analytic functions, the FTCC states that the integral is independent of the path in the \( s \) plane. Based on the chain rule, the derivative must also be independent of direction at \( s_0 \). This directly follows from the FTCC. If the integral of a function of a complex variable is to be independent of the path, the derivative of a function with respect to a complex variable must be independent of the direction. This follows from Taylor’s formula, Eq. 4.2.1.5, for the coefficients of the complex analytic formula.

**The Cauchy-Riemann conditions:** The FTC defines the area as an integral over a real differential \((dx \in \mathbb{R})\), while the FTCC relates an integral over a complex function \( F(s) \in \mathbb{C} \), along a complex interval (i.e., path) \((ds \in \mathbb{C})\). For the FTC the area under the curve only depends on the end points of the anti-derivative \( f(x) \). But what is the meaning of an “area” along a complex path? The Cauchy-Riemann conditions provide the answer.

### 4.2.2 Cauchy-Riemann conditions

For the integral of \( Z(s) = R(\sigma, \omega) + jX(\sigma, \omega) \) to be independent of the path, the derivative of \( Z(s) \) must be independent of the direction of the derivative. As we show next, this leads to a pair of equations known as the Cauchy-Riemann conditions. This is an important generalization of Eq. 1.1.1.1 (p. 15), which goes from real integration \((x \in \mathbb{R})\) to complex integration \((s \in \mathbb{C})\) based on lengths, thus on area.

To define \( \frac{d}{ds}Z(s) = \frac{d}{ds}[R(\sigma, \omega) + jX(\sigma, \omega)] \), take partial derivatives of \( Z(s) \) with respect to \( \sigma \) and \( j\omega \), and equate them:

\[
\frac{\partial Z}{\partial \sigma} = \frac{\partial R}{\partial \sigma} + j\frac{\partial X}{\partial \sigma} \quad \text{and} \quad \frac{\partial Z}{\partial j\omega} = \frac{\partial R}{\partial j\omega} + j\frac{\partial X}{\partial j\omega}.
\]

This says that a horizontal derivative, with respect to \( \sigma \), is equivalent to a vertical derivative, with respect to \( j\omega \). Taking the real and imaginary parts gives the two equations

\[
\text{CR-1: } \frac{\partial R(\sigma, \omega)}{\partial \sigma} = j\frac{\partial X(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \text{CR-2: } \frac{\partial R(\sigma, \omega)}{\partial j\omega} = -j\frac{\partial X(\sigma, \omega)}{\partial \sigma},
\]

known as the Cauchy-Riemann (CR) conditions. The \( j \) cancels in CR-1, but introduces a \( j^2 = -1 \) in CR-2. They may also be written in polar coordinates \((s = re^{j\theta})\) as

\[
\frac{\partial R}{\partial r} = \frac{1}{r} \frac{\partial X}{\partial \theta} \quad \text{and} \quad \frac{\partial X}{\partial r} = \frac{1}{r} \frac{\partial R}{\partial \theta}.
\]

If you are wondering what would happen if we took a derivative at 45 degrees, then we only need to multiply the function by \( e^{j\pi/4} \). But doing so will not change the derivative. Thus we may take the derivative in any direction by multiplying by \( e^{j\theta} \), and the CR conditions will not change.

The CR conditions are necessary conditions that the integral of \( Z(s) \), and thus its derivative, be independent of the path, expressed in terms of conditions on the real and imaginary parts of \( Z \). This is a very strong condition on \( Z(s) \), which follows assuming that \( Z(s) \) may be written as a Taylor series in \( s \):

\[
Z(s) = Z_o + Z_1s + \frac{1}{2}Z_2s^2 + \cdots,
\]

where \( Z_n \in \mathbb{C} \) are complex constants given by the Taylor series formula (Eq. 4.2.1.5). As with the real Taylor series, there is the convergence condition, that \(|s| < 1\), called the radius of convergence (RoC). This is an important generalization of the region of convergence (RoC) for real \( s = x \).
Every function that may be expressed as a Taylor series in \( s - s_0 \) about point \( s_0 \in \mathbb{C} \) is said to be complex analytic at \( s_0 \). This series, which must be single valued, is said to converge within a radius of convergence (RoC). This highly restrictive condition has significant physical consequences. For example, every impedance function \( Z(s) \) obeys the CR conditions over large regions of the \( s \) plane, including the entire right half-plane (RHP) \( (\sigma > 0) \). This condition is summarized by the Brune condition \( \Re\{Z(\sigma > 0)\} \geq 0 \), or alternatively \( \angle Z(s) < \angle s \) (Eq. 4.3.2.11, p. 124).

When the CR condition is generalized to volume integrals, it is called either Gauss’s Law or Green’s theorem, used in the solution of boundary value problems in engineering and physics (Kusse and Westwig, 2010).

We may merge these equations into a pair of second-order equations by taking a second round of partials. Specifically, eliminating the real part \( R(\sigma, \omega) \) of Eq. 4.2.2.6 gives

\[
\frac{\partial^2 R(\sigma, \omega)}{\partial \sigma \partial \omega} = \frac{\partial^2 X(\sigma, \omega)}{\partial^2 \omega} = -\frac{\partial^2 X(\sigma, \omega)}{\partial^2 \sigma},
\]

which may be compactly written as \( \nabla^2 X(\sigma, \omega) = 0 \). Eliminating the imaginary part gives

\[
\frac{\partial^2 X(\sigma, \omega)}{\partial \omega \partial \sigma} = \frac{\partial^2 R(\sigma, \omega)}{\partial^2 \sigma} = -\frac{\partial^2 R(\sigma, \omega)}{\partial^2 \omega},
\]

which may be written as \( \nabla^2 R(\sigma, \omega) = 0 \).

In summary, for a function \( Z(s) \) to be complex analytic, the derivative \( \frac{dZ}{ds} \) must be independent of direction (path), which requires that the real and imaginary parts of the function obey Laplace’s equation, i.e.,

\[
\begin{align*}
\text{CR-3: } & \nabla^2 R(\sigma, \omega) = 0 \quad \text{and} \quad \text{CR-4: } \nabla^2 X(\sigma, \omega) = 0.
\end{align*}
\]

The CR equations are easier to work with because they are first-order, but the physical intuition is best understood by noting two facts (1) the derivative of a complex analytic function is independent of its direction, and (2) the real and imaginary parts of the function both obey Laplace’s equation. Such relationships are known as harmonic functions.\(^1\)

As we shall see in the next few sections, complex analytic functions must be smooth since every analytic function may be differentiated an infinite number of times, within the RoC. The magnitude must attain its maximum and minimum on the boundary. For example, when you stretch a rubber sheet over a jagged frame, the height of the rubber sheet obeys Laplace’s equation. Nowhere can the height of the sheet rise above or below its value at the boundary.

Harmonic functions define conservative fields, which means that energy (like a volume or area) is conserved. The work done in moving a mass from \( a \) to \( b \) in such a field is conserved. If you return the mass from \( b \) back to \( a \), the energy is retrieved, and zero net work has been done.

### 4.3 Complex analytic Brune admittance

It is rarely stated that the variable that we are integrating over, either \( x \) (space) or \( t \) (time), is real \( (x, t \in \mathbb{R}) \), since that fact is implicit, due to the physical nature of the formulation of the integral. But this intuition must be refined once complex numbers are included with \( s \in \mathbb{C} \), where \( s = \sigma + \omega j \).

That time and space are real variables is more than an assumption: it is a requirement that follows from the order property. Real numbers have order. For example, if \( t = 0 \) is now (the present), then \( t < 0 \) is the past and \( t > 0 \) is the future. Since time and space are real \( (t, x \in \mathbb{R}) \), they obey this order property. To have time travel, time and space would need to be complex (they are not), since if the space axis were complex the order property would be invalid.

\(^1\)When the function is the ratio of two polynomials, as in the cases of the Brune impedance, they are also related to Möbius transformations, also known as bi-harmonic operators.
Interestingly, it was shown by d’Alembert (1747) that time and space are related by the pure delay, due to the wave speed $c_o$. To obtain a solution to the governing wave equation, which d’Alembert first proposed for sound waves, \( x, t \in \mathbb{R} \) may be functionally combined as

$$\zeta_{\pm} = t \pm x/c_o,$$

where \( c_o \in \mathbb{R} \) is the wave phase velocity. The d’Alembert solution to the wave equation, describing waves on a string under tension is

$$u(x, t) = f(t - x/c_o) + g(t + x/c_o), \quad (4.3.0.1)$$

which describes the transverse velocity (or displacement) of two independent waves \( f(\zeta_{-}), g(\zeta_{+}) \in \mathbb{R} \) on the string, which represent forward and backward traveling waves.\(^2\) For example, starting with a string at rest, if one displaces the left end, at \( x = 0 \), by a step function \( u(t) \), then that step displacement will propagate to the right as \( u(t - x/c_o) \), arriving at location \( x_o \) \([\text{m}]\), at time \( x_o/c_o \) \([\text{s}]\). Before this time, the string will not move to the right of the wave-front, at \( x_o \) \([\text{m}]\), and after \( t_o \) \([\text{s}]\) it will have a non-zero displacement. Since the wave equation obeys superposition (postulate P2, p. 106), it follows that the “plane-wave” eigen-function of the wave equation for \( x, k \in \mathbb{R}^3 \) are given by

$$\psi_{\pm}(x, t) = \delta(t \mp k \cdot x) \leftrightarrow e^{\pm st} e^{i k \cdot x}, \quad (4.3.0.2)$$

where \(|k| = 2\pi/|\lambda| = \omega/c_o \) is the wave number, \(|\lambda|\) is the wavelength, and \( s = \sigma + \omega j \), the Laplace frequency.

When propagation dispersion and losses are considered, we must replace the wave number \( jk \) with a complex analytic vector wave number \( \kappa(s) = k_x(s) + jk_y(s) \), which is denoted as either the complex propagation function or the dispersion relation. The vector propagation function is a subtle and significant generalization of the scalar wave number \( k = 2\pi/\lambda \).

Forms of energy loss, which include viscosity and radiation, require \( \kappa(s) \in \mathbb{C} \). Physical examples include acoustic plane waves, electromagnetic wave propagation, antenna theory, and one of the most difficult cases, that of 3D electron wave propagating in crystals (e.g., silicon), where electrons and electromagnetic (EM) waves are in a state of quantum mechanical equilibrium.

Even if we cannot solve these more difficult problems, we can still appreciate their qualitative solutions. One of the principles that allows us to do that is the causal nature of \( \kappa(s) \). Namely the \( \mathcal{L}^{-1} \mathcal{T} \) of \( \kappa(s) \) must be causal, thus Eq. 4.3.0.2 must be causal. The group delay then describes the nature of the frequency dependent causal delay. For example, if the group delay is large at some frequency, then the solutions will have the largest causal delay at that frequency (Brillouin, 1953; Papoulis, 1962). Qualitatively this gives a deep insight into the solutions, even when we cannot compute the exact solution.

Electrons and photons are simply different EM states, where \( \kappa(x, s) \) describes the crystal’s dispersion relations as functions of both frequency and direction, famously known as Brillouin zones. Dispersion is a property of the medium such that the wave velocity is a function of frequency and direction, as in silicon.\(^3\) which determines the dispersion relation (Papasimakis et al., 2018).

### 4.3.1 Generalized admittance/impedance

The most elementary examples of Brune admittance and impedance are those made up of resistors, capacitors and inductors. Such discrete element circuits arise not only in electrical networks but in mechanical, acoustical and thermal networks as well (Table 3.1, p. 97). These lumped-element networks can always be represented by ratios of polynomials. This gives them a similar structure, with easily classified properties. Such circuits are called Brune admittances (or impedances)\(^4\). An example of a

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\(^2\) d’Alembert’s solution is valid for functions that are not differentiable, such as \( \delta(t - c_o x) \).

\(^3\) In case you missed it, what I’m suggesting is that photons (propagating waves) and electrons (evanescent waves) are different EM wave “states” (Jaynes, 1991). This difference depends on the medium.

\(^4\) Some texts prefer the term imittance, to include both admittance or impedance.
special structure is that the degrees of the numerator and denominator polynomials cannot differ by more than one. This restriction on the degrees comes about because the real part of the admittance/impedance must be positive.

But there is a much broader class of admittances which come from transmission lines and other physical structures, which we refer to as generalized admittances. An interesting example is an admittance of the form $1/\sqrt{s}$, called a semi-capacitor, or $\sqrt{s}$ called a semi-inductor. Generalized admittance/impedance is not the ratio of two polynomials. As a result they are more difficult to characterize.

When a generalized admittance $Y(s)$ or its impedance $Z(s) = 1/Y(s)$ is transformed into the time domain it must have a real and positive surge admittance $Y_r \in \mathbb{R}$ or surge impedance $Z_r \in \mathbb{R}$, followed by the residual response $\nu(t), \zeta(t)$. We define the following notation for the admittance

$$Y(s) = Y_r + Y(s) \leftrightarrow y(t) = Y_r \delta(t) + \nu(t)$$

(4.3.1.3)

and impedance

$$Z(s) = Z_r + Z_i(s) \leftrightarrow z(t) = Z_r \delta(t) + \zeta(t).$$

(4.3.1.4)

The complexity of the notation is necessary, and follows from the fact that $z(t) \leftrightarrow Z(s)$ and $y(t) \leftrightarrow Y(s)$ are positive real and thus minimum phase.

When we are dealing with a transmission line (i.e., wave guides), the generalized admittance is defined as the ratio of the flow over the force. For an electrical system (Voltage $\Phi$, current $I$), the input admittance looking to the right from location $x$ is

$$Y_{in}^+(x > 0, s) = \frac{I^+(x, \omega)}{\Phi^+(x, \omega)},$$

and looking to the left is

$$Y_{in}^-(x < 0, s) = \frac{I^-(x, \omega)}{\Phi^-(x, \omega)}.$$

These two admittances are typically different.

**Generalized reflectance:** A function related to the generalized impedance is the reflectance $\Gamma(s)$, defined as the ratio of a reflected wave waves normalized by the incident wave. For the case of acoustics (pressure $P$, volume velocity $V$)

$$Y_{in}(x, s) \equiv \frac{V(\omega)}{P(\omega)} = \frac{V^+ - V^-}{P^+ + P^-}$$

(4.3.1.5)

$$= \frac{V^+}{P^+} \frac{1 - V^-/V^+}{1 + P^-/P^+}$$

(4.3.1.6)

$$= \frac{Y_r^+}{1 + \Gamma(x, s)} \tag{4.3.1.7}$$

When the physical system is continuous at the measurement point $x$, $Y_r^+(x) = Y_r^-(x) \in \mathbb{R}$. The reflectance $\Gamma(x, s)$ depends on either the area function, boundary conditions, or both.

There is a direct relationship between a transmission lines area function $A(x) \in \mathbb{R}$, its characteristic impedance $Y_r(x) \in \mathbb{R}$, and its eigenfunctions. We shall provide specific examples as they arise during the analysis of transmission lines (e.g., Fig. 5.3, p. 156).

A few papers that deal with the relation between $Y_{in}(s)$ and the area function $A(x)$ include Youla (1964); Sondhi and Gopinath (1971); Rasetshwane et al. (2012). However the general theory of this important and interesting problem is beyond the scope of this text (See “train problem” in exercise DE3, problem 2).
4.3. COMPLEX ANALYTIC BRUNE ADMITTANCE

Complex analytic \( \Gamma(s) \) and \( Y_{in}(s) = Z_{in}^{-1}(s) \)

When defining the complex reflectance \( \Gamma(s) \) a key assumption has been made: even though \( \Gamma(s) \) is defined by the ratio of two functions of real (radian) frequency \( \omega \), like the impedance, the reflectance must be causal (postulate P1, p. 106). That \( \gamma(t) \rightarrow \Gamma(s) \) and \( \zeta(t) \leftrightarrow Z_{in}(s) = 1/Y_{in}(s) \) are causal is required by the physics.

4.3.2 Complex analytic impedance

Conservation of energy (or power) is a cornerstone of modern physics. It may first have been under consideration by Galileo Galilei (1564-1642) and Marin Mersenne (1588-1648). Today the question is not whether it is true, but why. Specifically, what is the physics behind conservation of energy? Surprisingly, the answer is straightforward, based on its definition and the properties of impedance. Recall that the power is the product of the force and flow, and impedance is their ratio.

The total energy \( E(t) \) is the integral of the power, since \( P(t) = dE/dt \). Thus if we start with all the elements at rest (no currents or voltages), then the energy as a function of time is always positive

\[
E(t) = \int_0^t P(t) dt \geq 0,
\]

and is simply the total energy applied to the network (Van Valkenburg, 1964a, p. 376). Since the voltage and current are related by either an impedance or an admittance, conservation of energy depends on the property of impedance. From Ohm’s law and P1 (every impedance is causal)

\[
v(t) = z(t) \ast i(t) = \int_{\tau=0}^t z(\tau)i(t - \tau) d\tau \leftrightarrow V(s) = Z(s)I(s).
\]

Example: Let \( i(t) = \delta(t) \). Then \( |w|^2(\tau) = i(t) \ast i(t) = \delta(\tau) \). Thus

\[
I_{xx}(t) = \int_{\tau=0}^t z(\tau)|w|^2(\tau) d\tau = \int_{\tau=0}^t z(\tau)\delta(\tau) d\tau = \int_0^t z(\tau) d\tau.
\]

The Brune impedance always has the form \( z(t) = r_o \delta(t) + \zeta(t) \). The characteristic impedance (aka surge impedance) may be defined as (Lundberg et al., 2007)

\[
r_o = \int_{0^-}^{\infty} z(t) dt.
\]

This definition requires that the integral of \( \zeta(t) \) is zero, a suspect conclusion that needs further investigation.

Note that convolution by an admittance or impedance, linearizes the expression for the power. Perhaps easier to visualize is when working in the Laplace frequency domain, where the total energy, equal to the integral of the real part of the power, is

\[
\frac{1}{s} \Re V I = \frac{1}{2s} (V^*I + VI^*) = \frac{1}{2s} (Z^*I^*I + ZI^*I) = \frac{1}{s} \Re Z(s) |I|^2 \geq 0.
\]

Mathematically this is called a positive definite operator, since the positive and real resistance is sandwiched between the current, forcing the definiteness.

In conclusion, conservation of energy is totally dependent on the properties of the impedance. Thus one of the most important and obvious applications of complex functions of a complex variable is the impedance function. This seems to be the ultimate example of the FTCC, applied to \( z(t) \).
**Every impedance must obey conservation of energy (P3):** The impedance function $Z(s)$ has resistance $R$ and reactance $X$ as a function of complex frequency $s = \sigma + j\omega$. From the causality postulate (P1) (p. 106), $z(t < 0) = 0$. Every impedance is defined by a Laplace transform pair

$$z(t) \leftrightarrow Z(s) = R(\sigma, \omega) + jX(\sigma, \omega),$$

with $R, X \in \mathbb{R}$.

According to Postulate P3 (p. 106), a system is passive if it does not contain a power source. Drawing power from an impedance violates conservation of energy. This property is also called *positive-real*, which was defined by Brune (1931b,a) in the following condition:

$$\mathcal{R}\{Z(s \geq 0)\} \geq 0.$$  \hspace{1cm} (4.3.2.10)

Positive-real systems cannot draw more power than is stored in the impedance. The region $\sigma \leq 0$ is called the *left half s plane* (LHP), and the complementary region $\sigma > 0$ is called the *right half s plane* (RHP). According to the Brune condition the real part of every impedance must be non-negative in the RHP.

It is easy to construct examples of second-order poles or zeros in the RHP, such that P3 is violated. Thus P3 implies that the impedance may not have more than simple (first-order) poles and zeros, strictly in the LHP. But there is yet more: These poles and zero in the LHP must have order, to meet the minimum phase condition. This minimum phase condition is easily stated

$$\angle Z(s) < \angle s,$$ \hspace{1cm} (4.3.2.11)

but difficult to prove. There seems to be no proof that second-order poles and zeros (e.g., second-order roots) are not allowed. However such roots must violate a requirement that the poles and zeros must alternate on the $\sigma = 0$ axis, which follows from P3. In the complex plane the concept of “alternate” is not defined (complex numbers cannot be ordered). What has been proved (i.e., Foster’s reactance theorem (Van Valkenburg, 1964a), is that if the poles are on the real or imaginary axis, they must alternate, leading to simple poles and zeros (Van Valkenburg, 1964a). The restriction on poles is sufficient, but not necessary, as $Z(s) = 1/\sqrt{s}$ is a positive real (PR) impedance, but is less than a first-degree pole (Kim and Allen, 2013). The corresponding condition in the LHP, and its proof, remains elusive (Van Valkenburg, 1964a).

For example, a series resistor $R_o$ and capacitor $C_o$ have an impedance given by (Table C.3, p. 208)

$$Z(s) = R_o + 1/sC_o \leftrightarrow R_o \delta(t) + \frac{1}{C_o} u(t) = z(t),$$ \hspace{1cm} (4.3.2.12)

with constants $R_o, C_o \in \mathbb{R} > 0$. In mechanics an impedance composed of a dash-pot (damper) and a spring has the same form. A resonant system has an inductor, resistor and a capacitor, with an impedance given by

$$Z(s) = \frac{sC_o}{1 + sC_o R_o + s^2 C_o M_o} \leftrightarrow C_o \frac{d}{dt} \left( e^{s+t} + e^{s-t} \right) = z(t),$$ \hspace{1cm} (4.3.2.13)

which is a second degree polynomial with two complex resonant frequencies $s_{\pm}$. When $R_o > 0$ these roots are in the left half s plane, with $z(t) \leftrightarrow Z(s)$.

Systems (networks) containing many elements, and transmission lines, can be much more complicated, yet still have a simple frequency domain representation. This is the key to understanding how these physical systems work, as will be described next.
4.3. COMPLEX ANALYTIC BRUNE ADMITTANCE

Poles and zeros of positive real functions must be first degree: The definition of positive real (PR) functions requires that the poles and the zeros of the impedance function be simple (only first degree). Second degree poles would have a reactive “secular” response of the form \( h(t) = t \sin(\omega_k t + \phi)u(t) \), and these terms would not average to zero, depending on the phase, as is required of an impedance. As a result, only single degree poles are possible. Furthermore, when the impedance is the ratio of two polynomials, where the lower degree polynomial is the derivative of the higher degree one, then the poles and zeros must alternate. This is a well-known property of the Brune impedance that has never been adequately explained except for very special cases, denoted as Foster’s theorem (Van Valkenburg, 1964b, p. 104-107). I believe that no one has ever reported an impedance having second degree poles and zeros. That would be a rare impedance. Network analysis books never report 2nd degree poles and zeros in their impedance functions. Nor has there ever been any guidance as to where the poles and zeros might lie in the left-hand \( s \) plane. Understanding the exact relationships between pairs of poles and zeros, to assure that the real part of the impedance is real, would resolve this longstanding unsolved problem (Van Valkenburg, 1964b).

Calculation on Complex analytic functions: To solve a differential equation, or integrate a function, Newton used the Taylor series to integrate one term at a time. However, he only used real functions of a real variable, due to the fundamental lack of appreciation of the complex analytic function. This same method is how one finds solutions to scalar differential equations today, but using an approach that makes the solution method less obvious. Rather than working directly with the Taylor series, today we use the complex exponential, since the complex exponential is an eigen-function of the derivative

\[
\frac{d}{dt} e^{st} = se^{st}.
\]

Since \( e^{st} \) may be expressed as a Taylor series, having coefficients \( c_n = 1/n! \), in some real sense the modern approach is a compact way of doing what Newton did. Thus every linear constant coefficient differential equation in time may be simply transformed into a polynomial in complex Laplace frequency \( s \), by looking for solutions of the form \( A(s)e^{st} \), transforming the differential equation into a polynomial \( A(s) \) in complex frequency. For example,

\[
\frac{d}{dt} f(t) + af(t) \leftrightarrow (s + a)F(s).
\]

The root of \( A(s_r) = s_r + a = 0 \) is the eigenvalue of the differential equation. A powerful tool for understanding the solutions of differential equations, both scalar and vector, is to work in the Laplace frequency domain. The Taylor series has been replaced by \( e^{st} \), transforming Newton’s real Taylor series into the complex exponential eigen-function. In some sense, these are the same method, since

\[
e^{st} = \sum_{n=0}^{\infty} \frac{(st)^n}{n!}.
\]

(4.3.2.14)

Taking the derivative with respect to time gives

\[
\frac{d}{dt} e^{st} = se^{st} = s \sum_{n=0}^{\infty} \frac{(st)^n}{n!},
\]

(4.3.2.15)

which is also complex analytic. Thus if the series for \( F(s) \) is valid (i.e., it converges), then its derivative is also valid. This was a very powerful concept, exploited by Newton for real functions of a real variable, and later by Cauchy and Riemann for complex functions of a complex variable. The key question here is: Where does the series fail to converge? This is the main message behind the FTCC (Eq. 4.2.1.3).

\(^7\)Secular terms result from second degree poles since \( u(t) \cdot u(t) = tu(t) \leftrightarrow 1/s^2 \).
The FTCC (Eq. 4.1.0.5) is formally the same as the FTC (Eq. 4.2.0.2) (Leibniz formula), the key (and significant) difference being that the argument of the integrand $s \in \mathbb{C}$. Thus this integration is a line integral in the complex plane. One would naturally assume that the value of the integral depends on the path of integration. And it does, but in a subtle way, as quantified by Cauchy’s various theorems. If the path says in the same RoC region, then the integral is independent of that path. If a path includes a different pole, then the integral depends on the path, as quantified by the Cauchy residue theorem. The test is to deform the path from the first to the second. If in that deformation the path crosses a pole, then the integral will depend on the path. All of this is dependent of the causal nature of the integral.

But, according to the FTCC, it does not. In fact they are clearly distinguishable from the FTC. And the reasoning is the same. If $F(s) = df(s)/ds$ is complex analytic (i.e., has a power series $f(s) = \sum_k c_k s^k$, with $f(s), c_k, s \in \mathbb{C}$), then it may be integrated, and yet integral does not depend on the path. At first blush, this is sort of amazing. The key is that $F(s)$ and $f(s)$ must be complex analytic, which means they are differentiable. This all follows from the Taylor series formula Eq. 4.2.1.5 (p. 118) for the coefficients of the complex analytic series. For Eq. 4.2.1.3 to hold, the derivatives must be independent of the direction (independent of the path), as discussed on page 119. The concept of a complex analytic function therefore has eminent consequences, in the form of several key theorems on complex integration discovered by Cauchy (c1820).

The use of the complex Taylor series generalizes the functions it describes, with unpredictable consequences, as nicely shown by the domain coloring diagrams presented on p. 110. Cauchy’s tools of complex integration were first exploited in physics by Sommerfeld (1952) to explain the onset (e.g., causal) transients in waves, as explained in detail in Brillouin (1960, Chap. 3).

Up to 1910, when Sommerfeld first published his results using complex analytic signals and saddle point integration in the complex plane, there was a poor understanding of the implications of the causal wave-front. It would be reasonable to say that his insights changed our understanding of wave propagation, for both light and sound. Sadly this insight has never been fully appreciated, even to this day. If you question this review, please read Brillouin (1960, Chap. 1).

The full power of the complex analytic function was first appreciated by Bernard Riemann (1826-1866) in his University of Gottingen PhD thesis of 1851, under the tutelage of Carl Friedrich Gauss (1777-1855), which drew heavily on the work of Cauchy.

The key definition of a complex analytic function is that it has a Taylor series representation over a region of the complex frequency plane $s = \sigma + j\omega$, that converges in a region of convergence (RoC) about the expansion point, with a radius determined by the nearest pole of the function. A further surprising feature of all analytic functions is that within the RoC, the inverse of that function also has a complex analytic expansion. Thus given $w(s)$, one may also determine $s(w)$ to any desired accuracy, critically depending on the RoC. Given the right software (e.g., zviz.m), this relationship may be made precise.

### 4.3.3 Multi-valued functions

![Figure 4.2: Here we show the mapping for the square root function $z = \pm \sqrt{x}$, the inverse of $x = z^2$. This function has two single-valued sheets of the $x$ plane corresponding to the two signs of the square root. The best way to view this function is in polar coordinates, with $x = |x|e^{j\theta}$ and $z = \sqrt{|x|}e^{j\theta/2}$. Figure from: https://en.wikipedia.org/wiki/Riemann_surface](https://en.wikipedia.org/wiki/Riemann_surface)
In the field of mathematics there seems to have been a tug-of-war regarding the basic definition of the concept of a function. The accepted definition today seems to be a single-valued (i.e., complex analytic) mapping from the domain to the codomain (or range). This makes the discussion of multi-valued functions somewhat awkward. In 1851 Riemann (working with Gauss) seems to have resolved this problem for the complex analytic set of multi-valued functions by introducing the geometric concept of single-valued sheets, delineated by branch-cuts.

Two simple yet important examples of multi-valued functions are the circle \( z^2 = x^2 + y^2 \) and \( w = \log(z) \). For example, assuming \( z \) is the radius of the circle, solving for \( y(x) \) gives the double-valued function

\[
y(x) = \pm \sqrt{z^2 - x^2}.
\]

The related function \( z = \pm \sqrt{x} \) is shown in Fig. 4.2 as a three-dimensional display, in polar coordinates, with \( y(r) \) as the vertical axis, as a function of the angle and radius of \( x \in \mathbb{C} \).

If we accept the modern definition of a function as the mapping from one set to a second, then \( y(x) \) is not a function, or even two functions. For example, what if \( x > z \)? Or worse, what if \( z = 2j \) with \( |x| < 1 \)? Riemann’s construction, using branch cuts for multivalued function, resolves all these difficulties (as best I know).

To proceed, we need definitions and classifications of the various types of complex singularities:

1. Poles of degree 1 are called simple poles. Their amplitude is called the residue (e.g. \( \alpha/s \) has residue \( \alpha \)). Simple poles are special (Eq. 4.4.1.3, p. 132), as they play a key role in mathematical physics, since their inverse Laplace transform defines a causal eigen-function.

2. When the numerator and denominator of a rational function (i.e., ratio of two polynomials) have a common root (i.e., factor), that root is said to be removable.

3. A singularity that is not (1) removable, (2) a pole or (3) a branch point is called essential.

4. A complex analytic function (except for isolated poles), is called meromorphic Boas (1987). Meromorphic functions can have any number of poles, even an infinite number. The poles need not be simple.

5. When the first derivative of a function \( Z(s) \) has a simple pole at \( a \), then \( a \) is said to be a branch point of \( Z(s) \). An important example is the logarithmic derivative

\[
d\ln(s-a)^\alpha/ds = \alpha/(s-a), \ \alpha \in \mathbb{I}.
\]

However, the converse does not necessarily hold.

6. I am not clear about the interesting case of an irrational pole (\( \alpha \in \mathbb{I} \)). In some cases (e.g., \( \alpha \in \mathbb{F} \)) this may be simplified via the logarithmic derivative operation, as mentioned on page 64.

Branch cuts: Up to this point we have only considered poles of degree \( \alpha \in \mathbb{N} \) of the form \( 1/s^\alpha \). The concept of a branch cut allows one to manipulate (and visualize) multi-valued functions, for which \( \alpha \in \mathbb{F} \). This is done by breaking each region into single-valued sheets, as shown in Fig. 4.3 (right). The branch cut is a curve \( \in \mathbb{C} \) that separates the various single-valued sheets of a multi-valued function. The concepts of branch cuts, sheets and the extended plane were first devised by Riemann, working with

\[\text{https://www.maths.ox.ac.uk/about-us/departmental-art/theory}\]

Multivalued functions: To study the properties of multivalued functions and branch cuts, we look at \( w(s) = s^2 \) and \( w(s) = \log(s) \), along with their inverse functions \( w(s) = s^2 \) and \( w(s) = e^s \). For uniformity we shall refer to the complex abscissa \( (s = \sigma + \omega j) \) and the complex ordinate \( w(s) = u + v j \). When the complex abscissa and domain are swapped, by taking the inverse of a function, multi-valued functions are a common consequence. For example, \( f(t) = \sin(t) \) is single valued, and analytic in \( t \), thus has a Taylor series. The inverse function \( t(f) \) is multivalued.

The best way to explore the complex mapping from the complex planes \( s \to w(s) \) is to master the single-valued function \( s = w^2(s) \) and its double-valued inverse \( w(s) = \sqrt[2]{s} \). Figure 4.3 shows the single-valued function \( w(s) = s^2 \) (left), and Fig. 4.4 (right) \( w : \mathbb{C} \to \mathbb{C} \) its inverse, the double-valued mapping of \( s(w) = \pm \sqrt[2]{w} \). Single valued functions such as \( W(s) = s^2 \) are relatively straightforward. Multivalued functions require the concept of a branch cut, defined in the image \( (w(s)) \) plane, which is a technique to render the multiple values as single valued on each of several sheets, defined by the sheet index in the domain \((s)\) plane, and delineated by a branch cut (in the \( w \) plane). The sheets are labeled in the domain \((s)\) plane by a sheet index \( k \in \mathbb{Z} \) while branch points and cuts are defined in the image \((w)\) plane \( \text{F.k.a., the range} \). It is important to understand that the path of every branch cut is not unique, and branch points are movable.

The multi-valued nature of \( w(s) = \sqrt[2]{s} \) is best understood by working with the function in polar coordinates. Let \( w = re^{\psi j} \), defined in \((4.3.3.16)\), where \( r = |s|, \theta = \angle s, \in \mathbb{R} \) and \( k \in \mathbb{Z} \) is the sheet-index.

This concept of analytic inverses becomes important only when the function is multi-valued. For example, since \( w(s) = s^2 \) has a period of \( 2 \), then \( s(w) = \pm \sqrt[2]{w} \) is multivalued. Riemann dealt with such extensions with the concept of a branch-cut with multiple sheets, labeled by a sheet number. Each sheet describes an analytic function (Taylor series) that converges within some RoC, having a radius out to the nearest pole. Thus Riemann’s branch cuts and sheets explicitly deal with the need to define unique single-valued inverses of multi-valued functions. Since the square root function has two overlapping regions, corresponding to the \( \pm \) due to the radical, there must be two connected regions, sort of like mathematical Siamese twins: distinct, yet the same.

Hue: By studying the output of \( \text{viz.m} \) (Fig. 3.10, p. 110), one may appreciate domain-coloring. The domain angles \( \angle s \) go from \(-90^\circ < \theta < 90^\circ \), with \( \theta = 0 \) being red and \( \pm 90^\circ \) being green (between yellow and purple). The angle to hue map is shown in the left panel of Fig. 4.4. For the \( w(s) = s^2 \) the \( \angle s \) is expanded by \( 2 \), since \( \psi = 2\theta \). For \( w(s) = \sqrt[2]{s} \) the \( \angle s \) is compressed by a factor of \( 2 \), since \( \psi = \theta / 2 \). Thus the principle angle \( k = 0, -180^\circ < \theta < 180^\circ \) maps half the \( w \) plane \((-90^\circ < \psi < 90^\circ)\) from purple to yellow, while the \( k = 1 \) branch maps to \( 90^\circ < \psi < 270^\circ \). Note how the panel on the right of Fig. 4.3 matches the right half of \( s \) (purple = \(-90^\circ\), yellow/green = \(+90^\circ\)) while the middle panel above comes from the left side of \( s \) (green to purple). The center panel is green at \(-180^\circ\), and purple at \(+180^\circ\), which matches the left panel at \( \pm 180^\circ \), respectively (i.e., \( e^{\pi j/2} \sqrt[2]{s} \)).

Furthermore let
\[
w = re^{\psi j} = \sqrt[2]{e^{\theta j} e^{mk}}, \tag{4.3.3.17}\]
Figure 4.4: Colorized plots of \( w_k = |s| e^{\theta_k} e^{\pm \pi k} \) and \( w_k(s) = \sqrt{|s|} e^{\theta_k} e^{\pm \pi k} \), as defined in polar coordinates by Eqs. 4.3.3.16 and 4.3.3.17. Left: Color map hue reference plane \( s = |s| e^{\theta} e^{\pm \pi} \). This function is a Brune impedance (it represents an inductor). Center: Sheet index \( k = 0 \), \( w(s) = \sqrt{|s|} e^{\theta/2} \) for \( -\pi < \theta < +\pi \) and \( \psi = \zeta w(s) = \theta/2 \) between \( \pm \pi/2 \) [rads]. This function is positive-real (Eq. 3.2.2.18, p. 75), but is not a Brune impedance, since it is not the ratio of polynomials. Thus it is the generalized-impedance, known as a semi-inductor. Right: Sheet index \( k = 1 \), \( w(s) = e^{\theta/2} e^{\pm \pi k} \sqrt{|s|} \) for \( -\pi < \theta < +\pi \) and \( \psi < -\pi/2 \) [rads]. The branch cut is at \( \theta = \zeta s = \pm 180^\circ \) [rads] where the hue of \( w \) changes abruptly from green to purple (center) and from blue back to green (right). Note how the hue matches between the center and right panels at the branch cut: in the center panel purple runs along -180° and along 180° on the right. Likewise, green runs along +180° in the center and along -180° on the right. Thus \( w(s) = \sqrt{3} \) is analytic on the branch cut connecting the two sheets (\( k = 0 \to k = 1 \)). This function is not an impedance, since it does not satisfy the positive-real condition.

Moving the branch cut: It is important to understand that the function is analytic on the branch cut, but not at the branch point. One is free to move the branch cut (at will). It does not need to be on a line: it could be cut in almost any connected manner, such as a spiral. The only rule is that it must start and stop at the matching branch points, or at \( \infty \), which must have the same degree.

Figure 4.3 shows the single-valued function \( w(s) = s^2 \) (Left), and Fig. 4.4 (right) its inverse, the double-valued mapping of \( s(w) = \pm \sqrt{w} \).

The location of the branch cut may be moved by rotating the \( s \) coordinate system of Fig. 4.2. For example, \( w(s) = \pm j \sqrt{3} \) and \( w(s) = \pm \sqrt{-s} \) have different branch cuts, as may be easily verified using the Matlab/Octave commands \( j \ast \text{viz} \left( s \right) \) and \( \text{viz} \left( -s \right) \), as shown in Fig. 4.4. Since the cut may be moved, every function is analytic on the branch cut, if a Taylor series is formed on the branch cut, it will describe the function on the two different sheets. Thus the complex analytic series (i.e., the Taylor formula, Eq. 4.2.1.5) does not depend on the location of a branch cut, as it only describes the function uniquely (as a single-valued function), valid in its local region of convergence.

The second sheet (\( k = 1 \)) (Fig. 4.4) picks up at \( \theta = \pi \) [rads] and continues on to \( \pi + 2 \pi = 3 \pi \). The first sheet maps the angle of \( w \) (i.e., \( \phi = \angle w = \theta/2 \)) from \( -\pi/2 < \phi < \pi/2 \) (\( w = \sqrt{\ell} e^{\theta/2} \)). This corresponds to \( u = \Re \{ w(s) \} > 0 \). The second sheet maps \( \pi/2 < \psi < 3\pi/2 \) (i.e., 90° to 270°), which is \( \Re \{ w \} = u < 0 \). In summary, twice around the \( s \) plane is once around the \( w(s) \) plane, because the angle is half due to the \( \sqrt{3} \).

Branch cuts emanate and terminate at branch points, defined as singularities (poles), that can even have fractional degree, as for example \( 1/\sqrt{3} \), and terminate at one of the matching roots, which includes...
the possibility of $\infty$. For example, suppose that in the neighborhood of the pole, at $s_o$ the function is

$$f(s) = \frac{w(s)}{(s - s_o)^k},$$

where $w, s, s_o \in \mathbb{C}$ and $k \in \mathbb{Q}$. When $k = 1$, $s_o = \sigma_o + \omega_o j$ is a first degree “simple pole”, having degree 1 in the $s$ plane, with residue $w(s_o)$. Typically the order and degree are positive integers, but fractional degrees and orders are common in modern engineering applications (Kirchhoff, 1868; Lighthill, 1978). Here we shall allow both the degree and order to be fractional ($\in F$). When $k \in F \subset R$ ($k = n/m$ is a real reduced fraction. Namely when $\text{GCD}(n, m) = 1$, $n \perp m$). This defines the degree of a fractional pole. In such cases there must be two sets of branch cuts of degree $n$ and $m$. For example, if $k = 1/2$, the singularity (branch cut) is of degree 1/2, and there are two Riemann sheets, as shown in Fig. 4.3

**Fractional-order Bessel function:** An important example is the Bessel function

$$\delta(t) + \frac{1}{t} J_1(t)u(t) \leftrightarrow \sqrt{s^2 + 1},$$

as shown in Fig. 4.5, which is related to the solution to the wave equation in two-dimensional cylindrical coordinates (Table C.4, p. 209). Bessel functions are the solutions (i.e., eigen-functions) of guided acoustic waves in round pipes, or surface waves on the earth (seismic waves), or waves on the surface of a pond (Table 5.2, p. 158).

There are a limited number of possibilities for the degree, $k \in Z$ or $\in F$. If the degree is drawn from $R \notin F$, the pole cannot have a residue. According to the definition of the residue, $k \in F$ will not give a residue. But there remains open the possibility of generalizing the concept of the Riemann integral theorem, to include $k \in F$. One way to do this is to use the logarithmic derivative which renders fractional poles to simple poles with fractional residues.

If the singularity has an irrational degree ($k \in I$), the branch cut has the same “irrational degree.” Accordingly there would be an infinite number of Riemann sheets, as in the case of the log function. An example is $k = \pi$, for which

$$F(s) = \frac{1}{s^\pi} = e^{-\log(s^\pi)} = e^{-\pi \log(s)} = e^{-\pi \log(\rho) e^{-\pi \theta}}.$$

---

10 This presumes that poles appear in pairs, one of which may be at $\infty$.

11 We shall refer to the order of a derivative, or differential equation, and the degree of a polynomial, as commonly used in engineering applications.
4.3. COMPLEX ANALYTIC BRUNE ADMITTANCE

where the domain is expressed in polar coordinates \( s = \rho e^{\theta j} \). When \( k \in \mathbb{F} \) it may be close (e.g., \( k = \pi_{152}/\pi_{153} = \pi_{152}/(\pi_{152} + 2) = 881/883 \approx 0.999883 \), or its reciprocal \( \approx 1.0023 \)). The branch cut could be very subtle (it could even go unnoticed), but it would have a significant impact on the function, and on its inverse Laplace transform.

**Example:** Find poles, zeros and residues of \( F(s) \).

1. \[ F(s) = \frac{d}{ds} \ln \frac{s + e}{s + \pi}, \]

   **Solution:**
   \[ F(s) = \frac{d}{ds} \left[ \ln(s + e) - \ln(s + \pi) \right] = \left( \frac{1}{s + e} - \frac{1}{s + \pi} \right). \]
   The the poles are at \( s_1 = -e \) and \( s_2 = -\pi \) with respective residues of \( \pm 1 \).

2. \[ F(s) = \frac{d}{ds} \ln \frac{(s + 3)^e}{(s + j)^{-\pi}}. \]

   **Solution:**
   \[ F(s) = \frac{d}{ds} (e \ln(s + 3) + \pi \ln(s + j)) = \frac{e}{s + 3} + \frac{\pi}{s + j}. \]
   There is a very important take-home message here regarding the utility of the logarithmic derivative, which “linearizes” the fractional pole.

3. \[ F(s) = e^{\pi \ln s}, \]

   **Solution:** Taking the derivative
   \[ \frac{d}{ds} F(s) = \frac{d}{ds} \ln s^\pi = \pi \frac{d}{ds} \ln s = \frac{\pi}{s}. \]
   Thus we see that \( F'(s) \) has a pole at \( s = 0 \) with residue \( \pi \). It follows that \( F(s) = \int F'(s) ds \) has a second order pole. Thus the residue must be zero.

4. \[ F(s) = \pi^{-s}. \]

   **Solution:** Taking the logarithmic derivative, \( d \ln F(s)/ds = F'(s)/F(s) = -\ln \pi. \)
   Thus \( F'(s) = -\ln \pi F(s) = -\ln \pi \pi^{-s}. \)

**Log function:** Next we discuss the multi-valued nature of the log function. In this case there are an infinite number of Riemann sheets, not well captured by Fig. 3.12 (p. 112), which displays only the principal sheet. However, if we look at the formula for the log function, the nature is easily discerned. The abscissa \( s \) may be defined as multi-valued since

\[ s_k = r e^{2\pi kj} e^{\theta j}. \]

Here we have extended the angle of \( s \) by \( 2\pi k \), where \( k \) is the sheet index \( \in \mathbb{Z} \). Taking the log

\[ \log(s) = \log(r) + (\theta + 2\pi k)j. \]

When \( k = 0 \) we have the principal value sheet, which is zero when \( s = 1 \). For any other value of \( k \), \( w(s) \neq 0 \), even when \( r = 1 \), since the angle is not zero, except for the \( k = 0 \) sheet.
CHAPTER 4. STREAM 3A: SCALAR CALCULUS

4.4 Three Cauchy integral theorems

4.4.1 Cauchy’s theorems for integration in the complex plane

There are three basic definitions related to Cauchy’s integral formula. They are closely related, and can greatly simplify integration in the complex plane. The choice of names seems unfortunate, if not totally confusing.

1. Cauchy’s (integral) theorem (CT-1):

\[ \oint_C F(s) ds = 0, \]  

(4.4.1.1)

if and only if \( F(s) \) is complex analytic inside of a simple closed curve \( C \). Boas, 1987, p. 45; (Stillwell, 2010, page 319). The FTCC (Eq. 4.2.1.3) says that the integral only depends on the end points if \( F(s) \) is complex analytic. By closing the path (contour \( C \)) the end points are the same, thus the integral must be zero, as long as \( F(s) \) is complex analytic.

2. Cauchy’s integral formula (CT-2):

\[ \frac{1}{2\pi j} \oint_{\mathcal{B}} \frac{F(s)}{s-s_0} ds = \begin{cases} F(s_0), & s_0 \in \mathbb{C} < \mathcal{B} \text{ (inside)} \\ 0, & s_0 \in \mathbb{C} > \mathcal{B} \text{ (outside)} \end{cases}, \]  

(4.4.1.2)

Here \( F(s) \) is required to be analytic everywhere within (and on) the boundary \( \mathcal{B} \) of integration (Greenberg, 1988, p. 1200); (Boas, 1987, p. 51); (Stillwell, 2010, p. 220). When the point \( s_0 \in \mathbb{C} \) is within the boundary, the value \( F(s_0) \in \mathbb{C} \) is the residue of the pole \( s_0 \) of \( F(s) \). When the point \( s_0 \) lies outside the boundary, the integral is zero.

3. The (Cauchy) residue theorem (CT-3): (Greenberg, 1988, p. 1241), (Boas, 1987, p. 73)

\[ \oint_C f(s) ds = 2\pi j \sum_{k=1}^{K} c_k, \]  

(4.4.1.3)

where the residues \( c_k \in \mathbb{C} \) corresponds to the \( k \)th pole of \( f(s) \) enclosed by the contour \( C \). By the use of Cauchy’s integral formula, the right-most form of the residue theorem is equivalent to the CT-1.12

How to calculate the residue: The case of first degree poles has special significance, because the Brune impedance only allows simple poles and zeros, increasing its utility. The residues for simple poles are \( F(s_k) \), which is complex analytic in the neighborhood of the pole, but not at the pole.

Consider the function \( f(s) = F(s)/(s-s_k) \), where we have factored \( f(s) \) to isolate the first-order pole at \( s=s_k \), with \( F(s) \) analytic at \( s_k \). Then the residue of the poles at \( c_k = F(s_k) \). This coefficient is computed by removing the singularity, by placing a zero at the pole frequency, and taking the limit as \( s \to s_k \), namely

\[ c_k = \lim_{s \to s_k} [(s-s_k)F(s)] \]  

(4.4.1.4)


When the pole is an \( N \)th degree, the procedure is much more complicated, and requires taking \( N-1 \) order derivatives of \( f(s) \), followed by the limit process (Greenberg, 1988, p. 1242). Higher degree poles are rarely encountered; thus, it is good to know that this formula exists, but perhaps it is not worth the effort to learn (i.e., memorize) it.

12This theorem is the same as a 2D version of Stokes’s theorem (Boas, 1987).
4.4.2 Cauchy Integral Formula and Residue Theorem

CT-2 (Eq. 4.4.1.2) is an important extension of CT-1 (Eq. 4.4.1.1) in that a pole has been explicitly injected into the integrand at $s = s_o$. If the pole location is outside of the curve $C$, the result of the integral is zero, in keeping with CT-1. When the pole is inside of $C$, the integrand is no longer complex analytic at the enclosed pole. When this pole is simple, the residue theorem applies. By a manipulation of the contour in CT-2, the pole can be isolated with a circle around the pole, and then taking the limit, the radius may be taken to zero, in the limit, isolating the pole.

For the related CT-3 (Eq. 4.4.1.3) the same result holds, except it is assumed that there are $K$ simple poles in the function $F(s)$. This requires the repeated application of CT-2, $K$ times, so it represents a minor extension of CT-2. The function $F(s)$ may be written as $f(s)/P_K(s)$, where $f(s)$ is analytic in $C$ and $P_K(s)$ is a polynomial of degree $K$, with all of its roots $s_k \in C$.

**Non-integral degree singularities:** The key point is that this theorem applies when $n \in \mathbb{I}$, including fractionals $n \in \mathbb{F}$; but for these cases the residue is always zero, since by definition, the residue is the amplitude of the $1/s$ term (Boas, 1987, p. 73).

**Examples:**

1. When $n \in \mathbb{F}$ (e.g., $n = 2/3$), the residue of $s^n$ is zero, by definition.
2. The function $1/\sqrt{s}$ has a zero residue (apply the definition of the residue Eq. 4.4.1.4).
3. When $n \neq 1 \in \mathbb{I}$, the residue is, by definition, zero.
4. When $n = 1$, the residue is given by Eq. 4.4.1.4.
5. CT-1, 2, 3 are essential when computing the inverse Laplace transform.

**Summary and examples:** These three theorems, all attributed to Cauchy, collectively are related to the fundamental theorems of calculus. Because the names of the three theorems are so similar, they are easily confused.

1. In general it makes no sense (nor is there any need) to integrate through a pole, thus the poles (or other singularities) must not lie on $C$.
2. Theorem CT-1 (Eq. 4.4.1.1) follows trivially from the fundamental theorem of complex calculus (Eq. 4.2.1.3, p. 118), since if the integral is independent of the path, and the path returns to the starting point, the closed integral must be zero. Thus Eq. 4.4.1.1 holds when $F(s)$ is complex analytic within $C$.
3. Since the real and imaginary parts of every complex analytic function obey Laplace’s equation (Eq. 4.2.2.10, p. 120), it follows that every closed integral over a Laplace field, i.e., one defined by Laplace’s equation, must be zero. In fact this is the property of a conservative system, corresponding to many physical systems. If a closed box has fixed potentials on the walls, with any distribution whatsoever, and a point charge (i.e., an electron) is placed in the box, then a force equal to $F = qE$ is required to move that charge, and thus work is done. However, if the point is returned to its starting location, the net work done is zero.
4. Work is done in charging a capacitor, and energy is stored. However, when the capacitor is discharged, all of the energy is returned to the load.
5. Soap bubbles and rubber sheets on a wire frame obey Laplace’s equation.
6. These are all cases where the fields are Laplacian, thus closed line integrals must be zero. Laplacian fields are commonly observed because they are so basic.
7. We have presented the impedance as the primary example of a complex analytic function. Physically, every impedance has an associated stored energy, and every system having stored energy has an associated impedance. This impedance is usually defined in the frequency $s$ domain, as a force over a flow (i.e., voltage over current). The power $P(t)$ is defined as the force times the flow and the energy $E(t)$ as the time integral of the power

$$E(t) = \int_{-\infty}^{t} P(t) \, dt,$$

which is similar to Eq. 4.2.0.1 (p. 117) [see §3.5.3, Eq. 3.5.4.9 (p. 96)]. In summary, impedance and power and energy are all fundamentally related.

### 4.5 Inverse Laplace transform $\mathcal{L}^{-1}$

The inverse Laplace transform $\mathcal{L}^{-1}$ (Eq. 3.7.0.1) transforms a function of complex frequency $F(s)$ and returns a causal function of time $f(t)$

$$f(t) \leftrightarrow F(s),$$

where $f(t) = 0$ for $t < 0$. Examples are provided in Table C.3 (p. 208). We next discuss the details of finding the inverse transform by use of CT-3, and how the causal requirement $f(t < 0) = 0$ comes about.

The integrand of the inverse transform is $F(s)e^{st}$ and the limits of integration are $\sigma - \infty \pm \sigma t$. To find the inverse we must close the curve at infinity, and specify that the integral at $\omega \to \infty$. There are two ways to close these limits—"to the right $\sigma > 0$ (RHP), and to the left $\sigma < 0$ (LHP)—but there needs to be some logical reason for this choice. That logic is determined by the sign of $t$. For the integral to converge, the term $e^{st}$ must go to zero as $\omega \to \infty$. In terms of the real and imaginary parts of $s = \sigma + \omega j$, the exponential may be rewritten as $e^{\sigma t}e^{\omega t}$. Note that both $t$ and $\omega$ go to $\infty$. Thus it is the interaction between these two limits that determines how we pick the closure, RHP vs. LHP.

#### 4.5.1 Case for negative time ($t < 0$) and causality:

Let us first consider negative time, including $t \to -\infty$. If we were to close $\mathcal{C}$ in the LHP ($\sigma < 0$), then the product $\sigma t$ is positive ($\sigma < 0$, $t < 0$, thus $\sigma t > 0$). In this case as $\omega \to \infty$, the closure integral $|s| \to \infty$ will diverge. Thus we may not close in the LHP for negative time. If we close in the RHP $\sigma > 0$ then the product $\sigma t < 0$ and $e^{st}$ will go to zero as $\omega \to \infty$. This then justifies closing the contour, allowing for the use of the Cauchy theorems.

If $F(s)$ is analytic in the RHP, the FTCC applies, and the resulting $f(t)$ must be zero, and the inverse Laplace transform must be causal. This argument holds for any $F(s)$ that is analytic in the RHP ($\sigma > 0$).

**Unstable poles:** An important but subtle point arises: If $F(s)$ has a pole in the RHP, then the above argument still applies if we pick $\sigma>$ to be to the right of the RHP pole. This means that the inverse transform may still be applied to unstable poles (those in the RHP). This explains the need for the $\sigma$ in the limits. If $F(s)$ has no RHP poles, then $\sigma = 0$ is adequate, and this factor may be ignored.

#### 4.5.2 Case for zero time ($t = 0$):

When time is zero, the integral does not, in general, converge, leaving $f(t)$ undefined. This is most clear in the case of the step function $u(t) \leftrightarrow 1/s$, where the integral may not be closed, because the convergence factor $e^{st} = 1$ is lost for $t = 0$.

The fact that $u(t)$ does not exist at $t = 0$ explains the Gibbs phenomenon in the inverse Fourier transform. At times where a jump occurs, the derivative of the function does not exist, and thus the time response function is not analytic. The Fourier expansion cannot converge at places where the function is not analytic. A low-pass filter may be used to smooth the function, but at the cost of temporal resolution.
4.5. INVERSE LAPLACE TRANSFORM \( \mathcal{L}^{-1} \)

4.5.3 Case for positive time \((t > 0)\)

Next we investigate the convergence of the integral for positive time \(t > 0\). In this case we must close the integral in the LHP \((\sigma < 0)\) for convergence, so that \(st < 0\) \((\sigma \leq 0 \text{ and } t > 0)\). When there are poles on the \(\omega_j = 0\) axis, \(\sigma_0 > 0\) assures convergence by keeping the on-axis poles inside the contour. At this point CT-3 is relevant. If we restrict ourselves to simple poles (as required for a Brune impedance), the residue theorem may be directly applied.

The simplest example is the step function, for which \(F(s) = 1/s\) and thus

\[
\begin{align*}
    u(t) &= \int_{\text{LHP}} \left. \frac{e^{st}}{s} \right|_{s \to \sigma} \, ds \leftrightarrow \frac{1}{s},
\end{align*}
\]

which is a direct application of the CT-3, Eq. 4.4.1.3 (p. 132). The forward transform of \(u(t)\) is straightforward, as discussed on p. 103. This is true of most if not all of the elementary forward Laplace transforms. In these cases, causality is built into the integral by its limits. An interesting problem is how to prove that \(u(t)\) is not defined at \(t = 0\).

The form \(\sqrt{s}\) is called a semi-inductor (Kim and Allen, 2013), also know as the skin effect in EM theory. The form \(1/\sqrt{s}\) is a semi-capacitor.

Two more examples are given in Fig. 4.7 to show Bessel functions \(J_0(\pi z)\) and the Hankel function \(H_0^{(1)}(\pi z/2)\) colorized maps. Note how the white and black contour lines are always perpendicular where they cross, just as in the calibration plots for the x and y axes, shown in Fig. 3.10 on p. 110.

Along the \(\sigma\) axis the \(\cos(\pi x)\) is the periodic with a period of \(\pi\). The dark spots are at the zeros at \(\pm \pi/2, \pm 3\pi/2, \ldots\). Off the \(\omega_j = 0\) axis the function either goes to zero (black) or \(\infty\) (white).
behavior carries the same $\pi$ periodicity as it has along the $\omega = 0$ line. On the right is the Hankel function $H_0^{(1)}(\pi z)$, which is a mixed and distorted version of $\cos(\pi z)$ with the zeros pushed downward and $e^{\pi z}$. This colorized plot shows that these two functions become the same for $x = \Re z > 0$. These figure are worthy of careful study to develop an intuition for complex functions of complex variables. On p. 110 we explored related complex mappings in greater detail.

**Some open questions:** Without the use of CT-3 it is difficult to see how to evaluate the inverse Laplace transform of $1/s$ directly. For example, how does one show that the above integral is zero for negative time (or that it is 1 for positive time)? CT-3 neatly solves this difficult problem by the convergence of the integral for negative and positive time. Clearly the continuity of the integral at $\omega \to \infty$ plays an important role. Perhaps the Riemann sphere plays a role in this that has not yet been explored.

### 4.5.4 Properties of the $\mathcal{L}T$

As shown in the table of Laplace transforms, there are integral (i.e., integration, not integer) relationships, or properties, that are helpful to identify. The first of these is a definition, not a property:

$$f(t) \leftrightarrow F(s).$$

**Causality:** When taking the LT, the time response is given in lower case (e.g., $f(t)$) and the frequency domain transform is denoted in upper case (e.g., $F(s)$). It is required, but not always explicitly specified, that $f(t < 0) = 0$, that is, the time function must be *causal*, as stated by Postulate P1 (106).

**Linearity:** The most basic property is the linearity (superposition) property of the $\mathcal{L}T$, stated by Postulate P2 (106).
Convolution property: The product of two $\mathcal{L}$'s in frequency results in convolution in time

\[ F(s)G(s) \leftrightarrow f(t) \ast g(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \]

where we use the $\ast$ operator to indicate the convolution of two time functions.

A key application of convolution is filtering, which takes many forms. The most basic filter is the moving average, the moving sum of data samples, normalized by the number of samples. Such a filter has very poor performance. It also introduces a delay of half the length of the average, which may or may not constitute a problem, depending on the application. Another important example is a low-pass filter that removes high-frequency noise, or a notch filter that removes line-noise (i.e., 60 [Hz] in the US, and its 2nd and 3rd harmonics, 120 and 180 [Hz]). Such noise is typically a result of poor grounding and ground loops. It is better to solve the problem at its root than to remove it with a notch filter. Still, filters are very important in engineering.

By taking the LT of the convolution we can derive this relationship:

\[
\int_0^\infty [f(t) \ast g(t)]e^{-st}dt = \int_0^\infty \left[ \int_0^t f(\tau)g(t-\tau)d\tau \right] e^{-st}dt \\
= \int_0^t f(\tau) \left( \int_0^\infty g(t-\tau)e^{-st}dt \right) d\tau \\
= \int_0^t f(\tau) \left( e^{-s\tau} \int_0^\infty g(t')e^{-s't'}dt' \right) d\tau \\
= G(s) \int_0^t f(\tau)e^{-s\tau}d\tau \\
= G(s)F(s).
\]

We first encountered this relationship on p. 79 in the context of multiplying polynomials, which was the same as convolving their coefficients. The parallel should be obvious. In the case of polynomials, the convolution was discrete in the coefficients, and here it is continuous in time. But the relationships are the same.

Time-shift property: When a function is time-shifted by time $T_0$, the LT is modified by $e^{sT_0}$, leading to the property

\[ f(t - T_0) \leftrightarrow e^{-sT_0}F(s). \]

This is easily shown by applying the definition of the LT to a delayed time function.

Time derivative: The key to the eigen-function analysis provided by the LT is the transformation of a time derivative on a time function, that is,

\[ \frac{d}{dt}f(t) \leftrightarrow sF(s). \]

Here $s$ is the eigenvalue corresponding to the time derivative of $e^{st}$. Given the definition of the derivative of $e^{st}$ with respect to time, this definition seems trivial. Yet that definition was not obvious to Euler. It needed to be extended to the space of complex analytic function $e^{st}$, which did not happen until at least Riemann (1851).

Given a differential equation of order $K$, the LT results in a polynomial in $s$, of degree $K$. It follows that this LT property is the corner-stone of why the LT is so important to scalar differential equations, as it was to the early analysis of Pell’s equation and the Fibonacci sequence, as presented in earlier chapters. This property was first uncovered by Euler. It is not clear if he fully appreciated its significance, but by the time of his death, it certainly would have been clear to him. Who first coined the terms eigenvalue and eigen-function? The word eigen is a German word meaning of one.
**Initial and final value theorems:** There are much more subtle relations between $f(t)$ and $F(s)$ that characterize $f(0^+)$ and $f(t \to \infty)$. While these properties can be very important in certain applications, they are beyond the scope of the present treatment. These relate to so-called initial value theorems. If the system under investigation has potential energy at $t = 0$, then the voltage (velocity) need not be zero for negative time. An example is a charged capacitor or a moving mass. These are important situations, but better explored in a more in-depth treatment.

### 4.5.5 Solving differential equations

Many differential equations may be solved by assuming a power series (i.e., Taylor series) solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n$$  \hspace{1cm} (4.5.5.1)

with $r \in \mathbb{Z}$ and coefficients $c_n \in \mathbb{C}$. The method of Frobenius is quite general (Greenberg, 1988, p. 193).

**Example:** When a solution of this form is substituted into the differential equation, a recursion relation in the coefficients results. For example, if the equation is

$$y''(x) = \lambda^2 y(x)$$

the recursion is $c_n = c_{n-1}/n$. The resulting equation is

$$y(x) = e^{\lambda x} = x^0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

namely $c_n = 1/n!$, thus $nc_n = 1/(n-1)! = c_{n-1}$.

**Exercise:** Find the recursion relation for $y(x) = J_\nu(x)$ of order $\nu$, that satisfies Bessel’s equation

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0.$$

**Solution:** If we assume a complex analytic solution of the form Eq. 4.5.5.1, we find the Bessel recursion relation for coefficients $c_k$ to be (Greenberg, 1988, p. 231).

$$c_k = -\frac{1}{k(k + 2\nu)} c_{k-2}.$$
Chapter 5

Vector Calculus

5.1 Properties of fields and potentials

Before we can define the vector operations $\nabla()$, $\nabla \cdot ()$, $\nabla \times ()$, $\nabla^2()$, we must define the objects they operate on: scalar and vector fields. The word field has two very different meanings: a mathematical one, which defines an algebraic structure, and a physical one, discussed next.

Ultimately we wish to integrate in $\in \mathbb{R}^3, \mathbb{R}^n$ and $\in \mathbb{C}^n$. Integration is quantified by several fundamental theorems of calculus, each about integration (p. 117-119).

5.1.1 Scalar and vector fields

Scalar fields: We use the term scalar field interchangeably with analytic in a connected region of the spatial vector $x = [x, y, z]^T \in \mathbb{R}^3$. In mathematics, functions that are piece-wise differentiable are called smooth, which is distinct from analytic. Every analytic function may be written as a single valued and infinitely differentiable power series. A smooth function has at least one or more derivatives, but need not be analytic.

Example: The simplest example of a scalar field is the voltage between two very large (think $\infty$) conducting parallel planes, bias to $V_o$ [V]. In this case the voltage varies linearly (the voltage is complex analytic) between the two plates. For example

$$\Phi(x, y, z) = V_o (1 - x) \quad [V] \quad (5.1.1.1)$$

is an example of a scalar field. At $x = 0$ the voltage is $V_o$ and at $x = 1$ the voltage is zero. Between 0 and 1 the voltage varies linearly. Thus $\Phi(x)$ defines a scalar field. The gradient of $\Phi(x)$ is a square pulse

$$\nabla \Phi(x) = -V_o (u(x) - u(x - 1)).$$

Example: The function $f(t) = tu(t)$ is smooth and has one smooth derivative

$$\frac{d}{dt} tu(t) = u(t) + t\delta(t), \quad \frac{d^2}{dt^2} tu(t) = \frac{d}{dt} u(t) = \delta(t),$$

but does not have a second derivative at $t = 0$. Thus $tu(t)$ is not analytic at $t = 0$. However it has a Laplace transform $f(t) \leftrightarrow F(s)$

$$tu(t) = u(t) * u(t) \leftrightarrow \frac{1}{s^2}$$

with a second-order pole at $s = 0$ with amplitude 1. The amplitude of $1/s$ must be zero, since $F(s)$ has no such term. Thus the $LT$ is analytic everywhere except at its second-order pole. The derivative $df(t)/dt \leftrightarrow sF(s) = 1/s$ has a simple pole, with residue 1.
Example: Next consider \((a, b, c, d \in \mathbb{C})\)

\[
G(s) = \frac{a + b F(\zeta)}{c + d F(\zeta)} = \frac{as^2 + b}{cs^2 + d^2}
\]

which has second order poles and zeros.

**Example:** The outbound eigen-function of the lossy scalar wave equation in spherical coordinates (i.e., the spherical Bessel function) is

\[
g(r, t) = \frac{e^{\imath \omega t - \kappa(s)r}}{r},
\]

where \(g(r, t) \in \mathbb{C}\) is the complex pressure, \(\kappa(s) = (s + \beta_o \sqrt{s})/c_o \in \mathbb{C}\) is the complex wave number (Eq. G.1.0.1, p. 233), \(c_o\) is the speed of sound and \(r = \sqrt{x^2 + y^2} \in \mathbb{R}\). If we ignore viscous and thermal losses, \(\beta_o = 0\) (Mason, 1928).

The pressure, a potential, is the solution to the acoustic wave equation in spherical coordinates. The \(1/r\) term compensates for the increasing area of the spherical wave as it propagates, to maintain constant energy. The area of the wavefront is proportional to \(r^2\), so that thus the integral of the energy \((\propto \rho^2 \propto 1/r^2)\) over the area \(\propto r^2\), remains constant, as the wave progresses outward. Note that the gradient of the potential (i.e., pressure) is proportional to the flow (mass flux) of the wave. The power flux is the product of the potential and the flux, and the ratio is the impedance. In the case of acoustics, this ratio is called the acoustic impedance, measured in acoustic ohms (see Table 3.1, p. 97).

Note that \(\ln g(r, s) = st - \kappa(s)r - \ln r\) is analytic everywhere except at \(r = 0\), but double-valued due to \(\beta_o \sqrt{s}\), forcing a branch cut, as required to fully describe it in the complex \(s\) plane.

To keep the discussion simple, initially we will limit the definition to an analytic surface \(S(x)\), as shown in Fig. 5.1, having height \(z(x, y) \in \mathbb{R}\), as a function of \(x, y \in \mathbb{R}^2\) (a plane)

\[
z(x, y, t) = \phi(x, y, t),
\]

where \(z(x, y, t)\) describes a surface that is analytic in \(x\). Optionally, one may allow the field to be a single-valued function of time \(t \in \mathbb{R}\), since that is the nature of the solutions of the equations we wish to solve.

For example picture the smooth single-valued potential a height \(z\) (e.g., constant temperature at height \(z\)) shown in Fig. 5.1, having isoclines (lines on a surface with constant slope).

**Vector fields:** A vector field is composed of three scalar fields. For example, the electric field used in Maxwell’s equations \(\mathbf{E}(x, t) = [E_x, E_y, E_z]^T [\text{V/m}]\) has three components, each of which is a scalar field. When the magnetic flux vector \(\mathbf{B}(x)\) is static (P5, p. 107), the potential \(\phi(x) [\text{V}]\) uniquely defines \(\mathbf{E}(x, t)\) via the gradient.

\[
\mathbf{E}(x, t) = -\nabla \phi(x, t). \quad [\text{V/m}]
\]

The electric force on a charge \(q\) is \(\mathbf{F} = q \mathbf{E}\), thus \(\mathbf{E}\) is proportional to the force, and when the medium is conductive, the current density (a flow) is \(\mathbf{J}_m = \sigma_o \mathbf{E} [\text{A/m}^2]\). The ratio of the potential to the flow is an impedance, thus \(\sigma_o\) is a conductance.

**Example:** Suppose we are given the vector field in \(\mathbb{R}^3\)

\[
\mathbf{A}(x) = [\phi(x), \psi(x), \theta(x)]^T, \quad [\text{Wb/m}]
\]
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where each of the three functions is a scalar field. As an example, \( \mathbf{A}(x) = [x, xy, xyz]^T \) is a legal vector field, having components analytic in \( x \).

**Example:** From Maxwell’s equations, the magnetic flux vector is given by

\[
\mathbf{B}(x, t) = \nabla \times \mathbf{A}(x, t). \quad [\text{Wb/m}^2]
\]

We shall see that this is always true because the magnetic charge \( \nabla \cdot \mathbf{B}(x, t) \) must be 0, which is always true given in-vacuo conditions.

To verify that a field is a potential, check out the units [V, A, °C]. However, a proper mathematical definition is that the potential must be an analytic function of \( x \) and \( t \), so that one may operate on it with \( \nabla() \) and \( \nabla \times () \). Note that the divergence of a scalar field is not a legal vector operation.

Feynman (1970b, p. 14-1 to 14.3) provides an extended tutorial on the vector potential, with many examples.

**Scalar potentials:** The above discussion describes the utility of potentials for defining vector fields (e.g., Eqs. 5.1.1.2 and 5.1.1.3). The key distinction between a potential and a scalar field is that potentials have units, and thus have a physical meaning. Scalar potentials (i.e., voltage \( \phi(x, t) \) [V], temperature \( T(x, t) \) [°C] and pressure \( p(x, t) \) [Pascals]) are examples of physical scalar fields. All potentials are composed of scalar fields, but not all scalar fields are potentials.

**Example:** The \( \hat{y} \) component of \( \mathbf{E} \), \( E_y(x, t) = \hat{y} \cdot E(x, t) \) [V/m], is not a potential. While \( \nabla E_y \) is mathematically defined, as the gradient of one component of a vector field, it has no physical meaning (as best I know).

**Vector potentials:** Vector potentials, like scalar potentials, are vector fields with physically meaningful units. They are more complicated than scalar potentials because they are composed of three scalar fields. Vector fields are composed of laminar and rotational flow, which are mathematically described by the fundamental theorem of vector calculus, i.e., Helmholtz’s decomposition theorem. One superficial, but helpful comparison, is the momentum of a mass, which may be decomposed into its forward (linear) and rotational momentum.

Since we find it useful to analyze problems using potentials (e.g., voltage), and then take the gradient (e.g., voltage difference) to find the flow (electric field \( \mathbf{E}(x, t) \)), the same logic and utility applies when using the vector potential, to describe the magnetic flux (flow) \( \mathbf{B}(x, t) \) (Feynman, 1970c). When operating on a scalar potential we use a gradient, whereas for the vector potential, we operate with the curl.

In Eq. 5.1.1.2 we assumed that the magnetic flux vector \( \mathbf{B}(x) \) was static, thus \( \mathbf{E}(x, t) \) is the gradient of the time-dependent voltage \( \phi(x, t) \). However, when the magnetic field is dynamic (not static), Eq. 5.1.1.2 is not valid due to magnetic induction: A voltage induced into a loop of wire is proportional to the time-varying flux cutting across that loop of wire. This is known as the Ampere-Maxwell law. In the static case the induced voltage is zero.

Thus the electric field strength includes both the scalar potential \( \phi(x, t) \) and magnetic flux vector potential \( \mathbf{A}(x, t) \) components, while the magnetic field strength only depends on the magnetic potential.

5.1.2 Gradient \( \nabla \), divergence \( \nabla \cdot \), curl \( \nabla \times \), and Laplacian \( \nabla^2 \)

Three key vector differential operators are required for understanding linear partial differential equations, such as the wave and diffusion equations. All of these begin with the \( \nabla \) operator:

\[
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}.
\]

As outlined in Table 5.1, the official name of this operator is *nabla*. It has three basic uses: 1) the gradient of a scalar field, the 2) divergence of a vector field, and 3) the curl of a vector field. The shorthand notation \text{CH:START} \nabla \phi(x, t) = (\hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z) \phi(x, t) \text{ is convenient.} \text{CH-END}

\text{1}https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates
Table 5.1: The three vector operators manipulate scalar and vector fields, as indicated. The gradient converts scalar fields into vector fields. The divergence maps vector fields to scalar fields. The curl maps vector fields to vector fields. Four second-order operators are defined (see page: 169).

<table>
<thead>
<tr>
<th>Name</th>
<th>Input</th>
<th>Output</th>
<th>Operator</th>
<th>Mnemonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient</td>
<td>Scalar</td>
<td>Vector</td>
<td>$\nabla (\ )$</td>
<td>grad</td>
</tr>
<tr>
<td>Divergence</td>
<td>Vector</td>
<td>Scalar</td>
<td>$\nabla \cdot (\ )$</td>
<td>div</td>
</tr>
<tr>
<td>Laplacian</td>
<td>Scalar</td>
<td>Scalar</td>
<td>$\nabla \cdot \nabla = \nabla^2 (\ )$</td>
<td>DoG</td>
</tr>
<tr>
<td>Wedgie</td>
<td>Vector</td>
<td>Scalar</td>
<td>$\nabla \wedge (\ )$</td>
<td>wedge</td>
</tr>
<tr>
<td>Curl</td>
<td>Vector</td>
<td>Vector</td>
<td>$\nabla \times (\ )$</td>
<td>curl</td>
</tr>
<tr>
<td>Little God</td>
<td>Vector</td>
<td>Vector</td>
<td>$\nabla^2 (\ ) = \nabla(\nabla \cdot (\ )$</td>
<td>GoD</td>
</tr>
<tr>
<td>Bull-DoG</td>
<td>Vector</td>
<td>Vector</td>
<td>$\nabla^2 (\ ) = \nabla \cdot \nabla (\ )$</td>
<td>Dog</td>
</tr>
<tr>
<td>Curl of Curl</td>
<td>Vector</td>
<td>Vector</td>
<td>$\nabla \times \nabla \times (\ ) = \nabla^2 (\ ) - \nabla^2 (\ )$</td>
<td>CoC</td>
</tr>
</tbody>
</table>

| Div of Curl | Vector    | 0         | $\nabla \cdot \nabla \times (\ )$ | DoC      |
| Curl of Grad| Scalar    | 0         | $\nabla \times \nabla (\ )$ | CoG      |

Operator TestCases:

$\nabla, \nabla \times, \nabla \cdot, \text{Laplacian: } \nabla^2 \text{ Vector.Laplacian: } \nabla^2$

- BigGod: $\nabla^2, \text{BigGod: } \nabla \nabla, \text{LittleGod: } \nabla^2, \text{LittleGod: } \nabla (\nabla \cdot (\ ), \text{CoC: } \nabla \times \nabla \times,$
- BullDog: $\nabla \cdot \nabla, \text{CH:START} \text{depreciated LapDog: } \nabla^2 \text{ CH:END}$

Gradient:

As shown in Fig. 5.1, the gradient transforms a complex scalar field $\Phi (x, s) \in \mathbb{C}$ into a vector field ($\mathbb{C}^3$)

$$\nabla \Phi (x, s) = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \Phi (x, s) = \hat{x} \frac{\partial \Phi}{\partial x} + \hat{y} \frac{\partial \Phi}{\partial y} + \hat{z} \frac{\partial \Phi}{\partial z}. $$

The gradient may also be factored into a unit vector $\hat{n}$, as defined in Fig. 5.1, defining the direction of the gradient, and the gradient’s length $||\nabla(\ )||$, defined in terms of the norm of the gradient. Thus the gradient of $\Phi (x)$ may be written in “polar coordinates” as $\nabla \Phi (x) = ||\nabla \Phi|| \hat{n}$, thus defining the unit vector

$$\hat{n} = \frac{\nabla (\Phi (x))}{||\nabla \Phi||}. $$

Example: Consider the paraboloid $z = 1 - (x^2 + y^2)$ as the potential, with iso-potential circles of constant $z$ that have radius of zero at $z = 1$, and unit radius at $z = 0$. The negative gradient

$$E (x) = -\nabla z (x, y) = 2(x \hat{x} + y \hat{y} + 0 \hat{z})$$

is $\perp$ to the circles of constant radius (constant $z$), and thus points in the direction of the radius.

If one were free-fall ski this surface, they would be the first one down the hill. Normally skiers try to stay close to the iso-clines (not in the direction of the gradient), so they can stay in control. If you ski an iso-cline, you must walk, since there is no pull due to gravity.
5.1. PROPERTIES OF FIELDS AND POTENTIALS

Divergence:

The divergence of a vector field results in a scalar field.

**Example:** The divergence of the electric field flux vector \( \mathbf{D}(x) \) \([\text{C/m}^2]\) equals the scalar field charge density \( \rho(x) \) \([\text{C/m}^3]\)

\[
\nabla \cdot \mathbf{D}(x) \equiv \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \mathbf{D}(x) = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho(x). \tag{5.1.2.4}
\]

Thus the divergence is analogous to the scalar (dot) product (e.g., \( \mathbf{A} \cdot \mathbf{B} \)) between two vectors.

**Example:** Recall that the voltage is the line integral of the electric field

\[
V(a) - V(b) = \int_a^b \mathbf{E}(x) \cdot dx = - \int_a^b \nabla V(x) \cdot dx = - \int_a^b \frac{dV}{dx} \, dx, \tag{5.1.2.5}
\]

which is simply the fundamental theorem of calculus (p. 117). In a charge free region, this integral is independent of the path from \( a \) to \( b \), which is a property of a conservative system.

When working with guided waves (narrow tubes of flux) having rigid walls that block flow, such that the diameter is small compared with the wavelength (P10, p. 107), the divergence simplifies to

\[
\nabla \cdot \mathbf{D}(x) = \nabla_r D_r = \frac{1}{A(r)} \frac{\partial}{\partial r} A(r) D_r(r), \tag{5.1.2.6}
\]

where \( r \) is the distance down the horn (range variable), \( A(r) \) is the area of the iso-response surface as a function of range \( r \), and \( D_r(r) \) is the radial component of vector \( \mathbf{D} \) as a function of range \( r \). In spherical, cylindrical and rectangular coordinates, Eq.5.1.2.6 provides the correct expression (Table 5.2, p. 158).

**Properties of the divergence:** The divergence is a direct measure of the flux density of the vector field. A vector field is said to be **incompressible** if the divergence of that field is zero. It is therefore **compressible** when the divergence is non-zero (e.g., \( \nabla \cdot \mathbf{D}(x, s) = \rho(x, s) \)).

**Example:** Compared to air, water is considered to be incompressible. The stiffness of a fluid (i.e., the bulk modulus) is a measure of its compressibility. At very low frequencies air may be treated as incompressible (just like water), since as \( s \to 0 \),

\[
- \nabla \cdot \mathbf{u}(x, s) = \frac{s}{\eta \rho} \nabla P(x, s) \to 0.
\]

The definition of compressible depends on the wavelength in the medium, so the terms must be used with some awareness of the frequencies being used in the analysis. As a rule of thumb, if the wavelength \( \lambda = c_0 / f \) is much larger than the size of the system, the medium may be modeled as an incompressible fluid.

Curl:

The curl \( \nabla \times (\cdot) \) takes a vector in \( \mathbb{C}^3 \) into a second vector in \( \mathbb{C}^3 \). For the case of fluids, the **Vorticity** is defined as \( \mathbf{\omega} = \nabla \times \mathbf{v} \) and rotation as \( \mathbf{\Omega} = \mathbf{\omega} / 2 \). The curl is a measure of the rotation of a vector field in a plane about the axis perpendicular to that plane. For the case of water, it would correspond to the angular momentum, such as in a whirlpool, or in air, a tornado. For a spinning solid body a top is a third example. While a gyroscope falls over if not spinning, once spinning, it can stably stand on its pointed tip. These systems are stable due to conservation of angular momentum. **CH:**

The curl and the divergence (\( \nabla \cdot (\cdot) \)), are both key when writing out Maxwell’s four equations. Without a full understanding of these two differential operators there is little hope of understanding Maxwell’s basic results, the most important equations of mathematical physics, and the starting point for Einstein’s relativity theories. CH:**
Example: The curl transforms a complex vector field \( \mathbf{H}(x, s) \in \mathbb{C}^3 \) into a complex vector field into a current density \( \mathbf{C}(x, s) \in \mathbb{C}^2 \):

\[
\nabla \times \mathbf{H}(x, s) \equiv \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial_x & \partial_y & \partial_z \\
H_x & H_y & H_z
\end{vmatrix} = \mathbf{C}(x, s). \quad [\text{A/m}^2] \quad (5.1.2.7)
\]

The notation \(| \cdot |\) indicates the determinant (Appendix A.3.1, p. 192), \(\partial_x\) is shorthand for \(\partial/\partial x\) and \(\mathbf{H} = [H_x, H_y, H_z]^T\).

Example: If we let Remove large section \(\mathbf{H} = -y\hat{x} + x\hat{y} + 0\hat{z}, \nabla \times \mathbf{H} = 2\hat{z}\), thus \(\mathbf{H}\) has a constant rotation; when \(\mathbf{H} = 0\hat{x} + 0\hat{y} + z^2\hat{z}, \nabla \times \mathbf{H} = 0\) has a curl of zero, and thus is irrotational. There are simple rules that precisely govern when a vector field is rotational versus irrotational, and compressible versus incompressible. These classes are dictated by Helmholtz’s theorem, the fundamental theorem of vector calculus (Eq. 5.7.0.5, p. 177).

A special case of the curl is the two-dimensional differential wedge products

\[
\nabla_x \wedge \mathbf{H}(x, t) = \begin{vmatrix}
\partial_y & \partial_z \\
H_y & H_z
\end{vmatrix} = C_x(x, s). \quad [\text{A/m}^2].
\]

The curl is made up of three such differential wedge products.\(^3\)

Laplacian \(\nabla^2()\):

The Laplacian \(\nabla^2 \equiv \nabla \cdot \nabla\) is defined as the divergence of the gradient (Table 5.1, page 142)

\[
\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (5.1.2.8)
\]

Since the Laplacian does so much work we nickname it DoG for Div of Grad.

Starting from a scalar field, the gradient produces a vector, which is then operated on by the divergence to take the output of the gradient back to a scalar field. Thus the Laplacian transforms a scalar field back to a scalar field. We have seen the Laplacian before when we defined complex analytic functions (Eq. 4.2.2.10, p. 120).

Example: Taking the divergence of the simple example Eq. 5.1.1.1 results in the Laplacian of the voltage

\[
\nabla^2 \Phi(x) = -V_o(\delta(x) - \delta(x - 1)) = 0
\]

for \(0 < x < 1\). Thus this example obeys Laplace’s equation.

Example: One of the classic examples of the Laplacian is that of a voltage scalar field \(\Phi(x) \ [\text{V}]\), which results in the electric field vector

\[
\mathbf{E}(x) = [E_x(x), E_y(x), E_z(x)]^T = -\nabla \Phi(x). \quad [\text{V/m}]
\]

When scaled by the permittivity one obtains the electric flux \(\mathbf{D} = \epsilon_o \mathbf{E} \ [\text{C/m}^2]\), the charge density per unit area. Here \(\epsilon_o \ [\text{F/m}]\) is the vacuum permittivity, which is \(\approx 8.85 \times 10^{-12} \ [\text{F/m}]\).

Taking the divergence of \(\mathbf{D}\) results in the charge density \(\rho(x) \ [\text{C/m}^3]\) at \(x\)

\[
\nabla \cdot \mathbf{D} = \nabla^2 \Phi(x) = \rho(x).
\]

Thus the Laplacian of the voltage, scaled by \(\epsilon_o\), results in the local charge density.

\(^2\)https://en.wikipedia.org/wiki/Triple_product#As_an_exterior_product

\(^3\)This notation suggests that \(||\nabla \cdot \mathbf{E} + j\nabla \times \mathbf{E}||^2 = ||\nabla \cdot \mathbf{E}||^2 + ||\nabla \times \mathbf{E}||^2\), which seems to be related to Helmholtz’s theorem. For example since \(\mathbf{E} = -\nabla \Phi + \nabla \times \mathbf{A}\). Then \(\nabla \cdot \mathbf{E} = -\nabla^2 \Phi\) and \(\nabla \times \mathbf{E} = -s\mathbf{B} = -s\nabla \times \mathbf{A}\).
Example: Another classic example is an acoustic pressure field $\rho(x, t) [\text{Pa}]$, which defines a vector force density $f(x, t) = -\nabla \rho(x, t) [\text{N/m}^2]$ (Eq. 5.2.1.5, p. 154). When this force density $[\text{N/m}^2]$ is integrated over an area, the net radial force $[\text{N}]$ is

$$F_r = -\int_S \nabla \rho(x) dA. \quad [\text{N}] \quad (5.1.2.9)$$

An inflated balloon with a static internal pressure of 3 [atm], in an ambient pressure of 1 [atm] (sea level), forms a sphere due to the elastic nature of the rubber, which acts as a stretched spring under tension. The net force on the surface of the balloon is its area times the pressure drop of 2 atm across the surface. Thus the static pressure is

$$\rho(x) = 3u(r_o - r) + 1, \quad [\text{Pa}]$$

where $u(r)$ is a step function of the radius $r = ||x|| > 0$, centered at the center of the balloon, having radius $r_o$.

Taking the gradient gives the negative of the radial force density (i.e., perpendicular to the surface of the balloon)

$$-f_r(r) = \nabla \rho(x) = \frac{\partial}{\partial r} 3u(r_o - r) + 1 = -2\delta(r_o - r). \quad [\text{Pa}]$$

This equation describes a static pressure that is 1 [atm] ($10^5$ [Pa]) outside the balloon and 3 [atm] inside. The net positive force density is the negative of the gradient of the static pressure.

Finally, taking the divergence of the force produces a double delta function at $r = r_o$, namely $\nabla^2 \rho(x) = -2\delta^{(1)}(r_o - r)$, were 2 is the pressure drop across the balloon. If we took the thickness of the rubber ($l$ [m]) into account, then $\nabla^2 \rho = -2(\delta(r_o) - \delta(r_o - l))$.

### Vector Laplacian $\nabla^2 ()$:

A second form of Laplacian (Table 5.1, page 142) is the vector Laplacian $\nabla^2 ()$, defined as the divergence of the gradient $\nabla^2 () \equiv \nabla \cdot \nabla ()$, thus nicknamed **Bull-Dog**, operates on a vector to produce a vector. We shall first see this when working with Maxwell’s equations.

### 5.1.3 Scalar Laplacian operator in $N$ dimensions

In general it may be shown that in $N = 1, 2, 3$ dimensions (Sommerfeld, 1949, p. 227)

$$\nabla^2 r P = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial P}{\partial r} \right). \quad (5.1.3.10)$$

For each value of $N$, the area $A(r) = A_o r^{N-1}$. This will turn out to be useful when working with the Laplacian in 1, 2, and 3 dimensions. This naturally follows from the Webster horn equation (WHEN) (Eq. 5.2.2.10, p. 155).

Example: When $N = 3$ (i.e., spherical geometry)

$$\nabla^2 r P \equiv \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial P}{\partial r} = \frac{1}{r} \frac{\partial^2}{\partial r^2} r P \quad (5.1.3.11)$$

resulting in the general d’Alembert solutions (Eq. 4.3.0.1 p. 121) for the spherical wave equation

$$P^\pm(r, s) = \frac{1}{r} e^{\mp \kappa(s)r}. \quad (5.1.3.12)$$

---

4The force is pointing out, stretching the balloon.
Exercise: Prove this last result by expanding Eq. 5.1.3.11, 5.1.3.12 using the chain rule.

Solution: Expanding Eq. 5.1.3.11:

\[
\frac{1}{r^2} \partial_r r^2 \partial_r \rho = \frac{1}{r^2} \left( 2r + r^2 \partial_r \right) \partial_r \rho \\
= \frac{2}{r} \rho + \rho_{rr}.
\]

Expanding Eq. 5.1.3.12:

\[
\frac{1}{r} \partial_r r \rho = \frac{1}{r} \partial_r (\rho + r \rho_r) \\
= \frac{1}{r} (\rho_r + \rho_r + r \rho_{rr}) \\
= \frac{2}{r} \rho_r + \rho_{rr}.
\]

Thus the two are equivalent. ■

Summary: The radial component of the Laplacian in spherical coordinates (Eq. 5.1.3.11) simplifies to

\[
\nabla^2 \rho(x) = \frac{1}{r^2} \partial_r r^2 \partial_r \rho(x) = \frac{1}{r} \partial_r^2 r \rho(x).
\]

Since \( \nabla^2 = \nabla \cdot \nabla \), it follows that the net force \( f(x) = [F_r, 0, 0]^T \), Eq. 5.1.2.9 in spherical coordinates has a radial component \( F_r \) and angular components of zero. Thus the force across a balloon may be approximated by a delta function across the thin sheet of stretched rubber.

We can extended the preceding example in an interesting way to the case of a rigid hose, a rigid tube, that terminates on the right in an elastic medium (the above example of a balloon), such as an automobile tire. On the far left let’s assume there is a pump injecting the fluid into the rigid hose. Consider two different fluids: air and water. Air is treated as a compressible fluid, whereas water is incompressible. However, such a classification is relative, determined by the relative compliance of the balloon (i.e., tire) at the relatively rigid pump and hose.

This is a special case of a more general situation: When a fluid is treated as incompressible (rigid), the speed of sound becomes infinite, and the wave equation is an invalid description. In this case the motion is best approximated by Laplace’s equation. This represents the transition from short to long wavelengths, from wave propagation, having delay, to quasi-statics, having no delay.

This example may be modeled as either an electrical or mechanical system. If we take the electrical analog, the pump is a current source, injecting charge \( Q_o \) into the hose, which being rigid cannot expand (has a fixed volume). The hose may be modeled as a resistor, and the tire as a capacitor \( C_o \), which fills with charge as it is delivered via the resistor, from the pump. The capacitor obeys the same equation as Hooke’s law for a spring, \( F = K_o \Delta \), where \( K_o \) is the stiffness of the spring, \( C_o = 1/K_o \) is the spring’s compliance, and \( \Delta \) is the displacement. In electrical terms, \( Q_o = C_o \Phi \) where \( \Phi \) is the voltage, which acts as a force, \( Q_o \) is the charge, which plays the role of the mass of the fluid. The charge is conserved, just as the mass of the fluid is conserved (they cannot be created or destroyed).

The flow of the fluid is called the flux, which is the general term for the mass flow, or electrical current. The two equations may be rewritten directly in terms of the force, either \( F, \Phi \), and flow, either \( I = dQ/dt \), the electrical current, or \( J = dM/dt \), the mass flux. In terms of impedance,

\[
I = C_o \frac{d}{dt} \Phi \quad [\text{A}] 
\]

for the electrical analog, and

\[
J = C_o \frac{d}{dt} F \quad [\text{kgm/s/m}].
\]
It is common to treat the stiffness of the balloon, which acts as a spring, as a compliance \( C_o = 1/K_o \), in which case the impedance reduces to a single form. The impedance \( Z \) is defined in the frequency domain as the ratio of the generalized force over the generalized flow
\[
Z(s) = \frac{1}{sc_o} \text{ [ohms]}
\]
In the case of the mechanical system \( Z_m(s) \equiv F/J \), while for the electrical system, \( Z_e(s) \equiv \Phi/I \). It is conventional to use the unit [ohms] when working with any impedance, allowing for uniform terminology for different physical situations and forms of impedance. This greatly simplifies the notation.

While the two systems are very different in their physical realization, they are mathematically equivalent, forming a perfect analog. The formula for the impedance is typically expressed in \( s \), the Laplace frequency, which of course is the \( \mathcal{L} \mathcal{T} \) of the time variables. In the frequency domain Ohm’s law becomes Eq. 5.1.3.14 for the case of a mechanical compliance \( C_o = 1/K_o \) and Eq. 5.1.3.13 for the electrical capacitor \( C \).

The final solution of this system is solved in the frequency domain. The impedance seen by the source is the sum of the resistance \( R \) and the impedance of the load, giving
\[
Z = R + \frac{1}{sC}.
\]
This results in a simple relationship between the force and the flow, as determined by the action of the source on the load \( Z(s) \). The results are given in terms of the voltage across the compliance in terms of the voltage \( \Phi_e \) (or current \( I_s \)) due to the source. Given some algebra, the voltage across the compliance \( \Phi_c \), divided by the voltage of the source, is
\[
\frac{\Phi_c}{\Phi_{source}} = \frac{R}{R + 1/sC}.
\]
Thus the problem reduces to some algebra in the frequency domain. The time domain response is found by taking the inverse \( \mathcal{L} \mathcal{T} \), which in this case has a single pole at \( s_p = 1/RC \). Cauchy’s residue theorem (p. 132) gives the final answer, which describes how the voltage across the compliance builds exponentially with time, from zero to the final value. Given the voltage, the current may also be computed as a function of time. This then represents the entire process of either blowing up a balloon, or charging a capacitor, the difference being only the physical notation, as the math is identical.

Note that the differential equation is first-order in time, which in frequency means the impedance has a single pole. Thus the equation for the charging of a capacitor, or pumping up a balloon, describes a diffusion process. If we had taken the impedance of the mass of the fluid in the hose into account, we would have a lumped-parameter model of the wave equation, with a second-order system. This is mathematically the same as the homework assignment (DE-3) about train cars (masses) connected together by springs (5.4, Homework DE-3, problem #2).

**Example:** The voltage
\[
\Phi(x, t) = e^{-\kappa \cdot x} u(t - x/c) \leftrightarrow \frac{1}{s} e^{-\kappa \cdot x} \text{ [V]} \quad \text{(5.1.3.15)}
\]
is an important case since it represents one of d’Alembert’s two solutions (Eq. 4.3.0.1, p. 121) of the wave equation (Eq. 3.1.0.5, p. 57), as well as an eigen-function of the gradient operator \( \nabla \). From the definition of the scalar (dot) product of two vectors (Fig. 3.4, p. 84),
\[
\kappa \cdot x = \kappa_x x + \kappa_y y + \kappa_z z = ||\kappa|| \cdot ||x|| \cos \theta_{\kappa x},
\]
where \( ||\kappa|| = \sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2} \) and \( ||x|| = \sqrt{x^2 + y^2 + z^2} \) are the lengths of vectors \( \kappa \), and \( x \) and \( \theta_{\kappa x} \) is the angle between them. As before, \( s = \sigma + \omega j \) is the Laplace frequency.

To keep things simple let \( \kappa = [\kappa_x, 0, 0]^T \) so that \( \kappa \cdot x = \kappa_x x \hat{x} \). We shall soon see that \( ||\kappa|| = 2\pi/\lambda \) follows from the basic relationship between a wave’s radian frequency \( \omega = 2\pi f \) and its wavelength \( \lambda \)
\[
\omega \lambda = c_o. \quad \text{(5.1.3.16)}
\]
As frequency increases, the wavelength becomes shorter. This key relationship may have been first researched by Galileo (c.1564), followed by (c.1627) Mersenne\(^5\) (Fig. 1.5, p. 23).

**Exercise:** Show that Eq. 5.1.3.15 is an eigen-function of the gradient operator \(\nabla\).

**Solution:** Taking the gradient of \(\phi(x, t)\) gives

\[
\nabla e^{-\kappa x} u(t) = -\nabla \kappa \cdot x e^{-\kappa x} u(t) = -\kappa e^{-\kappa x} u(t),
\]

or in terms of \(\phi(x, t)\)

\[
\nabla \phi(x, t) = -\kappa \phi(x, t) \leftrightarrow -s e^{-\kappa x}.
\]

Thus \(\phi(x, t)\) is an eigen-function of \(\nabla\), having the vector eigenvalue \(\kappa\). As before, \(\nabla \phi\) is proportional to the current since \(\phi\) is a voltage, and the ratio, i.e., the eigenvalue, may be thought of as a mass, analogous to the impedance of a mass (or inductor). In general the units provide the physical interpretation of the eigenvalues and their spectra.

**Exercise:** Compute \(\mathbf{n}\) for \(\phi(x, s)\) as given by Eq. 5.1.3.15.

**Solution:** \(\mathbf{n} = \kappa/||\kappa||\) represents a unit vector in the \(\kappa\) direction.

**Exercise:** If the sign of \(\kappa\) is negative, what are the eigenvectors and eigenvalues of \(\nabla \phi(x, t)\)?

**Solution:**

\[
\nabla e^{-\kappa x} u(t) = -\kappa \cdot \nabla (x) e^{-\kappa x} u(t) = -\kappa e^{-\kappa x} u(t).
\]

Nothing changes other than the sign of \(\kappa\). Physically this means the wave is traveling in the opposite direction, corresponding to the forward and retrograde d’Alembert waves.

Prior to this section, we had considered the Taylor series in only one variable, such as for polynomials \(P_N(x), x \in \mathbb{R}\) (Eq.3.1.1.7 p. 58) and \(P_N(s), s \in \mathbb{C}\) (Eq.3.2.4.20 p. 77). The generalization from real to complex analytic functions led to the \(L^T\), and the hosts of integration theorems (FTCC, Cauchy CT-1, CT-2, CT-3). What is in store when we generalize from one spatial variable (\(\mathbb{R}\)) to three (\(\mathbb{R}^3\))?  

**Exercise:** Find the velocity \(v(t)\) of an electron in a field \(E\).

**Solution:** From Newton’s 2nd law, \(-qE = m_e \dot{v}(t)\) [Nt], where \(m_e\) is the mass of the electron. Thus we must solve this first-order differential equation to find \(v(t)\). This is easily done in the frequency domain \(v(t) \leftrightarrow V(\omega)\).

**Role of Potentials:** Note that the scalar fields (e.g., temperature, pressure, voltage) are all scalar potentials, summarized in Table 3.1 (p. 97). In each case the gradient of the potential results in a vector field, just as in the electric case above (Eq. 5.1.1.2).

It is important to understand the physical meaning of the gradient of a potential, which is typically a generalized force (electric field, acoustic force density, temperature flux), that in turn generates a flow (current, velocity, heat flux). The ratio of the potential over flow determines the impedance. Four examples follow:

\(^5\)https://www-history.mcs.st-and.ac.uk/Biographies/Mersenne.html

"In the early 1620s, Mersenne listed Galileo among the innovators in natural philosophy whose views should be rejected. However, by the early 1630s, less than a decade later, Mersenne had become one of Galileo’s most ardent supporters.” (Garber, 2004)
The voltage drop across a resistor causes a current to flow, as described by Ohm’s law. The difference in voltage between two points is a crude form of gradient when the frequency \( f \) [Hz] is low, such that the wavelength is much larger than the distance between the two points. This is the essence of the quasistatic approximation (Postulate P10, p. 107).

The gradient of the pressure gives rise to a force density in the fluid medium (air, water, oil, etc.), that causes a flow (velocity vector) in the medium.

The gradient of the temperature also causes a flow of heat, that is proportional to the thermal resistance, given Ohm’s law for heat (Feynman, 1970b, p. 3-7).

When a solution contains charged ions, it defines an electrochemical Nernst potential \( N(x, t) \) (Scott, 2002). This electrochemical potential is similar to a voltage or temperature field, the gradient of which defines a virtual force on the charged ions, resulting in a current.

Thus in the above examples there is a potential, the gradient of which is a force, that when applied to the medium (an impedance) causes a flow (flux or current) proportional to that impedance due to the medium. These general concepts are worthy of some thought. The product of the force and flow is a power.

**Exercise:** Show that the integral of Eq. 5.1.1.2 is an anti-derivative.

**Solution:** The solution uses the definition of the anti-derivative, defined by the FTC (Eq. 4.2.0.2, p. 118):

\[
\phi(x, t) - \phi(x_0, t) = \int_{x_0}^{x} E(x, t) \cdot dx
\]

\[
= -\int_{x_0}^{x} \nabla \phi(x, t) \cdot dx
\]

\[
= -\int_{x_0}^{x} \left( \hat{x} \frac{\partial \phi}{\partial x} + \hat{y} \frac{\partial \phi}{\partial y} + \hat{z} \frac{\partial \phi}{\partial z} \right) \cdot (\hat{x} dx + \hat{y} dy + \hat{z} dz)
\]

\[
= -\int_{x_0}^{x} \frac{\partial \phi}{\partial x} dx - \int_{y_0}^{y} \frac{\partial \phi}{\partial y} dy - \int_{z_0}^{z} \frac{\partial \phi}{\partial z} dz
\]

\[
= -\left( \phi(x, t) - \phi(x_0, t) \right).
\]

This may be verified by taking the gradient of both sides

\[
\nabla \phi(x, t) - \nabla \phi(x_0, t) = -\nabla \int_{x_0}^{x} E(x, t) \cdot dx = E(x, t).
\]

Applying the FTC (Eq. 4.2.0.2, p. 118), the anti-derivative must be \( \phi(x, t) = E_x x \hat{x} + 0\hat{y} + 0\hat{z} \). This very same point is made by Feynman (1970b, p. 4-1, Eq. 4.28).

Given that the force on a charge is proportional to the gradient of the potential, from the above exercise showing that the integral of the gradient only depends on the end points, the work done in moving a charge only depends on the limits of the integral, which is the definition of a conservative field, but which only holds in the ideal case where \( E \) is determined by Eq. 5.1.1.2, i.e., the medium has no friction (i.e., there are no other forces on the charge).
The **conservative field**: An important question is: “When is a field conservative?” A field is conservative when the work done by the motion is independent of the path of motion. Thus the conservative field is related to the FTC, which states that the integral of the work only depends on the end points.

A more complete answer must await the introduction of the fundamental theorem of vector calculus, discussed in Eq. 5.7.0.5, p. 177. A few specific examples provide insight:

**Example:** The gradient of a scalar potential, such as the voltage (Eq. 5.1.1.2), defines the electric field, which drives a current (flow) across a resistor (impedance). When the impedance is infinite, the flow will be zero, leading to zero power dissipation. When the impedance is lossless, the system is conservative.

**Example:** At audible frequencies the viscosity of air is quite small and thus, for simplicity, it may be zero, leading to zero power dissipation. When the impedance is lossless, the system is conservative.

**Example:** The force of gravity is given by the gradient of Newton’s gravitational potential (Eq. 3.1.0.1, p. 56) 

\[ F = -\nabla_r \phi_N(r) = -\frac{\partial}{\partial r} \frac{1}{r} = \frac{1}{r^2}. \]

Historically speaking \( \phi_N(r) \) was the first conservative field, used to explain the elliptic orbits of the planets around the sun. **Galileo’s law** says that bodies fall with constant acceleration, giving rise to a parabolic path and a time of fall proportional to \( t^2 \). This behavior of falling objects directly follows from the Galilean potential

\[ \phi_G(r) = \frac{1}{r-r_o} = \frac{-r_o}{1-r/r_o} \quad r<r_o \approx -r_o(1-r/r_o + (r/r_o)^2 + \cdots) \approx \frac{r_o}{r-o} - r, \]

which given the large radius \( r_o \) of the earth, and the small distance of the object from the surface of the earth \( r-r_o \), is equal to the distance above the ground. Thus Galileo’s law says that the force a falling body sees is constant

\[ F_G = -\nabla_r \phi_G(r) = 1. \]

This can be scaled by a constant to account for the magnitude of the gravitational force.

**CH:START** 
**Example:** Galileo discovered that the height of a falling object is proportional to the square of the time it falls. Based on Newton’s follow-up analysis, today we would say this height \( h(t) \) is

\[ h(t) = \frac{1}{2} m G_o (t-t_o)^2 \quad [\text{m}], \]

where \( m \) is the object’s mass and \( G_o \) is the gravitational constant for the earth at its surface \( r_o \). Show that \( h(t) \) directly follows from the potential \( \phi_G = r_o - r \). This formula applies if you toss a ball into the air, or if you drop it from a high place.

**Solution:** Given Galileo’s potential \( \phi_G(r) \approx \frac{r_o}{r-o} m G_o (r_o - r) \), show that the force is constant, thus that \( \ddot{h}(t) = m G_o \). Given Galileo’s formula for the height \( h(t) \), the velocity is \( v(t) = \dot{v}(t) = m G_o t \), and the acceleration is \( \ddot{v}(t) = m G_o \).

**Example:** Find the time that it takes to fall from a distance \( r = L \). Namely, solve \( h(t) = L \) for the time the object takes to fall the distance \( L \). **Solution:** Setting \( t_o = 0 \) gives \( t^2 = 2L/mG_o \). Thus the time to fall is \( T(L) = \sqrt{2L/mG_o} \).

**CH:START**

### 5.2 Partial differential equations and field evolution:

**CH:END**
The three main classes of partial differential equations (PDEs) are: elliptic, parabolic, and hyperbolic, distinguished by the order of the time derivative. These categories seem to have little mathematical utility (the categories appear as labels).

**The Laplacian** \( \nabla^2 \): In the most important case the space operator is the Laplacian \( \nabla^2 \), the definition of which depends on the dimensionality of the waves, that is, the coordinate system being used. We first discussed the Laplacian as a 2D operator on p. 119, when we studied complex analytic functions, and again on p. 141. An expression for \( \nabla^2 \) for 1, 2 and 3 dimensions was provided as Eq. 5.1.3.10 (p. 145). In 3D rectangular coordinates it is defined as (see p. 144)

\[
\nabla^2 T(x) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T(x). \tag{5.2.0.1}
\]

The Laplacian operator is ubiquitous in mathematical physics, starting with simple complex analytic functions (Laplace’s equation) and progressing to Poisson’s equation, the diffusion equation, and finally the wave equation. Only the wave equation expresses delay. The diffusion equation “wave” has an instantaneous spread (the effective “wavefront” velocity is infinite, yet the wavelength is long: i.e., it’s not a traveling wave).

Examples of elliptic, parabolic, and hyperbolic equations follow:

1. **Laplace’s equation:** The equation

\[
\nabla^2 \Phi(x) = 0 \tag{5.2.0.2}
\]

that describes, for example, the voltage inside a closed chamber that has various voltages on the walls, or the steady–state temperature within a closed container given a specified temperature distribution on the walls. There are no dynamics to the potential, even when it is changing, since the potential instantaneously follows the potential on the walls.

2. **Poisson’s equation:** In the steady state, the diffusion equation degenerates to either Poisson’s or Laplace’s equation, which are classified as elliptic equations (second-order in space, zero-order in time). Like the diffusion equation, the evolution has a wave velocity that is functionally infinite.

\[
\nabla^2 \Phi(x, t) = \rho(x, t),
\]

which holds for gravitational fields, or the voltage around a charge.

3. **Fourier diffusion equation:** Equation 5.2.0.3 describes the evolution of the scalar temperature \( T(x, t) \) (a scalar potential), gradients of solution concentrations (i.e., ink in water) and Brownian motion. Diffusion is first-order in time, which is categorized as parabolic (first-order in time, second-order in space). When these equations are Laplace transformed, diffusion has a single real root, resulting in a real solution (e.g., \( T \in \mathbb{R} \)). There is no wave-front for the case of the diffusion equation. As soon as the source is turned on, the field is non-zero at every point in the bounded container. As an example

\[
\text{CH : START} \nabla^2 T(x, t) = \kappa_o \frac{\partial T(x, t)}{\partial t} \iff s\kappa_o T(x, s), \text{CH : END} \tag{5.2.0.3}
\]

describes the temperature \( T(x, t) \leftrightarrow T(x, \omega) \), as proposed by Fourier in 1822, or the diffusion of two miscible liquids (Fick, 1855) or Brownian motion (Einstein, 1905). The diffusion equation is not a wave equation since the temperature wavefront propagates instantaneously. The diffusion equation does a poor job of representing the velocity of molecules banging into each other, since such collisions have a mean free path, and thus the velocity cannot be infinite.

4. Two types of wave equations
There are two basic categories of field evolution: diffusion and propagation. The Taylor series of \( \varrho \) is analytic, and for the same reasons, \( T \) is analytic, and for the same reasons, \( \varrho(x, t \in \mathbb{R}) \). The wave equation is classified as hyperbolic (second-order in time and space).

(b) Vector wave equations: Maxwell’s equations describe the propagation of the EM electric \( E(x, t) \) and magnetic \( H(x, t) \) field strength vectors, along with the electric \( D(x, t) = \varepsilon_0 E(x, t) \) and magnetic \( B(x, t) = \mu_0 H(x, t) \) flux vectors.

**Solution evolution:** The partial differential equation defines the “evolution” of the scalar field [pressure \( \varrho(x, t) \) and temperature \( T(x, t) \)], or vector field \( (E, D, B, H) \), as functions of space \( x \) and time \( t \). There are two basic categories of field evolution: diffusion and propagation.

1. **Diffusion:** The simplest and easiest PDE example, easily visualized, is a static \(^6\) (time invariant) scalar temperature field \( T(x) \) [°C]. Just like an impedance or admittance, a field has regions where it is analytic, and for the same reasons, \( T(x, t) \) satisfies Laplace’s equation
   \[ \nabla^2 T(x, t) = 0. \]
   Since there is no current when the field is static, such systems are lossless, and thus are conservative.

   When \( T(x, t) \) depends on time (is not static), it is described by the diffusion equation (Eq. 5.2.0.3), a rule for how \( T(x, t) \) evolves with time from its initial state \( T(x, 0) \). Constant \( \kappa_o \) is called the thermal conductivity, which depends on the properties of the fluid in the container, with \( s \kappa_o \) being the thermal admittance per unit area. The conductivity is a measure of how the heat gradients induce heat currents \( J = -\kappa_o \nabla T \), analogous to Ohm’s law for electricity.

   Note that when \( T(x, t \to \infty) \) the temperature reaches a steady state, \( J = 0 \) and \( \nabla^2 T = 0 \). This all depends on what is happening at the boundaries. When the wall temperature of a container is a function of time, then so will the internal temperature \( T(x, t) \) continue to change, but with a delay that depends on the thermal conductivity \( \kappa_o \).

   Such a system is analogous to an electrical resistor-capacitor series circuit, connected to a battery. For example: the wall temperature (voltage across the battery) represents the potential driving the system.

2. **Propagation:** Pressure and electromagnetic waves are described by a scalar potential (pressure) (Eq. 3.1.0.3, p. 57) and a vector potential (electromagnets) (Eq. 5.6.2.4, p. 173), resulting in scalar and vector wave equations.

   All these partial differential equations, scalar and vector wave equations, and the diffusion equation, depend on the Laplacian \( \nabla^2 \), which we first saw with the Cauchy–Riemann conditions (Eqs. 4.2.2.10, p. 120).

   The Taylor series of \( f(x) \): Next we expand the concept of the Taylor series of one variable, to vector \( x \in \mathbb{R}^3 \). Just as we generalized the derivative with respect to a real frequency variable \( \omega \in \mathbb{R} \), to complex frequency \( s = \sigma + \omega j \in \mathbb{C} \), here we generalize the derivative with respect to \( x \in \mathbb{R} \), to the vector \( x \in \mathbb{R}^3 \).

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Postulate P3, p. 107.
5.2. PARTIAL DIFFERENTIAL EQUATIONS AND FIELD EVOLUTION:

Since the scalar field is analytic in \( x \), it is a perfect place to start. Assuming we have carefully defined the Taylor series (p. 72) in one and two variables, the Taylor series of \( f(x) \) in \( x \in \mathbb{R}^3 \) about \( x = 0 \) may be defined as

\[
 f(x + \delta x) = f(x) + \nabla f(x) \cdot \delta x + \frac{1}{2!} \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial^2 f(x)}{\partial x_k \partial x_l} \delta x_k \delta x_l + \text{HOT} \tag{5.2.0.4}
\]

where \( \text{HOT} \) stands for Higher Order Terms (Greenberg, 1988, p. 639). From this definition it is clear that the gradient is the generalization of the second term in the 1D Taylor series expansion.

**Summary:** For every potential \( \phi(x, t) \) there exists a force density \( f(x, t) = -\nabla \phi(x, t) \), proportional to the potentials, which drives a generalized flow \( u(x, t) \). If the normal component of the force and flow are averaged over a surface, the mean-force and volume-flow (i.e., volume-velocity for the acoustic case) are defined. In such cases the impedance \( Z(s) = F(x, s)/|\nabla F(x, s)| \) is the net force through the surface force over the net flow, and Gauss’s law and quasi-statics (P10, p. 107) come into play (Feynman, 1970a). We call this the generalized impedance.

Assuming linearity (P2, p. 106), the product of the force and flow is the power, and the ratio (force/flow) is an impedance (Table 3.1, p. 97). This impedance statement is called either Ohm’s law, Kirchhoff’s laws, Laplace’s law, or Newton’s laws. In the simplest of cases, they are all linearized (proportional) complex relationships between a force and a flow. Very few impedance relationships are inherently linear over a large range of force or current, but for physically useful levels, they can be treated as linear. Nonlinear interactions require a more sophisticated approach, typically involving numerical methods.

In electrical circuits it is common to define a zero potential *ground* point that all voltages use as the referenced potential. The ground is a useful convention, as a simplifying rule, but it obscures the physics, and obscures the fact that the voltage is *not* the force. Rather, the force is the voltage difference, referenced to the ground, which is defined as zero volts. This results in abstracting away (i.e., hiding) the difference in voltage. It seems misleading (more precisely it is wrong) to state Ohm’s law as the voltage over the current, since Ohm’s law actually says that the voltage difference (i.e., voltage gradient) over the current defines an impedance (Kennelly, 1893).

When one measures the voltage between two points, it is a crude approximation to the gradient, based on the quasi-static approximation (P10). The pressure is also a potential, the gradient of which is a force density, which drives the volume-velocity (flow).

On page 169 we first introduce the fundamental theorem of vector calculus (page 167) (other wise known as Helmholtz’ decomposition theorem), which generalizes Ohm’s law to include circulation (e.g., angular momentum, vorticity and the related magnetic effects). To understand these generalizations in flow one needs to understand compressible and rotational fields (Table 5.3, p. 167), complex analytic functions, and more mathematical-physics history.

In summary, it is the *difference* in the potential (i.e., voltage, temperature, pressure) that is proportional to the flux. This can be viewed as a major simplification of the gradient relationship, justified by the quasi-static assumption P10 (p. 107).

The roots of the impedance are related to the eigenmodes of the system equations.

### 5.2.1 Scalar wave equation (Acoustics)

In this section we discuss the general solution to the wave equation. The wave equation has two forms: scalar waves (acoustics) and vector waves (electromagnetics). These have an important mathematical distinction but a similar solution space, one scalar and the other vector. To understand the differences we start with the scalar wave equation.

**The scalar wave equation:** A good starting point for understanding PDEs is to explore the scalar wave equation (Eq. 3.1.0.3, p. 57). Thus, we shall limit our analysis to acoustics, the classic case of scalar
waves. Acoustic wave propagation was first analyzed mathematically by Isaac Newton (electricity had yet to be discovered) in his famous book *Principia* (1687), in which he first calculated the speed of sound based on the conservation of mass and momentum.

**Early history:** The study of wave propagation begins at least as early as Huygens (ca. 1678), followed soon after (ca. 1687) by Sir Isaac Newton’s calculation of the speed of sound (Pierce, 1981, p. 15). The acoustic variables are the pressure

\[ p(x, t) \leftrightarrow P(x, \omega) \]

and the particle velocity

\[ u(x, t) \leftrightarrow U(x, \omega). \]

To obtain a wave, one must include two basic components: the stiffness of air, and its mass. These two equations shall be denoted (1) Newton’s second law \((F = ma)\) and (2) Hooke’s law \((F = kx)\), respectively. In vector form these equations are (1) *Euler’s equation* (i.e., conservation of momentum density)

\[ -\nabla \cdot u(x, t) = \rho_o \partial_t \nabla \cdot u(x, t) \leftrightarrow \rho_o s U(x, s), \tag{5.2.1.5} \]

which assumes the time-average density \(\rho_o\) to be independent of time and position \(x\), and (2) the *continuity equation* (i.e., conservation of mass density)

\[ -\nabla \cdot \mathbf{u}(x, t) = \frac{1}{\eta_o P_o} \partial_t p(x, t) \leftrightarrow \frac{s}{\eta_o P_o} P(x, s) \tag{5.2.1.6} \]

(Pierce, 1981; Morse, 1948, p. 295). Here \(P_o = 10^5 \text{[Pa]}\) is the barometric pressure, \(\eta_o P_o\) is the dynamic (adiabatic) stiffness, with \(\eta_o = 1.4\). Combining Eqs. 5.2.1.5 and 5.2.1.6 (removing \(u(x, t)\)) results in the 3-dimensional (3D) scalar pressure wave equation

\[ \nabla^2 p(x, t) = \frac{1}{c_o^2} \partial^2_t p(x, t) \leftrightarrow \frac{s^2}{c_o^2} P(x, s) \tag{5.2.1.7} \]

with \(c_o = \sqrt{\eta_o P_o / \rho_o}\) being the sound velocity. Because the merged equations describe the pressure, which is a scalar field, this is an example of the *scalar wave equation*.

**Exercise:** Show that Eqs. 5.2.1.5 and 5.2.1.6 can be reduced to Eq. 5.2.1.7. **Solution:** Taking the divergence of Eq. 5.2.1.5 gives

\[ -\nabla \cdot \nabla p(x, t) = \rho_o \partial_t \nabla \cdot u(x, t). \tag{5.2.1.8} \]

Note that \(\nabla \cdot \nabla = \nabla^2\) (Table 5.1, 142). Next, substituting Eq. 5.2.1.6 into the above relation results in the scalar wave equation Eq. 5.2.1.7, since \(c_o = \sqrt{\eta_0 P_0 / \rho_o}\). ■

### 5.2.2 The Webster horn equation (WHEN)

There is an important generalization of the problem of lossless plane-wave propagation in 1-dimensional (1D) uniform tubes, known as *transmission line theory*. As depicted in Fig. 5.2, by allowing the area \(A(r)\) (e.g., for the conical horn \(A(r) = A_o(r/L)^2\) with \(L = 1\) [m] and \(A_0 \leq 4\pi\)) of an acoustical waveguide (aka, horn) to vary along the range axis \(r\) (the direction of wave propagation) general solutions to the wave equation may be explored. Classic applications of horns include vocal tract acoustics, loudspeaker design, cochlear mechanics, quantum mechanics (e.g., the hydrogen atom), and wave propagation in periodic media (Brillouin, 1953).

One must be precise when defining the area \(A(x)\): The area is *not* the cross-sectional area of the horn, rather it is the wave-front (iso-pressure) area, which is related to Gauss’ Law, since the gradient of the pressure defines the force that drives the mass flow (aka, volume velocity).
The Webster Laplacian is based on the quasi-static approximation (P10: p. 107), which requires that the frequency lie below the critical value $f_c = c_o/2d$, namely that a half wavelength be greater than the horn diameter $d$ (i.e., $d < \lambda/2$). For the case of the adult human ear canal, $d = 7.5$ [mm], $f_c = (343/2 \cdot 7.5) \times 10^{-3} \approx 22.87$ [kHz].

The term on the right of Eq. 5.2.2.9, which is identical to Eq. 5.1.3.10 (p. 145), is also the Laplacian for thin tubes (e.g., rectangular, spherical, and cylindrical coordinates). Thus the Webster horn “wave” equation is

$$\nabla^2 \varphi(r, t) = \frac{1}{A(r)} \frac{\partial}{\partial r} \left[ A(r) \frac{\partial}{\partial r} \right] \varphi(r, t), \quad (5.2.2.9)$$

For the scalar wave equation (Eq. 5.1.3.10, p. 145), the Webster Laplacian is

$$\nabla^2 \varphi(r, t) = \frac{1}{A(r)} \frac{\partial}{\partial r} \left[ \frac{\partial}{\partial r} \varphi(r, t) \right], \quad (5.2.2.10)$$

where $\varphi(r, t) \leftrightarrow \mathcal{P}(r, s)$ is the acoustic pressure in Pascals [Pa] (Hanna and Slepian, 1924; Mawardi, 1949; Eisner, 1967; Morse, 1948); Olson (1947, p. 101); Pierce (1981, p. 360). Extensive experimental analyses for various types of horns (conical, exponential, parabolic) along with a review of horn theory may be found in Goldsmith and Minton (1924). Of special interest is Eisner (1967) due to his history section and long list of relevant article.

The limits of the Webster horn equation: It is commonly stated that the Webster horn equation (WHEN) is fundamentally limited and thus is an approximation that only applies to frequencies much lower than $f_c$ (Morse, 1948; Shaw, 1970; Pierce, 1981). However, in all these discussions it is assumed that the area function $A(r)$ is the horn’s cross-sectional area, not the area of the iso-pressure wave-front.

In the next section it is shown that this “limitation” may be avoided (subject to the $f < f_c$ quasi-static limit, P10, p. 107), making the Webster horn theory an “exact” solution for the lowest-order “plane-wave” eigen-functions of Eq. 5.2.2.10. The limiting nature of the quasi-static approximation is that it “ignores” higher-order evanescent modes, which are naturally small since, being evanescent modes below their cutoff frequency, the wave number is real, thus they do not propagate (Hahn, 1941; Karal, 1953). This is the same approximation that is required to define an impedance, since every eigenmode has an impedance (Miles, 1948). This method is frequently called a modal-analysis or eigen analysis. These modes define a Hilbert “vector” space (aka, eigen-space).

As derived in Appendix F (p. 229), the acoustic variables (eigen-functions) are redefined on the iso-pressure wave-front boundary for the pressure and the corresponding volume velocity (Hanna and Slepian, 1924; Morse, 1948; Pierce, 1981). The resulting acoustic impedance is then the ratio of the pressure over the volume velocity. This approximation is valid up to the frequency where the first cross

---

7This condition may be written several ways, the most common being $ka < 1$, where $k = 2\pi/\lambda$ and $a$ is the horn radius. This may be expressed in terms of the diameter as $\pi^2/2 < 1$, or $d < \lambda/\pi < \lambda/2$. Thus $d < \lambda/2$ may be a more precise metric by the factor $\pi/2 \approx 1.6$. This is called the half-wavelength assumption, a synonym for the quasi-static approximation.
CHAPTER 5. STREAM 3B: VECTOR CALCULUS

Figure 5.3: Throat acoustical resistance $r_A$ and acoustical reactance $x_A$, frequency characteristics of infinite eigen-functions of the parabolic, conical, exponential, hyperbolic and cylindrical horns, having a throat area of $1 \text{ cm}^2$. Note how the “critical” frequency (defined here as the frequency where the reactive and real parts of the radiation impedance are equal) of the horn reduces dramatically with the type of horn. For the uniform horn, the reactive component is zero, so there is no cutoff frequency. For the parabolic horn (1) the cutoff is around 3 kHz. For the conical horn (2) the cutoff is at 0.6 [kHz]. For the exponential horn (3) the critical frequency is around 0.18 [kHz], which is one 16th that of the parabolic horn. For each horn the cross-sectional area is defined as $100 \text{ cm}^2$ at a distance of $L = 1 \text{ m}$ from the throat (Olson, 1947, p. 101), (Morse, 1948, p. 283).

mode begins to propagate ($f > f_c$), which may be estimated from the roots of the Bessel eigen-functions (Morse, 1948). Perhaps it should be noted that these ideas, which come from acoustics, apply equally well to electromagnetics, or any other wave phenomena described by eigen-functions.

Visco-thermal losses: When losses are to be included, the wave number $\kappa(s) = s/c_o$ must be replaced with Eq. G.1.0.1 (p. 233). This introduces dispersion in the wave front due to the very small term $\beta_0 \sqrt{s}$, which contains a branch cut. When calculating the losses one must be careful that they are always on the correct sheet. In cases where precise estimates of the wave properties and input impedance are required, this term is critical.

The best known examples of wave propagation are electrical and acoustic transmission lines. Such systems are loosely referred to as the telegraph or telephone equations, referring back to the early days of their discovery (Heaviside, 1892; Campbell, 1903; Brillouin, 1953; Feynman, 1970a). In acoustics, wave-guides are known as horns, such as the horn connected to the first phonographs from around the turn of the century (Webster, 1919). Thus the names reflect the historical development, to a time when the mathematics and the applications were running in close parallel.

5.2.3 Matrix formulation of the WHEN

Newton’s laws of conservation of momentum (Eq. 5.2.1.5) and mass (Eq. 5.2.1.6) are modern versions of Newton’s starting point for calculating the horn lowest-order plane-wave eigenmode wave speed.

The acoustic equations for the average pressure $P(r, \omega)$ and the volume velocity are derived in Appendix F, where the pressure and particle velocity equations (Eqs. F.1.1.4, F.1.2.6) are transformed into a $2 \times 2$ matrix of acoustical variables, average pressure $P'(r, \omega)$ and volume velocity $V'(r, \omega)$

$$- \frac{d}{dr} \begin{bmatrix} P'(r, \omega) \\ V'(r, \omega) \end{bmatrix} = \begin{bmatrix} 0 & \frac{s \rho_0}{\eta_0 P_o} \\ s A(r) & 0 \end{bmatrix} \begin{bmatrix} P(r, \omega) \\ V(r, \omega) \end{bmatrix}. \quad (5.2.3.11)$$

Define

$$M(r) = \rho_0 / A(r) \quad \text{and} \quad C(r) = A(r) / \eta_0 P_o \quad (5.2.3.12)$$

as the per-unit-length mass and compliance of the horn (Ramo et al., 1965, p. 213). The product of $P'(r, \omega)$ and $V'(r, \omega)$ define the acoustic power while their ratio defines the horns admittance admittance $Y_{im}^+(r, s)$, looking in the two directions (Pierce, 1981, p. 37-41).
5.3. THREE EXAMPLES OF FINITE LENGTH HORNS

To obtain the Webster horn pressure equation Eq. 5.2.2.10 from Eq. 5.2.3.11, take the partial derivative of the top equation

\[
\frac{\partial^2 P}{\partial r^2} = s \frac{\partial M(r)}{\partial r} \nu' + s M(r) \frac{\partial \nu}{\partial r},
\]

and then use the lower equation to remove \( \partial \nu / \partial r \)

\[
\frac{\partial^2 P}{\partial r^2} - s \frac{\partial M(r)}{\partial r} \nu = s^2 M(r) C(r) P = \frac{s^2}{c_o^2} P.
\]

Note that \( c_o^2 = MC = \left( \frac{\rho_o}{3\pi} \right) \cdot \left( \frac{\Delta \eta}{\rho_o \nu_o} \right). \)

Then use the upper equation a second time to remove \( \nu \)

\[
\frac{\partial^2}{\partial r^2} P + \frac{1}{A(r)} \frac{\partial A(r)}{\partial r} \frac{\partial}{\partial r} P = \frac{s^2}{c_o^2} P(r, s).
\] (5.2.3.13)

By use of the chain rule, equations of this form may be directly integrated, since

\[
- \nabla_r P = \frac{1}{A(r)} \frac{\partial}{\partial r} \left[ A(r) \frac{\partial}{\partial r} P(r, s) \right] = \frac{\partial^2}{\partial r^2} P(r, s) + \frac{1}{A(r)} \frac{\partial A(r)}{\partial r} P(r, s).
\] (5.2.3.14)

This is equivalent to integration by parts, with integration factor \( A(r) \). Next we set \( \kappa(s) \equiv s / c_o \), which later will be generalized to include visco-thermal losses (Eq. G.1.0.1, p. 233).

Merging Eqs. 5.2.3.13 and 5.2.3.14 results in the Webster horn equation (WHEN) (Eq. 5.2.2.10, p. 155):

\[
\frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} P(r, s) \right) = \kappa^2(s) P(r, s) \leftrightarrow \frac{1}{c_o^2} \frac{\partial^2}{\partial r^2} \nu(r, t).
\] (5.2.3.15)

Equations having this form are known as Sturm-Liouville equations.

This important class of ordinary differential equations follows from the use of separation of variables of the Laplacian, in any (i.e., every) separable coordinate system (Morse and Feshbach, 1953, Ch. 5.1, p. 494-523). The frequency domain eigen-solutions are denoted \( P^\pm(r, s) \), which have a corresponding volume velocities, denoted \( \nu^\pm(r, s) \).

Summary: We transformed the 3D acoustic wave equation into acoustic variables (Eq. 5.2.1.7, p. 154) in Appendix F by the application of Gauss’s law, resulting in the 1D Webster horn equation (aka WHEN) (Eq. 5.2.2.10, p. 155), which is a non-singular Sturm-Liouville equation. Thus we demonstrated that Eqs. 5.2.1.7 and 5.2.3.11 reduce to to Eq. 5.2.3.15 in a horn.

5.3. Three examples of finite length horns

Summary of four classic horns: Figure 5.3 (p. 156) is taken from the classic book of Olson (1947, p. 101), showing the radiation impedance \( Z_{rad}(r, \omega) \) for five horns. Table 5.2 (p. 158) summarizes the properties of four of these: the uniform (cylindrical) \( A(x) = A_o \), parabolic \( A(r) = A_o r \), conical (spherical) \( A(r) = A_o r^2 \) and the exponential \( A(r) = A_o e^{2mr} \), three of which are discussed next.

5.3.1 Uniform horn

The 1D wave equation \([A(r) = A_o]\)

\[
\frac{d^2 P}{dr^2} = \kappa^2(s) P,
\]

where we set \( \kappa^2(s) \equiv s^2 / c_o^2 \), which later will be generalized to include visco-thermal losses (Eq. G.1.0.1, p. 233).

8The Webster horn equation is closely related to the Schrödinger’s equation (Salmon, 1946).
Table 5.2: Table of horns and their properties for \( N = 1, 2 \) or 3 dimensions, along with the exponential horn (EXP). In this table the horn’s range variable is \( \xi [m] \), having area \( A(\xi) [m^2] \), diameter \( \xi_o = \sqrt{A(\xi_o)/\pi} [m] \). \( F(r) \) is the coefficient on \( q_r, \, \kappa(s) \equiv s/c_o \), where \( c_o \) is the speed of sound and \( s = \sigma + \omega \) is the Laplace frequency. The range variable \( \xi \) may be rendered dimensionless (see Fig. 5.3) if scaled by \( L \) (i.e., \( r \equiv \xi L \)), with \( \xi \) the linear distance along the horn axis, from \( r_o/L \leq \xi \leq 1 \), corresponding to \( r_o \leq r \leq L \), having area \( A(\xi_o) [m^2] \leq A(\xi) [m^2] \leq 4\pi L^2 \). The horn’s eigen-functions are \( P^\pm(\xi, \omega) \leftrightarrow q^\pm(\xi, t) \). When \( \pm \) is indicated, the outbound solution corresponds to the negative sign. Eigen-functions \( H^\pm(\xi, s) \) are outbound and inbound Hankel functions. The last column is the input radiation admittance, normalized by the characteristic admittance \( \gamma^+_c(r) = A(r)/\rho_o c_o \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>Name</th>
<th>radius</th>
<th>Area/( A_o )</th>
<th>( F(r) )</th>
<th>( P^\pm(r, s) )</th>
<th>( q^\pm(r_o, t) )</th>
<th>( Y^\pm_{\text{rad}}/\gamma^+_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D</td>
<td>uniform</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( e^{\pm\kappa(s)r} )</td>
<td>( \delta(t) )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>2D</td>
<td>parabolic</td>
<td>( \sqrt{r/r_o} )</td>
<td>( r/r_o )</td>
<td>1/( r )</td>
<td>( H^+(\mp j\kappa(s)r) )</td>
<td>( \delta(t) )</td>
<td>( -\frac{2\pi H^+_e}{H^+_o} )</td>
</tr>
<tr>
<td>3D</td>
<td>conical</td>
<td>( r )</td>
<td>( r^2 )</td>
<td>2/( r )</td>
<td>( e^{\mp\kappa(s)r}/r )</td>
<td>( \delta(t) \pm \frac{\omega}{\kappa L} u(t) )</td>
<td>( 1 \pm c_o/sr_o )</td>
</tr>
<tr>
<td>EXP</td>
<td>exponential</td>
<td>( e^{\eta r} )</td>
<td>( e^{2\eta r} )</td>
<td>2( m )</td>
<td>( e^{-(m^2+\kappa^2)\xi}/\kappa )</td>
<td>( \gamma^+_c )</td>
<td>Eq. 5.3.1.1</td>
</tr>
</tbody>
</table>

Solution: The two eigen-functions of this equation are the two d’Alembert waves (Eq. 4.3.0.1, p. 121)

\[
\phi(x, t) = \alpha e^{\kappa x} - \beta e^{-\kappa x} \leftrightarrow \alpha e^{-\kappa(x-L)} + \beta e^{\kappa(x-L)},
\]

where \( \kappa(s) = s/c_o = \omega/c \) is denoted the propagation function (aka: wave-evolution function, propagation constant, and wave number) and \( \alpha, \beta \) are the amplitudes of the two waves.

Note that for the uniform lossless horn \( \omega/c_o = 2\pi/\lambda \). It is convenient to normalize \( P^+_0 = 1 \) and \( P^-_L = 1 \), as was done for the general case.

The characteristic admittance \( \gamma^+_c(x) \) (Eq. F.1.4.9) is independent of direction. The signs must be physically chosen, with the velocity \( \gamma^\pm \) into the port, to assure that \( \gamma^+_c > 0 \).

Applying the boundary conditions: The general solution in terms of the eigenvector matrix, evaluated at \( x = L \), is

\[
\begin{bmatrix}
P(x) \\
Q(x)
\end{bmatrix}_L =
\begin{bmatrix}
e^{-\kappa L} & e^{\kappa(x-L)} \\
\gamma^+_c e^{-\kappa L} & -\gamma^+_c e^{\kappa(x-L)}
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}_L
= \begin{bmatrix}
e^{-\kappa L} & 1 \\
\gamma^+_c e^{-\kappa L} & -\gamma^+_c
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}_L,
\]

where \( \alpha, \beta \) are the relative weights on the two unknown eigen-functions, to be determined by the boundary conditions at \( x = 0, L \). \( \kappa = s/c \) and \( \gamma^+_c = 1/Z_o = A_o/r_o c_o \).

Solving Eq. 5.3.1.1 for \( \alpha \) and \( \beta \) with determinant \( \Delta = -2\gamma^+_c e^{-\kappa L} \),

\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}_L =
\begin{bmatrix}
-1 & -\gamma^+_c \\
-\gamma^+_c e^{-\kappa L} & e^{-\kappa L}
\end{bmatrix}
\begin{bmatrix}
P(x) \\
Q(x)
\end{bmatrix}_L
= \frac{1}{2}
\begin{bmatrix}
e^{\kappa L} & -Z e^{\kappa L} \\
1 & Z
\end{bmatrix}
\begin{bmatrix}
P \\
Q
\end{bmatrix}_L.
\]

In the final step we swapped all the signs, including on \( Q \), and moved \( Z_o = 1/\gamma^+_c \) inside the matrix.

We may uniquely determine these two weights given the pressure and velocity at the boundary \( x = L \), which is typically determined by the load impedance \( (P_L/Q_L) \).

The weights may now be substituted back into Eq. 5.3.1.1 to determine the pressure and velocity amplitudes at any point \( 0 \leq x \leq L \).

\[
\begin{bmatrix}
P \\
Q
\end{bmatrix}_x
= \frac{1}{2}
\begin{bmatrix}
e^{-\kappa L} & e^{\kappa(x-L)} \\
\gamma^+_c e^{-\kappa L} & -\gamma^+_c e^{\kappa(x-L)}
\end{bmatrix}_x
\begin{bmatrix}
e^{\kappa L} & -Z e^{\kappa L} \\
1 & Z
\end{bmatrix}
\begin{bmatrix}
P \\
Q
\end{bmatrix}_L.
\]

Setting \( x = 0 \) and multiplying these out gives the final transmission matrix

\[
\begin{bmatrix}
P \\
Q
\end{bmatrix}_0
= \frac{1}{2}
\begin{bmatrix}
e^{\kappa L} + e^{-\kappa L} & Z_o (e^{\kappa L} - e^{-\kappa L}) \\
\gamma^+_c (e^{\kappa L} - e^{-\kappa L}) & e^{\kappa L} + e^{-\kappa L}
\end{bmatrix}_x
\begin{bmatrix}
P \\
Q
\end{bmatrix}_L.
\]
Note the diagonal terms are $\cosh \kappa L$ and off-diagonal terms are $\sinh \kappa L$.

Applying the last boundary condition, we evaluate Eq. 5.3.1.2 to obtain the ABCD matrix at the input ($x = 0$) (Pipes, 1958)

$$\begin{bmatrix} P \\ q' \end{bmatrix}_0 = \begin{bmatrix} \cosh \kappa L & \hat{Z}_r \sinh \kappa L \\ \gamma_r \sinh \kappa L & \cosh \kappa L \end{bmatrix} \begin{bmatrix} P \\ -q' \end{bmatrix}_L.$$  \hspace{1cm} (5.3.1.5)

Note that the determinant is 1, thus the system is reciprocal.

**Exercise:** Evaluate this expression in terms of the load impedance.

**Solution:** Since $Z_{\text{load}} = -\frac{P}{V}$, \hspace{1cm} (5.3.1.6)

\[ \frac{P}{V} \bigg|_0 = \frac{Z_{\text{load}} \cosh \kappa L - \hat{Z}_r \sinh \kappa L}{Z_{\text{load}} \gamma_r \sinh \kappa L - \cosh \kappa L}. \]

**Impedance matrix:** Expressing Eq. 5.3.1.5 as an impedance matrix gives (algebra required)

$$\begin{bmatrix} p_0 \\ p_L \end{bmatrix} = \frac{\hat{Z}_r}{\sinh(\kappa L)} \begin{bmatrix} \cosh(\kappa L) & 1 \\ 1 & \cosh(\kappa L) \end{bmatrix} \begin{bmatrix} q_0 \\ q'_L \end{bmatrix}. \hspace{1cm} \text{(5.3.1.5)}$$

**Exercise:** Write out the short-circuit ($V_L = 0$) input impedance $Z_{\text{in}}(s)$ for the uniform horn.

**Solution:**

\[ Z_{\text{in}}(s) = \left. \frac{P}{V} \right| = \hat{Z}_r \frac{\cosh \kappa L}{\sinh \kappa L} = \hat{Z}_r \tanh \kappa L \bigg|_{V_L = 0}. \]

**Input admittance $Y_{\text{in}}$:** Given the input admittance of the horn, it is possible to determine if it is uniform, without further analysis. Namely, if the horn is uniform and infinite in length, the input admittance at $x = 0$ is

\[ Y_{\text{in}}(x = 0, s) = \left. \frac{V(0, \omega)}{P(0, \omega)} \right| = \gamma_r, \]

since $\alpha = 1$ and $\beta = 0$. That is, for an infinite uniform horn, there are no reflections.

When the uniform horn is terminated with a fixed impedance $\hat{Z}_r$ at $x = L$, one may substitute pressure and velocity measurements into Eq. 5.3.1.2 to find $\alpha$ and $\beta$, and given these, one may calculate the pressure reflectance at $x = L$ (Eq. 3.3.2.6, p. 82)

\[ \Gamma_L(s) = \frac{\beta}{\alpha} = \frac{\hat{P}(L, \omega) - \hat{Z}_r V'(L, \omega)}{\hat{P}(L, \omega) + \hat{Z}_r V'(L, \omega)} = \frac{Z_L - \hat{Z}_r}{Z_L + \hat{Z}_r}, \]

given sufficiently accurate measurements of the throat pressure $P(0, \omega)$, velocity $V'(0, \omega)$, and the characteristic impedance of the input $\hat{Z}_r = \rho_o c/A(0)$.

### 5.3.2 Conical horn

Using the conical horn area $A(r) \propto r^2$ in Eq. 5.2.2.10, p. 155 [or Eq. 5.2.3.11 (p. 156)] results in the spherical wave equation

\[ \mathcal{P}_{rr}(r, \omega) + \frac{2}{r} \mathcal{P}_r(r, \omega) = \kappa^2 \mathcal{P}(r, \omega). \hspace{1cm} (5.3.2.7) \]

where $\kappa^2(s) \equiv s^2/c_0^2$. 

Radiation admittance for the conical horn: The conical horn’s acoustic input admittance $Y_{in}(r, s)$ at any location $r$ is found by dividing Eq. F.1.1.4 (p. 229) by $P(r, s)$

$$Y_{in}^{\pm}(r, s) = \frac{q_{r}^{\pm}}{p_{r}^{\pm}} = -\frac{A(r)}{s\rho_o} \frac{d}{dr} \ln p_{r}^{\pm}(r, s)$$

$$= \gamma_{r}(r) \left[ 1 \pm \frac{c_{0}}{sr} \right] \leftrightarrow \frac{A(r)}{\rho_o c_{0}} \left( \delta(t - r/c_{o}) \pm \frac{c_{0}}{r} u(t - r/c_{o}) \right).$$

Note how the pressure pulse is delayed by $r/c_{o}$ due to $e^{-\kappa(s)r}$, as it travels down the horn. As the area of the horn increases, the pressure decreases as $1/r = 1/\sqrt{A(r)}$. This results in the uniform back-flow due to conservation of mass and the characteristic admittance $\gamma_{r}(r)$ variation with $r$.

### 5.3.3 Exponential horn

If we define the area as $A(r) = e^{2mr}$, the eigen-functions of the horn are

$$p_{r}^{\pm}(r, \omega) = e^{-mr} \frac{e^{\pm j\sqrt{\omega^2 - \omega_c^2} r/c}}{r},$$

which may be shown by the substitution of $P_{c}^{\pm}(r, \omega)$ into Eq. 5.2.2.10 (p. 155), with $A(r) = e^{2mr}$.

This case is of special interest because the radiation impedance is purely reactive below the horn’s cutoff frequency ($\omega < \omega_c = mc_{o}$), as may be seen from curves 3 and 4 of Fig. 5.3 (p. 156). As a result, no energy can radiate from an open horn for $\omega < \omega_c$ because

$$\kappa(s) = -m \pm \frac{j}{c_{o}} \sqrt{\omega^2 - \omega_c^2}$$

is purely real (this is the case of non-propagating evanescent waves).

Using Eq. 4.3.1.7 (p. 122) the input admittance is

$$Y_{in}^{\pm}(x, s) = -\frac{A(x)}{s\rho_o} \left( m \pm \sqrt{m^2 + \kappa^2} \right) x.$$  

Kleiner (2013) gives an equivalent expression for $Y_{in}(x, \omega)$ having area $S(x) = e^{mx}$

$$Y_{in}(x, \omega) = \frac{S(x)}{\omega \rho} \left[ \frac{m}{2} + j \frac{\sqrt{4\omega^2 - (mc)^2}}{2c} \right],$$

and impedance

$$Z_{in}(r, \omega) = \frac{\rho c}{S_T} \left[ j \frac{\omega_c}{\omega} + \sqrt{1 - \left( \frac{\omega_c}{\omega} \right)^2} \right],$$

where $\omega_c(r)$ is the cutoff frequency.

One may use expansions of $A(r)$ in a Fourier-like exponential series along with superposition, to find the general solution for an arbitrary analytic $A(r)$.

### 5.4 Solution methods

Two distinct mathematical techniques are described to model the wave equation in physical systems: 1) partial differential equations (PDEs) and 2) lumped-element models (i.e., quasi-statics). We shall describe both these methods for the case of the scalar wave equation. The first of these is called seperation of variables. This method is limited to a small and restrictive number of separable coordinate systems (SCS). Once the SCS is chosen, the eigen-functions are known, for any SCS. The differential equations that result from the separation of variables are known as Sturm-Liouville equations, which are always scalar (ordinary) differential equations (ODEs).

A second method of solution is to limit an upper frequency limit, so that quasi-statics can be assumed (the wavelength must be larger than the size of the object being modeled). This method lends itself to the lumped-element transmission method.
1a. **Separable coordinate systems:** Classically PDEs are solved by *separation of variables*. Morse (1948, p. 296-7) shows this method is limited to a few coordinate systems, such as rectangular, cylindrical and spherical coordinates. Even a slight deviation from separable specific coordinate systems represents a major barrier toward further analysis and understanding, blocking insight into more general cases. Separable coordinate systems have a high degree of symmetry. Note that the solution of the wave equation is not tied to a specific coordinate system.

1b. **Sturm-Liouville methods and eigenvectors:** When the coordinate system is separable the resulting PDEs are always reduced to *Sturm-Liouville* equations. The important special class of Sturm-Liouville equations are solved since their eigen-functions are tabulated. Webster horn theory (Webster, 1919; Morse, 1948; Pierce, 1981) is a generalized Sturm-Liouville equation, which adds physics, in the form of the horn’s area-function. The Webster equation side-steps the seriously limiting problem of separation of variables by using the alternative quasi-static solution, which ignores high-frequency higher order evanescent modes. This is essentially a 1-dimensional lowpass approximation to the wave equation.

Mathematics provides rigor, while physics provides understanding. While both are important, it is the physical applications that make a theory useful.

2. **Lumped-element method:** As previously described (see page 92) a system may be represented in terms of lumped-elements, as either electrical inductors, capacitors and resistors or their mechanical counterparts, masses, springs and dash-pots. Such systems are represented by $2 \times 2$ transmission-matrices in the $s$ (i.e., Laplace) domain (Ramo et al., 1965).

When the system of lumped-element networks contains only resistors and capacitors, or resistors and inductors, the system does not support waves, and is related to the diffusion equation in its solution. Depending on the elements in the system of equations, there can be an overlap between a diffusion process and scalar waves, represented as transmission lines, both modeled as lumped-element networks of $2 \times 2$ matrices (Eq. 3.5.0.1, p. 92) (Campbell, 1922; Brillouin, 1953; Ramo et al., 1965).

**Example:** Quasistatic methods provide band-limited solutions below a critical frequency $f_c$, where a half wavelength approaches the element spacing $\Delta$, for a much wider class of geometries, by avoiding higher-order, high-frequency cross-modes. The model of the train is depicted in Fig. 5.4. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL.

Below the mechanical mass-sprint system it is the electrical equivalent circuit. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is $\Delta$ [m].

When the wavelength $\lambda = c_o/f_c$ is greater than twice the physical distance $\Delta$ between the elements

$$\lambda > \lambda_c = 2\Delta \quad [m],$$

the approximation is mathematically equivalent to a transmission line. As is described in DE-3, problem #2, the velocity is $c_o = 1/\sqrt{MC}$ [m/s]. As the frequency increases, the wavelength becomes shorter. When the frequency is equal to the critical frequency $f_c$, the critical wavelength $\lambda_c = c_o/f_c = 2\Delta$. Above the critical frequency the quasi-static (lumped-element) model breaks down and switches from a delay line to a lowpass filter, as discussed in DE-3, problem #2.

The frequency is under the control of the modeling process, as more elements may be added, to represent higher frequencies (shorter wavelengths). If the nature of the solution at high frequencies ($f > f_c$) is desired, one must add more sections, thereby increasing $f_c$. For many (perhaps most) problems, lumped elements are easy to use, and accurate Brillouin (1953); Ramo et al. (1965) as long as you don’t violate the upper frequency limit.
Figure 5.4: Depiction of a train consisting of cars, treated as a mass $M$ and linkages, treated as springs of stiffness $K$ or compliance $C = 1/K$. The equivalent electrical circuit is shown below the mass-spring system, with the masses modeled by inductors ($M$) and springs modeled as capacitors ($C$). For this model to accurately represent a transmission line the frequency must be less than the equivalent Nyquist frequency $f_c$. As the frequency is increased above $f_c$, the wavelength becomes shorter than the critical wavelength $\lambda_c = 2\Delta$, and the model becomes a lowpass filter, strongly departing from a transmission line, which has a flat frequency response.

5.4.1 Eigenfunctions $\varrho^\pm(r,t)$ of the WHEN

Because the wave equation (Eq. 5.2.1.7) is 2nd order in time, there are two causal independent eigenfunction solutions of the homogeneous (i.e., un-driven) Webster horn equation: an outbound (right-traveling) $\varrho^+(r,t)$ and an inbound (left-traveling) $\varrho^-(r,t)$ wave. These causal eigen-solutions may be Laplace transformed into the frequency-domain

$$\varrho^\pm(r,t) \leftrightarrow \mathcal{F}^\pm(r,s) = \int_0^\infty \varrho^\pm(r,t)e^{-st}dt.$$  

They may be normalized so that $\mathcal{F}^\pm(r_o,s) = 1$, where $r_o$ is the source excitation reference point.

Every eigen-function depends on an area function $A(r)$ (Eq. 5.2.2.10, p. 155). In theory then, given one it should be possible to find the other. This is known as the inverse problem, which is generally believed to be an unsolved problem. For example, given the eigenvalues $\lambda_k$, how does one determine the corresponding area function $A(r)$?

Because the characteristic impedance $\gamma_c(r)$ of the wave in the horn changes with location, there must be local reflections due to these area variations. Thus there are fundamental relationships between the area change $dA/dr$, the horn's eigen-functions $\mathcal{F}^\pm(r,s)$, eigenmodes and input impedance.

Complex vs. real frequency: We shall continue to maintain the distinction that functions of $\omega$ are Fourier transforms and causal functions of Laplace frequency $s$ correspond to Laplace transforms, which are necessarily complex analytic in $s$ in the right half-plane (RHP) region of convergence (RoC). This distinction is critical since we typically describe impedance $Z(s)$, and admittance $Y(s)$, as complex analytic functions in $s$, in terms of their poles and zeros. The eigen-functions $\mathcal{F}^\pm(r,s)$ of Eq. 5.2.2.10 are also causal complex analytic functions of $s$.

Plane-wave eigen-function solutions: Huygens (1690), three years after Newton’s publication of Principia, was the first to gain insight into wave propagation, today known as “Huygens’s principle.” While his concept showed a deep insight, we now know it was seriously flawed, as it ignored the backward traveling wave (Miller, 1991). In 1747 d’Alembert published the first correct solution for the plane-wave scalar wave equation

$$\varrho(x,t) = f(t-x/c_o) + g(t+x/c_o),$$  

where $f(\cdot)$ and $g(\cdot)$ are general functions of their argument. Why this is the solution may be easily shown by use of the chain rule, by taking partials with respect to $x$ and $t$.

In terms of the physics, d’Alembert’s general solution describes two arbitrary wave-forms $f(\cdot)$, $g(\cdot)$ traveling at a speed $c_o$, one forward and one reversed. Thus his solution is quite easily visualized.
5.5. INTEGRAL DEFINITIONS OF $\nabla()$, $\nabla \cdot ()$, $\nabla \times ()$ AND $\nabla \wedge ()$

**Exercise:** By the use of the chain rule, prove that d’Alembert’s formula satisfies the 1D wave equation.

**Solution:** Taking a derivative with respect to $t$ and $r$ gives

- $\partial_t \varrho(r, t) = -c_0 f'(r - c_0 t) + c_0 g'(r + c_0 t)$
- $\partial_r \varrho(r, t) = f'(r - c_0 t) + g'(r + c_0 t)$,

and a second derivative gives

- $\partial_{tt} \varrho(r, t) = c_0^2 f''(r - c_0 t) + c_0^2 g''(r + c_0 t)$
- $\partial_{rr} \varrho(r, t) = f''(r - c_0 t) + g''(r + c_0 t)$.

From these last two equations we have the 1D wave equation

$$\partial_{rr} \varrho(r, t) = \frac{1}{c_0^2} \partial_{tt} \varrho(r, t),$$

having solutions Eq. 5.4.1.1.

**Example:** Assuming $f(\cdot), g(\cdot)$ are $\delta(\cdot)$, find the Laplace transform of the solution corresponding to the uniform horn $A(x) = 1$.

**Solution:** Using Table C.3 (p. 208) of Laplace transforms on Eq. 5.4.1.1 gives

$$\varrho(x, t) = \delta(t - x/c_0) + \delta(t + x/c_0) \leftrightarrow e^{-sx/c_0} + e^{sx/c_0}. \quad (5.4.1.2)$$

Note that the delay $T_o = \pm x/c_0$ depends on the range $x$.

**3D d’Alembert spherical eigen-functions:** The d’Alembert solution generalizes to spherical waves by changing the area function of Eq. 5.2.2.10 to $A(r) = A_o r^2$ (See Eq. 5.1.3.10, p. 145 and Table 5.2, p. 158).

The wave equation then becomes

$$\nabla^2 r \varrho(r, t) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \varrho(r, t) = \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \varrho(r, t).$$

Multiplying by $r$ results in the general spherical (3D) d’Alembert wave equation solution

$$\varrho(r, t) = \frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}$$

for arbitrary wave-forms $f(\cdot)$ and $g(\cdot)$. These are the eigen-functions for the spherical scalar wave equation.

5.5 **Integral forms of $\nabla()$, $\nabla \cdot ()$ and $\nabla \times ()$**

The vector wave equation describes the evolution of a vector field, such as Maxwell’s electric field vector $E(x, t)$. When these fields are restricted to a one-dimensional domain they are known as guided waves constrained by wave guides.

These equations use thee differential vector operators, the gradient, divergence and the curl. There are two forms of definitions for each of these three operators: differential and integral. The integral form provides a more intuitive view of the operator, which in the limit converges to the differential form. Following a discussion of the gradient, divergence and curl integral operators, these two forms are discussed.

In addition there are three fundamental vector theorems: Gauss’s law (divergence theorem), Stokes’s law (curl theorem) and Helmholtz’s decomposition theorem. Without the use of these very fundamental vector calculus theorems, Maxwell’s equations cannot be understood.
5.5.1 Gradient: \( E = -\nabla \phi(x) \)

As shown in Fig. 5.1 (p. 140) the gradient maps \( \mathbb{R}^1 \rightarrow \mathbb{R}^3 \). The gradient is defined as the unit-normal \( \hat{n} \), weighted by the potential \( \phi(x) \), averaged over a closed surface \( S \)

\[
\nabla \phi(x) \equiv \lim_{S, V \to 0} \left\{ \frac{\int_S \phi(x) \hat{n} \, dS}{V} \right\} [V/m] \tag{5.5.1.1}
\]

having area \( S \) and volume \( V' \), centered at \( x \) (Greenberg, 1988, p. 773).\(^9\) Here \( \hat{n} \) is a dimensionless unit vector, perpendicular to the surface \( S \)

\[
\hat{n} = \frac{\nabla \phi}{\| \nabla \phi \|}. \tag{5.5.1.2}
\]

The dimensions of Eq. 5.5.1.1 are in the units of the potential times the area, divided by the volume, as needed for a gradient (e.g., \([V/m]\)). The units are potential-dependent. If \( \phi \) were temperature, the units would be \([\text{deg/m}]\).

Exercise: Justify the units of Eq. 5.5.1.1. Solution: The units depend on \( \phi \) per unit length. If \( \phi \) is voltage, then the gradient has units of \([V/m]\). Under the limit \( d|S|/||S|| \) must have units of \( m^{-1} \).

The natural way to define the surface and volume is to place the surface on the iso-potential surfaces, forming either a cube or pill-box shaped volume. As the volume \( ||S|| \) goes to zero, so must the area \( \|S\| \). One must avoid irregular volumes where the area is finite as the volume goes to zero (Greenberg, 1988, footnote p. 762).

A well-known example is the potential

\[
\phi(x, y, z) = \frac{Q}{\varepsilon_o \sqrt{x^2 + y^2 + z^2}} = \frac{Q}{\varepsilon_o R} \ [V]
\]

around a point charge \( Q \) [SI Units of Coulombs]. The constant \( \varepsilon_o \) is the permittivity \([\text{F/m}^2]\). A second well-known example is the acoustic pressure potential around an oscillating sphere, which has the same form (Table 5.2, p. 158).

How does this work? To better understand what Eq. 5.5.1.1 means, consider a three-dimensional Taylor series expansion of the potential in \( x \) about the limit point \( x_o \)

\[
\phi(x) \approx \phi(x_o) + \nabla \phi(x) \cdot (x - x_o) + \text{HOT}.
\]

We could define the gradient using this relationship as

\[
\nabla \phi(x_o) = \lim_{x \to x_o} \frac{\phi(x) - \phi(x_o)}{x - x_o}.
\]

For this definition to apply, \( x \) must approach \( x_o \) along \( \hat{n} \). To compute the higher order terms (HOT) one needs the Hessian matrix.\(^{10}\)

The natural way to define a surface \( |S| \) is to take find the iso-potential contours. The gradient is in the direction of maximum change in the potential, thus perpendicular to the iso-potential contours. The secret to the integral definition is in taking the limit. As the volume \( ||S|| \) shrinks to zero, the HOT terms are small, and the integral reduces to the first-order term in the Taylor expansion since the constant term integrates to zero. Such a construction was used in the proof of the Webster horn equation (Appendix F, p. 229; Fig. F.1, p. 230).

The problem with Eq. 5.5.1.1 is that it is recursive since \( \hat{n} \) is based on the gradient and is the kernel of the integral. Thus the integral definition of the gradient is based on the gradient itself. Equation 5.5.1.1 is actually a statement of the mean value theorem for the gradient.


\(^{10}\) \( H_{i,j} = \partial^2 \phi / \partial x_i \partial x_j \), which will exist if the potential is analytic in \( x \) at \( x_o \).
5.5. INTEGRAL DEFINITIONS OF $\nabla()$, $\nabla\cdot()$, $\nabla\times()$ AND $\nabla\wedge()$

5.5.2 Divergence: $\nabla\cdot\mathbf{D} = \rho [\text{C/m}^3]$

As briefly summarized by Eq. 5.1.2.4 on p. 143, the definition of the divergence at $x = [x, y, z]^T$ is

$$\nabla\cdot\mathbf{D}(x, t) \equiv [\partial_x, \partial_y, \partial_z] \cdot \mathbf{D}(x, t) = \left[ \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right](x, t) = \rho(x, t),$$

which maps $\mathbb{R}^3 \rightarrow \mathbb{R}$.

![Figure 5.5](image)

**Figure 5.5:** Left: On the left is the physical layout of the integral. Right: The top equation is the integral definition of the divergence of $\mathbf{D}$ as an integral over the closed surface $S$, of the normal component of vector $\mathbf{D}$, in the limit as the surface and volume shrink to 0. The middle equation states the enclosed charge $Q_{\text{enc}}$ in terms of the surface integral of the normal component of $\mathbf{D}$. The bottom equation gives the charge enclosed in terms of a volume integral over the enclosed charge density $\rho_{\text{enc}}$.

5.5.3 Divergence and Gauss’s law

Like the gradient, the divergence of a vector field may be defined as the surface integral of a *compressible* vector field, as a limit as the volume, enclosed by the surface, goes to zero. As for the case of the gradient, for this definition to make sense, the surface $S$ must be closed, defining volume $\mathcal{V}'$. The difference is that the surface integral is over the normal component of the vector field being operated on (Greenberg, 1988, p. 762-763)

$$\nabla\cdot\mathbf{D} = \lim_{\mathcal{V}',S\to 0} \left\{ \oint_{\mathcal{S}} \hat{n} \cdot \mathbf{D} \, dS \right\} [\text{C/m}^3]$$

As defined previously (Eq. 5.5.1.2) and shown here in Fig. 5.5, $\hat{n}$ is a unit vector normal to a closed iso-potential surface $S$. The limit, as the volume and surface simultaneously go to zero, defines the total flux across the surface. Thus the surface integral is a measure of the total flux $\perp$ to the surface. It is helpful to compare this formula with that for the gradient Eq. 5.5.1.1.

**Gauss’s law:** The definitions in Fig. 5.5 resulted in Gauss’s Law, a major breakthrough in vector calculus. As summarized by Feynman (1970b, p. 13-2):

The current leaving the closed surface $S$ equals the rate of the charge leaving that volume $\mathcal{V}'$, defined by that surface.

For the electrical case this is equivalent to the observation that the total flux across the surface is equal to the net charge enclosed by the surface. Since the volume integral over the charge density $\rho(x, y, z)$ is the total charge enclosed $Q_{\text{enc}}$,

$$Q_{\text{enc}} = \iiint_{\mathcal{V}'} \nabla\cdot\mathbf{D} \, d\mathcal{V} = \iiint_{\mathcal{V}'} \mathbf{D} \cdot \hat{n} \, d\mathcal{S} \quad [\text{C}].$$

When the surface integral over the normal component of $\mathbf{D}(x)$ is zero, the total charge is zero. If there is only positive (or negative) charge inside the surface, $\nabla\cdot\mathbf{D} = \rho(x) = 0$ charge density must also
CHAPTER 5. STREAM 3B: VECTOR CALCULUS

\[ \nabla \times \mathbf{H} \equiv \lim_{\mathcal{S},S \to 0} \left\{ \int_{\mathcal{S}} \hat{n} \times \mathbf{H} \, dS \right\} \quad [\text{A/m}^2] \]

\[ I_{\text{enc}} = \int_{\mathcal{S}} (\nabla \times \mathbf{H}) \cdot \hat{n} \, dS = \oint_{\mathcal{B}} \mathbf{H} \cdot d\mathbf{l} \quad [\text{A}] \]

Figure 5.6: Left: The integral definition of the curl is related to that of the divergence (Eq. 5.5.3.3, page 165) as an integration over the tangent to the surface, except: (1) the curl is defined as the cross-product \( \hat{n} \times \mathbf{H} \) [A/m\(^2\)] of unit vector \( \hat{n} \) with the current density \( \mathbf{H} \) (Greenberg, 1988, page 823), and (2) the surface is open, leaving a boundary \( \mathcal{B} \) along the open edge. As with the divergence, which leads to Gauss’s law, this definition leads to a second fundamental theorem of vector calculus: Stokes’s law (also called the curl theorem). Right: Equations that summarize Stokes theorem (law).

be zero. It is clear that this result only holds in the quasi-static limit, which is always satisfied because \( \mathcal{S} \to 0 \).

Taking the derivative with respect to time gives the total current normal to the surface:

\[ I_{\text{enc}} = \int_{\mathcal{S}} D \cdot \hat{n} \, dS = \dot{Q}_{\text{enc}} = \iint_{\mathcal{S}} \dot{\rho}_{\text{enc}} \, d\mathcal{V}. \quad [\text{A}] \quad (5.5.3.5) \]

Of course to define a volume, the surface must be closed, a necessary condition for Gauss’s law. This reduces to a common-sense summary that can be grasped intuitively.

5.5.4 Integral definition of the curl: \( \nabla \times \mathbf{H} = C \)

As briefly summarized on page 144, the differential definition of the curl maps \( \mathbb{R}^3 \mapsto \mathbb{R}^3 \). The curl of the magnetic field strength \( \mathbf{H}(x) \) is the current density \( C = \sigma \mathbf{E} + \dot{\mathbf{D}} \)

\[ \nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{vmatrix} = C \quad [\text{A/m}^2]. \]

Curl and Stokes’s law: As in the cases of the gradient and divergence, the curl also may be written in integral form, allowing for the physical interpretation of its meaning:

The surface integral definition of \( \nabla \times \mathbf{H} = C \) [A/m\(^2\)], where the current density \( C \) is \( \perp \) to the rotation plane of \( \mathbf{H} \).

Stokes’s law states that the open surface integral over the normal component of the curl of the magnetic field strength (\( \hat{n} \cdot \nabla \times \mathbf{H} \) [A/m\(^2\)]) is equal to the line integral \( \oint_{\mathcal{B}} \mathbf{H} \cdot d\mathbf{l} \) along the boundary \( \mathcal{B} \). As summarized in Fig. 5.6, Stokes’s law is

\[ I_{\text{enc}} = \int_{\mathcal{S}} (\nabla \times \mathbf{H}) \cdot \hat{n} \, dS \quad [\text{A}] \]

\[ = \iint_{\mathcal{S}} C \cdot \hat{n} \, dS \]

\[ = \oint_{\mathcal{B}} \mathbf{H} \cdot d\mathbf{l}, \quad [\text{A}] \quad (5.5.4.6) \]

namely

The line integral of \( \mathbf{H} \) along the open surface’s boundary \( \mathcal{B} \) is equal to the total current enclosed \( I_{\text{enc}} \).
5.5. **Integral Definitions of \( \nabla() \), \( \nabla \cdot () \), \( \nabla \times () \) And \( \nabla \wedge () \)**

In many texts the normalization (denominator under the integral) is a volume \( V \) (e.g., Greenberg (1988, p. 778, 823-4)). However because the surface is open, this volume does not exist (when defining a volume the a surface must be closed). The definition must hold even in the limit when the curved surface \( S \) degenerates to a plane, with the boundary \( B \) enclosing \( S \). In this limit there is no volume.

To resolve this problem, we have taken the normalization to be the surface \( S \). Note that in the limit \( B \to 0 \), the limiting definition is independent of any curvature, since the integral is over the normal component of \( H \) (i.e., \( \hat{n} \perp H(x,t) \)). The net flux is independent of the curvature of \( S \) as \( B \to 0 \).

**Summing it up:** Since integration is a linear process (suns of smaller elements), one may tile (tessellate) the surface, breaking it up into smaller surfaces and their boundaries, the sum of which is equal to the integral over the original boundary. This is an important concept which leads to the proof of Stokes’s law.

Table 5.1 (p. 142) provides a description of the three basic integration theorems, along with their mapping domains. The integral formulations of Gauss’s and Stokes’s laws use \( \hat{n} \cdot D \) and \( H \times \hat{n} \) in the integrands. The key distinction between the two laws naturally follows from the properties of the scalar \((A \cdot B)\) and vector \((A \times B)\) products, as discussed in Fig. 3.4, p. 84. To fully appreciate the differences between Gauss’s and Stokes’s laws, these two types of vector products must be mastered.

Paraphrasing Feynman (1970b, 3-12),

1. \( \Phi_2 = \Phi_1 + \int_1^2 \nabla \Phi \cdot dS \)
2. \( \oint D \cdot \hat{n} dS = \oint \nabla \cdot D dV \)
3. \( \oint B \cdot dL = \oint S (\nabla \times E) \cdot \hat{n} dS \)

5.5.5 **Helmholtz’s decomposition theorem**

We must now rethink everything defined above, in terms of the two types of vector fields, that decompose every analytic vector field (Table 5.3). The *irrotational field* is defined as one that is “curl free.” An *incompressible field* is one that is “divergence free.” According to Helmholtz’s decomposition, every analytic vector field may be decomposed into independent rotational and compressible components (Helmholtz, 1978). Another name for Helmholtz decomposition is the fundamental theorem of vector calculus (FTVC); Gauss’s and Stokes’s theorems, along with Helmholtz’s decomposition form the three key fundamental theorems of vector calculus. Portraits of Helmholtz and Kirchhoff are provided in Fig. 5.8.

Table 5.3: The four possible classifications of scalar and vector potential fields: rotational/irrotational and compressible/incompressible. Rotational fields are generated by the vector potential (e.g., \( A(x,t) \)), while compressible fields are generated by the scalar potentials (e.g., voltage \( \phi(x,t) \), velocity \( \psi \), pressure \( \rho(x,t) \) or temperature \( T(x,t) \)).

<table>
<thead>
<tr>
<th>Field: ( v(x,t) )</th>
<th>Compressible ( \nabla \cdot v \neq 0 )</th>
<th>Incompressible ( \nabla \cdot v = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotational ( \nabla \times v \neq 0 )</td>
<td>( v = \nabla \phi + \nabla \times \omega )</td>
<td>( v = \nabla \times \omega )</td>
</tr>
<tr>
<td>Irrotational ( \nabla \times v = 0 )</td>
<td>Laplace’s Eq. ( \nabla^2 v = \frac{1}{c^2} \ddot{v} )</td>
<td>Boundary layers</td>
</tr>
<tr>
<td>Conservative ( \nabla \cdot v \neq 0 )</td>
<td>Acoustics ( v = \nabla \psi )</td>
<td>Statics ( \nabla^2 \phi = 0 )</td>
</tr>
<tr>
<td>( \nabla \times v = 0 )</td>
<td>( \nabla^2 \rho(x,t) = \frac{1}{c^2} \ddot{\rho}(x,t) )</td>
<td>Laplace’s Eq. ( (c \to \infty) )</td>
</tr>
</tbody>
</table>

A magnetic solenoidal field is a uniform flux field \( \mathbf{B}_z(x) \) that is generated by a solenoidal coil, and to an excellent approximation, is uniform inside the coil, making it similar to that of a permanent

---

11 These theorems are mathematical relationships that follow from physical principles.
magnet. As a result, the divergence of a solenoidal field is, to a good approximation, zero, making it incompressible ($\nabla \cdot \mathbf{B} = 0$) and rotational ($\nabla \times \mathbf{B} \neq 0$).

I recommend you know this term, since it is widely used, but suggest the preferred terms are incompressible and rotational. Strictly speaking, the term “solenoidal field” only applies to a magnetic field produced by a solenoid, making the term specific to that case.

Figure 5.7: A solenoid is a uniform coil of wire. When a current is passed through the wire, a uniform magnetic field intensity $\mathbf{B}$ is created. From a properties point of view, this coil is indistinguishable from a permanent bar magnet, having north and south poles. Depending on the direction of the current, one end of a finite solenoidal coil is the north pole of the magnet, and the other end is the south pole. The uniform field inside the coil is called solenoidal, a confusing synonym for irrotational. (Figure from Wikipedia.)

**Helmholtz’s decomposition of a differentiable vector field:** This theorem is easily stated (and proved), but less easily appreciated (Heras, 2016). A physical description facilitates: Every vector field may be split into two independent parts: dilation and rotation. We have seen this same idea appear in vector algebra, where the scalar and wedge–products of two vectors are perpendicular (Fig. 3.4, p. 84).

For example, think of linear versus angular momentum, which are independent in that they represent different ways of delivering kinetic energy, via different modalities (degrees of freedom, DOF). Linear and rotational motions are a common theme in physics, rooted in geometry. Thus it seems a natural extension to split a vector field into independent dilation and rotational parts. For the case of fluid mechanics these modalities can couple through friction, due to viscosity.

A fluid, with mass and momentum can be moving along a path, and independently be rotating. These independent modes of motion correspond to different types (modes) of kinetic energy, such as translational, compressional and rotational. Each eigenmode of vibration can be viewed as an independent degree of freedom (DoF).

Helmholtz’s decomposition theorem (aka FTVC) quantifies these degrees of freedom. Second order vector identities $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ and $\nabla \times (\nabla \mathbf{A}) = 0$ may be used to verify the FTVC. The role of the FTVC is especially powerful when applied to Maxwell’s Eqs.

**The four categories of linear fluid flow:** The following is a summary of the four cases for fluid flow, as summarized in Fig. 5.3:

1.1 Compressible, rotational fluid (general case): $\nabla \psi \neq 0, \nabla \times \mathbf{w} \neq 0$. This is the case of wave propagation in a medium where viscosity cannot be ignored, as in the case of acoustics close to the boundaries, where viscosity contributes to losses (Batchelor, 1967).

1.2 Incompressible, rotational fluid (Lubrication theory): $\mathbf{v} = \nabla \times \mathbf{w} \neq 0, \nabla \cdot \mathbf{v} = 0, \nabla^2 \psi = 0$. In this case the flow is dominated by the walls, while the viscosity and heat transfer introduce shear. This is typical of lubrication theory and solenoidal fields.

2.1 Compressible, irrotational fluid (acoustics): $\mathbf{v} = \nabla \psi, \nabla \times \mathbf{w} = 0$. Here losses (viscosity and thermal diffusion) are small (assumed to be zero). One may define a velocity potential $\psi$, the gradient of which gives the air particle velocity, thus $\mathbf{v} = -\nabla \psi$. Thus for an irrotational fluid $\nabla \times \mathbf{v} = 0$ (Greenberg, 1988, p. 826). This is the case of the conservative field, where $\int_{\partial V} \mathbf{v} \cdot \mathbf{n} dS$ only depends on the end points, and $\oint_{\partial V} \mathbf{v} \cdot \mathbf{n} dS = 0$. When a fluid may be treated as having no viscosity, it is typically assumed to be irrotational, since it is the viscosity that introduces the shear (Greenberg, 1988, p. 814). A fluid’s angular velocity is $\Omega = \frac{1}{2} \nabla \times \mathbf{v}$, thus irrotational fluids have zero angular velocity ($\Omega = 0$).

2.2 Incompressible, irrotational fluid (statics): $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = 0$ thus $\mathbf{v} = \nabla \psi$ and $\nabla^2 \psi = 0$. An example of such a case is water in a small space at low frequencies, where the wavelength is long compared to the size of the container; the fluid may be treated as incompressible. When
\[ \nabla \times \mathbf{v} = 0 \], the effects of viscosity may be ignored, as it is the viscosity that creates the shear leading to rotation. This is the case of modeling the cochlea, where losses are ignored and the quasi-static limit is justified.

In summary, each of the cases is an approximation that best applies in the low frequency limit. This is why it is called quasi-static, meaning low, but not zero frequency, where the wavelength is large compared with the dimensions (e.g., diameter).

### 5.5.6 Second-order operators

Besides the first-order vector derivatives are second-order operators, the most important being the scalar Laplacian \( \nabla^2() = \nabla \cdot \nabla() \) and the vector Laplacian \( \nabla^2() = \nabla \cdot \nabla() \), which operates on vectors.\(^\text{12}\)

**Terminology:**

There are six second-order combinations of \( \nabla \), requiring six mnemonics (Table 5.1, p. 142):

1. **DoG**: Divergence of the gradient (scalar Laplacian) operates on scalar potentials: (Greenberg, 1988, p. 779)

\[
\nabla^2 \phi = (\nabla \cdot \nabla) \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}
\]


\[
\nabla^2 \mathbf{A} = (\nabla \cdot \nabla) \mathbf{A} = \frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} + \frac{\partial^2 \mathbf{A}}{\partial z^2} = \text{BullDogA} - \nabla \times \nabla \times \mathbf{A}
\]

3. **GoD**: (little-GoD) Gradient of the Divergence (\( \nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) \))

\[
\nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) = \nabla \left( \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) = \hat{x} \frac{\partial}{\partial x} \nabla \cdot \mathbf{A} + \hat{y} \frac{\partial}{\partial y} \nabla \cdot \mathbf{A} + \hat{z} \frac{\partial}{\partial z} \nabla \cdot \mathbf{A}
\]

4. **CoC**: Curl of the curl (\( \text{GoDA} - \text{DoG A} \)) (Sommerfeld, 1952, page 33, Eq. 2b).

\[
\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla) \mathbf{A} = \nabla^2 \mathbf{A} - \nabla^2 \mathbf{A}
\]

5. DoC: Divergence of the curl ($\nabla \cdot \nabla \times = 0$)

6. CoG: Curl of the gradient ($\nabla \times \nabla = 0$)

DoC(⋅) and CoG(⋅) are special because they are always zero

$$\nabla \cdot \nabla \times A = 0, \quad \nabla \times \nabla \phi = 0,$$

making them useful in proving the fundamental theorem of vector calculus, aka, Helmholtz’ decomposition theorem (Eq. 5.7.0.5, p. 177). A third special vector identity CoC is

$$\nabla \times \nabla \times A = \text{LapDog} A - \nabla^2 A,$$  \hspace{1cm} (5.5.6.7)

which operates on vector fields and is useful for defining the vector Laplacian DoG as the difference between little GoD (gOd) and CoC (i.e., GoD = gOd - CoC)

$$\nabla^2 () = \nabla^2 () - \nabla \times \nabla \times () .$$

The role of gOd ($\nabla^2$) is commonly ignored because it is zero for the magnetic wave equation, due to there being no magnetic charge ($\nabla \cdot B(x, t) = 0$ thus $\nabla^2 B(x, t) \equiv 0$). However for the electric vector wave equation it plays a role

$$\nabla^2 \phi(x, t) = -\nabla E(x, t) = -\frac{1}{\epsilon_0} \nabla^2 D(x, t) = -\frac{1}{\epsilon_0} \nabla \rho(x, t).$$

or since $\nabla \cdot D = \rho$,

$$\nabla^2 D(x, t) = \nabla \nabla \cdot D = -\nabla \rho(x, t).$$

When the charge density is inhomogeneous, such as the case of a plasma (e.g., the sun) this term will play an important role as a source term to the electric wave equation. This case needs to be further explored via some physical examples.

Exercise: Show that GoD and gOd differ.  \hspace{1cm} Solution: Use CoC on $A(x, t)$ to explore this relationship।

Discussion: It is helpful to split these six identities into two groups: the utility operators DoG, gOd, GoD, and the identity operators CoC (Eq. 5.5.6.7), DoC=0 and CoG=0. It is helpful to view these two groups as playing fundamentally different roles.

When using second-order differential operators one must be careful with the order of operations, which can be subtle. Most of this is common sense. For example, don’t operate on a scalar field with $\nabla \times$, and don’t operate on a vector field with $\nabla \times$. GoD acts on each vector component $\nabla^2 A = \nabla^2 A_x \hat{x} + \nabla^2 A_y \hat{y} + \nabla^2 A_z \hat{z}$, which is very different from the action of gOd.

### 5.6 The unification of electricity and magnetism

Once you have mastered the three basic vector operations – the gradient, divergence and curl – you are ready to appreciate Maxwell’s equations. Like the vector operations, these equations may be written in integral or differential form. An important difference is that with Maxwell’s equations, we are dealing with well defined physical quantities. The scalar and vector fields take on meaning, and units. Thus to understand these important equations, one must master both the names and units of the four fields $E, H, B, D$, as described in Table 5.4.

13This operation defines a dyadic tensor, the generalization of a vector.
5.6. THE UNIFICATION OF ELECTRICITY AND MAGNETISM

Figure 5.8: Left: von Helmholtz portrait taken from Helmholtz (1978). Right: Gustav Kirchhoff. Together they were the first to account for viscous (Helmholtz, 1858, 1978, 1863b) and thermal (Kirchhoff, 1868, 1974) losses in the acoustic propagation of airborne sound, as first experimentally verified by Mason (1928) (p. 177).

Table 5.4: The variables of Maxwell’s equations have names (e.g., EF, MI) and units (in square brackets [SI Units]). The units are necessary to obtain a full understanding of each of the four variable and their corresponding equation. For example, Eq. EF has units [V/m]. By integrating E from x = 0, 1, one obtains the voltage difference between the two points. The speed of light in-vacuo is \( c = 3 \times 10^8 \) [m/s], and the characteristic resistance of light \( r_o = 377 = \sqrt{\mu_o/\varepsilon_o} \) (i.e., ohms).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Units</th>
<th>Maxwell’s Eq.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>EF: Electric field strength</td>
<td>[V/m]</td>
<td>( \nabla \times E = -\partial_t B )</td>
</tr>
<tr>
<td>( D = \varepsilon_o E )</td>
<td>ED: Electric displacement (flux density)</td>
<td>[C/m²]</td>
<td>( \nabla \cdot D = \rho )</td>
</tr>
<tr>
<td>( H )</td>
<td>MF: Magnetic field strength</td>
<td>[A/m]</td>
<td>( \nabla \times H = J_m + \partial_t D )</td>
</tr>
<tr>
<td>( B = \mu_o H )</td>
<td>MI: Magnetic induction (flux density)</td>
<td>[Wb/m²]</td>
<td>( \nabla \cdot B = 0 )</td>
</tr>
</tbody>
</table>

Field strength \( E, H \): As summarized by Eqs. 5.6.1.1 there are two field strengths, the electric \( E \) with units of [V/m] and the magnetic \( H \) having units of [A/m]. The ratio \( |E|/|H| = \sqrt{\mu_o/\varepsilon_o} = 377 \) [ohms] for in-vacuo plane-waves (\( \mu_o, \varepsilon_o \)).

To understand the meaning of \( E \), if two conducting plates are placed 1 [m] apart, with 1 [V] across them, the electric field is \( E = 1 \) [V/m]. If a charge (i.e., an electron) is placed in an electric field, it feels a force \( f = qE \), where \( q \) is the magnitude of the charge [C].

To understand the meaning of \( H \), consider the solenoid made of wire, as shown in Fig. 5.7, which carries a current of 1 [A]. The magnetic field \( H \) inside such a solenoid is uniform and is pointed along the long axis, with a direction that depends on the polarity of the applied voltage (i.e., direction of the current in the wire).

Flux \( D, B \): Flux is a flow, such as the mass flux of water flowing in a pipe [kg/s], driven by a force (pressure drop) across the ends of the pipe, or the heat flux in a thermal conductor, having a temperature drop across it (i.e., a window or a wall). The flux is the same as the flow, be it charge, mass or heat (Table 3.1, p. 97). In Maxwell’s equations there are also two fluxes: the electric flux \( D \), and the magnetic flux \( B \). The flux density units for \( D \) are [A/m²] (flux in [A]) and the magnetic flux \( B \) is measured in Weber’s [Wb] [A/m²] or [Tesla] (henry-amps/area) [H-A/m²].
5.6.1 Maxwell’s equations

Maxwell’s equations (ME) consist of two curl equations Eqs. 5.6.1.1 operating on the field strengths $E$ and $H$, and two divergence equations, operating of the field fluxes $D$ and $B$. In matrix format ME are

$$
\nabla \times \begin{bmatrix} E(x, t) \\ H(x, t) \end{bmatrix} = \partial_t \begin{bmatrix} -B(x, t) \\ D(x, t) \end{bmatrix}
= \begin{bmatrix} 0 & -\mu_o \\ \epsilon_o & 0 \end{bmatrix} \partial_t \begin{bmatrix} E(x, t) \\ H(x, t) \end{bmatrix}
\leftrightarrow \begin{bmatrix} 0 & -s\mu_o \\ \sigma_o + s\epsilon_o & 0 \end{bmatrix} \begin{bmatrix} E(x, \omega) \\ H(x, \omega) \end{bmatrix}.
$$

(5.6.1.1)

When the medium is conducting, $\partial_t D$ must be replaced by $C = \sigma_o E + \partial_t D \leftrightarrow (\sigma_o + s\epsilon_o)E(x, \omega)$ where $\sigma_o + s\epsilon_o$ is an admittance density $[\Omega/m^2]$.

There are also two auxiliary equations:

$$
\nabla \cdot \begin{bmatrix} D \\ B \end{bmatrix} = -\partial_t \begin{bmatrix} \rho(x) \\ 0 \end{bmatrix}.
$$

(5.6.1.2)

The top equation states conservation of charge while the lower states there is no magnetic charge. When expressed in integral format, Stokes’s law follows from the curl equations and Gauss’s law from the divergence equations.

**Example:** When a static current is flowing in a wire in the $\hat{z}$ direction, the magnetic flux is determined by Stokes’s theorem (Fig. 5.6, p. 5.6). Thus, just outside of the wire we have

$$
I_{enc} = \iint_S (\nabla \times H) \cdot \hat{n} \, dS = \oint_B H \cdot d\ell. \quad [A]
$$

For this simple geometry, the current in a wire is related to $H(x, t)$ by

$$
I_{enc} = \oint_B H \cdot d\ell = H_\phi 2\pi r.
$$

Here $H_\phi$ is perpendicular to both the radius $r$ and the direction of the current $\hat{z}$. Thus

$$
H_\phi = \frac{I_{enc}}{2\pi r},
$$

where $H_\phi$ is attenuated by $1/r$ (Ramo et al., 1965, Eq. 9, page 244).

**Exercise:** Explain how Stokes’s theorem may be applied to $\nabla \times E = -\dot{B}$, and explain what it means. Hint: This is the identical argument given above for the current in a wire, but for the electric case.

**Solution:** Integrating the left side of equation Eq. 5.6.1.3 over an open surface results in a voltage (emf) induced in the loop closing the boundary $\partial S$ of the surface

$$
\phi_{induced} = \iint_S (\nabla \times E) \cdot \hat{n} \, dS = \oint_{\partial S} E \cdot d\ell \quad [\text{V}].
$$

(5.6.1.3)

The emf (electromagnetic force) is the same as the Thévenin source voltage induced by the rate of change of the flux. Integrating the Eq. 5.6.1.3 over the same open surface $S$ results in the source of the induced voltage $\phi_{induced}$, which is proportional to the rate of change of the flux [Webers]

$$
\phi_{induced} = -\frac{\partial}{\partial t} \iint_S B \cdot \hat{n} \, dA = L \dot{\psi} \quad [\text{Wb/s}] \text{ or } [\text{V}],
$$

where $L$ is the inductance of the wire. The area integral on the left is in [Wb/m²] resulting in the total flux crossing normal to the surface $\psi$ [Wb]. Thus the rate of change of the total flux [Wb/s] is a voltage [V].

If we apply Gauss’s theorem to the divergence equations, we find the total flux crossing the closed surface.
Exercise: Apply Gauss’s theorem to equation \( \mathbf{ED} \) and explain what it means in physical terms.

Solution: The area of the normal component of \( \mathbf{D} \) is equal to the volume integral over the charge density: thus, Gauss’s theorem says that the total charge within the volume \( Q_{\text{enc}} \), found by integrating the charge density \( \rho(x) \) over the volume \( \Omega' \), is equal to the normal component of the flux \( \mathbf{D} \) crossing the surface \( S \)

\[
Q_{\text{enc}} = \iiint_{\Omega'} \nabla \cdot \mathbf{D} \, dV = \iint_{S} \mathbf{D} \cdot \hat{n} \, dA.
\]

In the case where equal amounts of positive and negative charge exist within the volume, the integral will be zero. ■

Summary: Maxwell’s four equations relate the field strengths to the flux densities. There are two types of variables: field strengths \( (\mathbf{E}, \mathbf{H}) \) and flux densities \( (\mathbf{D}, \mathbf{B}) \). There are two classes: electric \( (\mathbf{E}, \mathbf{D}) \) and magnetic \( \mathbf{H}, \mathbf{B} \). One might naturally view this as a 2 × 2 matrix, with rows being electric and magnetic strengths, and columns being electric and magnetic and flux densities, defining a total of four variables (see boxed table).

<table>
<thead>
<tr>
<th>Type</th>
<th>Strength</th>
<th>Flux density</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electric</td>
<td>( E ) [V/m]</td>
<td>( D ) [C/m(^2)]</td>
</tr>
<tr>
<td>Magnetic</td>
<td>( H ) [A/m]</td>
<td>( B ) [Wb/m(^2)]</td>
</tr>
</tbody>
</table>

Applying Stokes’s “curl” theorem to the forces induces a Thévenin voltage (emf) or Norton current source. Applying Gauss’ “divergence” theorem to the flows gives the total charge enclosed. The magnetic charge is zero \( (\nabla \cdot \mathbf{B} = 0) \) because magnetic mono-poles do not exist. However, magnetic dipoles do exist, as in the example of the electron which contains a magnetic dipole.

5.6.2 Derivation of the vector wave equation

Next we provide the derivation of the vector wave equation starting from Maxwell’s equations (Eq. 5.6.1.1), which is reminiscent of the derivation of the Webster horn equation (Eq. 5.2.1, p. 153). Working in the frequency domain, and taking the curl of both sides, gives

\[
\nabla \times \nabla \times \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \begin{bmatrix} 0 & -s\mu_0 \\ s\epsilon_0 & 0 \end{bmatrix} \nabla \times \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \frac{s^2}{c_0^2} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}
\]

Using the CoC identity \( \nabla \times \nabla \times (\mathbf{E}) = \nabla^2 (\mathbf{E}) - \nabla^2 (\mathbf{H}) \) (Eq. 5.5.6.7, p. 170) gives

\[
\nabla^2 \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} - \nabla^2 \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = s^2 \frac{1}{c_0^2} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix},
\]

or finally Maxwell’s vector wave equation

\[
\nabla^2 \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} - \frac{s^2}{c_0^2} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = \nabla \left[ \frac{1}{\epsilon_0} \nabla \cdot \mathbf{D} \right] \left[ \frac{1}{\mu_0} \nabla \cdot \mathbf{B} \right]^0
\]

\[
= \frac{1}{\epsilon_0} \nabla \rho (x, s) \tag{5.6.2.4}
\]
with the electric excitation term $\nabla \rho(x, s)$. Note that if $\mu$ and $\epsilon$ depended on $x$, the terms on the right would not be zero. In deep outer space with its black holes and plasma everywhere (e.g., inside the sun) this seems quite likely.

Recall the d’Alembert solutions of the scalar wave equation (Eq. 4.3.0.1, p. 121)

$$E(x, t) = f(x - ct) + g(x + ct),$$

where $f, g$ are arbitrary vector fields. This result applies to the vector case since it represents three identical, yet independent, scalar wave equations, in the three dimensions.

**Poynting vector:** The EM power flux density $\mathcal{P}$ [W/m$^2$] is perpendicular to $E$ and $B$, denoted as

$$\mathcal{P} = \frac{1}{\mu_0} E \times B = E \times H. \quad [\text{W/m}^2]$$

The corresponding EM momentum flux density $\mathcal{M}$ (hence ME are related to mass, thus gravity) is

$$\mathcal{M} = \epsilon_0 E \times B = D \times B. \quad [\text{C/m}^2 \cdot \text{Wb/m}^2]$$

Since the speed of light is $c_0 = 1/\sqrt{\mu_0 \epsilon_0}$, divided by the momentum flux density is (Sommerfeld, 1952)

$$\mathcal{P} = \frac{c_0^2 \mathcal{M}}{\text{[W/m}^2].$$

which is clearly related to the Einstein energy–mass equivalence formula $E = mc_0^2$.

**Examples:** The power emitted by the sun is about $1360 \ \text{W/m}^2$, with a radiation pressure of $4 \times 10^{-6} \ \text{[N/m}^2]$ [i.e., 4 $\mu$Pa] (Fitzpatrick, 2008). By way of comparison, the threshold audible acoustic pressure at the human eardrum at 1 [kHz] is 20 $\mu$Pa. Also

The lasers used in Inertial Confinement Fusion (e.g., the NOVA experiment in Lawrence Livermore National Laboratory) typically have energy fluxes of $10^{18} \ \text{[W/m}^2$. This translates to a radiation pressure of about $10^4$ atmospheres!


**Electrical impedance seen by an electron:** Up to now we have only considered the Brune impedance which is a special case with no branch points or branch cuts. We can define impedance for the case of diffusion, as in the case of the diffusion of heat. There is also the diffusion of electrical and magnetic fields at the surface of a conductor, where the resistance of the conductor dominates the dielectric properties, which is called the electrical skin effect, where the conduction currents are dominated by the conductivity of the metal rather than the displacement currents. In such cases the impedance is proportional to $\sqrt{s}$, implying that it has a branch cut. Still in this case the real part of the impedance must be positive in the right half-plane, the required condition of all impedances, such that postulate P3 is satisfied (p. 106). The same effect is observed in acoustics (Appendix G).

**Example:** When we deal with Maxwell’s equations the force is defined by the Lorentz force

$$f = qE + qv \times B = qE + C \times B,$$

which is the force on a charge (e.g., electron) due to the electric $E$ and magnetic $B$ fields. The magnetic field plays a role when the charge has a velocity $v$. When a charge is moving with velocity $v$, it may be viewed as a current $C = qv$ (See discussion on p. 3.7.1, p. 107).

The complex admittance density is

$$Y(s) = \sigma_0 + s\epsilon_0 \quad [(\Omega/\text{m}^2)],$$

(Feynman, 1970b, p. 13-1). Here $\sigma_0$ is the electrical conductivity and $\epsilon_0$ is the electrical permittivity. Since $\omega \epsilon_0 \ll \sigma_0$ this reduces to the resistance of the wire, per unit length.$^{14}$

$^{14}$For copper $\omega \ll \omega_c = \sigma_0/\epsilon_0 \approx 6 \times 10^7/9 \times 10^{-12} \approx 6.66 \times 10^{18} \ [\text{rad/s}], \ or \ f_c = 10^{18} \ [\text{Hz}]. \ This \ corresponds \ to \ a \ wavelength \ of \ \lambda_0 \approx c_0/f_c = 0.30 \ [\text{nm}]. \ For \ comparison, \ the \ Bohr \ radius \ (hydrogen) \ is \ \approx 0.053 \ [\text{nm}] \ (5.66 \ times \ smaller) \ and \ the \ Lorentz \ radius \ (of \ the \ electron) \ is \ estimated \ to \ be \ 2.8 \times 10^{-15} \ [\text{m}] \ (2.8 \ [\text{femto meters}]).$
5.7 Potential solutions of Maxwell’s equations

The primary purpose for using potentials is for generating solutions to Maxwell’s equations. For example, extending Eq. 5.1.1.2 (p. 140), Maxwell’s equations may be expressed in terms of scalar and vector potentials. These relations are [Sommerfeld (1952, p. 146); Feynman (1970d, p. 18-10)]:

\[ E(x, t) = -\nabla \phi(x, t) - \frac{\partial A(x, t)}{\partial t}, \]  
\[ H(x, t) = \frac{1}{\mu_0} \left[ \nabla \times A(x, t) + \frac{\partial D(x, t)}{\partial t} \right]. \]  

Here we have extended \( H(x, t) \) to include the electric potential term

\[ D(x, t) = \varepsilon(x, t) E(x, t) = -\varepsilon(x, t) \nabla \phi(x, t), \]

normally taken to be zero, because taking the Curl of \( H(t) \) would naturally remove any electrical potential term, due to CoG=0.15

When the permittivity (\( \varepsilon_0(x, t) \) [F/m²]) is both inhomogeneous and time dependent,

\[ \nabla \cdot E = -\nabla^2 \Phi - \nabla \cdot \dot{A} = \rho(x, t)/\varepsilon_0(x, t) \]

and

\[ \nabla \times [\varepsilon(x, \omega) \nabla \phi(x, \omega)] = \varepsilon(x, t) \nabla \times \phi + \nabla \varepsilon(x, t) \times \nabla \phi \neq 0. \]

The extension makes the potential solutions symmetric so that \( E \) and \( H \) each have electrical and magnetic excitation.

Exercise: Explain why some dependence of \( \phi(x, t) \) does not appear in Eq. 5.7.0.2, but does in 5.7.0.1. Solution: For \( H(x, t) \) to depend on \( \phi(x, t) \) it must appear through the electric strength, as \( E(x, t) = -\nabla \Phi(x, t) \). But then \( \nabla \times H(x, t) \) would mean applying CoG=0 (i.e., \( \nabla \times \nabla \phi = 0 \)) on the right side of the equation. Since this term would be zero, it is assumed to be zero, thus \( H(x, t) \) is only the dependent on \( A(x, t) \). To fill out the symmetry we have added \( \partial_t D(x, t) \) to Eq. 5.7.0.2, to see what might happen in the general case.

Use of Helmholtz theorem on potential solutions: The generalized solution to Maxwell’s equations (Eqs. 5.7.0.1 and 5.7.0.2), pp. 175-175) have been expressed in terms of EM potentials \( \phi(x) \) and \( A(x) \) and Helmholtz’s theorem. These are “solutions” to Maxwell’s equations, expressed in terms of the potentials \( \phi(x, s) \) and \( A(x, s) \), as determined at the boundaries (Sommerfeld, 1952, p. 146). These relations are invariant to certain functions added to each potential, as shown below. They are equivalent to Maxwell’s equations following the application of \( \nabla \) and \( \nabla \times \).

Next we show that the potential equations (Eqs. 5.7.0.1, 5.7.0.2, p. 175) are consistent with Maxwell’s equations (Eq. 5.6.1.1 p. 172).

**ME for \( E(x, t) \):** Taking the curl of Eq. 5.7.0.1, applying CoG=0 and using Eq. 5.7.0.2

\[ \nabla \times E = -\nabla \times \nabla \phi - \nabla \phi \frac{\partial A}{\partial t} \]
\[ = -\frac{\partial B}{\partial t} \]  
(5.7.0.3)

---

15-in-vacuo \( \varepsilon_0 = 8.85 \times 10^{-12} \) [F/m²] is the capacitance, and \( s \varepsilon_0 \) is the electric compliance-density of light. The related magnetic mass-density is the permeability \( \mu_0 = 4\pi \times 10^{-7} \) [H/m²] having an inductive impedance of \( s\mu_0 \) [Ω/m]. It is helpful to think of \( s \varepsilon_0 \) as a capacitance per unit area and \( \mu_0 \) as a inductance per unit area (consistent with their units). The speed of light is \( c = 1/\sqrt{s \varepsilon_0 \mu_0} = 3 \times 10^8 \) [m/s].
reverses Maxwell’s equation for $E(x)$ (Eq. 5.6.1.1, p. 172).

Taking the Div of 5.7.0.2 and apply DoC=0 gives Eq. 5.6.1.2 (p. 172) for $B(x)$

$$\nabla \cdot B(x) = \nabla \cdot \nabla \cdot A(x) = 0.$$  

**ME for $H(x, t)$:** To recover Maxwell’s equation for $H(x)$ (Eq. 5.6.1.1, $\nabla \times H = C$) from the potential equation Eq. 5.7.0.2, we take the Curl and use $B = \epsilon_o H$ (Table 5.4, p. 171):

$$\nabla \times B(x) = \mu_o \nabla \times H(x)$$

$$= \nabla \times \nabla \times A(x)$$

$$= \nabla^2 A(x, t) - \nabla^2 A(x, t)$$

$$= \nabla^2 A(x, t) - \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} A(x, t)$$

$$= -\frac{1}{c_o^2} \left( \dot{A} + \nabla \dot{\Phi} \right) + \mu_o J$$

This last equation may be split into two independent equations by the use of Helmholtz theorem

$$\nabla^2 A - \frac{1}{c_o^2} \dot{A} = -\mu_o J \quad \text{and} \quad \nabla \cdot A + \frac{1}{c_o^2} \dot{\Phi} = 0.$$  

Taking the Div of 5.7.0.2 and apply DoC=0 gives Eq. 5.6.1.2 ($\nabla \cdot D = -\dot{\rho}$). Alternatively

$$\nabla^2 \Phi - \frac{1}{c_o^2} \dot{\Phi} = -\frac{\rho}{\epsilon_o}$$

which is the scalar potential wave equation, driven by the charge (Sommerfeld, 1952, p. 146).

**In summary:** In conclusion, Eq. 5.7.0.1, along with DoC=0 and CoG=0, gives Maxwell’s Eq. 5.6.1.1 and Eq. 5.6.1.2 for $E$. Likewise, Eq. 5.7.0.2, along with DoC=0 and CoG=0, gives Maxwell’s Eq. 5.6.1.1 and Eq. 5.6.1.2 for $H$. This equation derives the magnetic component of the field, expressed in terms of its vector potential, in the same way as Eq. 5.6.1.1 describes $E(x, t)$ in terms of the potentials.

We may view the potential equations (Eq 5.7.0.1, 5.7.0.2) as equivalent to Maxwell’s equations, thus they are the solutions to ME.

**Exercise:** Starting from the values of the speed of light $c_o = 3 \times 10^8$ [m/s] and the characteristic resistance of light waves $r_o = 377$ [ohms], use the formula for $c_o = 1/\sqrt{\mu_o \epsilon_o}$ and $r_o = \sqrt{\epsilon_o/\mu_o}$ to find values for $\epsilon_o$ and $\mu_o$. **Solution:** Squaring $c_o^2 = 1/\mu_o \epsilon_o$ and $r_o^2 = \mu_o/\epsilon_o$ we may solve for the two unknowns: $c_o^2 r_o^2 = 1/\mu_o \epsilon_o \mu_o/\epsilon_o = 1/\epsilon_o^2$, thus $\epsilon_o = 1/c_o r_o = 377/2.998 \approx 125.75 \times 10^{-8}$. The value of $\mu_o$ is defined in the international SI standard as $4 \pi 10^{-7} \approx 12.56610^{-7}$ [H/m].

It is easier to memorize $c_o$ and $r_o$, from which $\epsilon_o$ and $\mu_o$ may be quickly derived.

**Exercise:** Take the divergence of Maxwell’s equation for the magnetic intensity

$$\nabla \times H(x, t) = J_m(x, t) + \frac{\partial}{\partial t} D(x, t)$$

and explain what results ($J_m = \sigma E$). **Solution:** The divergence of the curl is zero (DoC=0),

$$\nabla \cdot \nabla \times H(x, t) = \nabla \cdot J_m(x, t) + \frac{\partial}{\partial t} \rho(x, t) = 0,$$  

which is conservation of charge (i.e., Gauss’s theorem).
5.8. THE QUASISTATIC APPROXIMATION

**Helmholtz’s decomposition theorem:** Helmholtz’s decomposition is expressed as the linear sum of a scalar potential \( \phi(x, y, z) \) (think voltage) and a vector potential (think magnetic vector potential). Specifically

\[
E(x, s) = -\nabla \phi(x, s) + \nabla \times A(x, s)
\]  
(5.7.0.5)

where \( \phi \) is the scalar and \( A \) is the vector potential, as a function of the Laplace frequency \( s \). Of course this decomposition is general (not limited to the electro-magnetic case). It applies to linear fluid vector fields, which include most liquids and air. When the rotational and dilatation become coupled, this relation must break down.\(^{16}\)

To show how this relationship splits the vector fields \( E \) into its two parts, we need DoC and CoG, the two key vector identities that are always zero for analytic fields: the curl of the gradient (CoG)

\[
\nabla \times \nabla \phi(x) = 0,
\]  
(5.7.0.6)

and the divergence of the curl\(^{17} \) (DoC)

\[
\nabla \cdot (\nabla \times A) = 0.
\]  
(5.7.0.7)

The above identities are easily verified by working out a few specific examples, based on the definitions of the three operators, gradient, divergence and curl, or in terms of the operator’s integral definitions, defined on p. 163. The identities have a physical meaning, as stated above: every vector field may be split into its transnational and rotational parts. If \( E \) is the electric field [V/m], \( \phi \) is the voltage and \( A \) is the induced rotational part, induced by a current.

By applying these two identities to Helmholtz’s decomposition, we can better appreciate the theorem’s significance. It is a form of proof actually, once you have satisfied yourself that the vector identities are true. In fact one can work backward using a physical argument, that rotational momentum (rotational energy) is independent of the transnational momentum. Once these forces are made clear, the vector operations all take on a very well defined meaning, and the mathematical constructions, centered around Helmholtz’s theorem, begin to provide some common-sense meaning. One could conclude that the physics is simply related to the geometry via the scalar and vector product.

Specifically, if we take the divergence of Eq. 5.7.0.5, and use the DoG, then

\[
-\frac{1}{\epsilon_0} \rho = \nabla \cdot E = \nabla \cdot \{ -\nabla \phi + \nabla \times A \} = -\nabla \cdot \nabla \phi = -\nabla^2 \phi,
\]

since the DoG zeros the vector potential \( A(x, y, z) \). If instead we use the CoG, then

\[
-\dot{B} = \nabla \times E = \nabla \times \{ -\nabla \phi + \nabla \times A \} = \nabla \times \nabla \times A = \nabla (\nabla \cdot A) - \nabla^2 A,
\]

since the CoG zeros the scalar field \( \phi(x, y, z) \). The last expression requires GoD.

5.8 The quasistatic approximation

A fundamental question that is begging for an answer, that I have not yet seen asked, is

*What is the mathematical description of quantum mechanics?*

First this question must have an answer, as quantum mechanics (QM) is a highly mathematical subject. Second we have seen that QM is related to the Webster Horn equation, thus may be defined through its Sturm-Liouville equation parameters, specifically the area function \( A(x) \). Third QM systems are nearly lossless. If there were zero loss, we would not be able to observe them. For all practical purposes, they can be considered by be lossless until they interact with outside forces or particles. It

\(^{16}\)The nonlinear Naiver–Stokes equations may be an example.

\(^{17}\)Helmholtz was the first person to apply mathematics in modeling the eye and the ear (Helmholtz, 1863a).
is likely that in their ideal unperturbed state, there is virtually zero loss. Yet we can see the tell-tail radiation signature, such as the Rydberg series, for the case of the Hydrogen atom.

To characterize a lossless system, such as the Hydrogen atom, as a Sturm-Liouville system, we need an area function that is exponential. In this case the propagation function $\kappa(s)$ has no real part, and the electromagnetic energy is trapped inside the area function (i.e., exponential horn). In this case $\kappa(s)$ has a cutoff frequency, below which the waves are trapped. The wave velocity in such cases is highly dispersive, giving rise to an accumulation point of the eigen frequencies. We see this in the analysis of the Rydberg series Appendix H.1.1, which is the quintessential example of the QM system. In such a system $\kappa(s)$ has two branches, corresponding to d’Alembert out and inbound electromagnetic waves, that are trapped by the exponential area function.

There are a number of assumptions and approximations that result in special cases, many of which are classic. These manipulations are typically done at the differential equation level, by making assumptions that change the basic equations that are to be solved. These approximations are distinct from assumptions made while solving a specific problem.\footnote{It is essential to watch this magical video by Carl Sagan about Einstein’s views on the speed of light. \url{https://www.youtube.com/watch?v=_pEiA0-r5A8}}

A few important examples include

1. **In-vacuo** waves (free-space scalar wave equation)
2. Expressing the vector wave equation in terms of scalar and vector potentials
3. Quasistatics
   - (a) scalar wave equation
   - (b) Kirchhoff’s low-frequency lumped-element approximation (LRC networks)
   - (c) Transmission line equations (telephone and telegraph equations)

One of the very first insights into wave propagation was due to Huygens (c1640) (Fig. 1.5, p. 23).

**Quasistatics and its implications:** *Quasistatics* (Postulate P10, p. 107) is an approximation used to reduce a partial differential equation to a scalar (one-dimensional) equation (Sommerfeld, 1952); that is, quasistatic is a way of reducing a three-dimensional problem to a one-dimensional problem. So that it is not miss-applied, it is important to understand the nature of this approximation, which goes to the heart of transmission line theory. The quasi-static approximation states that the wavelength $\lambda$ is greater than the dimensions of the object $\Delta$ (e.g., $\lambda \gg \Delta$). The best known examples, Kirchhoff’s current and voltage laws, KCL and KVL, almost follow from Maxwell’s equations given the quasi-static approximation (Ramo \textit{et al.}, 1965). These laws, based on Ohm’s law, state that the sum of the currents at a node must be zero (KCL) and the sum of the voltages around a loop must be zero (KCL).

These well-known laws are the analog of Newton’s laws of mechanics. The sum of the forces at a point is the analog of the sum of the loop voltages. Voltage $\phi$ is the force potential, since the electric field $E = -\nabla \phi$. The sum of the currents is the analog of the vector sum of velocities (mass) at a point, which is zero.

The acoustic wave equation describes how the scalar field pressure $p(x, t)$ and the vector force density potential $f(x, t) = -\nabla p(x, t) \text{[N/m}^2\text{]}$ propagate in three dimensions. The net force is the integral of the pressure gradient over an area. If the wave propagation is restricted to a pipe (e.g., organ pipe), or to a string (e.g., an guitar or lute), the transverse directions may be ignored, due to the quasi-static approximation. What needs to be modeled by the equations is the wave propagation along the pipe (string). Thus we may approximate the restricted three-dimensional wave by a one-dimensional wave.

However, if we wish to be more precise about this reduction in geometry ($\mathbb{R}^2 \rightarrow \mathbb{R}$), we need to consider the quasi-static approximation, as it makes assumptions about what is happening in the other directions, and quantifies the effect ($\lambda \gg \Delta$). Taking the case of wave propagation in a tube, say the
ear canal, there is the main wave direction, down the tube. But there is also wave propagation in the transverse direction, perpendicular to the direction of propagation. As shown in Table E.1 (p. 226), the key statement of the quasi-static approximation is that the wavelength in the transverse direction is much larger than the radius of the pipe. This is equivalent to saying that the radial wave reaches the walls and is reflected back, in a time that is small compared to the distance propagated down the pipe. Clearly the speed of sound down the pipe and in the transverse direction is the same if the medium is homogeneous (i.e., air or water). Thus the sound reaches the walls and is returned (reflected) to the center line in a time that the axial wave traveled about 1 diameter along the pipe. So if the distance traveled is several diameters, the radial parts of the wave have time to come to equilibrium. So the question one must ask is: What are the properties of this equilibrium? The most satisfying answer is provided by looking at the internal forces on the air, due to the gradients in the pressure.

The pressure \( p(x, t) \) is a potential, thus its gradient is a force density \( f(x, t) = -\nabla p(x, t) \). This equation tells us how the pressure wave evolves as it propagates down the horn. Any curvature in the pressure wave-front induces stresses, which lead to changes (strains) in the local wave velocity, in the directions of the force density. The main force is driving the wave-front forward (down the horn), but there are radial (transverse) forces as well, which tend to rapidly go to zero.

For example, if the tube has a change in area (or curvature), the local forces will create radial flow, which is immediately reflected by the walls, due to the small distance to the walls, causing the forces to average out. After traveling a few diameters, these forces will come to equilibrium and the wave will trend towards a plane wave (or satisfy Laplace’s equation if the distortions of the tube are severe). The internal stress caused by this change in area will quickly equilibrate.

There is a very important caveat, however: only at low frequencies, such that \( ka < 1 \), can the plane wave mode dominate. At higher frequencies (\( ka \geq 1 \)) where the wavelength is small compared to the diameter, the distance traveled between reflections is much greater than a few diameters. Fortunately the frequencies where this happens are so high that they play no role in frequencies that we care about in the ear canal. This effect results from cross-modes, which are radial and angular standing waves.

Of course such modes exist in the ear canal above 20 [kHz]. However, they are much more obvious on the eardrum where the sound wave speed is much slower than that in air (Parent and Allen, 2010; Allen, 2014). Because of the slower speed, the ear drum has low-frequency cross-modes, and these may be seen in the ear canal pressure, and are easily observable in ear canal impedance measurements. Yet they seem to have a negligible effect on our ability to hear sound with high fidelity. The point here is that the cross modes are present, but we call upon the quasi-static approximation as a justification for ignoring them, to get closer to the first-order physics.

### 5.8.1 Quasistatics and Quantum Mechanics

It is important to understand the meaning of Planck’s constant \( h \), which appears in the relations of both photons (light “particles”) and electrons (mass particles). If we could obtain a handle on what exactly Planck’s constant means, we might have a better understanding of quantum mechanics, and physics in general. By cataloging the dispersion relations (the relation between the wavelength \( \lambda(\nu) \) and the frequency \( \nu \)) between electrons and photons, this may be attainable.

Basic relations from quantum mechanics for photons and electrons include:

1. Photons (mass=0, velocity = \( c \))
   
   (a) \( c = \lambda\nu \): The speed of light \( c \) is the product of its wavelengths \( \lambda \) times its frequency \( \nu \). This relationship is only for mono-chromatic (single frequency) light.
   
   (b) The speed of light is
   
   \[
   c_o = \frac{1}{\sqrt{\mu_o/\epsilon_o}} = 3 \times 10^8 \text{ [m/s]}.
   \]
   
   (c) The characteristic resistance of light
   
   \[
   r_o = \sqrt{\mu_o/\epsilon_o} = |E|/|H| = 377 \text{ [ohms]}
   \]
is defined as the magnitude of the ratio of the electric $E$ and magnetic $H$ field, of a plane wave in-vacuo.

(d) $E = h \nu$: the photon energy is given by Planck’s constant

$$h \approx 6.623 \times 10^{-34} \text{ [joule \cdot s]}$$

times the frequency (i.e., bandwidth) of the photon.

2. Electrons (mass = $m_e$, velocity $V = 0$):

(a) $E_e = m_e c^2 \approx 0.91 \cdot 10^{-30} \cdot 0.3^2 \cdot 10^{12} = 8.14 \times 10^{-20} \text{ [J]}$ is the electron rest energy (velocity $V = 0$) of every electron, of mass $m_e = 9.1 \times 10^{-31} \text{ [kgm]}$, where $c_o$ is the speed of light.

(b) $p = h/\lambda$: The momentum $p$ of an electron is given by Planck’s constant $h$ divided by the wavelength of an electron $\lambda$. It follows that the bandwidth of the photon is given by

$$\nu_e = \frac{E_e}{h}$$

and the wavelength of an electron is

$$\lambda_e = \frac{h}{p_e}.$$

One might reason that QM obeys the quasi-static (long wavelength) approximation. If we compare the velocity of the electron $V$ to the speed of light $c$, then we see that

$$c_o = E/p \gg V = E/p = mV^2/mV.$$  

Models of the electron: It is helpful to consider the physics of the electron, a negatively charged particle that is frequently treated as a single point in space. If the size were truly zero, there could be no magnetic moment (spin). The accepted size of the electron is known as the Lorentz radius, $R = 2.8 \times 10^{-15} \text{ [m]}$. One could summarize the Lorentz radius as follows: Here lie many unsolved problems in physics. More specifically, at dimensions of the Lorentz radius, what exactly is the structure of the electron?

Ignoring these difficulties, if one integrates the charge density of the electron over the Lorentz radius and places the total charge at a single point, then one may make a grossly oversimplified model of the electron. For example, the electric displacement ($D = \epsilon_o E$) (flux density) around a point charge is

$$D = -\epsilon_o \nabla \phi(R) = -Q \nabla \left\{ \frac{1}{R} \right\} = -Q \delta(R). \quad \text{[C/m}^2]\]$$

This is a formula taught in many classic texts, but one should remember how crude a model of an electron it is. But it does describe the electric flux in an easily remembered form. However, computationally, it is less nice, due to the delta function. The main limitation of this model is that the electron has a magnetic dipole moment (spin), which a simple point charge model does not capture. When placed in a magnetic field, due to the magnetic dipole, the electron will align itself with the field.

One may apply a similar analysis to the gravitational potential. At the surface of the earth we are so far from the center of the earth that the potential appears to be linear, because the height is a tiny fraction of the radius of the earth.
5.8.2 Conjecture on photon energy

Photons are seen as quantized because they are commonly generated by atoms, which freely radiate photons (light-particles) having the difference in two energy (quantum, or eigen-states) levels. The relation \( E = h\nu \) does not inherently depend on \( \nu \) being a fixed frequency. Planck’s constant \( h \) is the EM energy density over frequency, and \( E(\nu_0) \) is the integral over frequency

\[
E(\nu_0) = h \int_{-\nu_0}^{\nu_0} d\nu = 2h\nu_0.
\]

When the photon is generated by an atom, \( \nu_0 \) is quantized by the energy level difference that corresponds to the frequency (energy level difference) of the photon jump.\(^{19}\)

Summary: Mathematics began as a simple way of keeping track of how many things there were. But eventually physics and mathematics cleverly and mysteriously evolved, to become tools to help us navigate our environment, both locally and globally, to: (1) solve daily problems such as food, water and waste management, (2) understand the solar system and the stars, (3) defend ourselves using tools of war, such as the hydrogen bomb. All powerful ideas have both bright and dark sides.

Based on the historical record of the abacus, one can infer that people precisely understood the concepts of counting, addition, subtraction, multiplication (recursive addition) and division (recursive subtraction with a fractional remainder). There is evidence that the abacus, a simple counting tool formalizing the addition of very large numbers, was introduced by the Romans to the Chinese, who used it for trade.

However, this working knowledge of arithmetic did not show up in written number systems. The Roman numerals were not useful for doing calculations done on the abacus. Only the final answer could be expressed in terms of the Roman number system.

According to the known written record, the number zero had no written symbol until the time of Brahmagupta (628 CE). One should not assume the concept of zero was not understood simply because there was no symbol for it in the Roman numeral system. Negative numbers and zero would have been obvious when using the abacus. Numbers between the integers would naturally be represented as fractional numbers (\( \mathbb{F} \)), since any irrational number (\( \mathbb{Q} \)) may be approximated with arbitrary accuracy using fractional (\( \mathbb{F} \)) numbers.

Mathematics is the science of formalizing a repetitive method into a set of rules (an algorithm), and then generalizing it as much as possible. Generalizing the multiplication and division algorithms to different types of numbers becomes increasingly more interesting as we move from integers to rational numbers, irrational numbers, real and complex numbers and, ultimately, vectors and matrices. How do you multiply two vectors, or multiply and divide one matrix by another? Is it subtraction as in the case of two numbers? Multiplying and dividing polynomials (by long division) generalizes these operations even further. Linear algebra is a further important generalization, fallout from the fundamental theorem of algebra, and essential for solving the generalizations of the number systems.

Many of the concepts about numbers naturally evolved from music, where the length of a string (along with its tension) determined the pitch (Stillwell, 2010, pp. 11, 16, 153, 261). Cutting the string’s length by half increased the frequency by a factor of 2. One fourth of the length increases the frequency by a factor of 4. One octave is a factor of 2 and two octaves a factor of 4 while a half octave is \( \sqrt{2} \). The musical scale was soon factored into rational parts. This scale almost worked, but did not generalize (sometimes known as the Pythagorean comma (Apel, 2003). resulting in today’s well tempered scale, which is based on 12 equal geometric steps along one octave, or 1/12 octave (\( \sqrt[12]{2} \approx 1.05946 \approx 18/17 = 1 + 1/17 \)).

But the concept of a factor was clear. Every number may be written as either a sum or a product (i.e., a repetitive sum). This led the early mathematicians to the concept of a prime number, which is based

\(^{19}\)There is no better example of this than the properties of very large Rydberg atoms, as beautifully articulated by MIT professor of physics Daniel Kleppler https://www.youtube.com/watch?v=e0IWPEhmMho.
on a unique factoring of every integer. At this same time (c5000 BCE), the solution of a second-degree polynomial was understood, which led to a generalization of factoring, since the polynomial, a sum of terms, may be written in factored form. If you think about this a bit, it is an amazing idea that needed to be discovered. This concept led to an important string of theorems on factoring polynomials, and how to numerically describe physical quantities. Newton was one of the first to master these tools with his proof that the orbits of the planets are ellipses, not circles. This led him to expanding functions in terms of their derivatives and power series. Could these sums be factored? The solution to this problem led to calculus.

So mathematics, a product of the human mind, is a highly successful attempt to explain the physical world. All aspects of our lives were, and are impacted by these tools. Mathematical knowledge is power. It allows one to think about complex problems in increasingly sophisticated ways. Does mathematics have a dark side? Perhaps no more than language itself. An equation is a mathematical sentence, expressing deep knowledge. Witness $E = mc^2$ and $\nabla^2 \psi = \ddot{\psi}$.

5.9 Further readings

The above concepts come straight from mathematical physics, as developed in the 17th–19th centuries. Much of this was first developed in acoustics by Helmholtz, Stokes and Rayleigh, following in Green’s footsteps, as described by Lord Rayleigh (1896). When it comes to fully appreciating Green’s theorem and reciprocity, I have found Rayleigh (1896) to be a key reference. If you wish to repeat my reading experience, start with Brillouin (1953, 1960), followed by Sommerfeld (1952) and Pipes (1958).

Second-tier reading contains many items: Morse (1948); Sommerfeld (1949); Morse and Feshbach (1953); Ramo et al. (1965); Feynman (1970a); Boas (1987). A third tier might include Helmholtz (1863a); Fry (1928); Lamb (1932); Bode (1945); Montgomery et al. (1948); Beranek (1954); Fagen (1975); Lighthill (1978); Hunt (1952); Olson (1947). Other physics writings include the impressive series of mathematical physics books by stalwart authors, J.C. Slater and Landau and Lifshitz.20

You must enter at a level that allows you to understand. Successful reading of these books critically depends on what you already know, after rudimentary (high school) level math has been mastered. Read in the order that helps you best understand the material.

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Appendices
Appendix A

Notation

A.1 Number systems

The notation used in this book is defined in this appendix so that it may be quickly accessed. Where the
definition is sketchy, page numbers are provided where these concepts are fully explained, along with
many other important and useful definitions. For example a discussion of \( \mathbb{N} \) may be found on page 28.
Math symbols such as \( \mathbb{N} \) may be found at the top of the index, since they are difficult to alphabetize.

A.1.1 Units

Strangely, or not, classical mathematics, as taught today in schools, does not seem to acknowledge the
concept of physical units. Units seem to have been abstracted away. This makes mathematics distinct
from physics, where almost everything has units. Presumably this makes mathematics more general
(i.e., abstract). But for the engineering mind, this is not ideal, or worse, as it necessarily means that
important physical meaning, by design, has been surgically removed. We shall use SI units whenever
possible, which means this book is not a typical book on mathematics. Spatial coordinates are quoted
in meters \([m]\), and time in seconds \([s]\). Angles in degrees have no units, whereas radians have units of
inverse-seconds \([s^{-1}]\). A complete list of SI units may be found at https://physics.nist.gov/
cuu/pdf/sp811.pdf and Graham et al. (1994) for a discussion of basic math notation.

A.1.2 Symbols and functions

We use \( \ln \) as the log function base \( e \), \( \log \) as base 2, and \( \pi_k \) to indicate the \( k \)th prime (e.g., \( \pi_1 = 2 \), \( \pi_2 = 3 \)).

When working with Fourier \( \mathcal{F} \) and Laplace \( \mathcal{L} \) transforms, lower case symbols are in the time
domain while upper case indicates the frequency domain, as \( f(t) \leftrightarrow F(\omega) \). An important exception is
Maxwell’s equations because they are so widely used as upper-case bold letters (e.g., \( \mathbf{E}(x, \omega) \)). It would
seem logical to change this to \( e(x, t) \leftrightarrow \mathbf{E}(x, \omega) \), to conform.

A.1.3 Common mathematical symbols

There are many pre-defined symbols in mathematics, too many to summarize here. We shall only use a
small subset, defined here.

- A set is a collection of objects that have a common property, defined by braces. For example, if
  set \( P = \{a, b, c\} \) such that \( a^2 + b^2 = c^2 \), then members of \( P \) obey the Pythagorean theorem. Thus
  we could say that \( \{1, 1, \sqrt{2}\} \in P \).

- Number sets: \( \mathbb{N}, \mathbb{P}, \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{I}, \mathbb{R}, \mathbb{C} \) are briefly discussed below, and in greater detail in §2.1
  (pp. 28–30).

\(^1\) https://en.wikipedia.org/wiki/List_of_mathematical_symbols_by_subject#Definition_symbols
• One can define sets of sets and subsets of sets, and this is prone (in my experience) to error. For example, what is the difference between the number 0 and the null set $\varnothing = \{\}$? Is $0 \in \varnothing$? Ask a mathematician. This seems a lackluster construction in the world of engineering.

• A vector is a column n-tuple. For example $[3, 5]^T = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

• The symbol $\perp$ is used in different ways to indicate two things are perpendicular, orthogonal, or in disjoint sets. In set theory $A \perp B$ is equivalent to $A \cap B = \varnothing$. If two vectors $E, H$ are perpendicular $E \perp H$, then their inner product $E \cdot H = 0$ is zero. One must infer the meaning of $\perp$ from its context.

Table A.1: List of all upper and lower case Greek letters used in the text.

| Greek letter | Frequently used with | Table A.2 indicates the symbol followed by a page number indication where it is discussed, and the Genus (class) of the number type. For example, $\omega$ [rad] is the radian frequency $2\pi f$, $\rho$ [kgm/m$^3$] is commonly the density. $\phi, \psi$ are commonly used to indicate angles of a triangle, and $\zeta(s)$ is the Riemann zeta function. Many of these are so well established it makes no sense to define new terms, so we will adopt these common terms (and define them).

Likely you do not know all of these Greek letters, commonly used in mathematics. Some of them are pronounced in strange ways. The symbol $\xi$ is pronounced “see,” $\zeta$ is “zeta,” $\beta$ is “beta,” and $\chi$ is “kie” (rhymes with pie and sky). I will assume you know how to pronounce the others, which are more phonetic in English. One advantage of learning \textsc{L\TeX} the powerful open-source math-oriented word-processing system used to write this book, is that math symbols are included, making them easily learned.

Double-Bold notation

Table A.2 indicates the symbol followed by a page number indication where it is discussed, and the Genus (class) of the number type. For example, $\mathbb{N} > 0$ indicates the infinite set of counting numbers $\{1, 2, 3, \cdots\}$, not including zero. Starting from any counting number, you get the next one by adding 1. Counting numbers are sometimes called the natural or cardinal numbers.

We say that a number is in the set with the notation $3 \in \mathbb{N} \subset \mathbb{R}$, which is read as “3 is in the set of counting numbers, which in turn is in the set of real numbers,” or in vernacular language “3 is a real counting number.”

Prime numbers ($\mathbb{P} \subset \mathbb{N}$) are taken from the counting numbers, but do not include 1.
Table A.2: Double-bold notation for the types of numbers. (#) is a page number. Symbol with an exponent denote the dimensionality. Thus $\mathbb{R}^2$ represents the real plane. An exponent of 0 denotes point, e.g., $j \in \mathbb{C}^0$. It is reasonable to consider negative primes to be primes.

<table>
<thead>
<tr>
<th>Symbol (p. #)</th>
<th>Genus</th>
<th>Examples</th>
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The signed integers $\mathbb{Z}$ include 0 and negative integers. Rational numbers $\mathbb{Q}$ are historically defined to include $\mathbb{Z}$, a somewhat inconvenient definition, since the more interesting class are the fractional $\mathbb{F}$, a subset of rationals $\mathbb{F} \in \mathbb{Q}$ that exclude the integers (i.e., $\mathbb{F} \perp \mathbb{Z}$). This is a useful definition because the rationals $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ are formed from the union of integers and fractionals.

The rationals may be defined, using set notation (a very sloppy notation with an incomprehensible syntax), as

$$\mathbb{Q} = \{p/q : q \neq 0 \& p, q \in \mathbb{Z}\},$$

which may be read as “the set ‘\{···\}’ of all $p/q$ such that ‘·’ $q \neq 0$, ‘and’ $p, q \subset \mathbb{Z}$. The translation of the symbols is in single (‘···’) quotes.

Irrational numbers $\mathbb{I}$ are very special: They are formed by taking a limit of fractionals, as the numerator and denominator $\rightarrow \infty$, and approach a limit point. It follows that irrational numbers must be approximated by fractionals.

The reals ($\mathbb{R}$) include complex numbers ($\mathbb{C}$) having a zero imaginary part (i.e., $\mathbb{R} \subset \mathbb{C}$).

The size of a set is denoted by taking the absolute value (e.g., $|\mathbb{N}|$). Normally in mathematics this symbol indicates the cardinality, so we are defining it differently from the standard notation.

A.1.4 Classification of numbers:

From the above definitions there exists a natural hierarchical structure of numbers:

$$\mathbb{P} \in \mathbb{N}, \quad \mathbb{Z} : \{\mathbb{N}, 0, -\mathbb{N}\}, \quad \mathbb{F} \perp \mathbb{Z}, \quad \mathbb{Q} : \mathbb{Z} \cup \mathbb{F}, \quad \mathbb{R} : \mathbb{Q} \cup \mathbb{I} \subset \mathbb{C}$$

1. The primes are a subset of the counting numbers: $\mathbb{P} \subset \mathbb{N}$.

2. The signed integers $\mathbb{Z}$ are composed of $\pm \mathbb{N}$ and $0$, thus $\mathbb{N} \subset \mathbb{Z}$.

3. The fractionals $\mathbb{F}$ do not include of the signed integers $\mathbb{Z}$.

4. The rationals $\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$ are the union of the signed integers and fractionals.

5. Irrational numbers $\mathbb{I}$ have the special properties $\mathbb{I} \perp \mathbb{Q}$.

6. The reals $\mathbb{R} : \mathbb{Q}, \mathbb{I}$ are the union of rationals and irrational $\mathbb{I}$.

7. Reals $\mathbb{R}$ may be defined as a subset of those complex numbers $\mathbb{C}$ having zero imaginary part.
A.1.5 Rounding schemes

In Matlab/Octave there are five different rounding schemes (i.e., mappings): \texttt{round}(x), \texttt{fix}(x), \texttt{floor}(x), \texttt{ceil}(x), \texttt{roundb}(x), with input \(x \in \mathbb{R}\) and output \(k \in \mathbb{N}\). For example \(3 = \lfloor x \rfloor, 3 = [e^1] = [2.7183]\) rounds to the nearest integer, whereas \(3 = \text{floor}(\pi)\) rounds down while \(3 = \text{ceil}(e^1) = [e^1]\) rounds up. Rounding schemes are use for quantizing a number and generating a remainder. For example: \(y = \text{rem}(x)\) is equivalent to \(y = x - \lfloor x \rfloor\). Note \(\text{round}(\pi) \equiv [\pi]\) introduces negative remainders in the remainder when ever a number rounds up \((\pi = [\pi] - 0.8541)\).

The \textit{continued fraction algorithm} (CFA), §2.3.2 (p. 45) is a recursive rounding scheme, operating on the reciprocal of the remainder. For example:

\[
\exp(1) = 3 + 1/(−4 + 1/(2 + 1/(5 + 1/((-2 + 1/((-7))))))) - o(1.75 \times 10^{-6}),
\]

\[
= [3; -4, 2, 5, -2, -7] - o(1.75 \times 10^{-6}).
\]

The expressions in brackets is a notation for the CFA integer coefficients. The Octave/Matlab function having \(x \in \mathbb{R}\), is either \texttt{rat}(x) with output \(\mathbb{N}\), or \texttt{rats}(x), with output \(\mathbb{F}\).

A.1.6 Periodic functions

Fourier series tells us that periodic functions are discrete in frequency, with frequencies given by \(nT_s\), where \(T_s\) is the sample period. The discrete \(\text{FFT}\) (DFT) is a good example. When using the DFT, the sample period is \(T_s = 1/2F_{\text{max}}\) and the minimum and maximum frequencies are given by \(F_{\text{min}} = F_{\text{max}}/\text{NFT}\) where \(\text{NFT}\) is the size of the DFT.

This concept is captured by the \textit{Fourier series}, which is a frequency expansion of a periodic function. This concept is quite general. Periodic in frequency implies discrete in time. Periodic and discrete in time requires periodic and discrete in frequency (the case of the DFT). The modulo function \(x = \text{mod} (x, y)\) is periodic with period \(y (x, y \in \mathbb{R})\).

A periodic function may be conveniently indicated using double-parentheses notation. This is sometimes known as modular arithmetic. For example,

\[
f((t))_{T} = f(t) = f(t \pm kT)
\]

is periodic on \(t, T \in \mathbb{R}\) with a period of \(T\) and \(k \in \mathbb{Z}\). This notation is useful when dealing with Fourier series of periodic functions such as \(\sin(\theta)\) where \(\sin(\theta) = \sin((\theta)2\pi) = \text{mod} (\sin(\theta), 2\pi)\).

When a discrete valued (e.g., time \(t \in \mathbb{N}\)) sequence is periodic with period \(N \in \mathbb{Z}\), we use square brackets

\[
f[[n]]_{N} = f[n] = f[n \pm kN],
\]

with \(k \in \mathbb{Z}\). This notation will be used with discrete-time signals that are periodic, such as the case of the DFT.

It is common for fractions to repeat. For example \(1/7 = 0.(142857)\) where the double brackets indicates this number repeats. That is \(1/7 = 0.142857, 142857, 142857, 142857, \ldots\).

A.2 Differential equations vs. polynomials

A polynomial has \textit{degree} \(N\) defined by the largest power. A quadratic equation is degree 2, and a cubic has degree 3. We shall indicate a polynomial by the notation

\[
P_N(z) = z^N + a_{N-1}z^{N-1} \cdots a_0.
\]

It is a good practice to normalize the polynomial so that \(a_N = 1\). This will not change the roots, defined by Eq. 3.1.1.7 (p. 58). The coefficient on \(z^{N-1}\) is always the sum of the roots \(z_n (a_{N-1} = \sum_{n} z_n)\), and \(a_0\) is always their product \((a_0 = \prod_{n} z_n)\).
Differential equations have order (polynomials have degree). If a second-order differential equation is Laplace transformed (Lec. 3.7, p. 103), one is left with a degree 2 polynomial. For example,

\[
\frac{d^2}{dt^2}y(t) + b\frac{dy(t)}{dt} + cy(t) = \alpha \left( \frac{d}{dt}x(t) + \beta x(t) \right) \leftrightarrow \\
(s^2 + bs + c)Y(s) = \alpha(s + \beta)X(s).
\]

(A.2.0.1)

\[
\frac{Y(s)}{X(s)} = \frac{s + \beta}{s^2 + bs + c} \equiv H(s) \leftrightarrow h(t).
\]

(A.2.0.2)

Using the same argument as for polynomials, the lead coefficient must always be 1. The coefficient \(\alpha \in \mathbb{R}\) is called the gain. The complex variable \(s\) is the Laplace frequency.

The ratio of the output \(Y(s)\) over the input \(X(s)\) is called the system transfer function \(H(s)\). When \(H(s)\) is the ratio of two polynomials in \(s\), the transfer function is said to be bilinear, since it is linear in both the input and output. The roots of the numerator are called the zeros and those of the denominator, the poles. The inverse Laplace transform of the transfer function is called the system impulse response, which describes the system’s output signal \(y(t)\) for any given input signal \(x(t)\), via convolution (i.e., \(y(t) = h(t) \ast x(t)\)).

### A.3 Matrix algebra: Systems

#### A.3.1 Vectors

Vectors as columns of ordered sets of scalars \(\in \mathbb{C}\). When we write them out in text, we typically use row notation, with the transpose symbol:

\[
[a, b, c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.
\]

This is strictly to save space on the page. The notation for conjugate transpose is \(\dagger\), for example

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix}^\dagger = \begin{bmatrix} a^* & b^* & c^* \end{bmatrix}.
\]

The above example is said to be a 3-dimensional vector because it has three components.

**Row vs. column vectors:** With rare exceptions, vectors are columns, denoted column-major.\(^2\) To avoid confusion, it is a good rule to make your mental default column-major, in keeping with most signal processing (vectorized) software.\(^3\) Column vectors are the unstated default of Matlab/Octave, only revealed when matrix operations are performed. The need for the column (or row) major is revealed as a consequence of efficiency when accessing long sequences of numbers from computer memory. For example, when forming the sum of many numbers using the Matlab/Octave command \(\text{sum}(A)\), where \(A\) is a matrix, Matlab/Octave operates on the columns, returning a row vector of column sums:

\[
\text{sum} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [4, 6].
\]

If the data were stored in row-major order, the answer would be the column vector \(\begin{bmatrix} 3 \\ 7 \end{bmatrix}\). Thus Matlab/Octave is column-major by default.

\(^2\)https://en.wikipedia.org/wiki/Row-_and_column-major_order

\(^3\)In contrast, reading words in English is ‘row-major.’
APPENDIX A. NOTATION

A.3.2 Vector products

A scalar product (aka dot product) is defined to “weight” vector elements before summing them, resulting in a scalar. The transpose of a vector (a row-vector) is typically used as a scale factor (i.e., weights) on the elements of a vector. For example,

\[
\begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix} \cdot \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}^T = \begin{bmatrix}
1 & 2 & -1
\end{bmatrix} \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = 1 + 2 \cdot 2 - 3 = 2.
\]

A more interesting example is

\[
\begin{bmatrix}
1 \\
2 \\
4
\end{bmatrix} \cdot \begin{bmatrix}
1 \\
1 \\
4
\end{bmatrix}^T = \begin{bmatrix}
1 & 2 & 4
\end{bmatrix} \begin{bmatrix}
1 \\
1 \\
4
\end{bmatrix} = 1 + 2s + 4s^2.
\]

Polar scalar product: The vector-scalar product in polar coordinates is (Fig. 3.4, p. 84)

\[B \cdot C = \|B\| \|C\| \cos \theta \in \mathbb{R},\]

where \(\cos \theta \in \mathbb{R}\) is called the direction-cosine between \(B\) and \(C\).

Polar wedge product: The vector wedge product in polar coordinates is (Fig. 3.4, p. 84)

\[B \wedge C = \|B\| \|C\| \sin \theta \in \mathbb{R},\]

where \(\sin \theta \in \mathbb{R}\) is therefore the direction-sine between \(B\) and \(C\).

Complex polar vector product: From these two polar definitions and \(e^{j\theta} = \cos \theta + j \sin \theta\),

\[B \cdot C + jB \wedge C = \|B\| \|C\| e^{j\theta}.
\]

Hence

\[|B \cdot C|^2 + |B \wedge C|^2 = \|B\|^2 \|C\|^2 \cos^2 \theta + \|B\|^2 \|C\|^2 \sin^2 \theta = \|B\|^2 \|C\|^2.
\]

This relationship holds true in any vector space, of any number of dimensions, containing vectors \(B\) and \(C\). In this case \(s = \sigma + \omega j \in \mathbb{C}\) can be the Laplace frequency. Jaynes (1991) has an relevant discussion about this type of vector product.

A.3.3 Norms of vectors

The norm of a vector is the scalar product of the vector with itself

\[\|A\| = \sqrt{A \cdot A} \geq 0,
\]

forming the Euclidean length of the vector.

Euclidean distance between two points in \(\mathbb{R}^3\): The scalar product of the difference between two vectors \((A - B) \cdot (A - B)\) is the Euclidean distance between the points they define

\[\|A - B\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.
\]
Triangle inequality

\[ \| A + B \| = \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2} \leq \| A \| + \| B \|. \]

In terms of a right triangle this says the the sum of the lengths of the two sides is greater to the length of
the hypotenuse, and equal when the triangle degenerates into a line.

Vector cross product: The vector product (aka cross product) \( A \times B = \| A \| \| B \| \sin \theta \) is defined
between the two vectors \( A \) and \( B \). In Cartesian coordinates

\[
A \times B = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.
\]

The triple product: This is defined between three vectors as

\[
A \cdot (B \times C) = \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.
\]

This may be indicated without the use of parentheses, since there can be no other meaningful interpreta-
tion. However for clarity, parentheses should be used. The triple product is the volume of the
parallelepiped (3D-crystal shape) outlined by the three vectors, as shown in Fig. ??, p. ??.

Dialects of vector notation: Physical fields are, by definition, functions of space \( x \) [m], and in the
most general case, time \( t \) [s]. When Laplace transformed, the fields become functions of space and
complex frequency (e.g., \( E(x, t) \leftrightarrow E(x, s) \)). As before, there are several equivalent vector notations.
For example, \( E(x, t) = \begin{bmatrix} E_x, E_y, E_z \end{bmatrix}^T = E_x(x, t)\hat{x} + E_y(x, t)\hat{y} + E_z(x, t)\hat{z} \) is “in-line,” to save
space. The same equation may written in “displayed” notation as:

\[
E(x, t) = \begin{bmatrix} E_x(x, t) \\ E_y(x, t) \\ E_z(x, t) \end{bmatrix} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}(x, t) = \begin{bmatrix} E_x, E_y, E_z \end{bmatrix}^T \equiv E_x\hat{x} + E_y\hat{y} + E_z\hat{z}.
\]

Note the three notations for vectors, bold font, element-wise columns, element-wise transposed rows
and dyadic format. These are all shorthand notations for expressing the vector. Such usage is similar to
a dialect in a language.

Complex elements: When the elements are complex (\( \in \mathbb{C} \)), the transpose is defined as the complex
conjugate of the elements. In such complex cases the transpose conjugate may be denoted with a \( ^\dagger \) rather
than \( T \)

\[
\begin{bmatrix} -2j \\ 3j \\ 1 \end{bmatrix}^\dagger = \begin{bmatrix} 2j & -3j & 1 \end{bmatrix} \in \mathbb{C}.
\]

For this case when the elements are complex, the dot product is a real number

\[
a \cdot b = a^\dagger b = \begin{bmatrix} a_1^* & a_2^* & a_3^* \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1^*b_1 + a_2^*b_2 + a_3^*b_3 \in \mathbb{R}.
\]
Norm of a complex vector: The dot product of a vector with itself is called the norm of a \[ \|a\| = \sqrt{a^\dagger a} \geq 0. \]
which is always non-negative.

Such a construction is useful when \( a \) and \( b \) are related by an impedance matrix
\[ V(s) = Z(s)I(s) \]
and we wish to compute the power. For example, the impedance of a mass is \( ms \) and a capacitor is \( 1/sC \). When given a system of equations (a mechanical or electrical circuit) one may define an impedance matrix.

Complex power: In this special case, the complex power \( P(s) \in \mathbb{R}(s) \) is defined, in the complex frequency domain \( (s) \), as
\[ P(s) = I^\dagger(s)V(s) = I^\dagger(s)Z(s)I(s) \leftrightarrow p(t) \quad [W]. \]
The real part of the complex power must be positive. The imaginary part corresponds to available stored energy.

The case of three-dimensions is special, allowing definitions that are not easily defined in more than three dimensions. A vector in \( \mathbb{R}^3 \) labels the point having the coordinates of that vector.

A.3.4 Matrices

When working with matrices, the role of the weights and vectors can change, depending on the context. A useful way to view a matrix is as a set of column vectors, weighted by the elements of the column-vector of weights multiplied from the right. For example,
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1M} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2M} \\
\vdots & & & & \vdots \\
a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NM}
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
\vdots \\
w_M
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & & & & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
a_{11} \\
a_{21} \\
a_{31} \\
\vdots \\
a_{N1}
\end{bmatrix}
+ \cdots +
\begin{bmatrix}
a_{12} \\
a_{22} \\
a_{32} \\
\vdots \\
a_{N2}
\end{bmatrix}
+ \cdots +
\begin{bmatrix}
a_{1M} \\
a_{2M} \\
a_{3M} \\
\vdots \\
a_{NM}
\end{bmatrix},
\]
where the weights are \( [w_1, w_2, \ldots, w_M]^T \). Alternatively, the matrix is a set of row vectors of weights, each of which is applied to the column vector on the right \( ([w_1, w_2, \ldots, W_M]^T) \).

The determinant of a matrix is denoted as either \( \det A \) or simply \( |A| \) (as in the absolute value). The inverse of a square matrix is \( A^{-1} \) or \( \inv A \). If \( |A| = 0 \), the inverse does not exist. \( AA^{-1} = A^{-1}A \).

Matlab/Octave’s notional convention for a row-vector is \( [a, b, c] \) and a column-vector is \( [a; b; c] \). A prime on a vector takes the complex conjugate transpose. To suppress the conjugation, place a period before the prime. The : argument converts the array into a column vector, without conjugation. A tacit notation in Matlab is that vectors are columns and the index to a vector is a row vector. Matlab defines the notation 1:4 as the “row-vector” \([1, 2, 3, 4] \), which is unfortunate as it leads users to assume that the default vector is a row. This can lead to serious confusion later, as Matlab’s default vector is a column. I have not found the above convention explicitly stated, and it took me years to figure this out for myself.

When writing a complex number we shall adopt \( 1j \) to indicate \( \sqrt{-1} \). Matlab/Octave allows either \( 1i \) or \( 1j \).

Units are SI; angles are in degrees [deg] unless otherwise noted. The units for \( \pi \) are always radians [rad]. For example \( \sin(\pi), e^{j\theta} = e^{\pi/2} \).
A.3.5 2 × 2 complex matrices

Here are some definitions to learn:

1. **Scalar**: A number – for example \{a, b, c, α, β, \ldots\} ∈ \{Z, Q, I, R, C\}

2. **Vector**: A quantity having direction as well as magnitude, often denoted by a bold letter \(\mathbf{x}\), or with an arrow over the top \(\vec{x}\). In matrix notation, this is typically represented as a single row \([x_1, x_2, x_3, \ldots]\) or single column \([x_1, x_2, x_3, \ldots]^T\) (where \(T\) indicates the transpose). In this class we will typically use column vectors. The vector may also be written out using unit vector notation \(\mathbf{x}\). In matrix notation, this is typically represented as a single row \([x, y, z]\). Unit vectors in the \(x, y, z\) Cartesian directions (here the vector’s subscript \(T\)) indicates the transpose. In this class we work only with \(R\) square matrices.

3. **Matrix**: \(A = [a_{11}, a_{12}, a_{31}, \ldots, a_{M1}]_{N \times M} \) can be a non-square matrix if the number of elements in each of the vectors (\(N\)) is not equal to the number of vectors (\(M\)). When \(M = N\), the matrix is square. It may be inverted if its determinant \(|A| = \prod \lambda_k \neq 0\) (where \(\lambda_k\) are the eigenvalues). In this text we work only with \(2 \times 2\) and \(3 \times 3\) square matrices.

4. **Linear system of equations**: \(Ax = b\) where \(x\) and \(b\) are vectors and matrix \(A\) is a square.

   (a) **Inverse**: The solution of this system of equations may be found by finding the inverse \(x = A^{-1}b\).

   (b) **Equivalence**: If two systems of equations \(A_0x = b_0\) and \(A_1x = b_1\) have the same solution (i.e., \(x = A_0^{-1}b_0 = A_1^{-1}b_1\)), they are said to be equivalent.

   (c) **Augmented matrix**: The first type of augmented matrix is defined by combining the matrix with the right-hand side. For example, given the linear system of equations of the form \(Ax = y\)

   \[
   \begin{bmatrix}
   a & b \\ c & d
   \end{bmatrix}
   \begin{bmatrix}
   x_1 \\ x_2
   \end{bmatrix}
   =
   \begin{bmatrix}
   y_1 \\ y_2
   \end{bmatrix},
   \]

   the augmented matrix is

   \[
   [A|y] = \begin{bmatrix}
   a & b \\ c & d
   \end{bmatrix}
   \begin{bmatrix}
   y_1 \\ y_2
   \end{bmatrix}.
   \]

   A second type of augmented matrix may be used for finding the inverse of a matrix (rather than solving a specific instance of linear equations \(Ax = b\)). In this case the augmented matrix is

   \[
   [A|I] = \begin{bmatrix}
   a & b \\ c & d
   \end{bmatrix}
   \begin{bmatrix}
   1 & 0 \\ 0 & 1
   \end{bmatrix}.
   \]

   Performing Gaussian elimination on this matrix, until the left side becomes the identity matrix, yields \(A^{-1}\). This is because multiplying both sides by \(A^{-1}\) gives \(A^{-1}A|A^{-1}I = I|A^{-1}\).

5. **Permutation matrix \((P)\)**: A matrix that is equivalent to the identity matrix, but with scrambled rows (or columns). Such a matrix has the properties \(\det(P) = \pm 1\) and \(P^2 = I\). For the \(2 \times 2\) case, there is only one permutation matrix:

   \[
   P = \begin{bmatrix}
   0 & 1 \\ 1 & 0
   \end{bmatrix},
   \]

   \[
   P^2 = \begin{bmatrix}
   0 & 1 \\ 1 & 0
   \end{bmatrix}
   \begin{bmatrix}
   0 & 1 \\ 1 & 0
   \end{bmatrix} = \begin{bmatrix}
   1 & 0 \\ 0 & 1
   \end{bmatrix}.
   \]

   A permutation matrix \(P\) swaps rows or columns of the matrix it operates on. For example, in the \(2 \times 2\) case, pre-multiplication swaps the rows,

   \[
   PA = \begin{bmatrix}
   0 & 1 \\ 1 & 0
   \end{bmatrix}
   \begin{bmatrix}
   a & b \\ α & β
   \end{bmatrix} = \begin{bmatrix}
   α & β \\ a & b
   \end{bmatrix}.
   \]
whereas post-multiplication swaps the columns,
\[
AP = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ \beta & \alpha \end{bmatrix}.
\]

For the $3 \times 2$ case there are $3 \cdot 2/2 = 3$ such matrices (swap a row with the other 2, then swap the remaining two rows).

6. **Gaussian elimination (GE) operations $G_k$**: There are three types of elementary row operations, which may be performed without fundamentally altering a system of equations (e.g. the resulting system of equations is equivalent). These operations are (1) swap rows (e.g. using a permutation matrix), (2) scale rows, or (3) perform addition/subtraction of two scaled rows. All such operations can be performed using matrices.

For lack of a better term, we’ll describe these as ‘Gaussian elimination’ or ‘GE’ matrices. We will categorize any matrix that performs only elementary row operations (but any number of them) as a ‘GE’ matrix. Therefore, a cascade of GE matrices is also a GE matrix.

Consider the GE matrix
\[
G = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.
\]

(a) This pre-multiplication scales and subtracts row (1) from (2) and returns it to row (2).
\[
GA = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} a & b \\ a-\alpha & b-\beta \end{bmatrix}.
\]

The shorthand for this Gaussian elimination operation is $(1) \leftarrow (1)$ and $(2) \leftarrow (1) - (2)$.

(b) Post-multiplication adds and scales columns.
\[
AG = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a-b & b \\ \alpha-\beta & \beta \end{bmatrix}.
\]

Here the second column is subtracted from the first, and placed in the first. The second column is untouched. This operation is not a Gaussian elimination. Therefore, to put Gaussian elimination operations in matrix form, we form a cascade of pre-multiply matrices.

Here $\det(G) = 1$, $G^2 = I$, which won’t always be true if we scale by a number greater than 1. For instance, if $G = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$ (scale and add), then we have $\det(G) = 1$, $G^n = \begin{bmatrix} 1 & 0 \\ n \cdot m & 1 \end{bmatrix}$.

**Exercise:** Find the solution to the following $3 \times 3$ matrix equation $Ax = b$ by Gaussian elimination. Show your intermediate steps. You can check your work at each step using Matlab.

\[
\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.
\]

---

The term ‘elementary matrix’ may also be used to refer to a matrix that performs an elementary row operation. Typically, each elementary matrix differs from the identity matrix by a single row operation. A cascade of elementary matrices could be used to perform Gaussian elimination.
1. Show (i.e., verify) that the first GE matrix \( G_1 \), which zeros out all entries in the first column, is given by

\[
G_1 = \begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}.
\]

Identify the elementary row operations that this matrix performs. **Solution:** Operate with GE matrix on \( A \)

\[
G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4
\end{bmatrix} = \begin{bmatrix}
1 & 1 & -1 \\
0 & -2 & 4 \\
0 & -2 & 5
\end{bmatrix}.
\]

The second row of \( G_1 \) scales the first row by -3 and adds it to the second row \((2) \Leftarrow -3(1) + (2)\).

The third row of \( G_1 \) scales the first row by -1 and adds it to the third row \([(3) \Leftarrow -(1) + (3)]\).

2. Find a second GE matrix, \( G_2 \), to put \( G_1A \) in upper triangular form. Identify the elementary row operations that this matrix performs. **Solution:**

\[
G_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix},
\]

or \([(2) \Leftarrow -(2) + (3)]\). Thus we have

\[
G_2G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 & -1 \\
3 & 1 & 1 \\
1 & -1 & 4
\end{bmatrix} = \begin{bmatrix}
1 & 1 & -1 \\
0 & -2 & 4 \\
0 & 0 & 1
\end{bmatrix}.
\]

3. Find a third GE matrix, \( G_3 \), which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs. **Solution:**

\[
G_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

which scales the second row by \(-1/2\). Thus we have

\[
G_3G_2G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 & -1 \\
0 & -2 & 4 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 1 & -1 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix}.
\]

4. Finally, find the last GE matrix, \( G_4 \), that subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix \((G_4G_3G_2G_1A = I)\). **Solution:**

\[
G_4 = \begin{bmatrix}
1 & -1 & -1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}.
\]
Thus we have

\[ G_4G_3G_2G_1[A|b] = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

5. Solve for \([x_1, x_2, x_3]^T\) using the augmented matrix format \(G_4G_3G_2G_1[A|b]\) (where \([A|b]\) is the augmented matrix). Note that if you’ve performed the preceding steps correctly, \(x = G_4G_3G_2G_1b\).

**Solution:** From the preceding problems, we see that \([x_1, x_2, x_3]^T = [3, -1, 1]^T.\]

### Inverse of the 2 × 2 matrix

We shall now apply Gaussian elimination to find the solution \([x_1, x_2]\) for the 2 × 2 matrix equation \(Ax = y\) (Eq. 3.4.2.8, left). We assume to know \([a, b, c, d]\) and \([y_1, y_2]\). We wish to show that the intersection (solution) is given by the equation on the right.

Here we wish to prove that the left equation (i) has an inverse given by the right equation (ii):

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (i); \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (ii). 
\]

To take the inverse we:

1. (1) swap the diagonal, (2) change the off-diagonal signs, and (3) normalize by the determinant \(\Delta\).

### Derivation of the inverse of a 2 × 2 matrix

1. Step 1: To derive (ii) starting from (i), normalize the first column to 1.

\[
\begin{bmatrix} 1 & \frac{b}{a} \\ 1 & \frac{c}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. 
\]

2. Step 2: Subtract row (1) from row (2): (2) ← (2) − (1)

\[
\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{a}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -1 \end{bmatrix}.
\]

3. Step 3: Multiply row (2) by \(ca\) and express result in terms of the determinate \(\Delta = ad - bc\).

\[
\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.
\]

4. Step 4: Solve row (2) for \(x_2\): \(x_2 = -\frac{c}{a}y_1 + \frac{a}{a}y_2\).

5. Step 5: Solve row (1) for \(x_1\):

\[
x_1 = \frac{1}{a}y_1 - \frac{b}{a}x_2 = \left[ \frac{1}{a} + \frac{b}{a} \right] y_1 - \frac{b}{a} \frac{c}{a} y_2.
\]

Rewritting in matrix format, in terms of \(\Delta = ad - bc\), gives:

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[ \frac{1}{a} + \frac{b}{a} \right] y_1 - \frac{b}{a} \frac{c}{a} y_2 = \begin{bmatrix} \Delta + bc \\ -c \Delta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},
\]

since \(d = (\Delta + bc)/a\).

**Summary:** This is a lot of messy algebra, so it is essential that you memorize the final result:

1. (1) swap the diagonal, (2) change the off-diagonal signs, and (3) normalize by the determinant \(\Delta\).
Appendix B

Eigenanalysis

Eigenanalysis is ubiquitous in engineering applications. It is useful in solving differential and difference equations, data-science applications, numerical approximation and computing, and linear algebra applications. Typically one must take a course in linear algebra to become knowledgeable in the inner workings of this method. In this appendix we intend to provide sufficient basics to allow one to read the text.

B.1 The eigenvalue matrix (Λ)

Given $2 \times 2$ matrix $A$, the related matrix eigen-equation is

$$AE = E\Lambda.$$  \hspace{1cm} (B.1.0.1)

Pre-multiplying by $E^{-1}$ diagonalizes $A$, resulting in the eigenvalue matrix

$$\Lambda = E^{-1}AE$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \hspace{1cm} (B.1.0.2)$$

Post-multiplying by $E^{-1}$ recovers $A$

$$A = E\Lambda E^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \hspace{1cm} (B.1.0.4)$$

Matrix product formula:

This last relation is the entire point of the eigenvector analysis, since it shows that any power of $A$ may be computed from powers of the eigenvalues. Specifically,

$$A^n = E\Lambda^n E^{-1}. \hspace{1cm} (B.1.0.5)$$

For example, $A^2 = AA = E\Lambda (E^{-1}E) \Lambda E^{-1} = E\Lambda^2 E^{-1}$.

Equations B.1.0.1, B.1.0.3 and B.1.0.4 are the key to eigenvector analysis, and you need to memorize them. You will use them repeatedly throughout this course.

Showing that $A - \lambda I_2$ is singular:

If we restrict Eq. B.1.0.1 to a single eigenvector (one of $e_{\pm}$), along with the corresponding eigenvalue $\lambda_{\pm}$, we obtain a matrix equations

$$Ae_{\pm} = e_{\pm}\lambda_{\pm} = \lambda_{\pm}e_{\pm}. \hspace{1cm} (B.1.0.6)$$
Note the swap in the order of $E_\pm$ and $\lambda_\pm$. Since $\lambda_\pm$ is a scalar, this is legal (and critically important), since this allows us to factor out $e_\pm$

$$ (A - \lambda_\pm I_2)e_\pm = 0. \quad \text{(B.1.0.6)} $$

The matrix $A - \lambda_\pm I_2$ must be singular because when it operates on $e_\pm$, having nonzero norm, it must be zero.

It follows that its determinant (i.e., $|(A - \lambda_\pm I_2)| = 0$) must be zero. This equation uniquely determines the eigenvalues $\lambda_\pm$.

### B.1.1 Calculating the eigenvalues $\lambda_\pm$

The eigenvalues $\lambda_\pm$ of $A$ may be determined from $|(A - \lambda_\pm I_2)| = 0$. As an example we let $A$ be Pell’s equation (Eq. 2.3.4.9, p. 50). In this case the eigenvalues may be found from

$$ \begin{vmatrix} 1 - \lambda_\pm & N \\ 1 & 1 - \lambda_\pm \end{vmatrix} = (1 - \lambda_\pm)^2 - N = 0, $$

thus $\lambda_\pm = (1 \mp \sqrt{N})$.\(^1\)

### B.1.2 Calculating the eigenvectors $e_\pm$

Once the eigenvalues have been determined, they are substituted into Eq. B.1.0.6, which determines the eigenvectors $E = \left[ e_+, e_- \right]$, by solving

$$ (A - \lambda_\pm I_2)e_\pm = \begin{bmatrix} 1 - \lambda_\pm & 2 \\ 1 & 1 - \lambda_\pm \end{bmatrix} e_\pm = 0, \quad \text{(B.1.2.7)} $$

where $1 - \lambda_\pm = 1 - (1 \mp \sqrt{N}) = \pm \sqrt{N}$, thus the Pell equation eigenvalues are

$$ \lambda_\pm = 1 \mp \sqrt{N}. $$

Recall that Eq. B.1.0.6 is singular because we are using an eigenvalue, and each eigenvector is pointing in a unique direction (this is why it is singular). You might expect that this equation has no solution. In some sense you would be correct. When we solve for $e_\pm$, the two equations defined by Eq. B.1.0.6 are co-linear (the two equations describe parallel lines so their wedge product is zero). This follows from the fact that there is only one eigenvector for each eigenvalue.

Since there is only one eigenvalue we are expecting trouble, yet we may proceeding to solve for $e_+ = [e_1^+, e_2^+]^T$ with eigenvalue $+\sqrt{N}$

$$ \begin{bmatrix} \sqrt{N} & 1 \\ 1 & \sqrt{N} \end{bmatrix} \begin{bmatrix} e_1^+ \\ e_2^+ \end{bmatrix} = 0. $$

If we divide the top row by $\sqrt{N}$ the two rows are identical, since the matrix must be singular. Thus this matrix equation gives two identical equations. This is the price of an over-specified equation (the singular matrix is degenerate).

We can determine each eigenvectors direction, but not their magnitudes.

Following the same procedure for $\lambda_- = -\sqrt{N}$, the equation for $e_-$ is

$$ \begin{bmatrix} -\sqrt{N} & N \\ 1 & -\sqrt{N} \end{bmatrix} \begin{bmatrix} e_1^- \\ e_2^- \end{bmatrix} = 0. $$

As before, this matrix is singular. Here $e_1^- - \sqrt{N}e_2^- = 0$, thus the eigenvector is $e^- = c [\sqrt{N}, 1]^T$ where $c$ is a normalization constant.

Thus the unnormalized eigenmatrix is

$$ E = \begin{bmatrix} e_1^+ & e_2^- \\ e_2^+ & e_2^- \end{bmatrix} = \begin{bmatrix} \sqrt{N} & -\sqrt{N} \\ 1 & 1 \end{bmatrix}. $$

\(^1\)It is a convention to order the eigenvalues from largest to smallest.
Normalization of the eigenvectors:
The constant $c$ may be determined by normalizing the eigenvectors to have unit length. Since we cannot determine the length, we set it to 1. In some sense the degeneracy is resolved by this normalization.

$$\left( \pm \sqrt{N} \right)^2 + 1^2 = N + 1 = 1/c^2.$$  

Thus the normalization factor to force each eigenvector to have length 1 is $c = 1/\sqrt{N+1}$.

### B.2 Pell equation solution example

§2.3.4 (p. 50) showed that the solution $[x_n, y_n]^T$ to Pell’s equation is given by powers of the Pell matrix $A$. For $N = 2$, in §2.3.4 we found the explicit formula for $[x_n, y_n]^T$, based on powers of the Pell matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

(B.2.0.1)

This recursive solution to Pell’s equation (Eq. 2.3.4.7) is Eq. 2.3.4.9 (p. 50). Thus we need powers of $A$, that is $A^n$, which gives an explicit expression for $[x_n, y_n]^T$. By the diagonalization of $A$, its powers are simply the powers of its eigenvalues.

From Matlab/Octave with $N = 2$ the eigenvalues of Eq. B.2.0.1 are $\lambda_{\pm} \approx [2.4142, -0.4142]$ (i.e., $\lambda_{\pm} = \sqrt{1 \pm \sqrt{2}}$). The solution for $N = 3$ is shown on page 200.

Once the matrix has been diagonalized, one may compute powers of that matrix as powers of the eigenvalues. This results in the general solution given by

$$[x_n, y_n] = 1^n A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1^n E \Lambda^n E^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$  

The eigenvalue matrix $D$ is diagonal with the eigenvalues sorted, largest first. The Matlab/Octave command $[E, D] = \text{eig}(A)$ is helpful to find $D$ and $E$ given any $A$. As we saw above,

$$\Lambda = 1^n \begin{pmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{pmatrix} \approx \begin{pmatrix} 2.4142 & 0 \\ 0 & -0.4142 \end{pmatrix}.$$ 

#### B.2.1 Pell equation eigenvalue-eigenvector analysis

Here we show how to compute the eigenvalues and eigenvectors for the $2 \times 2$ Pell matrix for $N = 2$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$  

The Matlab/Octave command $[E, D] = \text{eig}(A)$ returns the eigenvector matrix $E$

$$E = [e_+, e_-] = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0.8165 & -0.8165 \\ 0.5774 & 0.5774 \end{pmatrix}.$$ 

and the eigenvalue matrix $\Lambda$ (Matlab/Octave’s D)

$$\Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = \begin{pmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2.4142 & 0 \\ 0 & -0.4142 \end{pmatrix}.$$ 

The factor $\sqrt{3}$ on $E$ normalizes each eigenvector to 1 (i.e., Matlab/Octave’s command $\text{norm}([\sqrt{2}, 1])$ gives $\sqrt{3}$).

In the following discussion we show how to determine $E$ and $D$ (i.e., $\Lambda$), given $A$. 
Table B.1: Summary of the solution of Pell’s equation due to the Pythagoreans using matrix recursion, for the case of $N=3$. The integer solutions are shown on the right. Note that $x_n/y_n \to \sqrt{3}$, in agreement with the Euclidean algorithm. The Matlab/Octave program for generating this data is PellSolve3.m. It seems likely that the powers of $\beta_0$ could be absorbed in the starting solution, and then be removed from the recursion.

Pell’s Equation for $N = 3$

Case of $N = 3$ & $[x_0, y_0]^T = [1, 0]^T$, $\beta_0 = j/\sqrt{2}$

Note: $x_n^2 - 3y_n^2 = 1$, $x_n/y_n \to \sqrt{3}$

\[
\begin{bmatrix}
x_1 \\
y_1 \\
x_2 \\
y_2 \\
x_3 \\
y_3 \\
x_4 \\
y_4 \\
x_5 \\
y_5 \\
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \beta_0^1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1\beta_0)^2 - 3(1\beta_0)^2 = 1
\]

\[
\begin{bmatrix} x_2 \\
y_2 \\
x_3 \\
y_3 \\
x_4 \\
y_4 \\
x_5 \\
y_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0^2 \\ 1 \end{bmatrix} = \begin{bmatrix} \beta_0^2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (4\beta_0^2)^2 - 3(4\beta_0^2)^2 = 1
\]

\[
\begin{bmatrix} x_3 \\
y_3 \\
x_4 \\
y_4 \\
x_5 \\
y_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0^3 \\ 10 \\ 2 \end{bmatrix} = \begin{bmatrix} \beta_0^3 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (10\beta_0^3)^2 - 3(10\beta_0^3)^2 = 1
\]

\[
\begin{bmatrix} x_4 \\
y_4 \\
x_5 \\
y_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0^4 \\ 28 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} \beta_0^4 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} = (28\beta_0^4)^2 - 3(28\beta_0^4)^2 = 1
\]

\[
\begin{bmatrix} x_5 \\
y_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0^5 \\ 76 \\ 28 \\ 16 \end{bmatrix} = \begin{bmatrix} \beta_0^5 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 28 \\ 16 \end{bmatrix} = (76\beta_0^5)^2 - 3(44\beta_0^5)^2 = 1
\]

Pell’s equation for $N = 3$

In Table B.1, Pell’s equation for $N = 3$ is given, with $\beta_0 = j/\sqrt{2}$. Perhaps try other trivial solutions such as $[-1, 0]^T$ and $[\pm j, 0]^T$, to provide clues to the proper value of $\beta_0$ for cases where $N > 3$.

Exercise: I suggest that you verify $E\Lambda \neq \Lambda E$ and $AE = E\Lambda$ with Matlab/Octave. Here is the Matlab/Octave program which does this:

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}; \ %define the matrix \\
[E, D] = eig(A); \ %compute the eigenvector and eigenvalue matrices \\
A*E-E*D \ %This should be $\approx 0$, within numerical error. \\
E*D-E*A \ %This is not zero
\]

Summary:

Thus far we have shown that for the case of Pell matrix with $N = 2$, the normalized eigenmatrix and its inverse is

\[
E = [e_+, e_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix}, \quad E^{-1} = \frac{\sqrt{6}}{4} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix}
\]

and the eigenmatrix is

\[
\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix}.
\]

Note that when working with numeric data it is not necessary to normalize $E$. For example, the form of $e_+^T = [1 \pm \lambda^+, 1]^T$ is very simple, and easy to work with. Once normalize it becomes ($N = 2$) $[\sqrt{2}/\sqrt{3}, 1/\sqrt{3}]^T = [0.8165, 0.57735]^T$, obscuring its natural simplicity. The normalization buys little in terms of function.

\[2\text{My student Kehan found the general formula for }\beta_n.\]
Exercise: Verify that $\Lambda = E^{-1} AE$.

Solution: We shall work with the unnormalized eigenmatrix $ce$, where $c = \sqrt{2^2 + 1} = \sqrt{3}$. To compute the inverse of $ce$, 1) swap the diagonal values, 2) change the sign of the off diagonals, and 3) divide by the determinant $\Delta$:

$$(ce)^{-1} = \frac{1}{2c\sqrt{2}} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} = \frac{1}{2c} \begin{bmatrix} 0.707 & 1 \\ -0.707 & 1 \end{bmatrix}.$$ 

We wish to show that $\Lambda = E^{-1} AE$

$$\frac{1}{2c} \begin{bmatrix} 0.707 & 1 \\ -0.707 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} e^{\frac{1}{2c} \left( \begin{array}{cc} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{array} \right) \left( \begin{array}{c} 1 + \sqrt{2} \\ 0 \end{array} \right)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 - \sqrt{2} \end{bmatrix}.$$ 

which is best verified with Matlab.

Exercise: Verify that $A = E\Lambda E^{-1}$. Solution: We wish to show that

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} e^{\frac{1}{\sqrt{3}} \left( \begin{array}{cc} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{array} \right) \left( \begin{array}{c} 1 + \sqrt{2} \\ 0 \end{array} \right)} \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 1 - \sqrt{2} \end{bmatrix}. $$

All the above solutions have been verified with Octave.

Eigenmatrix diagonalization is helpful in generating solutions for finding the solutions of Pell’s and Fibonacci’s equations using transmission matrices.

Example: If the matrix corresponds to a transmission line, the eigenvalues have units of seconds [s]

$$\begin{bmatrix} V^+ \\ V^- \end{bmatrix}_n = \begin{bmatrix} e^{-sT_o} & 0 \\ 0 & e^{sT_o} \end{bmatrix} \begin{bmatrix} V^+ \\ V^- \end{bmatrix}_{n+1}. \quad (B.2.1.2)$$

In the time domain the forward traveling wave $v^+_n(t - (n + 1)T_o) = v^+_n(t - nT_o)$ is delayed by $T_o$. Two applications of the matrix delays the signal by $2T_o$.

### B.3 Symbolic analysis of $T E = E\Lambda$

#### B.3.1 The $2 \times 2$ transmission matrix

Here we assume

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with $\Delta_T = 1$.

The eigenvectors $e_\pm$ of $T$ are

$$e_\pm = \frac{1}{2c} \begin{bmatrix} (A - D) \mp \sqrt{(A - D)^2 + 4BC} \\ 1 \end{bmatrix} \quad (B.3.1.1)$$

and eigenvalues are

$$\lambda_\pm = \frac{1}{2} \left( (A + D) \pm \sqrt{(A - D)^2 + 4BC} \right) \quad (B.3.1.2)$$

The term under the radical (i.e., the discriminant) may be rewritten in terms of the determinant of $T$ (i.e., $\Delta_T = AD - BC$), since

$$(A - D)^2 - (A + D)^2 = -4AD.$$
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The for the ABCD matrix the expression under the radical becomes

\[(A - D)^2 + 4BC = A^2 + D^2 - 4AD + 4BC = A^2 + D^2 - 4\Delta_T.\]

Rewriting the eigenvectors and eigenvalues in terms of \(\Delta_T = \pm 1\), we find

\[e_\pm = \frac{1}{2} \left( (A - D) \mp \sqrt{(A + D)^2 - 4\Delta_T} \right) \]  \hspace{1cm} (B.3.1.3)

and

\[\lambda_\pm = \frac{1}{2} \left( (A + D) \mp \sqrt{(A + D)^2 - 4\Delta_T} \right). \]  \hspace{1cm} (B.3.1.4)

B.3.2 Matrices with symmetry

For the case of the ABCD matrix the eigenvalues depend on reciprocity, since \(\Delta_T = 1\) if \(T(s)\) is reciprocal, and \(\Delta_T = -1\) if it is anti-reciprocal. Thus it is helpful to display the eigenfunctions and values in terms of \(\Delta_T\) so this distinction is explicit.

Reversible systems:

When \(A = D\)

\[E = \begin{bmatrix} -\sqrt{\frac{2}{C}} & \sqrt{\frac{2}{C}} \\ 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} A - \sqrt{BC} & 0 \\ 0 & A + \sqrt{BC} \end{bmatrix} \]  \hspace{1cm} (B.3.2.5)

the transmission matrix is said to be reversible, and the properties greatly simplify.

Reciprocal systems

When the matrix is symmetric (\(B = C\)), the corresponding system is said to be reciprocal. Most physical systems are reciprocal. The determinant of the transmission matrix of a reciprocal network \(\Delta_T = AD - BC = 1\). For example, electrical networks composed of inductors, capacitors and resistors are always reciprocal. It follows that the complex impedance matrix is symmetric (Van Valkenburg, 1964a).

Magnetic systems such as dynamic loudspeakers are anti-reciprocal, and correspondingly \(\Delta_T = -1\). The impedance matrix of a loudspeaker is skew symmetric (Kim and Allen, 2013). All impedance matrices are either symmetric or anti-symmetric, depending on whether they are reciprocal (LRC networks) or anti-reciprocal (magnetic networks). These systems have complex eigenvalues with negative real parts, corresponding to lossy systems. In some sense, all of this follows from conservation of energy, but the precise general case is waiting for enlightenment. The impedance matrix is never Hermitian. It is easily proved that Hermitian matrices have real eigenvalues, which correspond to lossless networks. Any physical system of equations that has any type of loss cannot be Hermitian.

In summary, given a reciprocal system, the \(T\) matrix has \(\Delta_T = 1\), and the corresponding impedance matrix is symmetric (not Hermitian).

B.3.3 Impedance matrix

As previously discussed in §3.5 (p. 92), the \(T\) matrix corresponding to an impedance matrix is

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= Z(s)
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
= \frac{1}{C}
\begin{bmatrix}
A & \Delta_T \\
1 & D
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}.
\]
Reciprocal systems have skew-symmetric impedance matrices, namely $z_{12} = z_{21}$ (i.e., $\Delta_T = 1$). This condition is best understood using the T form of the impedance matrix, as shown in Fig. 3.7 (p. 95). When the system is both reversible $A = D$ and reciprocal, the impedance matrix simplifies to

$$Z(s) = \frac{1}{C} \begin{bmatrix} A & 1 \\ 1 & A \end{bmatrix}.$$ 

For such systems there are only two degrees of freedom, $A$ and $C$. As discussed previously in §3.5 (p. 92), each of these has a physical meaning: $1/A$ is the Thévenin source voltage given a voltage drive and $B/A$ is the Thévenin impedance (§3.5.1.3, p. 94).

**Impedance is not Hermitian:** By definition, when a system is Hermitian its matrix is conjugate symmetric

$$Z(s) = Z^\dagger(s),$$

a stronger condition than reciprocal, but not the symmetry of the Brune impedance matrix. A reciprocal Brune impedance is symmetric (not Hermitian).
Appendix C

Laplace transforms \( \mathcal{L} \mathcal{T} \)

Laplace transforms are discussed in §3.7 (p. 103), with the definition of the \( \mathcal{L} \mathcal{T} \) in Eq. 3.7.0.1 (p. 104). The Table of level-I (basic) \( \mathcal{L} \mathcal{T} \)s is in Table 3.2 (p. 100).

C.1 Properties of the Laplace Transform

The following is a summary description of the \( \mathcal{L} \mathcal{T} \):

1. Time \( t \in \mathbb{R} \) [s] and Laplace frequency [rad] are defined as \( s = \sigma + \omega \mathrm{j} \in \mathbb{C} \).

2. Given a \( \mathcal{L} \mathcal{T} \) pair \( f(t) \leftrightarrow F(s) \), in the engineering literature, the time domain is always lower case \( [f(t)] \) and causal (i.e., \( f(t < 0) = 0 \)) and the frequency domain is upper-case [e.g. \( F(s) \)]. Maxwell’s venerable equations are the unfortunate exception to this otherwise universal rule.

3. The target time function \( f(t < 0) = 0 \) (i.e., it must be causal). The time limits are \( 0^- < t < \infty \). Thus the integral must start from slightly below \( t = 0 \) to integrate over a delta function at \( t = 0 \). For example if \( f(t) = \delta(t) \), the integral must include both sides of the impulse. If you wish to include non-causal functions such as \( \delta(t + 1) \), it is necessary to extend the lower time limit. In such cases simply set the lower limit of the integral to \( -\infty \), and let the integrand \( (f(t)) \) determine the limits.

4. When taking the forward transform \( (t \rightarrow s) \), the sign of the exponential is negative. This is necessary to assure that the integral converges when the integrand \( f(t) \rightarrow \infty \) as \( t \rightarrow \infty \). For example, if \( f(t) = e^t u(t) \) (i.e., without the negative \( \sigma \) exponent), the integral does not converge.

5. The limits on the integrals of the reverse LTs are \( [\sigma_o - \infty \mathrm{j}, \sigma_o + \infty \mathrm{j}] \in \mathbb{C} \). These limits are further discussed in §4.5.4 (p. 136).

6. When taking the inverse Laplace transform, the normalization factor of \( 1/2\pi \mathrm{j} \) is required to cancel the \( 2\pi \mathrm{j} \) in the differential \( ds \) of the integral.

7. The frequencies for the LT must be complex, and in general \( F(s) \) is complex analytic for \( \sigma > \sigma_o \). It follows that the real and imaginary parts of \( F(s) \) are related by the CR conditions. Given \( \Re\{F(s)\} \) it is possible to find \( \Im\{F(s)\} \) (Boas, 1987). More on this in §4.2.2 (p. 119).

8. To take the inverse Laplace transform, we must learn how to integrate in the complex \( s \) plane. This will be explained in §4.4-4.5.4 (p. 132-136).
9. The Laplace Heaviside step function is defined as

\[ u(t) = \int_{-\infty}^{t} \delta(t) \, dt = \begin{cases} 1 & \text{if } t > 0 \\ \text{NaN} & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases} \]

Alternatively one could define \( \delta(t) = \frac{du(t)}{dt} \).

10. It is easily shown that \( u(t) \leftrightarrow \frac{1}{s} \) by direct integration

\[ F(s) = \int_{0}^{\infty} u(t) e^{-st} \, dt = -\frac{e^{-st}}{s} \bigg|_{0}^{\infty} = \frac{1}{s}. \]

With the LT step (\( u(t) \)) there is no Gibbs ringing effect.

11. The Laplace transform of a Brune impedance takes the form of a ratio of two polynomials. In such case the roots of the numerator polynomial are called the **zeros** while the roots of the denominator polynomial are called the **poles**. For example the LT of \( u(t) \leftrightarrow \frac{1}{s} \) has a pole at \( s = 0 \), which represents integration, since

\[ u(t) \ast f(t) = \int_{-\infty}^{\infty} f(\tau) \, d\tau \leftrightarrow \frac{F(s)}{s}. \]

12. The LT is quite different from the FT in terms of its analytic properties. For example, the step function \( u(t) \leftrightarrow \frac{1}{s} \) is complex analytic everywhere, except at \( s = 0 \). The FT of \( 1 \leftrightarrow 2\pi \delta(\omega) \) is not analytic anywhere.

13. Dilated step function \( (a \in \mathbb{R}) \)

\[ u(at) \leftrightarrow \int_{-\infty}^{\infty} u(at)e^{-st} \, dt = \frac{1}{a} \int_{-\infty}^{\infty} u(\tau)e^{-s(a/\tau)} \, d\tau = \frac{a}{|a|} \frac{1}{s} = \pm \frac{1}{s}, \]

where we have made the change of variables \( \tau = at \). The only effect that \( a \) has on \( u(at) \) is the sign of \( t \), since \( u(t) = u(2t) \). However \( u(-t) \neq u(t) \), since \( u(t) \cdot u(-t) = 0 \), and \( u(t) + u(-t) = 1 \), except at \( t = 0 \), where it is not defined.

Once complex integration in the complex plane has been defined (§4.2.2, p. 119), we can justify the definition of the inverse LT (Eq. 3.7.0.1).\(^1\)

### C.2 Tables of Laplace transforms

In the following are tables of \( LT \) and \( LT^{-1} \) which are a convenient summary of their properties and evaluations for many different functions. Table C.1 gives basic function properties such as convolution and the properties of step functions and frequency scaling. Table C.2 provides the commands for doing symbolic (computer algebra and calculus) transformations, which includes some unusual \( LT \)s and Taylor series of the \( \Gamma(s) \) function (Graham et al., 1994), a complex analytic extension of the factorial. Table C.3 gives the basic transforms typically used for more common calculations. Table C.4 provides extended less common transforms such as the half-derivative and integration and Bessel functions.

These tables are available in most books in differential equations and remain a core technology for analytic methods for solving differential equations.

\(^1\)https://en.wikipedia.org/wiki/Laplace_transform#Table_of_selected_Laplace_transforms
C.2. TABLES OF LAPLACE TRANSFORMS

Table C.1: Functional relationships between Laplace transforms.

<table>
<thead>
<tr>
<th>$LT$ functional properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t) \ast g(t) = \int_{t=0}^{t} f(t-\tau)g(\tau)d\tau \leftrightarrow F(s)G(s)$</td>
</tr>
<tr>
<td>$u(t) \ast f(t) = \int_{0}^{t} f(t)dt \leftrightarrow \frac{F(s)}{s}$</td>
</tr>
<tr>
<td>$f(at)u(at) \leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right)$</td>
</tr>
<tr>
<td>$f(t)e^{-at}u(t) \leftrightarrow F(s+a)$</td>
</tr>
<tr>
<td>$f(t-T)e^{-a(t-T)}u(t-T) \leftrightarrow e^{-sT}F(s+a)$</td>
</tr>
<tr>
<td>$f(-t)u(-t) \leftrightarrow F(-s)$</td>
</tr>
<tr>
<td>$f(-t)e^{-at}u(-t) \leftrightarrow F(a-s)$</td>
</tr>
<tr>
<td>$\frac{d}{dt}f(t) = \delta'(t) \ast f(t) \leftrightarrow sF(s)$</td>
</tr>
<tr>
<td>$\frac{\sin(t)u(t)}{t} \leftrightarrow \tan^{-1}(1/s)$</td>
</tr>
</tbody>
</table>

Table C.2: Symbolic relationships between Laplace transforms. $K_3$ is a constant.

<table>
<thead>
<tr>
<th>symts</th>
<th>command</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>symst</td>
<td>laplace$(t^{(p-1)})$</td>
<td>$\frac{\Gamma(p)s^{-p}}{a}$</td>
</tr>
<tr>
<td>symss</td>
<td>ilaplace(gamma(s))</td>
<td>$e^{-t}$</td>
</tr>
<tr>
<td>symsta</td>
<td>ilaplace(exp(-a*s)/s,s,t)</td>
<td>Heaviside$(t-a)$</td>
</tr>
<tr>
<td>symGamma</td>
<td>taylor(Gamma,s,t)</td>
<td>$\frac{1}{s} - \gamma + s \left( \frac{\gamma^2}{2} + \frac{\gamma^3}{12} \right) + s^2 \left( \frac{1}{6} \text{polygamma}(2,1) - \frac{\gamma^4}{12} - \frac{\gamma^5}{6} \right) + s^3K_3 + \cdots$</td>
</tr>
</tbody>
</table>
Function | Transform | Comment
--- | --- | ---
\( \delta(t) \) | \( 1 \) | Dirac
\( \delta(|a|) \) | \( \frac{1}{|a|} \) | time-scaled Dirac\( a \neq 0 \)
\( \delta(t - T_0) \) | \( e^{-st_0} \) | delayed Dirac
\( \sum_{n=0}^{\infty} \delta(t - nT_0) = \frac{1}{1 - \delta(t - T_0)} \leftrightarrow \frac{1}{1 - e^{-sT_0}} = \sum_{n=0}^{\infty} e^{-snT_0} \) | one-sided impulse train
\( u(t) \) | \( \frac{1}{s} \) | Heavistep step
\( u(-t) \) | \( -\frac{1}{s} \) | anti-causal step
\( u(at) \) | \( \frac{a}{s} \) | dilated or reversed step\( a \neq 0 \in \mathbb{R} \)
\( e^{at}u(-t) \) | \( \frac{1}{s + a} \) | anticausal damped step
\( e^{-at}u(t) \) | \( \frac{1}{s + a} \) | damped step\( a > 0 \in \mathbb{R} \)
\( \cos(at)u(t) \) | \( \frac{1}{2} \left( \frac{1}{s - a} + \frac{1}{s + a} \right) \) | \( a \in \mathbb{R} \)
\( \sin(at)u(t) \) | \( \frac{1}{2i} \left( \frac{1}{s - a} - \frac{1}{s + a} \right) \) | \( a \in \mathbb{C} \)
\( u(t - T_0) \) | \( \frac{1}{T_0} e^{-st_0} \) | time delay\( T_0 > 0 \in \mathbb{R} \)
\( \text{rect}(t) = \frac{1}{T_0} [u(t) - u(t - T_0)] \leftrightarrow \frac{1}{T_0} \left( 1 - e^{-st_0} \right) \) | rect-pulse
\( u(t) \star u(t) = tu(t) \leftrightarrow 1/s^2 \) | ramp
\( u(t) \star u(t) \star u(t) = \frac{1}{2} t^2 u(t) \leftrightarrow 1/s^3 \) | double ramp
\( \frac{1}{\sqrt{t}} u(t) \leftrightarrow \sqrt{\frac{s}{\pi}} \)
\( t^pu(t) \leftrightarrow \frac{\Gamma(p+1)}{s^{p+1}} \) | \( \Re p > -1, q \in \mathbb{C} \)
Table C.4: The following table provides an extended table of Laplace Transforms. \( J_0, K_1 \) are Bessel functions of the first and second kind.

<table>
<thead>
<tr>
<th>( f(t) \leftrightarrow F(s) )</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d^{1/2}}{dt^{1/2}} f(t) u(t) \leftrightarrow \sqrt{s} F(s) )</td>
<td>“half” derivative</td>
</tr>
<tr>
<td>( \frac{d^{1/2}}{dt^{1/2}} u(t) \leftrightarrow \sqrt{s} )</td>
<td>“half” derivative</td>
</tr>
<tr>
<td>( \frac{d}{dt} \frac{1}{\sqrt{\pi t}} u(t) \leftrightarrow \frac{s}{\sqrt{s}} = \sqrt{s} )</td>
<td>semi-inductor</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{\pi t}} u(t) \leftrightarrow \frac{1}{\sqrt{s}} )</td>
<td>“half” integration</td>
</tr>
<tr>
<td>( \text{erfc}(\alpha \sqrt{t}) \leftrightarrow \frac{1}{s} e^{-2\alpha \sqrt{s}} )</td>
<td>(Morse-Feshbach-II, p. 1582) ( \alpha &gt; 0 ); erfc</td>
</tr>
<tr>
<td>( J_0(at) u(t) \leftrightarrow \frac{1}{\sqrt{s^2 + a^2}} )</td>
<td>Bessel</td>
</tr>
<tr>
<td>( J_n(\omega_o t) u(t) \leftrightarrow \frac{(\sqrt{s^2 + \omega_o^2} - s)^n}{\omega_o^n \sqrt{s^2 + \omega_o^2}} )</td>
<td></td>
</tr>
<tr>
<td>( J_1(t) u(t)/t \leftrightarrow \sqrt{s^2 + 1} )</td>
<td></td>
</tr>
<tr>
<td>( J_1(t) u(t)/t + 2u(t) \leftrightarrow \sqrt{s^2 + 1 + s} = e^{\sinh^{-1}(s)} )</td>
<td></td>
</tr>
<tr>
<td>( \delta(t) + J_1(t) u(t)/t \leftrightarrow \sqrt{s^2 + 1} )</td>
<td></td>
</tr>
<tr>
<td>( J_n(\omega_o t) u(t) \leftrightarrow \frac{(\sqrt{s^2 + \omega_o^2} - s)^n}{\omega_o^n \sqrt{s^2 + \omega_o^2}} )</td>
<td></td>
</tr>
<tr>
<td>( I_0(t) u(t) \leftrightarrow 1/\sqrt{s^2 - 1} )</td>
<td></td>
</tr>
<tr>
<td>( u(t)/\sqrt{t+1} \leftrightarrow e^{s} \sqrt{\frac{\pi}{s}} \text{erfc}(\sqrt{s}) )</td>
<td></td>
</tr>
<tr>
<td>( \sqrt{t} u(t) * \sqrt{1 + t} u(t) \leftrightarrow e^{s/2} K_1(s/2)/2s )</td>
<td></td>
</tr>
</tbody>
</table>
C.2.1 \(\mathcal{L}^{-1}\) of the Riemann zeta function

The analytic properties of the zeta function have long been sought by mathematicians, starting with Euler, and then several others, who may have made their reputation working on this important function. For the neophyte, \(\zeta(s)\) is important because it is an analytic extension of the sieve, which is a method for identifying prime numbers.

Understood are the locations of the poles of zeta, which depend on the prime numbers. Not so well understood are the remaining analytic properties over the entire plane, such as the zeros of \(\zeta(x)\), namely the poles of \(1/\zeta(s)\). This section is a brief review of how \(\zeta(s)\) for beginners, building on the developments of analytic functions from Chapter 3, especially sections 3.2.4 and 3.1.1, page 32.

Let \(z \equiv e^{\delta T}\) where \(T\) is the “sample period” at which data is taken (every \(T\) seconds). For example if \(T = 22.676 \times 10^{-6} = 1/44100\) seconds then the data is sampled at 44.10 [kHz]. This is how a CD player works with high quality music. Thus the unit-time delay operator \(z^{-1}\) as

\[\delta(t - T) \leftrightarrow e^{-\delta T}\]

When dealing with the Euler and Riemann zeta function, the only sampling period that makes sense is \(T = 1\) [s] or 1 [Hz]. In this case, the samples of interest are (i.e. \(n \in \mathbb{Z}\)). Based on the sieve of Eratosthenes, Euler showed, that the counting numbers may be uniquely reduced to multiples of the primes. This is the basis for the Fundamental theorem of arithmetic which states that every integer may be uniquely factored into a product of prime numbers. This is a fundamental idea and the source of the concept of the prime number.

**The zeta function** The zeta function depends explicitly on the primes, which makes it a special function. In 1737 Euler proposed the real-valued function \(\zeta(x) \in \mathbb{R}\) and \(x \in \mathbb{R}\), to prove that the number of primes is infinite (Goldstein, 1973). Euler’s definition of \(\zeta(x) \in \mathbb{R}\) is given by the power series

\[
\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \quad \text{for } x > 1 \in \mathbb{R}.
\]

This series converges for \(x > 0\), since \(R = n^{-x} < 1, n > 1 \in \mathbb{N}\).

In 1860 Riemann extended the zeta function into the complex plane, resulting in \(\zeta(s)\), defined by the complex analytic power series, identical to the Euler formula, except \(x \in \mathbb{R}\) has been replaced by \(s \in \mathbb{C}\)

\[
\zeta(s) \equiv \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s} \quad \text{for } \Re\{s\} = \sigma > 1.
\]

This formula converges for \(\Re\{s\} > 1\) (Goldstein, 1973). To determine the formula in other regions of the \(s\) plane one must extend the series via analytic continuation. As it turns out, Euler’s formulation provided detailed information about the structure of primes, going far beyond his original goal.

**Euler product formula**

As was first published by Euler in 1737, one may recursively factor out the leading prime term, resulting in Euler’s product formula. Euler’s procedure is an algebraic implementation of the sieve of Eratosthenes (Fig. 2.3, page 40).

Multiplying \(\zeta(s)\) by the factor \(1/2^s\), and subtracting from \(\zeta(s)\), removes all the powers of \(2\): \(1/2^s + 1/2^{2s} + 1/2^{3s} + \cdots\)

\[
\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \frac{1}{16^s} + \cdots\right),
\]

---

2Sanity check: For example let \(n = 2\) and \(x > 0\). Then \(R = 2^{-x} < 1\), where \(\epsilon \equiv \lim x \to 0^+\). Taking the log gives \(\log_2 R = -\epsilon \log_2 2 = -\epsilon < 0\). Since \(\log R < 0\), \(R < 1\).
which results in
\[ \zeta_1(s) = (1 - 2^{-s}) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots. \] (C.2.1.4)

Repeating this with a lead factor 1/3\(s\) applied to Eq. C.2.1.4 gives
\[ \frac{1}{3^s} (1 - 2^{-s}) \zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \cdots. \] (C.2.1.5)

Subtracting Eq. C.2.1.5 from Eq. C.2.1.4 cancels the RHS terms of Eq. C.2.1.4, giving
\[ \zeta_2(s) = (1 - 3^{-s}) (1 - 2^{-s}) \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} \cdots. \]

If we express this in terms the primes \(\pi_k\) we can better visualize the structure
\[ \zeta_2(s) = (1 - \pi^{-2}_2) (1 - \pi^{-1}_1) \zeta(s) = 1 + \frac{1}{\pi_3^s} + \frac{1}{\pi_4^s} + \frac{1}{\pi_5^s} + \frac{1}{\pi_6^s} + \frac{1}{\pi_7^s} + \frac{1}{\pi_8^s} \cdots. \]

Thus \(\zeta_2\) has removed primes \(\pi_1, \pi_2\), leaving \(\pi_3\) as the lead term in the series on the RHS.

This leads to a recursion in \(\zeta_k\)
\[ \zeta_k(s) = \zeta(s) \prod_{l=1}^{k} \zeta_l(s) = 1 + \sum_{l=k+1}^{\infty} \pi_i^{-s}. \]

The series on the RHS converges rapidly to 1 as each prime is removed, because the RoC is becoming much larger with each recursion. Each recursive step in this construction assures that the lead term, along with all of its multiplicative factors, are subtracted out as with the cancellations with the sieve of Eratosthenes. It is instructive to compare each iteration with that of the sieve (Fig. 2.3, p. 40).

Repeating this process with the remaining primes removes all the terms on the RHS but the first (leaving 1), which results in Euler’s analytic product formula \((s = x \in \mathbb{R})\), or Riemann’s complex analytic product formula \((s \in \mathbb{C})\)
\[ 1 = \zeta(s)(1 - 2^{-s}) \cdot (1 - 3^{-s}) \cdot (1 - 5^{-s}) \cdot (1 - 7^{-s}) \cdot \cdots \cdot (1 - \pi_n^{-s}) \cdot \cdots. \]
\[ = \zeta(s) \prod_{k=1}^{\infty} (1 - \pi_k^{-s}) \quad \text{(C.2.1.6)} \]
\[ \zeta(s) = \frac{1}{\prod_k \mathcal{P}_k(s)} \quad \Re\{s\} = \sigma > 0, \text{ verify!} \] (C.2.1.7)

where \(\mathcal{P}_k(s) = 1 - \pi_k^{-s}\) defines the poles of \(\zeta(s)\) for prime \(\pi_k\).

**Finding the RoC of the product formula:** It would be interesting to find the RoC for \(\mathcal{P}_k(s)\), and for rigor, this question demands further investigation. To find the RoC we need to evaluate
\[ |\pi_k^{-s}| = |e^{-sT_k}| = |e^{-\sigma T_k}| = \left( \frac{1}{\pi_k} \right)^\sigma < 1 \quad \text{for} \quad \sigma > 0, \text{ verify} \]
where \(T_k = \ln \pi_k\). For example
\[ \frac{1}{\mathcal{P}_5(s)} = \frac{1}{1 - \left( \frac{1}{5} \right)^s} = 1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} \cdots, \quad \Re\{s\} = \sigma > 0. \]

The RoC for each root is \(\sigma \geq 0\) since when \(\sigma < 0\).

Since \(1/\pi_k < 1\) for all \(k \in \mathbb{N}\), the Taylor series of \(\zeta_k(x)\) is entire except at its poles. Note that the RoC of a Taylor series in powers of \(\pi_k^{-s}\) increases with \(k\).

---

3This is known as Euler’s sieve, as distinguish from the Eratosthenes sieve.
Exercise: Work out the RoC for $k = 2$.

Solution: The formula for the RoC is given above, which for $\pi_2 = 3$ is

$$|\pi_k^{-\sigma_r}| = \begin{cases} \left(\frac{1}{2}\right)^{\sigma_r} < 1 & \text{for } \sigma_r > 0, \\ \left(\frac{1}{2}\right)^{-\sigma_r} < 1 & \text{for } \sigma_r < 0, \end{cases}$$

where $\sigma_r$ is the boundary of the RoC.

Exercise: Show how to construct $\zeta_2(t) \leftrightarrow \zeta_2(s)$ by working in the time domain.

Solution: The procedure for building a sieve is to sum the integers

$$S_1 = \sum_{n=1}^{\infty} n2^{n-1} = 1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots,$$

while the sieve for the $k^{th}$ prime $\pi_k$ is

$$S_k = \sum_{n=1}^{\infty} n\pi_k^{n-1} = 1 \cdot \pi_k^0 + \pi_k \cdot 2^1 + \pi_k \cdot 2^2 + \cdots.$$ 

This sum may be written in terms of the convolution with the Heaviside step function $u_k$ since

$$u_k \ast u_k = nu_k = 0 \cdot u_0 + 1 \cdot u_1 + 2u_2 + \cdots + ku_k + \cdots.$$ 

Poles of $\zeta_k(s)$

Riemann proposed that Euler’s zeta function $\zeta(s) \in \mathbb{C}$ have a complex argument [first actually explored by Chebyshev in 1850, (Bombieri, 2000)], extending $\zeta(s)$ into the complex plane, $(s \in \mathbb{C})$, thus making it a complex analytic function. Thus one might presume that $\zeta(s)$ has an inverse Laplace transform. There seems to be very little written on this topic (Hill, 2007). We shall explore this question further here.

Figure C.1: Plot of $w(s) = \frac{1}{1 - \exp(-s \pi i)}$. Here $w(s)$ has poles where $e^{\omega_n \ln 2} = 1$, namely where $\omega_n \ln 2 = n2\pi$, as seen in the colorized-map ($s = \sigma + \omega j$ is the Laplace frequency [rad]).

One may now identify the poles of $\zeta_k(s) (p \in \mathbb{N})$, which are required to determining the RoC. For example, the $k^{th}$ factor of Eq. C.2.1.7 expressed as an exponential, is

$$\zeta_k(s) \equiv \frac{1}{1 - \pi_k^{-s}} = \frac{1}{1 - e^{-sT_k}} = \sum_{k=0}^{\infty} e^{-sT_k},$$

(C.2.1.8)
where $T_k = \ln \pi_k$. Thus $\zeta_p(s)$ has poles at $-s_T = 2\pi n \lambda$ (when $e^{-sT} = 1$), thus

$$\omega_n = \frac{2\pi n}{T_k},$$

with $-\infty < n \in \mathbb{Z} < \infty$. These poles are the eigenmodes of the zeta function. A domain-colorized plot of this function is provided in Fig. C.1. It is clear that the RoC of $\zeta_k$ is $>0$. It would be helpful to determine why $\zeta(s)$ as such a more restrictive RoC than each of its factors.

**Inverse Laplace transform**

The inverse Laplace transform of Eq. C.2.1.8 is an infinite series of delays of delay $T_p$ (Table C.3, p. 208)\(^4\)

$$Z_{\text{eta}}^e(t) = \delta(t)T_p \equiv \sum_{k=0}^{\infty} \delta(t - kT_p) \leftrightarrow \frac{1}{1 - e^{-sT_p}}.$$ \hspace{1cm} (C.2.1.9)

**Inverse transform of Product of factors**

The time domain version of Eq. C.2.1.7 may be written as the convolution of all the $Z_{\text{eta}}^e(t)$ factors

$$Z_{\text{eta}}^e(t) \equiv Z_{\text{eta}}^2(t) \ast Z_{\text{eta}}^3(t) \ast Z_{\text{eta}}^5(t) \ast Z_{\text{eta}}^7(t) \ast \cdots \ast Z_{\text{eta}}^p(t) \ast \cdots,$$ \hspace{1cm} (C.2.1.10)

where $\ast$ represents time convolution (Table C.1, p. 207).

**Physical interpretation**

Such functions may be generated in the time domain as shown in Fig. C.2, using a feedback delay of $T_p$ [s] as described by the equation in the figure, with a unity feedback gain $\alpha = 1$.

$$Z_{\text{eta}}^e(t) = Z_{\text{eta}}^e(t - T_p) + \delta(t).$$

Taking the Laplace transform of the system equation we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $Z_{\text{eta}}^e(t)$. Taking the $\mathcal{L}T$, we see that $\zeta(s)$ is an all-pole function

$$\zeta_p(s) = e^{-sT_p} \zeta_p(s) + 1(t), \text{ or } \zeta_p(s) = \frac{1}{1 - e^{-sT_p}}.$$ \hspace{1cm} (C.2.1.11)

If we closing the feed-forward path, a second transfer function is defined as

$$Y_p(s) = \frac{1 + e^{-sT_p}}{1 - e^{-sT_p}}.$$ \hspace{1cm} (C.2.1.12)

\(^4\)Here we use a shorthand double-parentheses notation $f(t)T \equiv \sum_{k=0}^{\infty} f(t - kT)$ to define the one-sided infinite sum.
The poles and zeros of the impedance interleave along the \( j \omega \) axis. By a minor modification, \( Y_p(s) \) may be rewritten as
\[
Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j \tan(sT_p/2),
\]
which is the input admittance of a uniform transmission line, terminated in a zero voltage condition (a zero-impedance condition).

It follows that if we take the input \( \delta(t) \) as a voltage and the output \( Y_p(s) \) as a current, the ratio of the output over the input is the admittance of a uniform transmission line \( Y_p(s) \leftrightarrow y_p(t) \) represents the input impedance at the input to the line.

Every impedance \( Z(s) \) has a corresponding \textit{reflectance} function given by a Möbius transformation, which may be read off of Eq. C.2.1.12 as
\[
\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}
\]
since impedance is also related to the round-trip delay \( T_p \) on the line. The inverse Laplace transform of \( \Gamma(s) \) is the round trip delay \( T_k = \log \pi_k \) on the line
\[
\gamma(t) = \delta(t - T_k) \leftrightarrow e^{-sT_k}.
\]

\textbf{Discussion:} In terms of the physics, these transmission line equations are telling us that \( \zeta(s) \) may be decomposed into an infinite cascade of transmission lines (Eq. C.2.1.10), each having a unique delay given by \( T_k = \ln \pi_k \), \( \pi_k \in \mathbb{P} \), the log of the primes. The input admittance of this cascade may be interpreted as an analytic continuation of \( \zeta(s) \) which defines the eigenmodes of that cascaded impedance function.

Working in the time domain provides a key insight, as it allows us to determine the analytic continuation of the infinity of possible continuations, which are not obvious in the frequency domain. Transforming to the time domain is a form of analytic continuation of \( \zeta(s) \), that depends on the assumption that \( Z_{ct}(t) \) is one-sided in time (causal).

\textbf{Additional relations:} Some important relations provided by both Euler and Riemann (1859) are needed when studying \( \zeta(s) \).

With the goal of generalizing his result, Euler extended the definition with the functional equation
\[
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \zeta(1-s).
\]
(C.2.1.16)
This seems closely related to Riemann’s time reversal symmetry properties (Bombieri, 2000)
\[
\pi^{-s/2} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-(1-s)/2} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s).
\]
This equation is of the form \( F \left( \frac{s}{2} \right) \zeta(s) = F \left( \frac{1-s}{2} \right) \zeta(1-s) \) where \( F(s) = \Gamma(s)/\pi^s \).

As shown in Table C.1 the \( \mathcal{LT}^{-1} \) of \( f(-t) \leftrightarrow F(-s) \) represents time-reversal. This leads to a causal and anti-causal function that are symmetric about \( \Re \{s\} = 1/2 \) (Riemann, 1859) leading to an interpretation of the Euler’s functional equation.

Riemann (1859, page 2) provides an alternate integral definition of \( \zeta(s) \), based on the complex contour integration\(^5\)
\[
2 \sin(\pi s) \Gamma(s - 1) \zeta(s) = \int_{-\infty}^{\infty} \frac{(-x)^{s-1}}{e^x - 1} dx. \quad -x \rightarrow yj \int_{-\infty}^{\infty} \frac{(yj)^{s-1}}{e^{-yj} - 1} dyj.
\]
Given the \( \zeta_k(s) \) it seems important to look at the inverse \( \mathcal{LT} \) of \( \zeta_k(1-s) \), to gain insight into the analytically extended \( \zeta(s) \)

\(^5\)Verify Riemann’s use of \( x \), which is taken to be real rather than complex. This could be more natural (i.e., modern Laplace transformation notation) if \( -x \rightarrow yj \rightarrow z \).
Integral definition of the complex Gamma function $\Gamma(s)$: The definition of the complex analytic Gamma function (p. 207)

$$\Gamma(s + 1) = s\Gamma(s) \equiv \int_0^\infty \xi^s e^{-\xi} d\xi,$$

which is a generalization of the real integer factorial function $n!$.

$$\xi(t) = \int_{-\infty}^{\infty} \Gamma(s + 1)e^{st} \frac{ds}{2\pi i}$$

What is the RoC of $\zeta(s)$? It is commonly stated that Euler’s and thus Riemann’s product formula is only valid for $\Re(s) > 1$, however this does not seem to be actually proved (I could be missing this proof). Here I will argue that the product formula is entire except at the poles. Namely that the formula is valid everywhere other that at the poles.

The argument goes as follows: Starting from the product formula (Eq. C.2.1.7 (p. 211)), form the log-derivative and study the poles and residues:

$$D(s) \equiv \frac{d}{ds} \ln \prod_k \frac{1}{1 - e^{-sT_k}} = -\sum_{k=1}^{\infty} \frac{1}{1 - e^{-sT_k}} \frac{d}{ds} \left(1 - e^{-sT_k}\right)$$

$$= -\sum_k T_k e^{-sT_k} \frac{1}{1 - e^{-sT_k}} \leftrightarrow \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \delta(t - nT_k).$$

Here $T_k = \ln \pi_k$, as previously defined, and $\leftrightarrow$ denotes the inverse Laplace transform, transforming $D(s) \leftrightarrow d(t)$, into the time domain. Note that $d(t)$ is a causal function, composed of an infinite number of delta functions (i.e., time delays), as outline by Fig. C.2 (p. 213).

Zeros of $\zeta(s)$ We are still left with the most important question of all “Where are the zeros of $\zeta(s)$?” Equation C.2.1.11 has no zeros, it is an all-pole system. The cascade of many such systems is also all-pole. As I see it, the issue is: What is the actual formula for $\zeta(s)$?

To answer this question me need to study the properties of the reflectance function $\Gamma(s)$ given by Eq. C.2.1.14. Frequency domain transfer functions having unity magnitude on the $j\omega$ axis are called all pass filters in the engineering literature. When the reflectance is loss-less, it is therefore all-pass since $|\Gamma(j\omega)| = 1$. An important property of all-pass filters is that they may be accurately approximated by pole-zero pairs straddling the $j\omega$ axis, with the poles to the left (as required by causality) and the zeros to the right. Given this placement, the phase of the poles and zeros add. The group delay gives the net delay of the all-pass filter, which is twice the delay of the poles alone. It would seem that this careful placement of the zeros exactly across from the poles, provides the requirement that the zeros all line up parallel to the $j\omega$ axis, as deemed by the Riemann hypothesis. Could this be the resolution of this long standing mystery? An alternative possibility is that the convergent product formula has zeros that are obscured by the lack of convergence of Eq. C.2.1.2.
Appendix D

Number theory applications

D.1 Division with rounding method

We want to show that the GCD for \( m, n, k \in \mathbb{N} \) (Eq. 2.3.1.2, p. 42) may be written in matrix form as

\[
\begin{bmatrix}
  m \\
  n
\end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & -\left\lfloor \frac{m}{n} \right\rfloor \end{bmatrix} \begin{bmatrix} m \\
  n
\end{bmatrix}_k.
\]

(D.1.0.1)

Eq. D.1.0.1 implements the \( \gcd(m,n) \) for \( m > n \).

This starts with \( k = 0, m_0 = a, \) and \( n_0 = b. \) With this method there is no need to test whether \( n_n < m_n, \) as it is built into the procedure. The method uses the floor function \( \lfloor x \rfloor, \) which finds the integer part of \( x \) (\( \lfloor x \rfloor \) rounds toward \(-\infty\)). After each step we will see that the value \( n_{k+1} < m_{k+1}. \)

The method terminates when \( n_{k+1} = 0 \) with \( \gcd(a,b) = m_{k+1}. \)

The following vectorized code is more efficient than the direct matrix method:

```matlab
function n = gcd2(a,b)

M = [abs(a); abs(b)]; %Save (a,b) in array M(2,1)

% done when M(1) = 0
while M(1) ~= 0
    disp(sprintf('M(1)=%g, M(2)=%g ',M(1),M(2)));
    M = [M(2) - M(1)*floor(M(2)/M(1)); M(1)]; %automatically sorted
end %done

n = M(2); %GCD is M(2)

```

With a minor extension in the test for “end,” this code can be made to work with irrational inputs (e.g., \((n\pi, m\pi)).\)

This method calculates the number of times \( n < m \) must subtract from \( m \) using the floor function. This operation is the same as the mod function.\(^1\) Specifically

\[
 n_{k+1} = m_k - \left\lfloor \frac{m}{n} \right\rfloor n_k
\]

(D.1.0.2)

so that the output is the definition of the remainder of modular arithmetic. This would have been obvious to anyone using an abacus, which explains why it was discovered so early.

Note that the next value of \( m = M(1) \) is always less than \( n = M(2) \), and must remain greater or equal to zero. This one-line vector operation is then repeated until the remainder \( M(1) \) is 0. The gcd is

\(^1\)https://en.wikipedia.org/wiki/Modulo_operation
then \( n = M(2) \). When we use irrational numbers the code still works except the error is never exactly zero due to IEEE 754 rounding. Thus the criterion must be that the error is within some small factor times the smallest number (which in Matlab/Octave is the number \( \text{eps} = 2.220446049250313 \times 10^{-16} \), as defined in the IEEE 754 standard).

Thus, without factoring the two numbers, Eq. D.1.0.2 recursively finds the gcd. Perhaps this is best seen with some examples.

The GCD is an important and venerable method, useful in engineering and mathematics, but, as best I know, is not typically taught in the traditional engineering curriculum.

**GCD applied to polynomials:** An interesting generalization is work with polynomials rather than numbers, and apply the Euclidean algorithm.

The GCD may be generalized in several significant ways. For example what is the GCD of two polynomials? To answer this question one must factor the two polynomials to identify common roots.

### D.2 Derivation of the CFA matrix

The CFA may be define starting from the basic definitions of the floor and remainder formulae. Starting from a decimal number \( x \) we split it into the decimal and remainder parts\(^2\) Starting with \( n = 0 \) and \( x_0 = x \in \mathbb{I} \), the integer part is

\[
m_0 = \lfloor x \rfloor \in \mathbb{N}
\]

and the remainder is

\[
r_0 = x - m_0.
\]

Corresponding the the CFA, the next target \( x_1 \) is for \( n = 1 \) is

\[
x_1 = r_0^{-1}
\]

and the integer part is \( m_1 = \lfloor x_1 \rfloor \). As in the case of \( n = 0 \), the integer part is

\[
m_1 = \lfloor x_1 \rfloor
\]

and the remainder is

\[
r_1 = x_1 - m_1.
\]

The recursion for \( n = 2 \) is similar.

To better appreciate what is happening it is helpful to write these recursion in a matrix format. Rewriting the case of \( n = 1 \) and using the remainder formula for the ratio of two numbers \( p \geq q \in \mathbb{N} \) with \( q \neq 0 \).

\[
\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} u_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}.
\]

From the remainder formula, \( u_1 = \lfloor p/q \rfloor \).

Continuing with \( n = 2 \)

\[
\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} u_2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.
\]

where \( u_1 = \lfloor r_0/r_1 \rfloor \).

Continuing with \( n = 3 \)

\[
\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} u_3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_2 \\ r_3 \end{bmatrix},
\]

\(^2\)The method presented here was developed by Yiming Zhang as a student project in 2019.
where \( u_2 = \lfloor r_1/r_2 \rfloor \).

For arbitrary \( n \) we find

\[
\begin{bmatrix}
  r_{n-2} \\
  r_{n-1}
\end{bmatrix}
= \begin{bmatrix}
  u_n & 1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  r_{n-1} \\
  r_n
\end{bmatrix}
\tag{D.2.0.1}
\]

where \( u_n = \lfloor r_{n-1}/r_n \rfloor \). This terminates when \( r_n = 0 \) in the above \( n \)th step when

\[
\begin{bmatrix}
  r_{n-2} \\
  r_{n-1}
\end{bmatrix}
= \begin{bmatrix}
  u_n & 1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  r_{n-1} \\
  r_n = 0
\end{bmatrix}.
\]

**Example:** Let \( p = 355 \) and \( q = 113 \), which are coprime, and set \( n = 1 \). Then Eq. D.2.0.1 becomes

\[
\begin{bmatrix}
  355 \\
  113
\end{bmatrix}
= \begin{bmatrix}
  3 & 1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  r_o \\
  r_1
\end{bmatrix},
\]

since \( u_1 = \lfloor 355/113 \rfloor = 3 \). Solving for the RHS gives \( \lfloor r_o; r_1 \rfloor = [113; 16] \) (355 = 113 · 3 + 16). To find \( \lfloor r_0; r_1 \rfloor \), take the inverse:

\[
\begin{bmatrix}
  r_o \\
  r_1
\end{bmatrix}
= \begin{bmatrix}
  0 & 1 \\
  1 & -3
\end{bmatrix}
\begin{bmatrix}
  355 \\
  113
\end{bmatrix}.
\]

For \( n = 2 \), with the RHS from the previous step,

\[
\begin{bmatrix}
  113 \\
  16
\end{bmatrix}
= \begin{bmatrix}
  u_2 & 1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  r_1 \\
  r_2
\end{bmatrix},
\]

since \( u_2 = \lfloor 113/16 \rfloor = 7 \). Solving for the RHS gives \( \lfloor r_1; r_2 \rfloor = [16; 1] \) (113 = 16 · 7 + 1). It seems we are done, but let’s go one more step.

For \( n = 3 \) we now have

\[
\begin{bmatrix}
  16 \\
  1
\end{bmatrix}
= \begin{bmatrix}
  u_3 & 1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  r_1 \\
  r_2
\end{bmatrix},
\]

since \( u_3 = \lfloor 16/1 \rfloor = 16 \). Solving for the RHS gives \( \lfloor r_1; r_2 \rfloor = [1; 0] \). This confirms we are done since \( r_2 = 0 \).

**Derivation of Eq. D.2.0.1:** Equation Eq. D.2.0.1 is derived as follows. Starting from the target \( x \in \mathbb{R} \), define

\[
p = \lfloor x \rfloor \quad \text{and} \quad q = \frac{1}{x - p} \in \mathbb{R}.
\]

These two relations for truncation and remainder allow us to write the general matrix recursion relation for the CFA, Eq. D.2.0.1. Given \( \{p, q\} \), continue with the above CFA method.

One slight problem with the above is that the output is on the right and the input on the left. Thus we need to take the inverse of these relationships to turn this into a composition.

**D.3 Taking the inverse to get the \( \gcd \).**

Variables \( p, q \) are the remainders \( r_{n-1} \) and \( r_n \) respectively. Using this notation with \( n - 1 \) gives Eq. D.2.0.1. Inverting this gives the formula for the GCD

\[
\begin{bmatrix}
  r_{n-1} \\
  r_n
\end{bmatrix}
= \begin{bmatrix}
  0 & 1 \\
  1 & -\frac{r_{n-2}}{r_{n-1}}
\end{bmatrix}
\begin{bmatrix}
  r_{n-2} \\
  r_{n-1}
\end{bmatrix}.
\]

This terminates when \( r_n = 0 \), and the \( \gcd(p, q) \) is \( r_{n-1} \). Not surprisingly these equations mirror Eq 2.3.1.3 (p. 44), but with different indexing scheme and interpretation of the variables.

This then explains why Gauss called the CFA the Euclidean algorithm. He was not confused. But since the equations have an inverse relationship, they are not strictly the same.
Appendix E

Eleven postulates of Systems of algebraic Networks

Physical systems obey basic rules that follow from the physics. It is helpful to summarize these restrictions as postulates presented in terms of a taxonomy, or categorization method, of the fundamental properties of physical systems. Eleven of these are listed below from an article by Kim and Allen (2013).

E.1 Representative system

A taxonomy of physical systems comes from a systematic summary of the laws of physics, which includes at least the eleven basic network postulates, described in Sect. 3.7.

To describe the network postulates, it is helpful to start from a two-port matrix representation as discussed in Sect. 3.5 (p. 92).

Figure E.1: The schematic representation of an algebraic network, defined by its two-port ABCD transmission, that has three elements called the Hunt parameters (Hunt, 1952): $Z_e(s)$, the electrical impedance, $z_m(s)$, the mechanical impedance, and $T(s)$, the transduction coefficient matrix of an electromechanic transducer network. The port variables are $\Phi(f)$ and $I(f)$: the frequency domain voltage and current, and $F(f)$ and $U(f)$: the force and velocity (Hunt, 1952; Kim and Allen, 2013). This matrix factors the two-port model into three $2 \times 2$ matrices, separating the three physical elements as matrix algebra. It is a standard impedance convention that the flows $I(f)$ and $U(f)$ are defined into each port. Thus it is necessary to apply a negative sign on the velocity $-U(f)$ so that it has an outward flow, as required to match the next cell with its inward flow.

As shown in Fig. E.1, the two-port transmission matrix for an acoustic transducer (loudspeaker) is characterized by the equation

$$ \begin{bmatrix} \Phi_i \\ I_i \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} F_i \\ -U_i \end{bmatrix} = \frac{1}{T} \begin{bmatrix} z_m(s) & Z_e(s)z_m(s) + T^2 \\ 1 & Z_e(s) \end{bmatrix} \begin{bmatrix} F_i \\ -U_i \end{bmatrix}, \quad (E.1.0.1) $$

shown as a product of three $2 \times 2$ matrices in the figure, with each factor representing one of the three Hunt parameters of the loudspeaker.

This figure represents the electromechanical motor of the loudspeaker, consisting of three elements, the electrical input impedance $Z_e(s)$, a gyrator, which is similar to a transformer that relates current to force, and the output mechanical impedance $z_m(s)$. This circuit describes what is needed to fully characterize its operation, from electrical input to mechanical (acoustical) output.

The input is electrical (voltage and current) $[\Phi_i, I_i]$ and the output (load) is the mechanical (force and velocity) $[F_i, U_i]$. The first matrix is the general case, expressed in terms of four unspecified functions $A(s), B(s), C(s)$ and $D(s)$, while the second matrix is for the specific example of Fig. E.1. The
three entries are the electrical driving point impedance \( Z_e(s) \), the mechanical impedance \( z_m(s) \) and the transduction \( T = B_o l \) where \( B_o \) is the magnetic flux strength and \( l \) is the length of the wire crossing the flux. Since the transmission matrix is antireciprocal, its determinate \( \Delta_T = -1 \), as is easily verified.

Other common examples of cross-modality transduction and current–thermal (thermoelectric effect) and force–voltage (piezoelectric effect). These systems are all reciprocal: thus the transduction has the same sign.

E.1.1 Impedance matrix

These eleven postulates describe the properties of a system having an input and an output. For the case of an electromagnetic transducer (loudspeaker) the system is described by the two-port transmission matrix, as shown in Fig. E.1. The electrical input impedance of a loudspeaker is \( Z_e(s) \), defined by

\[
Z_e(s) = \frac{V(\omega)}{I(\omega)} \bigg|_{U=0}.
\]

Note that this driving-point impedance must be causal since it is a function of \( s \); thus it has a Laplace transform. The corresponding two-port impedance matrix for Fig. E.1 is

\[
\begin{bmatrix}
\Phi_i \\
F_l
\end{bmatrix} = \begin{bmatrix}
z_{11}(s) & z_{12}(s) \\
z_{21}(s) & z_{22}(s)
\end{bmatrix}
\begin{bmatrix}
I_i \\
U_l
\end{bmatrix}
= \begin{bmatrix}
Z_e(s) & -T(s) \\
T(s) & z_m(s)
\end{bmatrix}
\begin{bmatrix}
I_i \\
U_l
\end{bmatrix}.
\]

Such a description allows one to define Thévenin parameters, a very useful concept used widely in circuit analysis and other network models from other modalities.

The impedance matrix is an alternative description of the system, but with generalized forces \([\Phi_i, F_l]\) on the left and generalized flows \([I_i, U_l]\) on the right. A rearrangement of terms allows one to go from the ABCD to the impedance parameters (Van Valkenburg, 1964b). The electromagnetic transducer is antireciprocal (Postulate P6), \( z_{12} = -z_{21} = T = B_o l \).

E.2 Taxonomy of algebraic networks

The postulates are extended beyond those defined by Carlin and Giordano (Sect. 3.7, p. 103), when there are interaction of waves and a structured medium, along with other properties not covered by classic network theory. Assuming quasistatics (QS), the wavelength must be large relative to the medium’s lattice constants. Thus the QS property must be extended to three dimensions, and possibly to the cases of an-isotropic and random media.

Causality: P1  As we stated above, due to causality the negative properties (e.g., negative refractive index) must be limited in bandwidth, as a result of the Cauchy–Riemann conditions. However even causality needs to be extended to include the delay, as quantified by the d’Alembert solution to the wave equation, which means that the delay is proportional to the distance. Thus we generalize P1 to include the space dependent delay. When we wish to discuss this property we denote it Einstein causality, which says that the delay must be proportional to the distance \( x \), with impulse response \( \delta(t - x/c) \).

Linearity: P2  The wave properties of may be nonlinear. This is not restrictive as most physical systems are naturally nonlinear. For example, a capacitor is inherently nonlinear: as the charge builds up on the plates of the capacitor, a stress is applied to the intermediate dielectric due to the electrostatic force \( F = qE \). In a similar manner, an inductor is nonlinear. Two wires carrying a current are attracted or repelled, due to the force created by the flux. The net force is the product of the two fluxes due to each current.

In summary, most physical systems are naturally nonlinear, it’s simply a matter of degree. An important counter example is a amplifier with negative feedback, with very large open-loop gain. There
are, therefore, many types of nonlinear, instantaneous and those with memory (e.g., hysteresis). Given the nature of P1, even an instantaneous nonlinearity may be ruled out. The linear model is so critical for our analysis, providing fundamental understanding, that we frequently take P1 and P2 for granted.

**Passive/Active: P3** This postulate is about conservation of energy and Otto Brune’s positive Real (PR) condition, that every passive impedance must obey. Following up on the earlier work of his primary PhD thesis advisor Wilhelm Cauer (1900–1945) and Ernst Guillemin, along with Norbert Weiner and Vannevar Bush at MIT, Otto Brune mathematically characterized the properties of every PR 1-port driving point impedance (Brune, 1931b).

When the input resistance of the impedance is real, the system is said to be passive, which means the system obeys conservation of energy. The real part of every PR impedance is nonnegative everywhere in the right half-plane (\(\Re Z_o\)). This means that the total energy absorbed by any PR impedance must remain positive for all time. Mathematically this may be stated as

\[
E(t) = \int_{-\infty}^{t} v(t)i(t) dt = \int_{-\infty}^{t} i(t) \ast z(t) i(t) dt > 0,
\]

where \(i(t)\) is any current, \(v(t) = z(t) \ast i(t)\) is the corresponding voltage and \(z(t)\) is the real causal impulse response of the impedance [e.g., \(z(t) \leftrightarrow Z(s)\) are a Laplace Transform pair]. In summary, if \(Z(s)\) is PR, then \(E(t)\) is PD.

**Energy and Power:** Since P3 requires impedance PR condition, it assures that every impedance is positive-definite (PD), thus guaranteeing that conservation of energy is obeyed (Schwinger and Saxon, 1968, p.17). This means that the total energy absorbed by any PR impedance must remain positive for all time. Mathematically this may be stated as
As discussed in detail by Van Valkenburg, any rational PR impedance can be represented as a partial fraction expansion, which can be expanded into first-order poles as

$$Z(s) = K \frac{\prod_{n=1}^{L} (s - n_i)}{\prod_{k=1}^{N} (s - d_k)} = \frac{\rho_n}{s - s_n} e^{j(\theta_n - \theta_d)}, \quad (E.2.0.2)$$

where $\rho_n$ is a complex scale factor (residue). Every pole in a PR function has only simple poles and zeros, which requires that $|L - N| \leq 1$ (Van Valkenburg, 1964b).

Whereas the PD property clearly follows from Postulate P3 (conservation of energy), the physics is not so clear. Specifically what is the physical meaning of the specific constraints on $Z(s)$? In many ways, the impedance concept is highly artificial, as expressed by Postulates P1–P7.

When the impedance is not rational, special care must be taken. An example of this is the semi-inductor $M/\sqrt{s}$ and semi-capacitor $K/\sqrt{s}$ due, for example, to the skin effect in EM theory and viscous and thermal losses in acoustics, both of which are frequency dependent boundary-layer diffusion losses (Vanderkooy, 1989). They remain positive-real but have a branch cut, thus are double valued in frequency.

**Real time response: P4** The impulse response of every physical system is real, vs. complex. This requires that the Laplace Transform have conjugate-symmetric symmetry $H(s) = H^*(s^*)$, where the * indicates conjugation [e.g., $R(\sigma, \omega) + X(\sigma, \omega) = R(\sigma, \omega) - X(\sigma, -\omega)$].

**Time invariant: P5** The meaning of time-invariant requires that the impulse response of a system does not change over time. This requires that the system coefficients of the differential equation describing the system are constant (independent of time).

**Rayleigh Reciprocity: P6** Reciprocity is defined in terms of the unloaded output voltage that results from an input current. Specifically (same as Eq. 3.5.2.6, p. 94)

$$\begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} = \left[ \frac{A(s)}{C(s)} \right] \begin{bmatrix} 1 & \Delta_T \\ D(s) \end{bmatrix} \quad (E.2.0.3)$$

where $\Delta_T = A(s) D(s) - B(s) C(s) = \pm 1$ for the reciprocal and antireciprocal systems respectively. This is best understood in terms of Eq. E.1.1.2. The off-diagonal coefficients $z_{12}(s)$ and $z_{21}(s)$ are defined as

$$z_{12}(s) = \frac{\Phi_i}{U_i} \bigg|_{I_i=0} \quad z_{21}(s) = \frac{F_l}{I_i} \bigg|_{U_i=0}.$$ 

When these off-diagonal elements are equal [$z_{12}(s) = z_{21}(s)$] the system is said to obey Rayleigh reciprocity. If they are opposite in sign [$z_{12}(s) = -z_{21}(s)$], the system is said to be antireciprocal. If a network has neither reciprocal or antireciprocal characteristics, then we denote it as nonreciprocal (McMillan, 1946). The most comprehensive discussion of reciprocity, even to this day, is that of Rayleigh (1896, Vol. I). The reciprocal case may be modeled as an ideal transformer (Van Valkenburg, 1964a) while for the antireciprocal case the generalized force and flow are swapped across the two-port. An electromagnetic transducer (e.g., a moving coil loudspeaker or electrical motor) is antireciprocal (Kim and Allen, 2013; Beranek and Mellow, 2012); it requires a gyrator rather than a transformer, as shown in Fig. E.1.

**Reversibility: P7** A second two-port property is the reversible/nonreversible postulate. A reversible system is invariant to the input and output impedances being swapped. This property is defined by the input and output impedances being equal.

Referring to Eq. E.2.0.3, when the system is reversible, $z_{11}(s) = z_{22}(s)$ or, in terms of the transmission matrix variables, $\frac{A(s)}{C(s)} = \frac{D(s)}{C(s)}$ or simply $A(s) = D(s)$, assuming $C(s) \neq 0$. 

---

**APPENDIX E. ELEVEN POSTULATES OF SYSTEMS OF ALGEBRAIC NETWORKS**
An example of a nonreversible system is a transformer where the turns ratio that is not 1. Also an ideal operational amplifier (when the power is turned on) is nonreversible due to the large impedance difference between the input and output. Furthermore, it is active; it has a power gain, due to the current gain at constant voltage (Van Valkenburg, 1964b).

Generalizations of this lead to group theory, and Noether’s theorem. These generalizations apply to systems that have many modes, whereas quasistatics holds when operate below a cutoff frequency (Table E.1), meaning that, as in the case of the transmission line, there are no propagating transverse modes. While this assumption is never exact, it leads to highly accurate results because the nonpropagating evanescent transverse modes are attenuated over a short distance, and thus, in practice, may be ignored (Montgomery et al., 1948; Schwinger and Saxon, 1968, Chaps. 9-11).

We extend the Carlin and Giordano postulate set to include reversibility, which was refined by Van Valkenburg (1964a). To satisfy the reversibility condition, the diagonal components in a system’s impedance matrix must be equal. In other words, the input force and the flow are proportional to the output force and flow, respectively (i.e., \( Z_e = z_m \)).

**Spatial invariant:** P8 The characteristic impedance and wave number \( \kappa(s, x) \) may be strongly frequency and/or spatially dependent, or even be negative over some limited frequency ranges. Due to causality, the concept of a negative group velocity must be restricted to a limited bandwidth (Brillouin, 1960). As makes clear Einstein’s theory of relativity, all materials must be strictly causal (Postulate P1), a view that must therefore apply to acoustics, but at a very different time scale. We first discuss generalized postulates, expanding on those of Carlin and Giordano.

**Deterministic (random):** P9 When the media are uniform and time invariant, the impedance and transfer functions will be deterministic. When the media are turbulent, the response will be random. When light propagates through the universe, it is strongly time varying. Thus astrophysics will be seen as random where as experiments in the shelter of the relatively uniform environment of the earth will be deterministic. Model calculation will also be deterministic unless one is trying to create a ransom time varying turbulent medium.

**The quasistatic constraint:** P10 When a system is described by the wave equation, delay is introduced between two points in space, which depends on the wave speed. When the wavelength is large compared to the delay, one may successfully apply the quasistatic approximation. This method has wide-spread application, and is frequency used without mention of the assumption. This can lead to confusion, since the limitations of the approximation may not be appreciated. An example is the use of quasistatics in Quantum Mechanics. The QS approximation has wide spread use when the signals may be accurately approximated by a band-limited signal. Examples include KCL, KVL, wave guides, transmission lines, and most importantly, impedance. The QS property is not mentioned in the six postulates of Carlin and Giordano (1964), thus they need to be extended in some fundamental ways.

When the dimensions of a cellular structure in the material are much smaller than the wavelength, can the QS approximation be valid? This effect can be viewed as a mode filter that suppresses unwanted (or conversely enhances the desired) modes (Ramo et al., 1965). QS may be applied to a 3 dimensional specification, as in a semiconductor lattice. But such applications fall outside the scope of this text (Schwinger and Saxon, 1968).

Although I have never seen the idea discussed in the literature, the QS approximation is applied when Green’s theorem is defined. For example, Gauss’s law is not true when the volume of the container violates QS, since changes in the distribution of the charge have not reached the boundary, when doing the integral. Thus such integral relationships assume that the system is in quasi-steady-state (i.e., that QS holds).

Formally, QS is defined as \( ka < 1 \), where \( k = 2\pi/\lambda = \omega/c \) and \( a \) is the cellular dimension or the size of the object. Other ways of expressing this include \( \lambda/4 > a \), \( \lambda/2\pi > a \), \( \lambda > 4a \), or \( \lambda > 2\pi a \). It is
APPENDIX E. ELEVEN POSTULATES OF SYSTEMS OF ALGEBRAIC NETWORKS

Table E.1: There are several ways of indicating the quasistatic (QS) approximation. For network theory there is only one lattice constant \( a \), which must be much less than the wavelength (wavelength constraint). These three constraints are not equivalent when the object may be a larger structured medium, spanning many wavelengths, but with a cell structure size much less than the wavelength. For example, each cell could be a Helmholtz resonator, or an electromagnetic transducer (i.e., an earphone).

<table>
<thead>
<tr>
<th>Measure</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ka &lt; 1 )</td>
<td>Wavenumber constraint</td>
</tr>
<tr>
<td>( \lambda &gt; 2\pi a )</td>
<td>Wavelength constraint</td>
</tr>
<tr>
<td>( f_c &lt; c/2\pi a )</td>
<td>Bandwidth constraint</td>
</tr>
</tbody>
</table>

not clear if it is better to normalize \( \lambda \) by \( 4 \) (quarter-wavelength constraint) or \( 2\pi \approx 6.28 > 4 \), which is more conservative by a factor of \( \pi/2 \approx 1.6 \). Also \( k \) and \( a \) can be vectors (e.g., Eq. 3.1.0.5, p. 57).

Sergei Schelkunoff may have been the first to formalize this concept (Schelkunoff, 1943), but he was not the first to use it, as exemplified by the Helmholtz resonator. George Ashley Campbell was the first to use the concept in the important application of a wave–filter, some 30 years before Schelkunoff (Campbell, 1903). These two men were 40 years apart and both worked for the telephone company (after 1929, called AT&T Bell Labs) (Fagen, 1975).

There are alternative definitions of the QS approximation, depending on the geometrical cell structure. The alternatives are outlined in Table E.1.

**The quasistatic approximation**: Since the velocity perpendicular to the walls of the horn must be zero, any radial wave propagation is exponentially attenuated \( (\kappa(s) \) is real and negative, i.e., the propagation function \( \kappa(s) \) will not describe radial wave propagation), with a space constant of about 1 diameter. The assumption that these radial waves can be ignored (i.e., more than 1 diameter from their source) is called the quasistatic approximation. As the frequency is increased, once \( f \geq f_c = 2c_o/\lambda \), the radial wave can satisfy the zero normal velocity wall boundary condition, and therefore will not be attenuated. Thus above this critical frequency, radial waves (higher order modes) are supported (\( \kappa \) becomes imaginary). Thus for Eq. 5.2.2.10 to describe guided wave propagation, \( f < f_c \). But even under this condition, the solution will not be precise within a diameter (or so) of any discontinuities (i.e., rapid variations) in the area.

Each horn, as determined by the area function \( A(r) \), has a distinct wave equation and thus a distinct solution. Note that the area function determines the upper cutoff frequency via the quasistatic approximation, since \( f_c = c_o/\lambda_c, \lambda_c/2 > d \), and \( A(r) = \pi(d/2)^2 \). Thus to satisfy the quasistatic approximation, the frequency \( f \) must be less than the cutoff frequency

\[
f < f_c(r) = \frac{c_o}{4} \sqrt{\frac{\pi}{A(r)}}. \tag{E.2.0.4}
\]

We have discussed two alternative matrix formulations of these equations: the ABCD transmission matrix, used for computation, and the impedance matrix, used when working with experimental measurements (Pierce, 1981, Chap. 7). For each formulation, reciprocity and reversibility show up as different matrix symmetries, as addressed in Sect. 3.7 (p. 103) (Pierce, 1981, p. 195-203).

**Periodic ↔ discrete: P11** As has been shown in the discussion on the Fourier transform, when the time (or frequency) domain response is periodic, the frequency (or time) domain is discrete. This is a fundamental symmetry property that must always be obeyed. This is closely related to the causal ↔ complex analytic property of the Laplace and \( z \) transforms.
Summary

A transducer converts between modalities. We propose the general definition of the eleven system postulates, that include all transduction modalities, such as electrical, mechanical, and acoustic. It is necessary to generalize the concept of the QS approximation (Postulate P9) to allow for guided waves.

Given the combination of the important QS approximation, and these space-time, linearity, and reciprocity properties, a rigorous definition and characterization a system can thus be established. It is based on a taxonomy of such materials formulated in terms of material and physical properties and extended network postulates.
Appendix F

Webster horn equation Derivation

F.1 Overview

In this appendix we transform the acoustic equations Eqs. 5.2.1.5 and 5.2.1.6 (p. 154) into their equivalent integral form, Eq. 5.2.2.10 (p. 155). This derivation is similar (but not identical) to that of Hanna and Slepian (1924) and Pierce (1981, p. 360).

F.1.1 Conservation of momentum:

The first step is an integration of the normal component of Eq. 5.2.1.5 (p. 154) over the isopressure surface \( S \), defined by \( \nabla p = 0 \)

\[- \int_S \nabla p(x,t) \cdot dA = \rho_o \frac{\partial}{\partial t} \int_S u(x,t) \cdot dA.\]

The average pressure \( \rho(x,t) \) is defined by dividing by the total area

\[\rho(x,t) \equiv \frac{1}{A(x)} \int_S p(x,t) \hat{n} \cdot dA. \quad (F.1.1.1)\]

From the definition of the gradient operator

\[\nabla p = \frac{\partial p}{\partial x} \hat{n}, \quad (F.1.1.2)\]

where \( \hat{n} \) is a unit vector perpendicular to the isopressure surface \( S \). Thus the left side of Eq. 5.2.1.5 reduces to \( \partial \rho(x,t) / \partial x \).

The integral on the right side defines the volume velocity

\[\nu(x,t) \equiv \int_S u(x,t) \cdot dA. \quad (F.1.1.3)\]

Thus the integral form of Eq. 5.2.1.5 becomes

\[- \frac{\partial}{\partial x} \rho(x,t) = \frac{\rho_o}{A(x)} \frac{\partial}{\partial t} \nu(x,t) \leftrightarrow Z(x,s) \nu', \quad (F.1.1.4)\]

where

\[Z(s,x) = sp_o/A(x) = sM(x) \quad (F.1.1.5)\]

and \( M(x) = \rho_o/A(x) \) [kgm/m\(^5\)] is the per-unit-length mass density of air.
Figure F.1: Derivation of horn equation using Gauss’s law: The divergence of the velocity $\nabla \cdot \mathbf{u}$, within $\delta x$, shown as the filled blue filled in region, is integrated over the enclosed volume. Next the divergence theorem is applied, transforming the integral to a surface integral normal to the surface of propagation. This results in the difference of the two volume velocities $\nu(x + \delta x) - \nu(x) = |\mathbf{u}(x + \delta x) \cdot \mathbf{A}(x + \delta x) - \mathbf{u}(x) \cdot \mathbf{A}(x)|$. The flow is always perpendicular to the isopressure contours.

F.1.2 Conservation of mass:

Integrating Eq. 5.2.1.6 over the volume $V$ gives

$$-\int_V \nabla \cdot \mathbf{u} \, dV = \frac{1}{\eta_o P_o} \frac{\partial}{\partial t} \int_V p(x, t) \, dV.$$

The volume $V$ is defined by two isopressure surfaces between $x$ and $x + \delta x$ (blue region of Fig. F.1). On the right-hand side we use the definition of the average pressure (i.e., Eq. F.1.1.1), integrated over the volume $dV$.

Applying Gauss’s law to the left-hand side,\(^1\) and using the definition of $\varrho$ (on the right), in the limit $\delta x \to 0$, gives

$$-\frac{\partial}{\partial x} \varrho(x, t) = \frac{A(x)}{\eta_o P_o} \frac{\partial}{\partial t} \varrho(x, t) \leftrightarrow \gamma(x, s) P(x, s)$$

(F.1.2.6)

where

$$\gamma(x, s) = sA(x)/\eta_o P_o = sC(x).$$

(F.1.2.7)

$C(x) = A(x)/\eta_o P_o$ [m\(^4\)/N] is the per-unit-length compliance of the air. Equations F.1.1.4 and F.1.2.6 accurately characterize the Webster plane-wave mode up to the frequency where the higher-order eigenmodes begin to propagate (i.e, $f > f_o$).

F.1.3 Horn properties

Speed of sound $c_o$

In terms of $M(x)$ and $C(x)$, the speed of sound and the acoustic admittance are

$$c_o = \sqrt{\frac{\text{stiffness}}{\text{mass}}} = \sqrt{\frac{1}{C(x)M(x)}} = \sqrt{\frac{\eta_o P_o}{\rho_o}}.$$  

(F.1.3.8)

This assumes the medium is lossless. For a discussion of lossy propagation, see Appendix G (p. 233).

F.1.4 Characteristic admittance $\gamma_r(x)$:

Since the horn equation (Eq. 5.2.2.10) is second-order, it has two eigenfunction solutions $\varphi^\pm$. The ratios of Eq. F.1.2.7 over Eq. F.1.1.5 are determined by the local stiffness $1/C(x)$ and mass $M(x)$. The ratio of $C/M$ determines the area dependent characteristic admittance $\gamma_r(x)$ ($\in \mathbb{R}$)

$$\gamma_r(x) = \frac{1}{\sqrt{\text{stiffness}} \cdot \text{mass}} = \sqrt{\frac{\gamma(x, s)}{Z(x, s)}} = \sqrt{\frac{C(x)}{M(x)}} = \sqrt{\frac{A(x)}{\rho_o}} \frac{sA(x)}{\eta_o P_o} = \frac{A(x)}{\rho_o c_o} > 0.$$  

(F.1.4.9)

(Campbell, 1903, 1910, 1922). The characteristic impedance is $Z_r(x) = 1/\gamma_r(x)$. Based on a physical argument, $\gamma_r(x)$ must be positive and real, thus only the positive square root is allowed. As long as $A(x)$ has no jumps (is continuous), $\gamma_r(x)$ must be the same in both directions. It is locally determined by the isopressure surface and its volume velocity.

\(^1\)As shown in Fig. F.1, taking the limit of the difference between the two volume velocities $\nu(x + \delta x) - \nu(x)$ divided by $\delta x$ results in $\partial \nu/\partial x$. 

Radiation admittance

The radiation admittance is defined looking into a horn with no termination (infinitely long) from the input at \( x = 0 \)

\[
Y_{\text{rad}}^{\pm}(s) = \frac{\mathcal{Y}^{\pm}}{\mathcal{P}^{\pm}} \in \mathbb{C}.
\]  
(F.1.4.10)

The impedance depends on the direction, with + looking to the right and − to the left.

The input admittance \( Y_{\text{in}}^{\pm}(x, s) \) is computed using the upper equation of Eq. 5.2.3.11 (p. 156) for \( \mathcal{V}(x, s) \) and then divide by the pressure eigenfunction \( \mathcal{P}^{\pm} \). This results in the logarithmic derivative of \( \mathcal{P}^{\pm}(x, s) \)

\[
Y_{\text{in}}^{\pm}(x, s) = \frac{\mathcal{V}^{\pm}}{\mathcal{P}^{\pm}} = \frac{-1}{s M(x)} \frac{\partial}{\partial r} \ln \mathcal{P}^{\pm}(x, s).
\]

For example, for the conical horn (last column of Table 5.2, p. 158)

\[
Y_{\text{in}}^{\pm} = \mathcal{Y}_r (1 \pm c_o/\rho_o \sqrt{s}).
\]  
(F.1.4.11)

Note that \( Y_{\text{in}}^{+}(x, s) + Y_{\text{in}}^{-}(x, s) = 2 \mathcal{Y}_r = 2 A_0 r^2/\rho_o c_o \in \mathbb{R} \), which shows that the frequency–dependent parts of the two admittances, begin equal and opposite in sign, exactly cancel.

As the wavefront travels down the variable area horn, there is a mismatch in the characteristic admittance due to the change in area. This mismatch creates a reflected wave, which in the case of the conical horn is \(-c_o/\rho_o \sqrt{s}\). Due to conservation of volume, there is a corresponding identical forward component that travels forward, equal to \(+c_o/\rho_o \sqrt{s}\). The sum of these two responses to the change in area must be zero, in order to conserve volume velocity.

The resulting equation for the velocity eigenfunctions is therefore

\[
\mathcal{V}^{\pm}(x, s) = Y_{\text{in}}^{\pm}(x, s) \mathcal{P}^{\pm}(x, s).
\]

Propagation function \( \kappa(s) \)  

The eigenfunctions of the lossless wave equation propagate as

\[
\mathcal{P}^{\pm}(x, s) = e^{\mp \kappa(s)x} \sqrt{A(x)},
\]

where \( \kappa(s) = \sqrt{Z(x, s)} \mathcal{V}(x, s) = \pm s \sqrt{MC} \). The velocity eigenfunctions \( \mathcal{V}^{\pm}(x, s) \) may be computed from Eq. F.1.1.4.

From the above definitions,

\[
\kappa(s) = \frac{s \rho_o}{A(x)} \frac{A(x)}{\eta_o P_o} = \frac{s}{c_o}.
\]

Thus \( \kappa(s) \) and \( s \) are the eigenvalues of the differential operations \( \partial/\partial x \) and \( \partial/\partial t \) operating on the pressure \( \mathcal{P}(x, s) \). See Appendix G for the inclusion of viscothermal losses.
Appendix G

Visco-thermal losses

G.1 Adiabatic approximation at low frequencies

At very low frequencies the adiabatic approximation must break down. Above this frequency, viscous
and thermal damping in air can become significant. In acoustics these two effects are typically ignored,
by assuming that wave propagation is irrotational, thus is described by the scalar wave equation. How-
ever this is an approximation.

As first explained by Kirchhoff (1868), following Helmholtz (1858), these two loss mechanisms are
related; but to understand why is somewhat complicated, due to both the history and the mathematics.
The full theory was first worked out by Kirchhoff (1868, 1974). Both forms of damping are due to two
different but coupled diffusion effects, first from viscous effects, due to shear at the container walls, and
second from thermal effects, due to small deviations from adiabatic expansion (Kirchhoff, 1868, 1974).
I believe ultimately Einstein was eventually involved as a result of his studies on Brownian motion
(Einstein, 1905).1

Newton’s early development understandably ignored viscous and thermal losses, which can be sig-
nificant in a thin region called the boundary layer, at acoustic frequencies when the radius of the con-
tainer (i.e., the horn) becomes less than the viscous boundary layer (e.g., much less than 1 mm) as
first described in Helmholtz (1858, 1863b, 1978). Helmholtz’s analysis (Helmholtz, 1863b) was soon
extended to include thermal losses by Kirchhoff (1868, 1974, English translation), as succinctly sum-
marized by Lord Rayleigh (1896), and then experimentally verified by Warren P. Mason (Mason, 1927,
1928).

The mathematical nature of damping is that the propagation function $\kappa(s)$ (i.e, complex wave num-
ber) is extended to

$$\kappa(s) = \frac{s + \beta_o \sqrt{s}}{c_o}.$$  \hfill (G.1.0.1)

where the forwarded $P_-$ and backward $P_+$ pressure waves propagate as

$$P_\pm(s, x) = e^{-\kappa(s)x}, e^{-\pi(s)x}$$ \hfill (G.1.0.2)

with $\pi(s)$ the complex conjugate of $\kappa(s)$, and $\Re \kappa(s) > 0$. The term $\beta_o \sqrt{s}$ effects both the real and
imaginary parts of $\kappa(s)$. The real part is a frequency dependent loss and the imaginary part introduces a
frequency dependent speed of sound (Mason, 1928).

G.1.1 Lossy wave-guide propagation

In the case of lossy wave propagation, the losses are due to viscous and thermal damping. The formu-
lation of viscous loss in air transmission was first worked out by Helmholtz (1863a), and then extended

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1See Ch. 3 of https://www.ks.uiuc.edu/Services/Class/PHYS498/.
APPENDIX G. VISCO- THERMAL LOSS

Figure G.1: This figure, taken from Mason (1928), compares the Helmholtz-Kirchhoff theory for $|\kappa(f)|$ to Mason’s 1928 experimental results measurement of the loss. The ratio of two powers ($P_1$ and $P_2$) are plotted (see Mason’s discussion immediately below his Fig. 4), and as indicated in the label: “$10\log_{10} P_1/P_2$ for 1 [cm] of tube length.” This is a plot of the transmission power ratio in [dB/cm].

by Kirchhoff (1868), to include thermal damping (Rayleigh, 1896, Vol. II, p. 319). These losses are explained by a modified complex propagation function $\kappa(s)$ (Eq. G.1.0.1). Following his review of these theories, Crandall (1926, Appendix A) noted that the “Helmholtz-Kirchhoff” theory had never been experimentally verified. Acting on this suggestion, Mason (1928) set out to experimentally verify their theory.

Mason’s specification of the propagation function

Mason’s results are reproduced here in Fig. G.1 as the solid lines for tubes of fixed radius between 3.7–8.5 [mm], having a power reflectance given by

$$|\Gamma_L(f)|^2 = |e^{-\kappa(f)}|^2 \text{[dB/cm]}. \quad (G.1.1.3)$$

The complex propagation function cited by Rayleigh (1896) is (Mason, 1928, Eq. 2)

$$\kappa(\omega) = \frac{P\eta_0\sqrt{\omega}}{2c_oS\sqrt{2\rho_o}} + \frac{i\omega}{c_o} \left(1 + \frac{P\eta_0'}{2\sqrt{2\omega\rho_o}}\right) \quad (G.1.1.4)$$

and the characteristic impedance is

$$z_o(\omega) = \sqrt{P_o\gamma\rho_o}\left[1 + \frac{P\gamma'}{2S\sqrt{2\omega\rho_o}} - \frac{P\gamma'}{2\sqrt{2\omega\rho_o}}\right], \quad (G.1.1.5)$$

where $S$ is the tube area and $P$ is its perimeter. Mason specified physical constants for air to be $\eta_o = 1.4$ (ratio of specific heats), $\rho_o = 1.2$ [kgm/m$^3$] (density), $c_o = \sqrt{P_o\eta_o/\rho_o} = 341.56$ [m/s] (lossless air velocity of sound at 23.5° [C]), $P_o = 10^5$ [Pa] (atmospheric pressure), $\mu_o = 18.6 \times 10^{-6}$ [Pa-s] (viscosity). Based on these values, $\eta_0'$ is defined as the composite thermodynamic constant (Mason, 1928)

$$\eta_0' = \sqrt{\mu_o} \left[1 + \sqrt{5/2} \left(\eta_o^{1/2} - \eta_o^{-1/2}\right)\right]$$

$$= \sqrt{18.6} \times 10^{-3} \left[1 + \sqrt{5/2} \left(\sqrt{1.4} - 1/\sqrt{1.4}\right)\right]$$

$$= 6.618 \times 10^{-3}.$$
G.1. ADIABATIC APPROXIMATION AT LOW FREQUENCIES

G.1.2 Impact of viscous and thermal losses

Assuming air at 23.5° [C], \( c_o = \sqrt{\gamma_o P_o/\rho_o} \approx 344 \text{ [m/s]} \) is the speed of sound, \( \gamma_o = c_p/c_v = 1.4 \) is the ratio of specific heats, \( \mu_o = 18.5 \times 10^{-6} \text{ [Pa-s]} \) is the viscosity, \( \rho_o \approx 1.2 \text{ [kg/m}^2] \) is the density, \( P_o = 10^5 \text{ [Pa]} \) (1 atm).

Equation G.1.1.4 and the measured data are compared in Fig. G.1, reproduced from Mason’s Fig. 4, which shows that the wave speed drops from 344 [m/s] at 2.6 [kHz] to 339 [m/s] at 0.4 [kHz], a 1.5% reduction in the wave speed. At 1 [kHz] the loss is 1 [dB/m] for a 7.5 [mm] tube. Note that the loss and the speed of sound vary inversely with the radius. As the radius approaches the boundary layer thickness, i.e., the radial distance such that the loss is \( e^{-1} \), the effect of the damping dominates the propagation.

With some significant algebra, Eq. G.1.1.4 may be greatly simplified to Eq. G.1.0.1.\(^5\)

\[
\beta_o = \frac{P \eta'_o}{2S \sqrt{\rho_o}} = \frac{P}{2S} \times \frac{\eta'_o}{\rho_o} \approx 6.0415 \times 10^{-3}
\]

For the case of a cylindrical waveguide the radius is \( R = 2S/P \). Thus

\[
\beta_o = \frac{1}{R} \sqrt{\frac{\eta'_o}{\rho_o}} = \frac{1}{R} \times 6.0415 \times 10^{-3}.
\]

Here \( \beta_o = P \eta'/2S \rho_o \) is a thermodynamic constant, \( P \) is the perimeter of the tube and \( S \) the area (Mason, 1928).

For a cylindrical tube having radius \( R = 2S/P \), \( \beta_o = \eta'_o / R \sqrt{\rho_o} \). To get a feeling for the magnitude of \( \beta_o \), consider a 7.5 [mm] tube (i.e., the average diameter of the adult ear canal). Then \( \eta' = 6.6180 \times 10^{-3} \) and \( \beta_o = 1.6110 \). Using these conditions the wave-number cutoff frequency is \( 1.611^2/2\pi = 0.4131 \text{ [Hz]} \). At 1 kHz the ratio of the loss over the propagation is \( \beta_o/\sqrt{|s|} = 1.6011/\sqrt{2\pi}10^3 \approx 2\% \).

At 100 [Hz] this is a 6.4% effect.\(^4\)

**Cut-off frequency** \( s_o \): The frequency where the loss-less part equals the lossy part is defined as \( \kappa(s_o) = 0 \), namely

\[
s_o + \beta_o \sqrt{s_o} = 0.
\]

Solving for \( s_o \) gives the real, and negative, pole frequency of the system

\[
\sqrt{s_o} = -\beta_o = -6.0415 \times 10^{-3} / R.
\]

To get a feeling for the magnitude of \( \beta_o \) let \( R = 0.75/2 \text{ [cm]} \) (i.e., the average radius of the adult ear canal). Then

\[
-\sqrt{s_o} = \beta_o = 6.0145 \times 10^{-3} / 3.75 \times 10^{-3} \approx 1.6.
\]

We conclude that the losses are insignificant in the audio range since for the case of the human ear canal \( f_o = \beta_o^2/2\pi \approx 0.407 \text{ [Hz]} \).\(^5\)

Note how the propagation function has a Helmholtz-Kirchhoff correction for both the real and imaginary parts. This means that both the speed of sound and the damping are dependent on frequency, proportional to \( \beta_o \sqrt{s/o} \). Note also that the smaller the radius, the greater the damping.

**Summary**: The Helmholtz-Kirchhoff theory of viscous and thermal losses results in a frequency dependent speed of sound, having a frequency dependence proportional to \( 1/\sqrt{\omega} \) (Mason, 1928, Eq. 4). This corresponds to a 2% change in the sound velocity over the decade from 0.2-2 [kHz] (Mason, 1928, Fig. 5), in agreement with experiment.

---

\(^5\)The real and imaginary parts of this expression, with \( s = i\omega \), give Eq. G.1.0.1, 233.
\(^4\)\url{home/jba/Mimosa/2C-FindLengths.16/doc.2-c_calib.14/m/MasonKappa.m}
\(^5\)\url{home/jba/Mimosa/2C-FindLengths.16/doc.2-c_calib.14/m/MasonKappa.m}
Appendix H

Quantum Mechanics and the WHEN

While it is clear that both Schrödinger’s equation and Dirac’s equations are highly accurate, after about 100 years, it is not clear why. Both of these theories seem to violate classical electromagnetics (EM) (i.e., Ohm’s law), since they are built on energy principles rather than electric and magnetic fields. Here we delve into this question, by providing a classical (i.e., EM based) derivation for the Hydrogen atom, one of the most important and obvious successes of quantum mechanics (QM). The problem with QM is not that it fails, rather that it succeeds, without obvious basis. The problem is that one cannot understand the basic principles, and it seems to be in contradiction with any any principles of a physical theory. Based on the Rydberg series, we determine the reflection coefficient, and thus the radiation impedance seen by the electron, in a radial coordinate centered on the proton. Since the electron and proton both have spin 1/2, their magnetic fields must attractively align, accounting for the near field vector potential, and complementing the far field attraction due to their opposite signs. As the electron and proton approach each other, due to their far-field potential attraction, the magnetic near field would further be attractive at close range, due to the magnetic dipoles of the two “particles,” causing them to merge with neutral net magnetic moment and neutral charge, giving a highly stable hydrogen atom. However given a sufficiently strong distorting field, this highly symmetric state could be disturbed, leading to photon radiation, constrained by the radial eigen states. It seems more clear than ever that photons and electrons are in a state of equilibrium at the outer skirt of very large Rydberg atoms.\(^1\)

\[
\begin{align*}
\text{Ly-}\alpha & \quad \text{Ba-}\alpha \\
100 \text{ nm} & \quad \text{1000 nm} \\
\text{visible} & \quad 10 \text{ 000 nm}
\end{align*}
\]

Figure H.1: Diagram of the wavelength spectrum of hydrogen for the Lyman, Balmer and Paschen series, as a function of each lines wavelength. The notation “La-α” indicates the longest wavelength $\lambda_{11} = 122$ [nm] (i.e., lowest frequency of 2.46 [GHz]) for the Lyman series. Figure citation: https://en.wikipedia.org/wiki/Hydrogen_spectral_series

H.1  Equation for Rydberg eigenmodes:

Like every tuned resonant circuits, atoms have well defined resonant frequencies, aka eigen-modes. Figure H.1 shows the observed radiation spectra for Hydrogen. From the very beginning, it was clear there is a pattern to these spectral lines. In 1880 Rydberg easily fit a formula that quantifies the observed

\(^1\)https://physics.aps.org/synopsis-for/10.1103/PhysRevLett.121.193401
Rydberg frequencies: \( f_{n,m} = \frac{cR}{\lambda^2_n} \)

Figure H.2: Rydberg frequencies in [GHz] and the corresponding wavelengths, computed from the Rydberg formula \( \frac{1}{\lambda_{nm}} = R \left( \frac{1}{n^2} - \frac{1}{m^2} \right) \), where integer \( n \) defines the series (Lyman: \( n = 1 \), Balmer: \( n = 2 \), Paschen: \( n = 3 \), etc.) and integer \( m > n \) defines the outer transition line (See Fig. H.3). For example, according to the lower panel (green series), the Lyman series line \( \lambda_{1,2} = 122 \) [nm] (\( n = 1 \) and \( m = 2 \)), in agreement with the lower panel of this figure, Figs. H.3 and H.1. The frequency of the Paschen series line (3,6) is at 1.094 [GHz] (0.3 [GHz]) (upper panel) (http://www.physics.drexel.edu/~tim/open/hydrofin/).

eigen spectral lines of hydrogen, in terms of the reciprocal of the radiated wavelengths

\[
\frac{1}{\lambda_{nm}} = R_\infty \left( \frac{1}{n^2} - \frac{1}{m^2} \right), \quad \frac{f_{nm}}{c_0 R_\infty} = \frac{1}{n^2} - \frac{1}{m^2},
\]

(H.1.0.1)

all based on these simple observations. Here \( R_\infty = 1.097 \times 10^7 \) [m\(^{-1}\)], \( c_0 = 3 \times 10^8 \) [m/s] is the speed of light, \( f_{nm} \) are the dimensionless Rydberg integer frequencies with \( n, m \in \mathbb{N} \) are positive integers \( \in \mathbb{N} \), where \( n \) labels the series, and \( n < m (\lambda > 0) \) describes the transition from orbit \( n \) to orbit \( m \), as described in the caption of Fig. H.3.

H.1.1 The Rydberg atom model

In 1909 Rutherford demonstrated that the atom consisted of a dense core (the proton) surrounded by electrons. This view was supported by the spectrum of the atom, which allows for a radiation spectrum caused by electrons jumping from one energy level to another. It was then noted by Bohr in 1913 (Bohr, 1954) and others that the wavelengths of hydrogen, as described by Eq. H.1.0.1, are consistent with Fig. H.2, where the reciprocal wavelength [m\(^{-1}\)] is given by Eq. H.1.0.1, having frequencies \( f_{nm} = c/\lambda_{nm} \) [Hz]. The challenge of the 1920s was to explain these intuitive and simple models of hydrogen. This gave rise to the birth of quantum mechanics, the history of which is nicely summarized in Condon and Morse (1929).

It was clear from the days of Bohr that the Rydberg formula did not follow the typical rules of eigen spectra, so much so that Arnold Sommerfeld wrote (Sommerfeld, 1949, p. 201):

The lines of this spectrum cumulate at the limit given by the Rydberg constant \( R \). The adjoining continuum lies in the near ultraviolet range. Both the discrete and the continuous spectrum are given by the Schrödinger equation. This equation reduces to a simple mathematical formula the enigma of the spectral lines, with their finite accumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.
**H.2. RYDBERG SOLUTION METHODS**

![Figure H.3: This diagram defines hydrogen’s allowed electron transitions, defining the Lyman (n=1), Balmer (n=2) and Paschen (n=3) series. The numbers represent the wavelengths $\lambda$ [nm] of the photons having frequencies $f_{n,m} = c_0/\lambda_{n,m}$, following an electron transition from level $n$ to $m$ (Taken from: https://en.wikipedia.org/wiki/Hydrogen_spectral_series).](image)

**H.1.2 Rydberg wave equation:**

The objective of this report is to demonstrate that one can define a classical Sturm-Liouville model of the enigmatic Rydberg atom, by the use of the Webster Horn equation

$$\frac{1}{A(r)} \frac{\partial}{\partial r} A(r) \frac{\partial}{\partial r} \psi(r,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(r,t),$$

which is a 1 dimensional wave equation for the electric potential $\psi(r,t)$ propagating in a wave-guide having area $A(r)$ as a function of the range, where $r$ is the range variable (the axis of wave propagation).

We shall show that given the Rydberg spectrum (Eq. H.1.0.1), one may accurately estimate the electric reflectance $\Gamma(s)$ looking out from the origin (i.e., the proton location, as indicated by the small red dot in Fig. H.3). The radiation impedance $Z_{rad}(s)$ seen by the proton is related to the reflectance $\Gamma(s)$ by the relation

$$Z_{rad}(s) = r_o \frac{1 + \Gamma(s)}{1 - \Gamma(s)}.$$  

(H.1.2.3)

This formula is the basis of the Smith chart used in both physics and engineering studies. It follows that once $\Gamma(s)$ is known (i.e., evaluated given Eq. H.1.0.1), the radiation impedance may be computed. It has been shown that the area function $A(r)$ may be found given the radiation impedance (Sondhi and Gopinath, 1971; Youla, 1964).

**H.2 Rydberg solution methods**

The basic idea behind the method is to use Eq. H.1.2.3, by noting that the poles of the impedance are determined by the roots denominator of $Z_{rad}$. Specifically, if $s_p$ is an impedance pole, then it must satisfy $\Gamma(s_p) \approx 1$. Since except for losses due to radiation, the atom is loss-less, thus $|\Gamma(s)| = 1$. Namely it must be of the form

$$\Gamma(s) = e^{-\phi(f)},$$

(H.2.0.1)

where the phase $\phi(f) \in \mathbb{R}$ and $s = \sigma + \omega j$ is the complex Laplace radian frequency, with $\omega = 2\pi f$ [Hz]. Since we know the eigen-mode frequencies, which obey $\phi(f_{n_o,m}) = 2\pi m$, we may find $\phi(f)$, as follows: For a given series index $n_o$, and given the eigen-frequencies $f_m$, we seek the phase mode function $\phi_{n_o}(f)$ that maps the eigen-frequencies to their mode index $m$, i.e.,

$$\phi_{n_o}(f_m) = 2\pi m.$$  

\(^2\)The first attempt at this analysis is /home/jba/DOCS/doc.RydbergModel.17/RydbergModel.pdf.
**APPENDIX H. QUANTUM MECHANICS AND THE WHEN**

![Lyman eigen-frequencies [GHz]](image)

**Complex Mode index vs frequency (Re: o, Im: x)**

\( \phi(f) = 3 \ln \Gamma(s) \)

**Figure H.4:** The top panel is a plot of Eq. H.1.0.1, showing how the eigen-mode frequencies \( f_l \) depend on the eigen-number index \( l \). As the mode number increases, the frequency reach an asymptote at \( c_o R \approx 3.29 \text{ [GHz]} \), with a wavelength limit near \( 1/R \approx 91.2 \text{ [nm]} \). The lower panel shows the inverse mapping from frequency to the mode index number \( \phi(f) \). This figure is for the Lyman series \((n = 1, m = 1, \cdots, 20)\). The inverse of this relationship is \( l = \phi^{-1}(f_l) \) may be derived from Eq. H.1.0.1, which provides the pole frequencies required to satisfy Eq. H.1.2.3. Note that for frequencies greater than \( c_o R \) the phase switches from real purely real to imaginary, accounting for the free electrons that must exist above the upper accumulation frequency (i.e., 3.29 [GHz] for this example).

### H.2.1 Group delay \( \tau(s) \)

The phase \( \phi(\omega) \) is related to the group delay \( \tau(\omega) \) by the relation

\[
\tau(\omega) = -\frac{\partial}{\partial \omega} \phi(\omega).
\]

Here one may assume that the phase if complex analytic,\(^3\) thus allowing a causal damping term into the reflectance phase Eq. H.2.0.1. This follows naturally because the reflectance must be causal (Postulate 3.7.1, p. 106). In the time domain the delay may be written in terms of the inverse \( \mathcal{L} \mathcal{T} \) of the group delay

\[
\Gamma(s) = e^{-\int_0^s \tau(s) \, ds}.
\]

Typically one uses the reflectance phase \( 2\pi\phi(f) \), thus the group delay is \( \tau(f) = -\partial\phi(f)/\partial f \) which is physically interpreted here as the frequency dependent delay from the proton to the radius of the electron’s orbit. Thus this delay is given by

\[
\tau(f) = n \frac{\partial}{\partial f} \left(1 - \frac{n^2}{c_o R f}\right)^{-1/2} = \frac{n^3}{2c_o R} \left(1 - \frac{n^2}{c_o R f}\right)^{-3/2}
\]

which is constant for low frequencies and then rises to \( \infty \) as frequency approaches the Rydberg frequency \( (f \to c_o R/n^2) \).

\(^3\)It follows that these relationships are related by a Hilbert transform.
One may solve Eq. H.1.0.1 for $m$, for the case of the Lyman series ($n_o = 1$), by the use of the following identity for the Rydberg eigen-frequencies $f_{nm}$, which follow directly from Eq. H.1.0.1, with $m = n_o + l$ (with $n_o, m, l \in \mathbb{N}$)

$$f_{nm} = \frac{c_o}{\lambda_{nm}} = c_o R \left( \frac{1}{n_o^2} - \frac{1}{(n_o + l)^2} \right)$$

$$= \frac{c_o R}{n_o^2} \left( 1 - \frac{1}{(1 + l/n_o)^2} \right) \quad (H.2.1.2)$$

Note that as $l \to \infty$, $f_{n_o,l} \to c_o R/n_o^2$, which is Sommerfeld’s finite accumulation point [Hz] $f_{n_o,\infty}$ for the Lyman series ($n_o = 1$).

The above equation may be solved for the mode number $l/n < 1$ as a function of mode frequency

$$n^2 \frac{f_{nl}}{c_o R} = 1 - \frac{1}{(1 + l/n)^2} \quad \text{Starting from Eq. H.2.1.2}$$

$$\frac{1}{(1 + l/n)^2} = 1 - n^2 \frac{f_{nl}}{c_o R} \quad \text{Solving for } l/n$$

$$(1 + l/n)^2 = \frac{1}{1 - n^2 \frac{f_{nl}}{c_o R}}$$

$$\frac{l}{n} = \pm \frac{1}{\sqrt{1 - n^2 \frac{f_{nl}}{c_o R}}} - 1 \quad \phi(f_{nl})/2\pi = l = m - n_o \in \mathbb{N}. \quad (H.2.1.3)$$

as summarized in the lower panel of Fig. H.4.

### H.2.2 Finding the area function

Once the phase has been determined, one may compute the impedance using Eq. H.1.2.3. One may then decompose the impedance by using the analytic continued fraction algorithm (aka Cauer synthesis), discussed in Sect. 3.5, p. 92.

### H.3 Euclid’s formula and the Rydberg atom model

Fundamental the quantum mechanics is the Rydberg series, which describes the quantized energy levels of atoms$^4$

$$\nu_{n,m} = c_o R_\infty Z_n^2 \left( \frac{1}{n^2} - \frac{1}{m^2} \right) \quad (H.3.0.1)$$

where $\nu_{n,m}$ are the possible eigenfrequencies, $c_o$ is the speed of light, $R_\infty \approx 1.097 \times 10^7$ is the Rydberg constant, $Z_n$ is the atomic number, along with positive integers $m > n \in \mathbb{N}$, which represent the principal quantum numbers that label all possible allowed atomic eigenstates. Integer $n$ indicates the lowest (rest) atomic eigenstate while $m$ lables the higher (excited) state.$^5$ When $n = 1$ the series is the Lyman series corresponding to Hydrogen ($Z_1 = 1$). When $n = 1, m = 2$ and $Z_1 = 1$, the frequency is

$$\nu_{1,1} = c_o R_\infty \left( \frac{1}{1^2} - \frac{1}{2^2} \right)^{3/4} = 2.5 \times 10^{15} \text{ [Hz]} \quad (H.3.0.2)$$

and the wavelength is $\lambda = c_o/\nu = 4 \times 10^8 / R_\infty = 36.36 \text{ [m]}$.

An open question in this model is: Why are states either empty or filled? The amplitudes of the modes of a string or organ pipe are not quantized. What is it about the atom that forces the energy to be quantized?

$^4$https://www.youtube.com/watch?v=e0IWPPEhmMho

Given observed frequencies $\nu_{n,m}$ it is possible to find the area function that traps the photons into the Rydberg eigenstates. Eq. H.3.0.1 may be rewritten as

$$\nu_{n,m} = c_o R Z_n^2 \left( \frac{m^2 - n^2}{(2mn)^2} \right)^{1/2}$$

It is interesting to compare Eq. H.3.0.1 to Euclid’s formula Eq. 2.3.3.6 (p. 49)

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2,$$

where $m > n \in \mathbb{N}$. Euclid’s formula is equivalent to the Pythagorean theorem for integers since

$$c^2 = a^2 + b^2,$$

with $\{a, b, c\} \in \mathbb{N}$. Here $a < b < c$.

If we interpret the quantum numbers as multiples of a quarter wavelength, then the Rydberg formula is congruent to the Pythagorean theorem. Given the symmetry, this cannot be an accident.

In terms of the lengths of the right triangle $\{a, b, c\}$, Rydberg’s formula becomes

$$\nu_{n,m} = c_o R Z_n^2 \left( \frac{a}{b^2} \right)^{1/2}$$

But since $b^2 = c^2 - a^2$,

$$\nu_{n,m} = \frac{R Z_n^2}{a} \left( \frac{a^2}{c^2 - a^2} \right)^{1/2} = \frac{R Z_n^2}{a} \frac{a^2}{c^2} \left( \frac{1}{1 - (a/c)^2} \right).$$

In terms of quantized (discrete) angles, $\sin(\theta_{n,m}) = a/c$

$$\nu_{n,m} = \frac{R Z_n^2}{a} \left( \frac{\sin^2 \theta}{1 - \sin^2 \theta} \right) = \frac{R Z_n^2}{a} \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{R Z_n^2}{a} 4 \tan^2 \theta_{n,m}.$$  

**H.3.1 Eigenmodes of the Ryderberg atom:**

One way to think of eigenmodes is to make an analogy to a piano string, or an organ pipe. In these much simpler systems, there is an almost constant delay, say $\tau$ due to a characteristic length, say $L = \tau c_o$, such that the eigenmodes of a string are given by integer multiples of a half wavelength $\nu_n = n c_o / 2L$ while the eigenmodes of the organ pipe is multiples of a quarter wavelength. The distinction is the boundary conditions. For the string the endpoint boundary conditions are pinned displacement (i.e., zero velocity). The organ pipe is closed at one end and open at the other, resulting in multiples of a quarter wavelength $\nu_n = n c_o / 4L$. In each case $\nu = n/\tau$ where $\tau = 2L/c_o$ is the round trip delay, thus $\nu = n c_o / 2L$.

We suggest looking at the Rydberg series in the say way, but with the very different eigen frequencies (Eq. H.3.0.1). Sommerfeld (1949, p. 201) makes a very interesting comment regarding Eq. H.3.0.1:

This equation reduces to a simple mathematical the enigma of the spectral lines, with their finite cummulation point, the behavior of which differs so fundamentally from that of all mechanical systems.
H.3.2 Discussion

The Rydberg frequencies \( f_{nl} \) \((n = 1, l = 1, \cdots, \infty)\) has poles in the radiation impedance (Eq. H.1.2.3) when \( \phi(f_{nl}) \in \mathbb{N} \). Working backwards from the Rydberg formula (Eq. H.2.0.1), we have solved for \( \phi(f_{nl}) \) indicating where this condition is valid (Eq. H.2.1.3). Since the reflectance and the impedance must be causal complex analytic functions of Laplace frequency \( s \), we must replace the discrete frequency \( f_{nl} \) with

\[
j2\pi f_{nl} \rightarrow s = \sigma + \omega j,
\]

thereby forcing \( l(s) \) to be a complex analytic function of \( s \). Then the poles of the radiation impedance must satisfy

\[
\Gamma(s_{nl}) = e^{j2\pi l(f_{nl})} = 1,
\]

resulting in eigen-frequencies at \( f_{nl} \).

The next step in this analysis is determine the area function \( A(r) \) given \( Z_{rad} \) (Eq. H.1.2.3). To do this one must solve an integral equation for \( A(r) \), as discussed by Sondhi and Gopinath (1971); Youla (1964).

Perhaps this could be done using an analytic representation for the area function

\[
A(r) = \sum_k a_k r^k.
\]

H.4 Relations between Sturm-Liouville and Quantum Mechanics

If we compare the Schrödinger equation (SE) for hydrogen with the corresponding Sturm-Liouville equation we can begin to appreciate their differences. The QM equation for hydrogen is

\[
\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m_o} \nabla^2 \psi(x,t) + V(r) \psi(x,t)
\]

\[
= -\frac{\hbar^2}{2m_o} \left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \psi(x,t) + V(r) \psi(x,t) \right) \quad (H.4.0.1)
\]

\[
= -\frac{\hbar^2}{2m_o} \left[ \frac{2}{r} \frac{\partial}{\partial r} \psi(x,t) + \frac{\partial^2}{\partial r^2} \psi(x,t) \right] + V(r) \psi(x,t) \quad (H.4.0.2)
\]

whereas the Horn equation is given by Eq. H.1.2.2.

There are several obvious and disturbing differences between these two equations: First the SE is, of course, first order in time. Diffusion equations have no delay, thus cannot have traditional eigen modes, which result from standing waves in a wave equation, due to boundary conditions. Second the EM horn equation is of Sturm-Liouville (SL) form, which is a true wave equation (vs. the SE, which is a diffusion equation). The obvious question arises: Is there a potential \( V \) that would allow these two formulations to be equivalent. If so, then this would provide an explanation as to why the SE is successful in explaining the properties of Rydberg atoms.

To explore this possibility we may expand the two differential equations, and directly compare them. Expanding Eq. H.1.2.2 gives

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(r,t) = \frac{1}{A(r)} \frac{\partial}{\partial r} A(r) \frac{\partial}{\partial r} \psi(r,t)
\]

\[
= \frac{\partial^2}{\partial r^2} \psi(r,t) + \frac{1}{A(r)} \frac{\partial A(r)}{\partial r} \psi(r,t) \quad (H.4.0.3)
\]

\[
= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(r,t) + \frac{1}{A(r)} \frac{\partial A(r)}{\partial r} \psi(r,t) \quad (H.4.0.4)
\]

Between these two equations we may remove \( \psi'' \)

\[
\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m_o} \left[ \frac{2}{r} \frac{\partial}{\partial r} \psi(x,t) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(r,t) - \frac{1}{A(r)} \frac{\partial A(r)}{\partial r} \psi(r,t) \right] + V(r) \psi(x,t).
\]

(H.4.0.5)

It seems that this may isolate the time and spatial parts (as in separation of variables).
APPENDIX H. QUANTUM MECHANICS AND THE WHEN

H.4.1 The exponential horn

A relevant and motivational example is the solution of the exponential horn, having area function \( A(r) = A_0 e^{2mr} \). This case is interesting because it has a closed form solution, which seems relevant, and perhaps even related to that of the hydrogen atom.

Assuming that \( \varrho(r,t) \leftrightarrow P(r,\omega) \) are a Fourier transform pair, with \( A(r) = A_0 e^{2mr} \), Eq. H.1.2.2 reduces to

\[
\frac{\partial^2 P(r,\omega)}{\partial r^2} + 2m \frac{\partial P(r,\omega)}{\partial r} = \kappa^2 P(r,\omega) \leftrightarrow \frac{1}{c^2_0} \frac{\partial^2 \varrho}{\partial t^2} \tag{H.4.1.6}
\]

with \( \kappa(s) = s/c_0 \).

**Exercise:** Show that Eq. H.4.1.6 follows from Eq. H.1.2.2.  **Solution:** Starting from Eq. H.1.2.2 with area \( A(r) = A_0 e^{2mr} \),

\[
\varrho_{rr}(r,t) + 2m \varrho_r(r,t) = \kappa^2 \varrho(r,\omega) \leftrightarrow \frac{1}{c^2_0} \frac{\partial^2 \varrho}{\partial t^2} \]

which is the time domain version of Eq. H.4.1.6. ■

Next consider the Fourier series (or Fourier transform) of the area function

\[
A(r) = \sum_k a_k e^{2mr}.
\]

It follows from the linearity of the wave equation that the general solution of Eq. H.4.1.6 is

\[
\varrho_{\pm}(r,\omega) = \sum_k a_k^\pm(\omega) e^{-mr} e^{\mp \sqrt{m^2 + \kappa^2} r}.
\]

Here we have combined \( \varrho_{\pm}(\omega) \) and \( a_k \) as coefficients \( a_k^\pm(\omega) \).
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Appendix I

Figure I.1:  Timeline from the early Asians to Bombelli (p. 15).

Figure I.2:  Timeline from Descartes to Cauchy (p. 18).
Chronological history: 16th to 19th centuries

<table>
<thead>
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<th>1525</th>
<th>1564</th>
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<td>Descartes</td>
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<td>Einstein</td>
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<tr>
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<td>Mozart</td>
<td>Kirchhoff</td>
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Figure I.3: Timeline from Bombelli to Gauss (p. 23).

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<tr>
<td>Mozart</td>
<td>Beethoven</td>
<td>Rayleigh</td>
<td>Heaviside</td>
<td>Poincare</td>
<td>Brillouin</td>
<td>Sommerfeld</td>
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</tbody>
</table>

Figure I.4: Timeline from Newton to Brillouin (p. 56).
Appendix J

Figure J.1: Venn diagram showing relations between Engineering, Mathematics and Physics (p. 10).

Figure J.2: Fibonacci spiral (p. 52).

Figure J.3: Newtons method, applied to two polynomials: real roots (left); complex roots (right) (p. 62).
Figure J.4: Colorized plots: Left: trivial case $w(s) = s$; Right: $w(s) = s - \sqrt{j}$ (p. 110).

Figure J.5: $w(s) = \sin(0.5\pi(s - j))$ (p. 111).

Figure J.6: Colorized plots of $w(z) = e^z$ (left) and $w(z) = \ln(z)$ (right) (p. 112).
\[ s = \arctan(z) \]

\[ s = \frac{i}{2} \ln \frac{1-i\cdot z}{1+i\cdot z} \]

Figure J.7: Plots of \( w(z) = \arctan(z) \) (left) and \( w(z) = \frac{i}{2} \ln \frac{1-i\cdot z}{1+i\cdot z} \) (right) (p. 116).

\[ w = s^2 \]
\[ w = -\sqrt{-s} \]

Figure J.8: Colorized plots of \( w(s) = s^2 \) (left) and \( w(s) = -\sqrt{-s} \) (right) (p. 128).

Figure J.9: Plot of 3D version of \( w(s) = \pm \sqrt{s} \) showing both \( \pm \)sheets and cut on \( x = 0 \) (p. 126).
Figure J.10: Plots of \( w(s) = s, w(s) = \sqrt{s} \) and \( w(s) = e^{\pi \cdot s} \sqrt{s} \) (p. 129).

Figure J.11: Plot of \( w(s) = \sqrt{\pi/s} \) (left) and \( w(s) = \sqrt{s^2 + 1} \) (right) (p. 130).
Figure J.12: Plot of $w(s) = \cos(\pi z)$ (right) and $w(z) = J_0(\pi z)$ (left) (p. 135).

Figure J.13: Plot of Bessel function $J_0(\pi z)$ and Hankel function $H_0^{(1)}(\pi z/2)$ (p. 136).
Figure J.14: Baffled conical horn (p. 155).

Figure J.15: Webster horn equation setup for derivation (p. 230).
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Chapter 2

Instructors manual: Number systems

2.1 Problems NS-1

Topic of this homework:

Introduction to MATLAB/OCTAVE (see the Matlab or Octave tutorial for help).

Deliverable: Report with charts and answers to questions. Hint: Use \LaTeX.¹

Plotting complex quantities in Octave/Matlab

Problem # 1: Consider the functions $f(s) = s^2 + 6s + 25$ and $g(s) = s^2 + 6s + 5$.

- 1.1: Find the zeros of functions $f(s)$ and $g(s)$ using the command \texttt{roots()}. 
  \textbf{Sol:} The roots of $f(s)$ are $-3 \pm 4i$ (in Matlab: \texttt{roots([1 6 25])}). The roots of $g(s)$ are $-1$ and $-5$ (in Matlab: \texttt{roots([1 6 5])}). You will find the program that generates all these figures at https://jontalle.web.engr.illinois.edu/uploads/298.17/NS1.m ■

- 1.2: Show the roots of $f(s)$ as red circles and of $g(s)$ as blue plus signs.
  The x-axis should display the real part of each root, and the y-axis should display the imaginary part. Use \texttt{hold on} and \texttt{grid on} when plotting the roots. \textbf{Sol:}

¹https://www.overleaf.com
1.3 Give your figure the title ‘Complex Roots of f(s) and g(s)’. Label the x- and y-axis ‘Real Part’ and ‘Imaginary Part.’

Hint: use xlabel and ylabel. Type ylim([-10 10]) and xlim([-10 10]), to expand the axes.

Problem # 2: Consider the function \( h(t) = e^{j2\pi ft} \) for \( f = 5 \) and \( t = [0:0.01:2] \).

- 2.1: Use subplot to show the real and imaginary parts of \( h(t) \)

Make two graphs in one figure. Label the x-axes ‘Time (s)’ and the y-axes ‘Real Part’ and ‘Imaginary Part’.

Sol: Breaking \( h(t) \) into real and imaginary parts gives \( e^{j2\pi 5t} = \cos(10\pi t) + j\sin(10\pi t) \).

- 2.2: Use subplot to plot the magnitude and phase parts of \( h(t) \).

Use the command angle or unwrap(angle()) to plot the phase. Label the x-axes ‘Time (s)’ and the y-axes ‘Magnitude’ and ‘Phase (radians)’. Sol:
2.1. PROBLEMS NS-1

Prime numbers, infinity, etc. in Octave/Matlab

Problem #3: Prime numbers, infinity, etc.

– 3.1: Use the Matlab/Octave function `factor` to find the prime factors of 123, 248, 1767, and 999,999.

Sol: Factors: 123 (3, 41), 248 (2,2,2,31), 1767 (3,19,31), 999999 (3,3,3,7,11,13,37)

– 3.2: Use the Matlab/Octave function `isprime` to check if 2, 3 and 4 are prime numbers. What does the function `isprime` return when a number is prime, or not prime? Why?

Sol: Function `isprime(2)` returns 1, `isprime(3)` returns 1, and `isprime(4)` returns 0. 1 means ‘yes’ and 0 means ‘no’.

– 3.3: Use the Matlab/Octave function `primes.m` to generate prime numbers between 1 and $10^6$.

Save them in a vector $x$. Plot this result using the command `hist(x)`. Sol:

```
0 2 4 6 8 10
x 10^5
0 200 400 600 800 1000 1200 1400
Primes between 1 and 10^6
Frequency of primes
Number
```

– 3.4: Now try `[n, bincenters] = hist(x)`. Use `length(n)` to find the number of bins. Sol: `length(n)` is 10.

– 3.5: Set the number of bins to 100 by using an extra input argument to the function `hist`. Show the resulting figure and give it a title and axes labels. Sol:

```
0 2 4 6 8 10
x 10^5
0 200 400 600 800 1000 1200 1400
Primes between 1 and 10^6
Frequency of primes
Number
```
Problem # 4: Inf, NaN and logarithms in Octave/Matlab

− 4.1: Try 1/0 and 0/0 in the Octave/Matlab command window.
What are the results? What do these ‘numbers’ mean in Octave/Matlab? Sol: 1/0 returns Inf (infinity) and 0/0 returns NaN (‘not a number’). ■

− 4.2: Try \( \log(0), \log(10(0)) \) and \( \log(2(0)) \) in the command window.
In Matlab/Octave, the natural logarithm \( \ln() \) is computed using the function \( \log \). Functions \( \log_{10} \), and \( \log_{2} \) are computed using \( \log(10) \) and \( \log(2) \). Sol: \( \log(0) \) is -Inf. Working in any base results in a scale factor, so the value does not change in these different bases. For example if \( 10^{x} = 2^{y} \) then \( x = \log_{10} 2^{y} = y \log_{10} 2 = 0.30103y \). ■

− 4.3: Try \( \log(1) \) in the command window. What you expect for \( \log(10(1)) \) and \( \log(2(1)) \)?
Sol: As with \( \log(0) \), changing base of \( \log(1) = 0 \) gives the same result, because scaling 0 always gives 0. ■

− 4.4: Try \( \log(-1) \) in the command window. What do you expect for \( \log(10(-1)) \) and \( \log(2(-1)) \)?
Sol: From Matlab/Octave \( \log(-1) = i\pi \). For the answer to the two other questions, see the next problems. ■

− 4.5: Show how Matlab/Octave arrives at the above answer because \( -1 = e^{i\pi} \).
Sol: \( \log(-1) \) is 0 + \( i\pi \), because \( \ln(-1) = \ln(e^{i\pi}) = i\pi \ln(e) = i\pi \). For base 10 let \( e^{x} = 10^{y} \) and \( y = \log_{10} e^{x} \). Thus \( \log(-1) = \log_{10}(e^{i\pi}) \). Likewise \( \log_{2}(-1) = \pi i \log_{2}(e) = 4.532i \). ■

− 4.6: Try \( \log(e^{\exp(j*sqrt(\pi))}) \) (i.e., \( \log e^{j\sqrt{\pi}} \)) in the command window. What do you expect?
Sol: \( \log e^{j\sqrt{\pi}} = j\sqrt{\pi} = 1.7725j \). because \( \ln(\cdot) \) is the inverse of \( e(\cdot) \). ■

− 4.7: What does inverse mean in this context? What is the inverse of \( \ln f(x) \)?
Sol: \( \ln f(x) = e^{\ln f(x)} \). Conclusion: \( e^{G} \) and \( \ln G \) are mutual inverses: that is: \( \ln(\cdot) \) of \( e^{(\cdot)} \) and \( e^{(\cdot)} \) of \( \ln(\cdot) \).
Or said another way: \( G = e^{\ln(G)}, \ln G = e^{G} \). ■

− 4.8: What is a decibel? (Look up decibels on the internet.)
Sol: The decibel is very important in engineering (and unused in mathematics). It is defined as the log of a power ratio. If a power ratio is 2, the dB value is 6 [dB]. A ratio of 10 is 20 [dB]. Thus the formula for the dB-ref is \( 10 \log_{10} \frac{P}{P_{ref}} \). The decibel is defined on the log (i.e., ratio) scale. Engineers quickly learn to “think” in dB units, because its so easy (once they learn to think in terms of ratios).
While the definition is in terms of power, it is practice, is almost always used in terms of voltage, pressure, current, velocity, (force and flux), etc. Of course power is the product of force and flux, and the log of the power is the sum of the log of the force plus the log of the flux. ■

Problem # 5: Very large primes on Intel computers

− 5.1: Find the largest prime number that can be stored on an Intel 64 bit computer, which we call \( \pi_{max} \).
Hint: As explained in the Matlab/Octave command \( \text{help flintmax} \), the largest positive integer is \( 2^{32} \), however the largest integer that can be factored is \( 2^{32} = \sqrt{2^{64}} \). Explain the logic of your answer.
Hint: help isprime(). Sol: Using Matlab/Octave, start with the largest integer \( 2^{32} \) and check if its prime. Then work down by subtracting 1, and again check. Stop when you get to the first prime below the largest integer. The answer I get is: \( 2^{32} - 5 = 4, 294, 967, 291 \), is the first prime below \( 2^{32} \) prime. ■
Problem #6: Suppose you are interested in primes that are greater than $\pi_{\text{max}}$. How can you find them on an Intel computer (i.e., one using IEEE-floating point)?

6.1: Thus consider a sieve containing only odd numbers, starting from 3 (not 2).

Hint 1: Since every prime number greater than 2 is odd, there is no reason to check the even numbers. $n_{\text{odd}} \in \mathbb{N}/2$ contain all the primes other than 2. Sol: At this time, I don’t see any way to do this, due to the Matlab/Octave limitation that it cannot factor numbers larger that $2^{32}$. ■

Problem #7: The following identity is interesting:

\[
\begin{align*}
1 & = 1^2 \\
1 + 3 & = 2^2 \\
1 + 3 + 5 & = 3^2 \\
1 + 3 + 5 + 7 & = 4^2 \\
1 + 3 + 5 + 7 + 9 & = 5^2 \\
\vdots
\end{align*}
\]

\[
\sum_{n=0}^{N-1} 2n + 1 = N^2.
\]

7.1: Can you find a proof?²

Sol: Subtracting any line from the line following it, gives:

\[
\begin{align*}
(1 - 1) + 3 & = 2^2 - 1^2 \\
5 & = 3^2 - 2^2 \\
7 & = 4^2 - 3^2 \\
9 & = 5^2 - 4^2 \\
\vdots
\end{align*}
\]

\[
\sum_{n=0}^{N-1} 2n + 1 - \sum_{n=0}^{N-2} 2n + 1 = N^2 - (N - 1)^2
\]

\[
2N - 1 = N^2 - (N^2 - 2N + 1)
\]

\[
2N - 1 = 2N - 1.
\]

Thus the two sides are equal, as suggested by the above formula.

Can you find a simpler more constructive “proof?” Hint: assuming you know what integration by parts is, can you devise a concept called Summation by parts? ■

²This problem came from an exam problem for Math 213, Fall 2016.
2.2 Problems NS-2

Topic of this homework:
Prime numbers, greatest common divisors, the continued fraction algorithm
Deliverable: Answers to questions.

Prime numbers

Problem # 1: Every integer may be written as a product of primes.

– 1.1: Put the numbers 1,000,000, 1,000,004 and 999,999 in the form \( N = \prod_{k} \pi_{k}^{\beta_{k}} \) (Hint: Use Matlab/Octave to find the prime factors).
Sol: 1,000,000 = 2^6 · 5^6
1,000,004 = 2^2 · 53^2 · 89
999,999 = 3^3 · 7 · 11 · 13 · 37

– 1.2: Give a generalized formula for the natural logarithm of a number, \( \ln(N) \), in terms of its primes \( \pi_{k} \) and their multiplicities \( \beta_{k} \). Express your answer as a sum of terms.
Sol: \( \ln N = \sum_{k} \beta_{k} \ln(\pi_{k}) \)

Problem # 2: Using the computer

– 2.1: Explain why the following brief Matlab/Octave program returns the prime numbers \( \pi_{k} \) between 1 and 100.

\begin{verbatim}
n=2:100;
k = isprime(n);
n(k)
\end{verbatim}
Sol: The first line \( n = 2 : 100 \) defines the row vector \( n = [2, 3, 4, \ldots, 100] \). The second line creates a row vector the same length as \( n \), with entries of 1 if the element is prime and zero if the element is not prime. The third line \( n(k) \) prints out \( n() \) if \( k = 1 \), namely it is a list of all the primes from 2 to 100. Run this program without the ‘;’ at the end of each line, and to see what it is doing.

– 2.2: How many primes are there between 2 and \( N = 100 \)?
Sol: \( \text{length}(n(k)) \) returns 25. Thus there are 25 primes less than 100 (\( N/4 \), on average).

Problem # 3: Prime numbers may be identified using a ‘sieve.’

– 3.1: By hand, perform the sieve of Eratosthenes for \( n = 1 \ldots 49 \). Circle each prime \( p \), then cross out each number which is a multiple of \( p \).
Sol: Note: 1 should not be circled as it is not a prime.
– 3.2: What is the largest number you need to consider before only primes remain?
Sol: $\lceil \sqrt{50} \rceil = \lceil 7.0711 \rceil = 7. ■

– 3.3: Generalize: for $n = 1 \ldots N$, what is the highest number you need to consider before only the primes remain?
Sol: $\text{floor}(\sqrt{N}) ■

– 3.4: Write each of these numbers as a product of primes:
$22 = \text{Sol: } 2 \cdot 11 = \pi_1 \pi_5 ■$
$30 = \text{Sol: } 2 \cdot 3 \cdot 5 = \pi_1 \pi_2 \pi_3 ■$
$34 = \text{Sol: } 2 \cdot 17 = \pi_1 \pi_7 ■$
$43 = \text{Sol: } \pi_{14} ■$
$44 = \text{Sol: } 4 \cdot 11 = \pi_1^2 \pi_5 ■$
$48 = \text{Sol: } 4 \cdot 12 = 4^2 \cdot 3 = \pi_1^4 \pi_2 ■$
$49 = \text{Sol: } 7^2 = \pi_2^2 ■$

– 3.5: Find the largest prime $\pi_k \leq 100$? Hint: Do not use Matlab/Octave other than to check your answer. Hint: Write out the numbers starting with 100 and counting backwards: 100, 99, 98, 97, · · · . Cross off the even numbers, leaving 99, 97, 95, · · · . Pull out a factor (only 1 is necessary to show that it is not prime).
Sol: 99 = 11 \cdot 9, \pi_{25} = 97. ■

– 3.6: Find the largest prime $\pi_k \leq 1000$? Hint: Do not use Matlab/Octave other than to check your answer.
Sol: Write out the numbers starting with 1000 and counting backwards: 1000, 999, 998, 997, · · · . Cross off the even numbers, leaving 999, 997, 995, · · · . Pull out a factor (only 1 is necessary to show that it is not prime). 9 \cdot 111, 997 = \pi_{168}, 5 \cdot 199 = \pi_3 \cdot \pi_{46}. ■

– 3.7: Explain why $\pi_k^{-a} = e^{-a \ln \pi_k}$.
Sol: This follows from the identity $z^a = e^{a \ln z}$ with $a, z \in \mathbb{C}$. ■

Greatest common divisors

Consider the Euclidean algorithm to find the greatest common divisor (GCD; the largest common prime factor) of two numbers. Note this algorithm may be performed using one of two methods:

<table>
<thead>
<tr>
<th>Method</th>
<th>Division</th>
<th>Subtraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>On each iteration...</td>
<td>$a_{i+1} = b_i$</td>
<td>$a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$</td>
</tr>
<tr>
<td></td>
<td>$b_{i+1} = a_i - b_i \cdot \text{floor}(a_i/b_i)$</td>
<td>$b_{i+1} = \min(a_i, b_i)$</td>
</tr>
<tr>
<td>Terminates when...</td>
<td>$b = 0 \ (\gcd = a)$</td>
<td>$b = 0 \ (\gcd = a)$</td>
</tr>
</tbody>
</table>

The division method (Eq. 2.1, §2.1.2, Ch. 2) is preferred because the subtraction method is much slower.

**Problem # 4: Understanding the Euclidean (GCD) algorithm**
- 4.1: Use the Octave/Matlab command \texttt{factor} to find the prime factors of $a = 85$ and $b = 15$.

\textbf{Sol:} From Octave’s \texttt{factor()} we find $85 = 17 \cdot 5$, $15 = 3 \cdot 5$. ■

- 4.2: What is the greatest common prime factor of these two numbers?

\textbf{Sol:} The largest common factor $\gcd(85, 15)$ is $5$. ■

- 4.3: By hand, perform the Euclidean algorithm for $a = 85$ and $b = 15$.

\textbf{Sol:} Division method:

\[
\begin{align*}
a_1 &= 15 & b_1 &= 85 - 15 \left\lfloor \frac{85}{15} \right\rfloor = 10 \\
a_2 &= 10 & b_2 &= 15 - 10 \left\lfloor \frac{15}{10} \right\rfloor = 5 \\
a_3 &= 5 & b_3 &= 10 - 5 \left\lfloor \frac{10}{5} \right\rfloor = 0 \\
\therefore \quad \gcd &= 5
\end{align*}
\]

\textbf{Subtraction method:}

\[
\begin{align*}
a_1 &= 85 - 15 = 70 & b_1 &= 15 \\
a_2 &= 70 - 15 = 55 & b_2 &= 15 \\
a_3 &= 55 - 15 = 40 & b_3 &= 15 \\
a_4 &= 40 - 15 = 25 & b_4 &= 15 \\
a_5 &= 25 - 15 = 10 & b_5 &= 15 \\
\text{swap} & & \\
a_6 &= 15 - 10 = 5 & b_6 &= 10 \\
a_7 &= 10 - 5 = 5 & b_7 &= 5 \\
\therefore \quad \gcd &= 5 \quad \blacksquare
\end{align*}
\]

- 4.4: By hand, perform the Euclidean algorithm for $a = 75$ and $b = 25$. Is the result a prime number?

\textbf{Sol:} Division method:

\[
\begin{align*}
a_1 &= 25 & b_1 &= 75 - 25 \left\lfloor \frac{75}{25} \right\rfloor = 0 \\
\therefore \quad \gcd &= 25
\end{align*}
\]

\textbf{Subtraction method:}

\[
\begin{align*}
a_1 &= 75 - 25 = 50 & b_1 &= 25 \\
a_2 &= 50 - 25 = 25 & b_2 &= 25 \\
\therefore \quad \gcd &= 25
\end{align*}
\]

The result is $25 = 5^2$, the \textit{square} of a prime number. ■

- 4.5: Consider the first step of the GCD division algorithm when $a < b$ (e.g. $a = 25$ and $b = 75$). What happens to $a$ and $b$ in the first step? Does it matter if you begin the algorithm with $a < b$ vs. $b < a$?

\textbf{Sol:} If $a < b$, the first step of the division algorithm swaps the terms ($a \rightarrow b$ and $b \rightarrow a$). ■
2.2. PROBLEMS NS-2

- 4.6: Describe in your own words how the GCD algorithm works. Try the algorithm using numbers which have already been separated into factors (e.g. \( a = 5 \cdot 3 \) and \( b = 7 \cdot 3 \)).

**Sol:** Division method:

\[
\begin{align*}
a_1 &= 5 \cdot 3 & b_1 &= 7 \cdot 3 - 5 \cdot 3 \left\lfloor \frac{7 \cdot 3}{5 \cdot 3} \right\rfloor &= 2 \cdot 3 \\
a_2 &= 2 \cdot 3 & b_2 &= 5 \cdot 3 - 2 \cdot 3 \left\lfloor \frac{5 \cdot 3}{2 \cdot 3} \right\rfloor &= 1 \cdot 3 \\
a_3 &= 1 \cdot 3 & b_3 &= 2 \cdot 3 - 1 \cdot 3 \left\lfloor \frac{2 \cdot 3}{1 \cdot 3} \right\rfloor &= 0
\end{align*}
\]

Subtraction method:

\[
\begin{align*}
a_1 &= 7 \cdot 3 - 5 \cdot 3 = 2 \cdot 3 & b_1 &= 5 \cdot 3 \\
a_2 &= 5 \cdot 3 - 2 \cdot 3 = 3 \cdot 3 & b_2 &= 2 \cdot 3 \\
a_3 &= 3 \cdot 3 - 2 \cdot 3 = 1 \cdot 3 & b_3 &= 2 \cdot 3 \\
a_4 &= 2 \cdot 3 - 1 \cdot 3 = 1 \cdot 3 & b_4 &= 1 \cdot 3
\end{align*}
\]

The algorithm iteratively converges on the GCD by subtracting out multiples of the GCD until only the GCD is left.

- 4.7: Find the GCD of \( 2 \cdot \pi_{25} \) and \( 3 \cdot \pi_{25} \).

**Sol:** \( \pi_{25} \).

**Problem # 5: Coprimes**

1. Define the term **coprime**. **Sol:** when two integers have no common factors they are said to be **coprime**

2. How can the Euclidean algorithm be used to identify coprimes? **Sol:** If \( \gcd(a, b) = 1 \) they only have 1 as a common factor, thus they are coprime.

3. Give at least one application of the Euclidean algorithm. **Sol:** Given two integers \( n, d \in \mathbb{Z} \), if we wish to reduce the fraction \( n/d \), we must cancel the common factors. Example: If \( n = 9, d = 6 \) then \( 9/6 = (3 \cdot 3)/(2 \cdot 3) = 3/2 \), where the GCD, 3, may be identified using the Euclidean algorithm. While this fraction may be easily simplified via inspection, the GCD algorithm could be very helpful for larger numbers \( n, d \).

- 5.1: Write a Matlab function, **function x = my_gcd(a, b)**, which uses the Euclidean algorithm to find the GCD of any two inputs \( a \) and \( b \). Test your function on the \( (a,b) \) combinations from parts (a) and (b). Include a printout (or handwrite) your algorithm to turn in.

**Hints and advice:**

- Don’t give your variables the same names as Matlab functions! Since \( \text{gcd} \) is an existing Matlab/Octave function, if you use it as a variable or function name, you won’t be able to use \( \text{gcd}() \) to check your \( \text{gcd}() \) function. Try \texttt{clear all} to recover from this problem.

- Try using a ‘while’ loop for this exercise (see Matlab documentation for help).

- You may need to make some temporary variables for \( a \) and \( b \) in order to perform the algorithm.

**Sol:** Division method:

```matlab
function x = my_gcd(a, b)
while b>0
```

\texttt{atmp = a; btmp = b;}
\texttt{a = btmp; b = atmp-btmp*floor(atmp/btmp);}
\texttt{end}

\textbf{Subtraction method:}
\begin{verbatim}
function x = mygcd(a,b)
    while a \neq b
        atmp = a; btmp = b;
        a = max(atmp,btmp) - min(atmp,btmp); b = min(atmp,btmp);
    end
end
\end{verbatim}

\textbf{Algebraic generalization of the GCD (Euclidean) algorithm}

\textbf{Problem \# 6:} In this problem we are looking for integer solutions \((m, n) \in \mathbb{Z}\) to the equations \(ma + nb = \gcd(a, b)\) and \(ma + nb = 0\) given positive integers \((a, b) \in \mathbb{Z}^+\).

Note that this requires that either \(m\) or \(n\) be negative. These solutions may be found using the Euclidean algorithm only if \((a, b)\) are coprime \((a \perp b)\). Note that integer (whole number) polynomial relations such as these are known as \textit{Diophantine equations.} The above equations are \textit{linear Diophantine equations}, possibly the simplest form of such relations.

\textbf{Example: }\texttt{gcd(2, 3) = 1):} For \((a, b) = (2, 3)\), the result is as follows:

\[
\begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & -2
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
2 \\
3
\end{bmatrix} = \begin{bmatrix}
-1 & 1 \\
3 & -2
\end{bmatrix} \begin{bmatrix}
2 \\
3
\end{bmatrix}
\]

Thus from the above equation we find the solution \((m, n)\) to the integer equation

\[2m + 3n = \gcd(2, 3) = 1,
\]

namely \((m, n) = (-1, 1)\) (i.e., \(-2+3 = 1\)). There is also a second solution \((3, -2)\) (i.e., \(3 \cdot 2-2 \cdot 3 = 0\)), which represents the terminating condition. Thus these two solutions are a pair and the solution only exists if \((a, b)\) are coprime \((a \perp b)\).

\textbf{Subtraction method:} This method is more complicated than the division algorithm, because at each stage we must check if \(a < b\). Define

\[
\begin{bmatrix}
a_0 \\
b_0
\end{bmatrix} = \begin{bmatrix}
a \\
b
\end{bmatrix}, \quad Q = \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix}, \quad S = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

where \(Q\) sets \(a_{i+1} = a_i - b_i\) and \(b_{i+1} = b_i\) assuming \(a_i > b_i\), and \(S\) is a \textit{\text{swap-matrix}} which swaps \(a_i\) and \(b_i\) if \(a_i < b_i\). Using these matrices, the algorithm is implemented by assigning

\[
\begin{bmatrix}
a_{i+1} \\
b_{i+1}
\end{bmatrix} = Q \begin{bmatrix}
a_i \\
b_i
\end{bmatrix} \text{ for } a_i > b_i, \quad \begin{bmatrix}
a_{i+1} \\
b_{i+1}
\end{bmatrix} = QS \begin{bmatrix}
a_i \\
b_i
\end{bmatrix} \text{ for } a_i < b_i.
\]

The result of this method is a cascade of \(Q\) and \(S\) matrices. For \((a, b) = (2, 3)\), the result is as follows:

\[
\begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
2 \\
3
\end{bmatrix} = \begin{bmatrix}
2 & -1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
2 \\
3
\end{bmatrix}.
\]

Thus we find two solutions \((m, n)\) to the integer equation \(2m + 3n = \gcd(2, 3) = 1\).

\textbf{– 6.1:} By inspection, find at least one integer pair \((m, n)\) that satisfies \(12m + 15n = 3\).

\textbf{Sol:} By inspection, \((m, n) = (-1, 1)\) is one solution.
Using matrix methods for the Euclidean algorithm, find integer pairs \((m, n)\) that satisfy \(12m + 15n = 3\) and \(12m + 15n = 0\). Show your work!!! Sol: Division method:

\[
\begin{bmatrix}
3 \\
0
\end{bmatrix} =
\begin{bmatrix}
-1 & 1 \\
5 & -4
\end{bmatrix}
\begin{bmatrix}
12 \\
15
\end{bmatrix}
\]

Subtraction method:

\[
\begin{bmatrix}
3 \\
3
\end{bmatrix} =
\begin{bmatrix}
-1 & 1 \\
4 & -3
\end{bmatrix}
\begin{bmatrix}
12 \\
15
\end{bmatrix}
\]

– 6.2: Does the equation \(12m + 15n = 1\) have integer solutions for \(n\) and \(m\)? Why, or why not? Sol: No, because \(\gcd(12, 15) = \gcd(3 \times 4, 3 \times 5) = 3\), not 1. Thus there are no Diophantine solutions to this equation.

**Problem # 7: Matrix approach:**

It can be difficult to keep track of the a’s and b’s when the algorithm has many steps. We need an alternative way to run the Euclidean algorithm, using matrix algebra. Matrix methods provide a more transparent approach to the operations on \((a, b)\). Thus the Euclidean algorithm can be classified in terms of standard matrix operations.

– 7.1: Write out the matrix approach, discussed at the end of §2.3.1 (Eq. 2.3.1.3, p. 44). Sol: Division method:

Define

\[
\begin{bmatrix}
a \\
b
\end{bmatrix}_0 =
\begin{bmatrix}
a_0 \\
b_0
\end{bmatrix},
\begin{bmatrix}
a \\
b
\end{bmatrix}_{i+1} =
\begin{bmatrix}
0 & 1 \\
1 & -\lfloor a/b \rfloor
\end{bmatrix}_i
\begin{bmatrix}
a \\
b
\end{bmatrix}_i
\]

**Continued fraction algorithm (CFA) (8 pts)**

**Problem # 8:CFA of ratios of large primes**

– 8.1: Starting from the primes below \(10^6\), form the CFA of \(\pi_j/\pi_k\) with \(j = 78498\) and \(k < j\).

Sol: First generate \(10^6\) primes with the matlab command \(\pi=\text{primes}(11+1e6)\).

The length of \(\pi\) is \(j = 78499\), \(\pi(j) = 1,000,003\), \(\pi(j - 1) = 999,983\) and \(\pi(j - 2) = 999,979\).

Let the target fraction be

\[
T = \frac{\pi(\text{end} - 1)}{\pi(\text{end} - 2)} = \frac{999983}{999979} = 1.000004000084002.
\]

Finding the CFA of \(T\) gives

\[\text{rat}(T) = 1 + 1/249995 = [1; 249995].\]

Factoring this integer gives \(\text.factor(249995) = 5 \times 49999\). ■

– 8.2: Look at other ratios of prime numbers and look for a pattern in the CFA of the ratios of large primes. What is the most obvious conclusion? Sol: The CFA terminates in only one term, as in the above example. ■
Chapter 2. Instructors Manual: Number Systems

8.3: (4pts) Expand $23/7$ as a continued fraction. Express your answer in bracket notation (e.g., $\pi = [3., 7, 16, \cdots]$). Show your work. Sol: $23/7 = (21 + 2)/7 = 3 + 2/7 = 3 + 1/(6 + 1)/2 = 3 + 1/(3 + 1/2)$. In bracket notation $23/7 = [3., 3, 2]$. Matlab gives $\text{rat}(23/7) = 3 + 1/(4 + 1/(-2))$, or $[1., 4, -2]$ because rounding $7/2$ can be taken as either $3 + 1/2$ or $4 - 1/2$. ■

8.4: (2pts) Can $\sqrt{2}$ be represented as a finite continued fraction? Why or why not? Sol: No, because it is irrational. ■

8.5: (2pts) What is the CFA for $\sqrt{2} - 1$? Hint: $\sqrt{2} + 1 = \frac{1}{\sqrt{2} - 1} = [2; 2, 2, 2, \cdots]$. Sol: $1 + \sqrt{2} = 2 + 1/(2 + 1/(2 + \cdots))$ or $[2., 2, 2, 2, \cdots]$, thus $\sqrt{2} - 1 = [2., 2, 2, 2, \cdots] - 2 = 0 + 1/(2 + 1/(2 + 1/(2 + \cdots)))).$ ■

8.6: Find the CFA for $1 + \sqrt{3}j$ Sol: $\text{rat}(1 + \text{sqrt}(3j)) = [2; 2, 2, 2, \cdots]$.

8.7: Show that $\frac{1}{1 - \sqrt{a}} = a + a^2 + a^3 + a^4 + a^5 + a^6 + a + 1 = 1 - a^6$

```matlab
syms a,b
b= taylor(1/( 1-sqrt(a) ))
simplify((1-sqrt(a))*b) = 1-a^6
```

Use symbolic analysis to show this, then explain. Sol: This is a taylor expansion of $1$ expressed in terms of removable singularities. See Cotes Theorem (1716) (Stillwell, 2010, p. 289). ■

Continued fractions

Problem #9: Here we explore the continued fraction algorithm (CFA), §2.3.2, (p. 45). In its simplest form the CFA starts with a real number, which we denote as $\alpha \in \mathbb{R}$. Let us work with an irrational real number, $\pi \in \mathbb{I}$, as an example, because its CFA representation will be infinitely long. We can represent the CFA coefficients $\alpha$ as a vector of integers $n_k$, $k = 1, 2 \cdots \infty$

$$\alpha = [n_1; n_2, n_3, n_4, \cdots] = n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \frac{1}{n_4 + \cdots}}}$$

As discussed in §2.3.1 (p. 42), the CFA is recursive, with three steps per iteration:

For $\alpha_1 = \pi$, $n_1 = 3$, $r_1 = \pi - 3$ and $\alpha_2 \equiv 1/r_1$.

$$\alpha_2 = 1/0.1416 = 7.0625 \cdots$$

$$\alpha_1 = n_1 + \frac{1}{\alpha_2} = n_1 + \frac{1}{n_2 + \frac{1}{\alpha_3}} = \cdots$$

In terms of a Matlab/Octave script
2.2. PROBLEMS NS-2

\[
\text{alpha0} = \pi; \\
K = 10; \\
n = \text{zeros}(1,K); \text{ alpha} = \text{zeros}(1,K); \\
\text{ alpha}(1) = \text{alpha0};
\]

for \( k = 2 : K \) \%k = 1 to K
  \( n(k) = \text{round}(\text{alpha}(k-1)); \)
  \( \text{n}(k) = \text{fix}(\text{alpha}(k-1)); \)
  \( \text{ alpha}(k) = 1 / (\text{alpha}(k-1) - n(k)); \)
  \( \text{disp}([\text{fix}(k), \text{round}(n(k)), \text{alpha}(k)]); \text{pause}(1) \)
end
\text{disp}([\text{n}; \text{ alpha}]);
\%
Now compare this to matlab’s rat() function
\text{rat(alpha0,1e-20)}

- 9.1: By hand (you may use Matlab/Octave as a calculator), find the first 3 values of \( n_k \) for \( \alpha = e^\pi \).
Sol: The CFA for this is: \( e^\pi = 23.1407 \cdots = [23; 7, 9, 4, \cdots]. \)

- 9.2: For part (1), what is the error (remainder) when you truncate the continued fraction after \( n_1, \ldots, n_3 \)? Give the absolute value of the error, and the percentage error relative to the original \( \alpha \).
Sol: The remainder is \( e^\pi - (23 + 1/(7 + (1/9))) \) which gives an error of \( \epsilon = |e^\pi - (23 + 1/(7 + (1/9)))|/e^\pi = 2.92 \cdot 10^{-6} = 0.0003\% \).

- 9.3: Use the Matlab/Octave program provided to find the first 10 values of \( n_k \) for \( \alpha = e^\pi \), and verify your result using the Matlab/Octave command \text{rat()}. 
Sol: \( e^\pi = 23.1407 \cdots = [23; 7, 9, 4, -2, -591, -2, -10, 3, -2, \cdots]. \)

- 9.4: Discuss the similarities and differences between the Euclidean algorithm (EA) and CFA.
Sol:
1. Both are recursive, meaning that the steps are repeated one after another.
2. The EA starts from two numbers (a,b). The output of the \text{gca}(a,b) is the GCD. The CFA starts with a single number and the output is a sequence of integers. If the sequence terminates the number was rational. If the sequence does not terminate, the number is irrational.
3. The EA works with the difference between the minimum and maximum of the two numbers whereas the CFA works with the rounding function and the reciprocal of the error.
4. It would seem that the goals of the two algorithms, the starting point, and the results are totally different. Both are very useful and powerful. Both generalize to more difficult situations than working with simple numbers.

- 9.5: Extra Credit Show that the CFA is the inverse operation (i.e., the CFA is the GCD, run in reverse) (Hint: see §2.3.1 (p. 42)).
Sol:
Starting from Eq. 2.3.1.2
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix}^{[\frac{m}{n}]} =
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -[\frac{m}{n}] \\
0 & +1
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
1 & -[\frac{m}{n}]
\end{bmatrix}.
\]
The matrix equation for the CFA is derived in §D, p. 217. We conclude that taking Eq. 2.3.1.2 to the ⌊mn⌋ power, and swapping rows, results in a CFA matrix. However I believe this must be iterated. It follows that the GCD and CFA are inverses because the matrix formulations are inverses.
2.3 Problems NS-3

Topic of this homework: Pythagorean triples, Pell’s equation, Fibonacci sequence

Deliverable: Answers to problems

Pythagorean triplets

**Problem #1:** Euclid’s formula for the Pythagorean triplets a, b, c is: \( a = p^2 - q^2 \), \( b = 2pq \), and \( c = p^2 + q^2 \).

1.1: What condition(s) must hold for \( p \) and \( q \) such that \( a, b, \) and \( c \) are always positive and nonzero?

Sol: \( p > q > 0 \) (strictly greater than)

1.2: Solve for \( p \) and \( q \) in terms of \( a, b \) and \( c \).

Sol:

**Method 1:** Given \( a, c \), one may find \( p, q \) via matrix operations by solving the nonlinear system of equations for \( p, q \).

First solve linear system of equations for \( p^2, q^2 \):

\[
\begin{bmatrix}
    a \\
    c
\end{bmatrix} = \begin{bmatrix}
    1 & -1 \\
    1 & 1
\end{bmatrix} \begin{bmatrix}
    p^2 \\
    q^2
\end{bmatrix}
\]

Inverting this 2x2 matrix gives (the determinant \( \Delta = 2 \))

\[
\begin{bmatrix}
    p^2 \\
    q^2
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
    1 & 1 \\
    -1 & 1
\end{bmatrix} \begin{bmatrix}
    a \\
    c
\end{bmatrix}.
\]

Thus \( p = \pm\sqrt{(a + c)/2}, q = \pm\sqrt{(c - a)/2}. \)

**Method 2:** The algebraic approach is:

\[
a + c = (p^2 - q^2) + (p^2 + q^2) = 2p^2
\]

\[
-a + c = -(p^2 - q^2) + (p^2 + q^2) = 2q^2
\]

Thus \( p = \sqrt{(a + c)/2}, q = \sqrt{(c - a)/2}, \) where \( p, q \in \mathbb{N}. \)

Method 1 seems more “transparent” than Method 2.

**Problem #2:** The ancient Babylonians (c2000BEC) cryptically recorded \( (a,c) \) pairs of numbers on a clay tablet, archeologically denoted Plimpton-322.
– 2.1: Find \( p \) and \( q \) for the first five pairs of \( a \) and \( c \) from the tablet entries. \textit{Table 1: First five \((a,c)\) pairs of Plimpton-322.}

<table>
<thead>
<tr>
<th>( a )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>169</td>
</tr>
<tr>
<td>3367</td>
<td>4825</td>
</tr>
<tr>
<td>4601</td>
<td>6649</td>
</tr>
<tr>
<td>12709</td>
<td>18541</td>
</tr>
<tr>
<td>65</td>
<td>97</td>
</tr>
</tbody>
</table>

Find a formula for \( a \) in terms of \( p \) and \( q \).

\[
(a, c) = (119, 169) \quad (p, q) = \pm (12, 5) \\
(a, c) = (3367, 4825) \quad (p, q) = \pm (64, 27) \\
(a, c) = (4601, 6649) \quad (p, q) = \pm (75, 32) \\
(a, c) = (12709, 18541) \quad (p, q) = \pm (125, 54) \\
(a, c) = (65, 97) \quad (p, q) = \pm (9, 4)
\]

– 2.2: Based on Euclid’s formula, show that \( c > (a, b) \).

\[
\text{Sol:} \quad c - a = (p^2 + q^2) - (p^2 - q^2) = 2q^2 \\
\text{Because} \ 2q^2 \text{ is always positive,} \ c > a \\
\text{Note that by the definition of} \ p, q \in \mathbb{N}, \ p > q. \]

– 2.3: What happens when \( c = a \)?

\[
\text{Sol:} \quad \text{Then it’s not a triangle since} \ b = 0. \ \text{The triangle is degenerate.}
\]

– 2.4: Is \( b + c \) a perfect square? Discuss.

\[
\text{Sol:} \quad b + c = p^2 + 2pq + q^2 = (p + q)^2. \ \text{Since} \ p \ \text{and} \ q \ \text{are integers,} \ b + c \ \text{will always be a perfect square} \\
\]

Pell’s equation:

\textbf{Problem # 3: Pell’s equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that} \( \sqrt{2} \in \mathbb{I} \). \textit{We seek integer solutions of}

\[
x^2 - Ny^2 = 1.
\]

As shown in §2.3.4 (p. 50) the solutions \( x_n, y_n \) for the case of \( N = 2 \) are given by the linear 2x2 matrix recursion

\[
\begin{bmatrix}
x_{n+1} \\
y_{n+1}
\end{bmatrix}
= 1J
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix}
\]

with \([x_0, y_0]^T = [1, 0]^T\) and \(1J = \sqrt{-1} = e^{\pi i/2}\). It follows that the general solution to Pell’s equation for \( N = 2 \) is

\[
\begin{bmatrix}
x_n \\
y_n
\end{bmatrix}
= (e^{\pi i/2})^n
\begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_0 \\
y_0
\end{bmatrix}
\]

To calculate solutions to Pell’s equation using the matrix equation above, we must calculate

\[
A^n = e^{\pi n i/2}
\begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix}
= e^{\pi n i/2}
\begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix}
\cdots
\begin{bmatrix}
1 & 2 \\
1 & 1
\end{bmatrix}
\]

which becomes tedious for \( n > 2 \), since it requires \( n \times 2 \times 2 \) matrix multiplications.
Diagonalization of a matrix ("eigenvalue/eigenvector decomposition"):

As derived in Appendix B, the most efficient way to compute $A^n$ is to diagonalize the matrix $A$, by finding its eigenvalues and eigenvectors.

The eigenvalues $\lambda_k$ and eigenvectors $\vec{e}_k$ of a square matrix $A$ are related by

$$ A\vec{e}_k = \lambda_k \vec{e}_k, \quad \text{(NS-3 2.1)} $$

such that multiplying an eigenvector $\vec{e}_k$ of $A$ by the matrix $A$ is the same as multiplying by a scalar, $\lambda_k \in \mathbb{C}$ (the corresponding eigenvalue). The complete eigenvalue problem may be written as

$$ AE = E\Lambda. $$

If $A$ is a $2 \times 2$ matrix, the matrices $E$ and $\Lambda$ (of eigenvectors and eigenvalues, respectively) are

$$ E = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. $$

Thus, the matrix equation $AE = \begin{bmatrix} A\vec{e}_1 & A\vec{e}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \vec{e}_1 & \lambda_2 \vec{e}_2 \end{bmatrix} = E\Lambda$ contains Eq. NS-3 2.1 for each eigenvalue-eigenvector pair.

The diagonalization of the matrix $A$ refers to the fact that the matrix of eigenvalues, $\Lambda$, has non-zero elements only on the diagonal. The key result is found by post-multiplication of the eigenvalue matrix by $E^{-1}$, giving

$$ AEE^{-1} = A = E\Lambda E^{-1}. \quad \text{(NS-3 2.2)} $$

If we now take powers of $A$, the $n^{th}$ power of $A$ is

$$ A^n = (E\Lambda E^{-1})^n $$

$$ = E\Lambda E^{-1} E\Lambda E^{-1} \cdots E\Lambda E^{-1} $$

$$ = E\Lambda^n E^{-1}. \quad \text{(NS-3 2.3)} $$

This is a very powerful result, because the $n^{th}$ power of a diagonal matrix is extremely easy to calculate:

$$ \Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}. $$

Thus, from Eq. NS-3 2.3 we can calculate $A^n$ using only two matrix multiplications

$$ A^n = E\Lambda^n E^{-1}. $$

Finding the eigenvalues:

The eigenvalues $\lambda_k$ are determined by Eq. NS-3 2.1, by factoring out $\vec{e}_k$

$$ A\vec{e}_k = \lambda_k \vec{e}_k $$

$$ (A - \lambda_k I) \vec{e}_k = \vec{0}. $$

Matrix $I = [1, 0; 0, 1]^T$ is the identity matrix, having the dimensions of $A$, with elements $\delta_{ij}$ (i.e., diagonal elements $\delta_{11,22} = 1$ and off-diagonal elements $\delta_{12,21} = 0$).

The vector $\vec{e}_k$ is not zero, yet when operated on by $A - \lambda_k I$, the result must be zero. The only way this can happen is if the operator is degenerate (has no solution), that is

$$ \det(A - \lambda I) = \det \begin{bmatrix} (a_{11} - \lambda) & a_{12} \\ a_{21} & (a_{22} - \lambda) \end{bmatrix} = 0. \quad \text{(NS-3 2.4)} $$

These concepts may be easily extended to higher dimensions.
This means that the two equations have the same roots (the equation is degenerate).

This determinant equation results in a second degree polynomial in \( \lambda \)

\[
(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0,
\]

the roots of which are the eigenvalues of the matrix \( A \).

**Finding the eigenvectors:**

An eigenvector \( \vec{e}_k \) can be found for each eigenvalue \( \lambda_k \) from Eq. NS-3 2.1,

\[
(A - \lambda_k I)\vec{e}_k = \vec{0}.
\]

The left side of the above equation becomes a column vector, where each element is an equation in the elements of \( \vec{e}_k \), set equal to 0 on the right side. These equations are always degenerate, since the determinant is zero. Thus the two equations have the same slope.

Solving for the eigenvectors is often confusing, because they have arbitrary magnitudes, \( ||\vec{e}_k|| = \sqrt{e_{k,1}^2 + e_{k,2}^2} = d \). From Eq. NS-3 2.1, you can only determine the relative magnitudes and signs of the elements of \( \vec{e}_k \), so you will have to choose a magnitude \( d \). It is common practice to normalize each eigenvector to have unit magnitude \( (d = 1) \).

- 3.1: Find the companion matrix, and thus the matrix \( A \), having the same eigenvalues as Pell’s equation.

*Hint: Use Matlab’s function \([E, \text{Lambda}] = \text{eig}(A)\) to check your results!*

*Sol:* The companion matrix is

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\]

- 3.2: Solutions to Pell’s equation were used by the Pythagoreans to explore the value of \( \sqrt{2} \). Explain why Pell’s equation is relevant to \( \sqrt{2} \).

*Sol:* As discussed §2.3.4 Chapter 2), as the iteration \( n \) increases, the ratio of the \( x_n/y_n \) approaches \( \sqrt{2} \).

- 3.3: Find the first 3 values of \((x_n, y_n)^T\) by hand and show that they satisfy Pell’s equation for \( N = 2 \).

*Sol:* See class notes (slide 9.4.2) for this calculation. • By hand, find the eigenvalues \( \lambda_{\pm} \) of the \( 2 \times 2 \) Pell’s equation matrix

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\]

*Sol:* The eigenvalues are given by the roots of the equation \( (1 - \lambda_\pm)^2 = 2 \). Thus \( \lambda_\pm = 1 \pm \sqrt{2} = \{2.1412, -0.4142\} \)

- 3.4: By hand, show that the matrix of eigenvectors, \( E \), is

\[
E = \begin{bmatrix} \vec{e}_+ & \vec{e}_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}
\]

*Sol:* The eigenvectors \( \vec{e}_\pm \) may be found by solving

\[
A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda_\pm \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \rightarrow (A - \lambda_\pm I) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0
\]
For $\lambda_+$, this gives

$$0 = \begin{bmatrix} 1 - (1 + \sqrt{2}) & 2 \\ 1 & 1 - (1 + \sqrt{2}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

which gives the relation between the elements of $\vec{e}_+, e_1, e_2$, as $e_1 = \sqrt{2}e_2$.

The eigenvectors are defined to be unit length and orthogonal, namely

1. $||\vec{e}_k||^2 = \vec{e}_k \cdot \vec{e}_k = 1$
2. $\vec{e}_+ \cdot \vec{e}_- = 0$.

Once we normalize $\vec{e}_+$ to have unit length, we obtain the first eigenvector

$$\vec{e}_+ = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

Repeating this for $\lambda_-$ gives

$$\vec{e}_- = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

Thus, the matrix of eigenvalues is

$$E = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$

---

3.5: Using the eigenvalues and eigenvectors you found for $A$, verify that

$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

Sol: Using the formula for a matrix inverse, we find

$$E^{-1} = \frac{1}{\det(E)} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} = \frac{3}{-2\sqrt{2} \cdot \sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} = -\frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}$$

Thus

$$E^{-1}AE = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$

---

3.6: Now that you have diagonalized $A$ (Equation NS-3 2.3), use your results for $E$ and $\Lambda$ to solve for the $n = 10$ solution $(x_{10}, y_{10})^T$ to Pell’s equation with $N = 2$.

Sol: $x_{10} = -3363$ and $y_{10} = -2378$. Note this formulation gives the negative solution, but since the values for $n = 10$ are real, when they are squared in Pell’s equation, it makes no difference whether they are negative or positive.
The Fibonacci sequence

The Fibonacci sequence is famous in mathematics, and has been observed to play a role in the mathematics of genetics. Let \( x_n \) represent the Fibonacci sequence,

\[
x_{n+1} = x_n + x_{n-1},
\]

where the current input sample \( x_n \) is equal to the sum of the previous two inputs. This is a ‘discrete time’ recurrence relation. To solve for \( x_n \), we require some initial conditions. In this exercise, let us define \( x_0 = 1 \) and \( x_{n<0} = 0 \). This leads to the Fibonacci sequence \( \{1, 1, 2, 3, 5, 8, 13, \ldots\} \) for \( n = 0, 1, 2, 3, \ldots \).

Equation NS-3 2.5 is equivalent to the \( 2 \times 2 \) matrix equation

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
\]

**Problem #4:** Here we seek the general formula for \( x_n \). Like the Pell’s equation, Eq. NS-3 2.5 has a recursive, eigen analysis solution. To find it we must recast \( x_n \) as a \( 2 \times 2 \) matrix relation, and then proceed as we did for the Pell case.

- **4.1:** By example, show that the Fibonacci sequence \( x_n \) as described above may be generated by

\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

- **4.2:** What is the relationship between \( y_n \) and \( x_n \)?

**Sol:** This equation says that \( x_n = x_{n-1} + y_{n-1} \) and \( y_n = x_{n-1} \). The latter equation may be rewritten as \( y_n = x_{n-1} \). Thus

\[
x_n = x_{n-1} + x_{n-2},
\]

which is Eq. NS-3 2.5.

- **4.3:** Write a Matlab/Octave program to compute \( x_n \) using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is \( x_{40} \)? Note: Consider using the eigen analysis of \( A \), described by Eq. NS-3 2.3 (p. 17).

**Sol:** You can try something like:

```matlab
function xn = fib(n)
A = [1 1; 1 0]; [E,D] = eig(A); xy = E*D^n*inv(E)*[1; 0];
xn = xy(1);
```

Given the initial conditions we defined, \( x_{40} = 165,580,141 \).

- **4.4:** Using the eigen analysis of the matrix \( A \) (and a lot of algebra), it is possible to obtain the general formula for the Fibonacci sequence

\[
x_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].
\]

- **4.5:** What are the eigenvalues \( \lambda_+ \) of the matrix \( A \)?

**Sol:** The eigenvalues of the Fibonacci matrix are given by

\[
det \begin{bmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0,
\]

thus \( \lambda_\pm = \frac{1 \pm \sqrt{5}}{2} = [1.618, -0.618] \).
– 4.6: How is the formula for \( x_n \) related to these eigenvalues? Hint: find the eigen vectors.

**Sol:** The eigenvectors (determined from the equation \((A - \lambda \pm I)e = 0\), and normalized to 1) are given by

\[
\vec{e}_+ = \begin{bmatrix} \frac{\lambda_+}{\sqrt{\lambda_+^2 + 1}} \\ \frac{1}{\sqrt{\lambda_+^2 + 1}} \end{bmatrix} \quad \vec{e}_- = \begin{bmatrix} \frac{\lambda_-}{\sqrt{\lambda_-^2 + 1}} \\ \frac{1}{\sqrt{\lambda_-^2 + 1}} \end{bmatrix} \quad E = [\vec{e}_+ \quad \vec{e}_-]
\]

From the eigen analysis, we find that

\[
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = E \begin{bmatrix} \lambda_+ \\ 0 \\ 0 \\ \lambda_- \end{bmatrix} E^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} \lambda_+ \\ 0 \\ 0 \\ \lambda_- \end{bmatrix} \frac{1}{e_{11}e_{22} - e_{12}e_{21}} \begin{bmatrix} e_{22} \\ -e_{21} \\ e_{11} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Solving for \( x_n \) we find that

\[
x_n = \frac{1}{(e_{11}e_{22} - e_{12}e_{21})} \left( \lambda_+ e_{11}e_{22} - \lambda_- e_{12}e_{21} \right)
\]

\[
= \frac{1}{\sqrt{\lambda_+^2 + 1}\lambda_- + \lambda_-^2 + \lambda_-} \lambda_+ \left( \frac{\lambda_+}{\sqrt{(\lambda_+^2 + 1)(\lambda_-^2 + 1)}} \right) - \lambda_- \left( \frac{\lambda_-}{\sqrt{(\lambda_+^2 + 1)(\lambda_-^2 + 1)}} \right)
\]

\[
= \frac{1}{\sqrt{5}} \left( \lambda_+^2 - \lambda_-^2 \right)
\]

– 4.7: What happens to each of the two terms 

\([(1 \pm \sqrt{5})/2]^n + 1? \) **Sol:** \([(1 - \sqrt{5})/2]^n \to 0 \) and \([(1 + \sqrt{5})/2]^n \to \infty \)

– 4.8: What happens to the ratio \( x_{n+1}/x_n \)?

**Sol:** \( x_{n+1}/x_n \to (1 + \sqrt{5})/2 \), because \( (1 - \sqrt{5})/2 \to 0 \) as \( n \to \infty \) thus for large \( n \), \( x_n \approx [(1 + \sqrt{5})/2]^n \).

**Problem #5:** Replace the Fibonacci sequence with

\[ x_n = \frac{x_{n-1} + x_{n-2}}{2}, \]

such that the value \( x_n \) is the average of the previous two values in the sequence.

– 5.1: What matrix \( A \) is used to calculate this sequence?

**Sol:**

\[ A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \]

– 5.2: Modify your computer program to calculate the new sequence \( x_n \). What happens as \( n \to \infty \)?

**Sol:** As \( n \to \infty \), \( x_n \to 2/3 \)

– 5.3: What are the eigenvalues of your new \( A \)? How do they relate to the behavior of \( x_n \) as \( n \to \infty \)? Hint: you can expect the closed-form expression for \( x_n \) to be similar to Eq. NS-3 2.8.

**Sol:** The eigenvalues are \( \lambda_+ = 1 \) and \( \lambda_- = -0.5 \). From Eq. NS-3 2.3, the expression for \( A^n \) is

\[ A^n = (E\Lambda E^{-1})^n = E\Lambda^n E^{-1} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}. \]
The solution is the sum of two sequences, one a constant and the other an oscillation that quickly fades. As \( n \to \infty, \lambda_+^n = 1^n \to 1 \) and \( \lambda_-^n = (-1/2)^n \to 0 \). The solution becomes

\[
x_n = \frac{2}{3} \left[ \lambda_+^n - \lambda_-^n \right] = \frac{2}{3} [1^n - (-1)^n] \to \frac{2}{3}.
\]

5.4: What matrix \( A \) is used to calculate this sequence? 

Sol: 

\[
A = \begin{bmatrix} 1 & 0.01 \\ 1 & 0 \end{bmatrix}
\]

5.5: Modify your computer program to calculate the new sequence \( x_n \). What happens as \( n \to \infty \)? 

Sol: As \( n \to \infty, x_n \to \infty \)  

5.6: What are the eigenvalues of your new \( A \)? How do they relate to the behavior of \( x_n \) as \( n \to \infty \)? Hint: you can expect the closed-form expression for \( x_n \) to be similar to Eq. NS-3 2.8.

Sol: The eigenvalues are \( \lambda_+ = 1.0033 \) and \( \lambda_- = -0.5033 \). As \( n \to \infty, \lambda_+^n \to \infty \) and \( \lambda_-^n \to 0 \). Because \( \lambda_+^n \) ‘blows up,’ the expression for \( x_n \) also ‘blows up.’

Problem # 6: Consider the expression

\[
\sum_{k=1}^{N} f_k^2 = f_N f_{N+1}.
\]

6.1: Find a formula for \( f_n \) that satisfies this relationship. Hint: It only holds for the Fibonacci recursion formula.

Sol: Write this out for \( N \) and \( N - 1 \):

\[
\begin{align*}
    f_1^2 + f_2^2 + \cdots + f_{N-1}^2 + f_N^2 &= f_N f_{N+1} \\
    f_1^2 + f_2^2 + \cdots + f_{N-1}^2 &= f_{N-1} f_N
\end{align*}
\]

Subtracting gives

\[
\begin{align*}
    f_N^2 &= f_N f_{N+1} - f_{N-1} f_N = f_N (f_{N+1} - f_{N-1}) \\
    f_N &= f_{N+1} - f_{N-1}
\end{align*}
\]

Thus the relation only holds for the Fibonacci recursion formula.

CFA as a matrix recursion

Problem # 7: The CFA may be written as a matrix recursion. For this we adopt a special notation, unlike other matrix notations,\(^5\) with \( k \in \mathbb{N} \)

\[
\begin{bmatrix} n \\ x \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & [x_k] \\ 0 & x_{k-1} - [x_k] \end{bmatrix} \begin{bmatrix} n \\ x \end{bmatrix}_k.
\]

\(^4\) I found this problem on a worksheet for Math 213 midterm (213practice.pdf).

\(^5\) This notation is highly non-standard, due to the nonlinear operations. The matrix elements are derived from the vector rather than multiplying them. These calculation may be done with the help of Matlab/Octave.
This equation says that \( n_{k+1} = \lfloor x_k \rfloor + 1 \) and \( x_{k+1} = 1/(x_k - \lfloor x_k \rfloor) \). It does not mean that \( n_{k+1} = \lfloor x_k \rfloor x_k \), as would be implied by standard matrix notation. The lower equation says that \( r_k = x_k - \lfloor x_k \rfloor \) is the remainder, namely \( x_k = \lfloor x-k \rfloor + r_k \) (Octave/Matlab’s \( \text{rem}(x, \text{floor}(x)) \) function), also known as \( \text{mod}(x, y) \).

- 7.1: Start with \( n_0 = 0 \in \mathbb{N}, x_0 \in \mathbb{I}, n_1 = \lfloor x_0 \rfloor \in \mathbb{N}, r_1 = x - \lfloor x \rfloor \in \mathbb{I} \) and \( x_1 = 1/r_1 \in \mathbb{I}, r_n \neq 0 \). For \( k = 1 \) this generates on the left the next CFA parameter \( n_2 = \lfloor x_1 \rfloor \) and \( x_2 = 1/r_2 = 1/(x_0 - \lfloor x_0 \rfloor) \) from \( n_0 \) and \( x_0 \).

Find \([n, x]_k^{T+1}\) for \( k = 2, 3, 4, 5 \). Sol: If \( x_0 = \pi \), then \( n_1 = \lfloor \pi \rfloor = 3, r_1 = \pi - n_1 = 0.14159 \cdots \) and \( x_1 = 1/r_1 \approx 7.06: \)

\[
\begin{bmatrix}
3 \\
7.06251
\end{bmatrix}_{1} = 
\begin{bmatrix}
0 \\
0
\end{bmatrix} 
\begin{bmatrix}
\lfloor \pi \rfloor \\
\pi — \lfloor \pi \rfloor
\end{bmatrix}_{0}
\]

and for \( n = 2 \)

\[
\begin{bmatrix}
7 \\
15.99659
\end{bmatrix}_{2} = 
\begin{bmatrix}
7 \\
1.006251
\end{bmatrix}_{1} = 
\begin{bmatrix}
0 \\
0
\end{bmatrix} 
\begin{bmatrix}
7 \\
1 — 7.06251
\end{bmatrix}_{0}
\]

For \( n = 3, \pi_3 = [n_1; n_2, n_3] = [3; 7, 15] \). Continuing \( n_4 = [1.003418] = 1 \) and \( n_5 = 292 \). ■
Chapter 3

Algebraic Equations

3.1 Problems AE-1

**Topic of this homework:** Fundamental theorem of algebra, polynomials, analytic functions and their inverse, convolution, Newton’s root finding method, Riemann zeta function.

**Deliverable:** Answers to problems

**Note:** The term ‘analytic’ is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

**Polynomials and the fundamental theorem of algebra (FTA)**

**Problem # 1: A polynomial of degree $N$ is defined as**

$$P_N(x) = a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N$$

– 1.1: How many coefficients $a_n$ does a polynomial of degree $N$ have?
**Sol:** $N + 1$

– 1.2: How many roots does $P_N(x)$ have?
**Sol:** $N$

**Problem # 2: The fundamental theorem of algebra (FTA)**

– 2.1: State and then explain the Fundamental Theorem of Algebra.

**Sol:** The FTA says that every polynomial has at least one root $x = x_r$.

– 2.2: Using the FTA, prove your answer to the question (2) above.

**Hint:** Apply the FTA to prove how many roots a polynomial $P_N(x)$ of order $N$ has. **Sol:** When a root is determined, it may be factored out, leaving a new polynomial of degree one less than the first. Specifically

$$P_{N-1}(x) = \frac{P_N(x)}{x - x_r}.$$

Thus it follows that by a recursive application of this theorem, a polynomial has a number of roots equal to its degree. All the roots must be counted, including repeated and complex roots, and roots at $\infty$. 

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Problem #3: Consider the polynomial function \( P_2(x) = 1 + x^2 \) of degree \( N = 2 \), and the related function \( F(x) = 1/P_2(x) \).

- 3.1: What are the roots (e.g. ‘zeros’) \( x_\pm \) of \( P_2(x) \)?

Hint: Complete the square on the polynomial \( P_2(x) = 1 + x^2 \) of degree 2, and find the roots. Sol: Solving for the roots by setting \( P_2(x) = 0 \) gives \( x_\pm^2 = -1 \) leading to \( x_\pm = \pm 1 \).

Problem #4: \( F(x) \) may be expressed as \( (A, B, x_\pm \in \mathbb{C}) \)

\[
F(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-},
\]

where \( x_\pm \) are the roots (zeros) of \( P_2(x) \), which become the poles of \( F(x) \), and \( A, B \) are the residues. The expression for \( F(x) \) is sometimes called a ‘partial fraction expansion’ or ‘residue expansion,’ and it appears in many engineering applications.

- 4.1

Find \( A, B \in \mathbb{C} \) in terms of the roots \( x_\pm \) of \( P_2(x) \).

Sol: The fastest (i.e., easiest) way to find the constants \( A, B \) is to cross-multiply

\[
\frac{1}{1 + x^2} = \frac{A(x - x_-) + B(x - x_+)}{(x - x_+)(x - x_-)} = \frac{(A + B)x - (Ax_- + Bx_+)}{(x - x_+)(x - x_-)},
\]

Since the numerator must equal 1, \( B = -A \) and \( A = 1/(x_+ - x_1) \).

In summary, in terms of the roots of Eq. AE-1.1

\[
A = -B = \frac{1}{(x_+ - x_-)}, \quad \text{thus} \quad F(x) = \frac{1}{1 + x^2} = \frac{1}{2j} \left( \frac{1}{x - 1j} - \frac{1}{x + 1j} \right).
\]

Verify your answers for \( A, B \) by showing that this expression for \( F(x) \) is indeed equal to \( 1/P_2(x) \). Sol: This is easily verified by cross-multiplying and simplifying. In the numerator the \( x \) terms cancel and Eq. AE-1.1 is recovered.

- 4.2

Give the values of the poles and zeros of \( P_2(x) \). Sol: The zeros are at \( x_z = \pm j \), and the poles are at \( x_p = \pm \infty \).

- 4.3

Give the values of the poles and zeros of \( F(x) = 1/P_2(x) \). Sol: The poles are at \( x_p = \pm j \), and the zeros are at \( x_z = \pm \infty \).

Analytic functions

Overview: Analytic functions are defined by infinite (power) series. The function \( f(x) \) is analytic at any value of \( x = x_0 \) where there exists a convergent power series

\[
P(x) = \sum_{n=0}^{\infty} a_n x^n
\]

such that \( P(x_0) = f(x_0) \). The local power series for \( f(x) \) near \( x = x_0 \) is often obtained by finding the Taylor series:

\[
f(x) \approx f(x_0) + \frac{df}{dx} \bigg|_{x=x_0} (x - x_0) + \frac{1}{2!} \frac{d^2 f}{dx^2} \bigg|_{x=x_0} (x - x_0)^2 + \ldots
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n} \bigg|_{x=x_0} (x - x_0)^n.
\]
The point $x = x_0$ is called the series expansion point.

When the expansion point is at $x_0 = 0$, the series is denoted a MacLaurin series. Two classic examples are the geometric series\(^1\) where $a_n = 1$

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \quad \text{(AE-1.2)}
\]

and the exponential function where $a_n = 1/n!$

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \quad \text{(AE-1.3)}
\]

The coefficients for both series may be derived from the Taylor formula (or MacLaurin formula, when the expansion point is zero).

**Problem # 5: The geometric series**

- **5.1:** What is the region of convergence (RoC) for the power series of $1/(1 - x)$ given above (e.g. where does the power series $P(x)$ converge to the function value $f(x)$)? State your answer as a condition on $x$.

  **Hint:** What happens to the power series when $x > 1$?

  **Sol:** $|x| < 1$, because for $|x| \geq 1$ the power series diverges to infinity.

- **5.2:** In terms of the pole, what is the RoC for the geometric series (Eq. AE-1.2)?

  **Sol:** The nearest pole relative to the expansion point, at $x = 0$ is at the nearest pole $x_p = 1$ to the expansion point at $x = 0$. Namely the RoC is $1 \text{ re } 0$.

- **5.3:** How does the RoC relate to the location of the pole of $1/(1 - x)$?

  **Sol:** The pole is at $x = 1$, on the border of the RoC. The nearest pole relative to the expansion point, at $x = 0$ is at $x = 1$. Thus the RoC is $1$.

- **5.4:**

  Where are the zeros, if any, in Eq. AE-1.2? **Sol:** There is a single zero at $x = \infty$.

- **5.5:** Assuming $x$ is in the RoC, prove that the geometric series correctly represents $1/(1 - x)$ by multiplying both sides of Eq. AE-1.2 by $(1 - x)$.

  **Sol:**

  \[
  1 = \frac{1 - x}{1 - x} = \frac{1}{1 - x} - x(1 + x + x^2 + \ldots), \quad |x| < 1
  \]

  \[
  = (1 - x)(1 + x + x^2 + x^3 \ldots), \quad |x| < 1
  \]

  \[
  = (1 + x + x^2 + x^3 \ldots) - x(1 + x + x^2 \ldots)
  \]

  \[
  = (1 + x + x^2 + x^3 \ldots) - (x + x^2 + x^3 \ldots)
  \]

  \[
  = (1 + (x - x) + (x^2 - x^2) + (x^3 - x^3) \ldots), \quad |x| < 1
  \]

  \[
  = 1
  \]

  The introduction of a removable singularity voids the solution at $x = 1$. When one expands the denominator in a series, the relation is good for $|x| < 1$ (not for $x = 1$).

  If one lets $z = 1/x$ the relation becomes

  \[
  1 = \frac{1 - z}{1 - z},
  \]

  which is valid for $z \neq 1$, which when expanded the RoC is $|z| < 1$, or $x > 1$. \[\text{\textcopyright} \]

\(^1\)The geometric series is not defined as the function $1/(1 - x)$, it is defined as the series $1 + x + x^2 + x^3 + \ldots$, such that the ratio of consecutive terms is $x$.\[\text{\textcopyright} \]
5.6: Use the geometric series to study the degree \( N \) polynomial (It is very important to note that all the coefficients of this polynomial are)

\[
P_N(x) = 1 + x + x^2 + \ldots + x^N = \sum_{n=0}^{N} x^n. \tag{AE-1.4}
\]

5.7: Prove that

\[
P_N(x) = \frac{1 - x^{N+1}}{1 - x} \tag{AE-1.5}
\]

Sol:

\[
P_N(x) = 1 + x + x^2 \ldots x^N
\]

\[
= \sum_{n=0}^{\infty} x^n - \sum_{n=N+1}^{\infty} x^n
\]

\[
= \sum_{n=0}^{\infty} x^n - x^{N+1} \sum_{n=0}^{\infty} x^n
\]

\[
= (1 - x^{N+1}) \sum_{n=0}^{\infty} x^n
\]

\[
= \frac{1 - x^{N+1}}{1 - x}
\]

5.8: What is the RoC for Eq. AE-1.3?

Sol: This series converges everywhere (The RoC is the entire plane \(|z| < \infty\)), due to the strong convergence effect of the factorial (\(a_n = 1/n!\)) of the coefficients of the series.

5.9: what is the RoC for Eq. AE-1.4?

Sol: There is no pole, thus the RoC is \(\infty\). The polynomial only has zeros.

5.10: What is the RoC for Eq. AE-1.5?

Sol: The pole nearest the expansion point is at \(x = 1\). Thus the RoC is 1.

5.11: Evaluate \(P_N(x)\) at \(x = 0\) and \(x = .9\) for the case of \(N = 100\), and compare the result to that from Matlab.

Sol: \(P_N(0) = 1\) and \(P_N(0.9) = \frac{1 - 0.9^{N+1}}{1 - 0.9} = 9.999760947410010\). According to Matlab \(P_{100}(0) = 1\) and \(P_{100}(0.9) = 9.999760947410014\), with a difference of \(-3.55271 \times 10^{-15}\) (i.e., -16\times\text{eps}).

\[
\text{clear all;close all;format long}
\text{N=100; x=0.9; S=0;}
\text{for n=0:N}
\text{S=S+x^n}
\text{end}
\text{P100=(1-x^*(N+1))/(1-x)};
\text{disp(sprintf('S= %g, P100= %g, error= %g',S,P100, S-P100))}
\]

5.12: How many poles does \(P_N(x)\) have? Where are they?

Sol: Since \(P_N(x)\) is defined by Eq. AE-1.4, there is no poles at \(x = 1\). However it still has a pole of order \(N\) at \(x = \infty\). To show this, define \(z = 1/x\) and study the zeros.

5.13: How many zeros does \(P_N(x)\) have? State where are they in the complex plane?

Sol: There are zeros at \(x = \frac{N+1}{\sqrt{2}} = e^{2\pi i/(N+1)}\). Total = \(N + 1\) zeros. However, the zero at 1 is removable because it is on top of the pole at 1. This is refered to as a removable singularity.
3.1. PROBLEMS AE-1

– 5.14: Does Eq. AE-1.4 have both poles and zeros? Explain.
Sol: Written in this way, every \( N \)th degree polynomial, with \( N \) zeros, has a single pole and \( N + 1 \) zeros.

– 5.15: Explain why Eq. AE-1.4 and AE-1.5 have different numbers of poles and zeros.
Sol: Eq. AE-1.5 has \( N + 1 \), known as the roots of unity, which exactly cancel the pole. Exactly how this happens is very interesting. This is similar to the GCD problem we saw previously where common factors in a rational number may be removed, but in this case, it is a common root that cancels.

– 5.16: Is the function \( \frac{1}{1-x} \) analytic outside of the RoC stated in part (a)?
Hint: Can it be represented by a different power series outside this RoC?
Sol: Yes, and we can use the geometric series to prove this. Consider \( x = \frac{1}{r} > 1 \), meaning \( r < 1 \).

\[
\frac{1}{1-x} = \frac{-r}{1-r} = -r \sum_{n=0}^{\infty} r^n = -\sum_{n=0}^{\infty} r^n = -\sum_{n=1}^{\infty} \frac{1^n}{x}
\]

Problem # 6 The exponential series.

– 6.1: What is the region of convergence (RoC) for the exponential series given above (e.g. where does the power series \( P(x) \) converge to the function value \( f(x) \))?
Sol: The exponential is convergent everywhere on the open real line.

– 6.2: Let \( x = j \) in Eq. AE-1.3, and write out the series expansion of \( e^x \) in terms of its real and imaginary parts.
Sol:

\[
e^j = \sum_{n=0}^{\infty} \frac{j^n}{n!}
= 1 + j - \frac{1}{2!} - j \frac{1}{3!} + \frac{1}{4!} + j \frac{1}{5!} - \frac{1}{6!} \cdots
= \left( 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \right) + j \left( \frac{1}{3!} - \frac{1}{5!} + \frac{1}{7!} - \cdots \right)
= \sum_{n=0,2,\ldots}^{n \text{ even}} \frac{(-1)^n}{n!} + j \sum_{n=1,3,\ldots}^{n \text{ odd}} \frac{(-1)^n}{n!}.
\]

– 6.3: Let \( x = j \theta \) in Eq. AE-1.3, and write out the series expansion of \( e^x \) in terms of its real and imaginary parts. How does your result relate to Euler’s identity \( (e^{j\theta} = \cos(\theta) + j \sin(\theta)) \)?
Sol:

\[
e^{j\theta} = \sum_{n=0}^{\infty} \frac{j^n \theta^n}{n!}
= 1 + j \theta - \frac{\theta^2}{2!} - j \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j \frac{\theta^5}{5!} - j \frac{\theta^6}{6!} \cdots
= \left( \cos \theta \right) + j \left( \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots \right)
= \cos(\theta) + j \sin(\theta).
\]
Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function \((x, y \in \mathbb{C})\)

\[ y(x) = \frac{1}{1 - x} \]

the inverse is

\[ x = \frac{y - 1}{y} = 1 - \frac{1}{y}. \]

Problem #7: Considering the inverse function described above

- 7.1: Where are the poles and zeros of \(x(y)\)?
  Sol: Pole at \(y = 0\), zero at \(y = 1\). There are no poles or zeros at \(\infty\) because \(\lim_{y \to \pm \infty} (y - 1)/y = 1\)

- 7.2: Where (for what condition on \(y\)) is \(x(y)\) analytic?
  Sol: Anywhere but the pole, at \(y = 0\).

Problem #8 Considering the exponential function \(z(x) = e^x\) \((x, z \in \mathbb{C})\).

- 8.1: Find the inverse \(x(z)\).
  Sol: Taking the natural log \((\ln)\) of both sides gives \(x = \ln(z)\). Thus the natural log is the inverse of the exponential.

- 8.2: Where are the poles and zeros of \(x(z)\)?
  Sol: Pole at \(z = 0\), zero at \(z = 1\). There is another pole at \(z = +\infty\) as well.

- 8.3: If \(y(x) = 1/(1 - x)\) and \(z(x) = e^x\), compose these two functions to obtain \((y \circ z)(x)\).
  Give the expression for \((y \circ z)(x) = y(z(x))\). Sol:

\[ (y \circ z)(x) = \frac{1}{1 - e^x} \]

- 8.4: Where are the poles and zeros of \((y \circ z)(x)\)?
  Sol: Pole at \(x = 0\), zero at \(x = +\infty\).

- 8.5: Where (for what condition on \(x\)) is \((y \circ z)(x)\) analytic?
  Sol: Everywhere except \(x = 0\).

Convolution

Multiplying two polynomials, when they are short or simple, is not demanding. However if they have many terms, it can become tedious. For example, multiplying two 10\(^{th}\) degree polynomials is not something one would want to do every day.

An alternative is a method called convolution, as described in §3.3 (p. 79).

Problem #9: Convolution of sequences. Practice convolution (by hand!!) using a few simple examples. Show you work!!! Check your solution using Matlab.

- 9.1: Convolve the sequence \(\{0 1 1 1 1\}\) with itself.
  Sol: \(\{0 0 1 2 3 4 3 2 1\}\).
9.2: Calculate \( \{1, 1\} \ast \{1, 1\} \ast \{1, 1\} \)

Sol:

\[ \{1, 1\} \ast \{1, 2, 1\} = \{1, 3, 3, 1\} \]

Problem #10: Multiplying two polynomials is the same as convolving their coefficients.

\[ f(x) = x^3 + 3x^2 + 3x + 1 \]
\[ g(x) = x^3 + 2x^2 + x + 2 \]

10.1: In Octave/Matlab, compute \( h(x) = f(x) \cdot g(x) \) two ways.

Use: (a) the commands roots and poly, and (b) the convolution command conv. Confirm that both methods give the same result. Sol: \( h(x) = [1, 3, 3, 1] \ast [1, 2, 1, 2] \).

10.2: What is \( h(x) \)?

Sol: \( h(x) = x^6 + 5x^5 + 10x^4 + 12x^3 + 11x^2 + 7x + 2 \).

Newton’s root-finding method

Problem #11: Use Newton’s iteration to find roots of the polynomial

\[ P_3(x) = 1 - x^3 \]

11.1: Draw a graph describing the first step of the iteration starting with \( x_0 = (1/2, 0) \).

Sol: Start with an \((x, y)\) coordinate system and put points at and the vertex of \( P_3(x) \).

11.2: Calculate \( x_1 \) and \( x_2 \). What number is the algorithm approaching?

Sol: First we must find \( P_3'(x) = -3x^2 \). Thus the equation we must iterate is Eq. 3.1.2.14 (p. 61)

\[ x_{n+1} = x_n + \frac{1 - x_n^3}{3x_n^2} \]

Given a first guess for the root \( x_0 \), the following are \( x_1 = x_0 + \frac{1 - x_0^3}{3x_0^2} \) and \( x_2 = x_1 + \frac{1 - x_1^3}{3x_1^2} \). Note that if \( x + 0 \) is the root, \( x_1 = x_0 \) and we are done. However if \( x_0 = 0 \), \( x_1 = \infty \) since \( x_0 = 0 \) is a root of \( P_3'(x) \). Thus one must not start at the roots of \( P_3'(x) = 0 \).

11.3: Here is an Octave/Matlab script for the \( P_2(x) \) case. Modify it to find \( P_3(x) \):

```octave
x(1)=1/2; %x(1)=0.9; %x(1)=-10
y(1)=x(1);
for n=2:10
    x(n) = x(n-1) + (1-x(n-1)^2)/(2*x(n-1));
    y(n) = (1+y(n-1)^2)/(2*x(n-1));
end
semilogy(abs(x)-1); hold on
semilogy(abs(7)-1,'or'); hold off
```

Sol:

```octave
x=1/2;
for n = 1:3
    x = x+(1-x*x*x)/(3*x*x);
end
```
– 11.4: For $n = 4$, what is the absolute difference between the root and the estimate, $|x_r - x_4|$?

**Sol:** 4.6E-8 (very small!) ■

– 11.5: What happens if $x_0 = -1/2$?

**Sol:** The solution converges to the root $x = 1$ in one step. ■

– 11.6: Does Newton’s method work for $P_2(x) = 1 + x^2$?

If so, why? Hint: What are the roots in this case?

**Sol:** Here $P_2'(x) = +2x$ thus the iteration gives

$$x_{n+1} = x_n - \frac{1 + x_n^2}{2x_n}.$$  

In this case the roots are $x_\pm = \pm 1\jmath$, namely purely imaginary. If we start with a real number for $x_0$, and use real arithmetic, obviously Newton’s method fails, because there is no way for the answer to become complex. Real in, real out. ■

– 11.7: What if you let $x_0 = (1 + \jmath)/2$ for the case of $P_2(x) = 1 + x^2$?

**Sol:** By starting with a complex initial value, we fix the Real in = Real out problem. ■

3.1.1 **Riemann zeta function** $\zeta(s)$

**Definitions and preliminary analysis:**

The zeta function $\zeta(s)$ is defined by the complex analytic power series

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.$$  \hspace{1cm} (AE-1.6)

This series converges, and thus is valid, only in the region of convergence (ROC) given by $\Re s = \sigma > 1$ since there $|n^{-\sigma}| \leq 1$. To determine its formula in other regions of the $s$ plane one must extend the series via analytic continuation.

**Euler product formula:** As was first published by Euler in 1737, one may recursively factor out the leading prime term, resulting in Euler’s product formula.\(^2\) Multiplying $\zeta(s)$ by the factor $1/2^s$, and subtracting from $\zeta(s)$, removes all the terms $1/(2n)^s$ (e.g., $1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots$)

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots\right),$$  \hspace{1cm} (AE-1.7)

which results in

$$\left(1 - \frac{1}{2^s}\right) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots.$$  \hspace{1cm} (AE-1.8)

**Problem #12:**

– 12.1: What is the RoC for Eq. AE-1.6

**Sol:** $|3^s| > 1$. This is an example of analytic continuation of the initial series. ■

– 12.2: What is the RoC for Eq. AE-1.8

**Sol:** $|5^s| > 1$. Thus we extended the RoC even further. ■

\(^2\)This is known as Euler’s sieve, as distinguish from the Eratosthenes sieve.
3.1. PROBLEMS AE-1

- 12.3: Repeat the algebra of Eq. AE-1.7 using the lead factor \( 1/3^s \).

\[
\frac{1}{3^s} \left( 1 - \frac{1}{2^s} \right) \zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \cdots \tag{AE-1.9}
\]

Subtracting Eq. AE-1.8 from Eq. AE-1.6 cancels the RHS terms of Eq. AE-1.6

\[
\left( 1 - \frac{1}{3^s} \right) \left( 1 - \frac{1}{2^s} \right) \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} + \cdots
\]

- 12.4: Repeat the algebra of Eq. AE-1.7 for all prime scale factors (i.e., \( 1/5^s, 1/7^s, \cdots, 1/\pi_k^s, \cdots \)) to show that

\[
\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1 - \pi_k^{-s}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s), \tag{AE-1.10}
\]

where \( \pi_p \) represents the \( p^{th} \) prime.

**Sol:**
The above defines each factor \( \zeta_k(s) \) as the \( k^{th} \) term of the product. Each recursive step in this construction assures that the lead term, along with all of its multiplicative factors, are subtracted out.

- 12.5: Given the product formula we may identify the poles of \( \zeta_p(s) \) \( (p \in \mathbb{Z}) \), which is important for defining the ROC of each factor.

For example, the \( p^{th} \) factor of Eq. AE-1.10, expressed as an exponential, is

\[
\zeta_p(s) \equiv \frac{1}{1 - \pi_p^{-s}} = \frac{1}{1 - e^{-sT_p}}, \tag{AE-1.11}
\]

where \( T_p \equiv \ln \pi_p \). **Sol:**
Thus factor \( \zeta_p(s) \) has poles at \( s_n(p) \) where \( 2\pi j n = s_n T_p \), giving

\[
s_n(p) = \frac{2\pi j n}{\ln \pi_p}
\]

with \(-\infty < n \in \mathbb{C} < \infty \). With each step the RoC is larger, resulting in an analytic function having its RoC approaching \( \infty \). These poles might be viewed as the eigen-modes of the zeta function.

- 12.6: Plot \( \zeta_p(s) \) using zviz for \( p = 1 \). Describe what you see.

**Sol:** \( \zeta_1(s) \) has poles at integral multiples of \( T_1 = \log 2 \), as shown below.

\[
\text{Figure 3.1: Plot of } w(s) = \frac{1}{1 - e^{-sT_1}} \text{ which is related to each factor } \zeta_p(s) \text{ (Eq. AE-1.10). Here } w_k(s) \text{ has poles where } 1 = e^{-sT_k} \text{, namely at } s_n = n2\pi j, \text{ as may be seen from the colorized plot.}
\]

\[3\text{Each factor (i.e., } \zeta_p(s) \text{) has poles at } s_n = j2\pi n/T_p, \ n \in \mathbb{C} \text{ (i.e., } e^{-sT_p} = 1 \text{).}\]
3.2 Problems AE-2

Topic of this homework:
Linear systems of equations; Gaussian elimination; Matrix permutations; Overspecified systems of equations; Analytic geometry; Ohm’s law; Two-port networks
Deliverable: Answers to problems

Nonlinear (quadratic) to linear equations

Problem # 1: Nonlinear (quadratic) to linear equations
In the following problems we deal with algebraic equations in more than one variable, that are not linear equations. For example, the circle \( x^2 + y^2 = 1 \) may be solved for \( y(x) = \pm \sqrt{1 - x^2} \).

Example: If we let \( z = x + yj = x + j\sqrt{1 - x^2} = e^{j\theta} \), we obtain the equation for half a circle \((y > 0)\). The entire circle is described by the magnitude of \( z \), as \(|z|^2 = (x + y)(x - y) = 1\).

– 1.1: Given the curve defined by the equation:

\[
x^2 + xy + y^2 = 1
\]

– 1.2: Find the function \( y(x) \).
Sol: Completing the square in \( y \) and solve for \( y(x) \):

\[
(y + x/2)^2 - x^2/4 + x^2 = 1
\]

\[
(y + x/2)^2 = 1 - \frac{3}{4}x^2
\]

\[
y + x/2 = \pm \sqrt{\frac{4 - 3x^2}{4}}
\]

\[
y = \frac{1}{2} \left(\pm \sqrt{4 - 3x^2} - x\right)
\]

– 1.3: Using Matlab/Octave, plot \( y(x) \), and describe the graph.
Sol:

Thus we find the equation is a rotated ellipse.
3.2. PROBLEMS AE-2

– 1.4: What is the name of this curve?
Sol: It is an ellipse, rotated by 45 degrees. ■

– 1.5: Find the solution (in $x$, $p$, and $q$) to the following equations:

\[
\begin{align*}
    x + y &= p \\
    xy &= q
\end{align*}
\]

Sol: Solve the first equation for $y$ as $y = p - x$, and then substitute it into the second equation

\[
x(p - x) = -x^2 + px = q.
\]

Thus we find the quadratic

\[
x^2 - px + q = 0
\]

having roots given by completing the square

\[
(x - p/2)^2 = (p/2)^2 - q.
\]

resulting in

\[
x = p/2 \pm \sqrt{(p/2)^2 - q},
\]

$y = p - x$.

Summary: Here we started with one linear and one quadratic (hyperbola). By the use of composition we found the roots. ■

– 1.6: Find an equation that is linear in $y$ starting from equations that are quadratic (2nd degree) in the two unknowns $x$, $y$:

\[
\begin{align*}
    x^2 + xy + y^2 &= 1 \\
    4x^2 + 3xy + 2y^2 &= 3
\end{align*}
\]

Sol: The goal is to obtain a linear equation in $y$.

Method 1: remove $xy$ term: Scale the upper equation by 3 and subtract from the lower:

\[
\begin{align*}
    4x^2 + 3xy + 2y^2 &= 3 \\
    3x^2 + 3xy + 3y^2 &= 3
\end{align*}
\]

giving $x^2 - y^2 = 0$, or $x = \pm y$.

This results in the two equations

\[
\begin{align*}
    x^2 &= 0 \\
    y^2 &= 0
\end{align*}
\]

Adding these gives $2x^2 + x^2 = 1$, which is $3x^2 = 1$ and $x^2 = 1$. Thus the final solutions are $x = \pm y = \pm 1/\sqrt{3}$ and $x = \pm y = \pm 1$.

– 1.7: Compose the two quadratic equations

\[
\begin{align*}
    x^2 + xy + y^2 &= 1 \\
    2x^2 + xy &= 1
\end{align*}
\]

and describe the results. Sol: By isolating $y$ from one of the two equations, we may remove it from the other equation, giving us a single 4th degree equation in $x$:

\[
x^2 + (1 - 2x^2) + (1 - 2x^2)^2 / x^2 = 1
\]
or

\[ x^4 + x^2 - 2x^4 + 1 - 4x^2 + 4x^4 - x^2 = 0 \]

Collecting terms

\[ 3x^4 - 4x^2 + 1 = 0 \]

This is a quartic, but is a quadratic in \( x^2 \). Of course \( x \) may be complex, rendering this very difficult to deal with in any detail. **Conclusion:** We started with two ellipses (they have an \( xy \) term which can be removed by a rotation, as we showed in Problem 1.1) This again demonstrates that composition of \( m \times n \) gives degree of \( mn \). When one multiplies polynomials the degree is \( m + n \). Thus composition gives the product and multiplication gives the sum of the degrees of the individual polynomials. ■

**Intersection and analytic geometry**

To derive Euclid’s formula, it was necessary to intersect a circle and a secant line. Consider the unit circle of radius 1, centered at \((x, y) = (0, 0)\)

\[ x^2 + y^2 = 1 \]

and the secant line through \((-1, 0)\)

\[ y = t(x + 1) \]

having slope \( t \) and intercept \( x = -1 \). If the slope \( 0 < t < 1 \), the line intersects the circle at a second point \((a, b)\) in the positive \( x, y \) quadrant. The goal is to find \( a, b \in \mathbb{N} \) and then show that \( c^2 = a^2 + b^2 \). Since the construction gives a right triangle with short sides \( a, b \in \mathbb{N} \), then it follows that \( c \in \mathbb{N} \).

1) \( t = \frac{p}{q} \in \mathbb{Q} \)
2) \( a = p^2 - q^2 \)
3) \( b = 2pq \)
4) \( c = p^2 + q^2 \)

**Problem #2: Derive Euclid’s formula**

2.1: **Draw the circle and the line, given a positive slope \( 0 < t < 1 \).**

**Sol:** Sol in given in Fig. 3.2 ■
3.2. PROBLEMS AE-2

**Problem # 3: Substitute** \( y = t(x + 1) \) (the line equation) into the equation for the circle, and solve for \( x(t) \).

*Hint: Because the line intersects the circle at two points, you will get two solutions for \( x \). One of these solutions is the trivial solution \( x = -1 \).*  
*Sol: \( x(t) = (1 - t^2)/(1 + t^2) \) ■

- **3.1: Substitute** the \( x(t) \) you found back into the line equation, and solve for \( y(t) \).  
*Sol: \( y(t) = 2t/(1 + t^2) \) ■

- **3.2: Let** \( t = q/p \) be a rational number, where \( p \) and \( q \) are integers. Find \( x(p, q) \) and \( y(p, q) \).  
*Sol: \( x(p, q) = 2pq/(p^2 + q^2) \) and \( y(p, q) = (p^2 - q^2)/(p^2 + q^2) \) ■

- **3.3: Substitute** \( x(p, q) \) and \( y(p, q) \) into the equation for the circle, and show how Euclid’s formula for the Pythagorean triples is generated.  
*Sol: Multiplying out gives \( p^2 + q^2 = (p^2 - q^2) + 2pq \) ■

For full points you must show that you understand the argument. Explain the mean of the comment “magic happens” when \( t^4 \) cancels.

**Gaussian elimination**

**Problem # 4: Gaussian elimination**

- **4.1: Find the inverse of**  
\[
A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.
\]
*Sol:  
\[
A^{-1} = \frac{1}{3 - 8} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}.
\]

- **4.2: Verify that** \( A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).  
*Sol: Multiply them to show this. ■

**Problem # 5: Find the solution to the following 3x3 matrix equation** \( Ax = b \) by Gaussian elimination. Show your intermediate steps. You can check your work at each step using Octave/Matlab.

\[
\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.
\]

- **5.1 Show** (i.e., verify) that the first GE matrix \( G_1 \), which zeros out all entries in the first column, is given by  
\[
G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.
\]
Identify the elementary row operations that this matrix performs. **Sol:** Operate with GE matrix on $A$

\[
G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 & 1 \\
3 & 1 & 1 & 9 \\
1 & -1 & 4 & 8
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & -2 & 4 & 6 \\
0 & -2 & 5 & 7
\end{bmatrix}
\]

It scales the first row by -3 and adds it to the second row, and scales the first row by -1 and adds it to the third row. ■

– 5.2 Find a second GE matrix, $G_2$, to put $G_1A$ in upper triangular form. Identify the elementary row operations that this matrix performs.

**Sol:**

\[
G_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

which scales the second row by -1 and adds it to the third row. Thus we have

\[
G_2G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 & 1 \\
3 & 1 & 1 & 9 \\
1 & -1 & 4 & 8
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & -2 & 4 & 6 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

– 5.3 Find a third GE matrix, $G_3$, which scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.

**Sol:**

\[
G_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

which scales the second row by -1/2. Thus we have

\[
G_3G_2G_1[A|b] = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -1 & 1 \\
3 & 1 & 1 & 9 \\
1 & -1 & 4 & 8
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 1 & -2 & -3 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

– 5.4: Finally, find the last GE matrix, $G_4$, that subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix $(G_4G_3G_2G_1A = I)$.

**Sol:**

\[
G_4 = \begin{bmatrix}
1 & -1 & -1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\]
Thus we find $G_4 G_3 G_2 G_1 [A|b]$ is

\[
\begin{bmatrix}
1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 1 \\
0 & 1 & 1/2 & 0 & 0 & 1 & 0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

\[= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\]

- 5.5: Solve for $\{x_1, x_2, x_3\}^T$ using the augmented matrix format $G_4 G_3 G_2 G_1 \{A|b\}$ (where $\{A|b\}$ is the augmented matrix). Note that if you've performed the preceding steps correctly, $x = G_4 G_3 G_2 G_1 b$.

Sol: From the preceding problems, we see that $[x_1, x_2, x_3]^T = [3, -1, 1]^T$.

- 5.6: Find the pivot matrix $G$ that rescales the second row of the augmented matrix $A|b$ by $1/3$.

Sol:

\[
G_1 = \begin{bmatrix} 1 \\ 1/3 \\ 1 \end{bmatrix}
\]

Proceeding

\[
G_1 A = \begin{bmatrix} 1 \\ 1/3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1/3 & 1/3 & 3 \\ 1 & -1 & 4 & 8 \end{bmatrix}
\]

Two linear equations

Problem #6 In this exercise we transition from a general pair of equations

\[
f(x, y) = 0 \\
g(x, y) = 0
\]

to the important case of two linear equations

\[
y = ax + b \\
y = \alpha x + \beta.
\]

Note that, to help keep track of the variables, Roman coefficients $(a, b)$ are used for the first equation and Greek $(\alpha, \beta)$ for the second.

- 6.1: What does it mean, graphically, if these two linear equations have

1. a unique solution,
2. a non-unique solution, or
3. no solution?

Sol: There are three possibilities:
1. When they have different slopes, they meet at one (x,y) point, which is the solution.

2. If the two lines are identical, any point on the line is a solution.

3. If they have the same slope but different intercepts (are parallel to each other) there is no solution.

6.2: Assuming the two equations have a unique solution, find the solution for x and y.

Sol: Since there must be one point where the two are equal, we may solve for that by setting the y values equal to each other:

\[ ax + b = \alpha x + \beta \]

Thus

\[
\begin{align*}
x &= \frac{\beta - b}{a - \alpha} \\
y &= a \frac{\beta - b}{a - \alpha} + b
\end{align*}
\]

6.3: When will this solution fail to exist (for what conditions on a, b, \( \alpha \), and \( \beta \))?

Sol: As stated above, if they have the same slope \( \alpha = a \) but different intercepts \( \beta \neq b \), there is no solution. When \( \beta = b \) and \( \alpha = a \) every point on the line is a solution.

6.4: Write the equations as a 2x2 matrix equation of the form \( A\vec{x} = \vec{b} \), where \( \vec{x} = \{x, y\}^T \).

Sol:

\[
\begin{bmatrix}
1 & -a \\
1 & -\alpha
\end{bmatrix}
\begin{bmatrix}
y \\
x
\end{bmatrix}
= 
\begin{bmatrix}
b \\
\beta
\end{bmatrix}
\]

6.5: Finding the inverse of the 2x2 matrix, and solve the matrix equation for x and y.

Sol:

\[
\begin{bmatrix}
y \\
x
\end{bmatrix}
= \frac{1}{\Delta}
\begin{bmatrix}
-\alpha & a \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
b \\
\beta
\end{bmatrix}
= \frac{1}{\Delta}
\begin{bmatrix}
-\alpha b + a\beta \\
-b + \beta
\end{bmatrix}
\]

where the determinant is \( \Delta \equiv a - \alpha \).

6.6: Discuss the properties of the determinant of the matrix (\( \Delta \)) in terms of the slopes of the two equations (a and \( \alpha \)).

Sol: When the slopes are the same there is no solution and \( \Delta = 0 \). Thus the matrix solution is consistent with the geometry. This is our first result in analytic geometry.

Problem # 7: The application of linear functional relationships between two variables:

2x2 matrices are used to describe 2-port networks, as will be discussed in §3.5 (p. 92). Transmission lines are a great example, where both voltage and current must be tracked as they travel along the line. Figure 3.8 (p. 97) shows an example segment of a transmission line.

Suppose you are given the following pair of linear relationships between the input (source) variables \( V_1 \) and \( I_1 \), and the output (load) variables \( V_2 \) and \( I_2 \) of the transmission line.

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}
\]
3.2. PROBLEMS AE-2

– 7.1: Let the output (the load) be \( V_2 = 1 \) and \( I_2 = 2 \) (i.e., \( V_2/I_2 = 1/2 \ (\Omega) \)). Find the input voltage and current, \( V_1 \) and \( I_1 \).

**Sol:** This case corresponds to

\[
\begin{bmatrix}
 V_1 \\
 I_1 
\end{bmatrix} = \begin{bmatrix} j & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1j + 2 \\ 1 - 2 \end{bmatrix}
\]

Thus \( V_1 = 2 + 1j \) and \( I_1 = -1 \).

– 7.2: Let the input (source) be \( V_1 = 1 \) and \( I_1 = 2 \). Find the output voltage and current \( V_2 \) and \( I_2 \).

**Sol:** With the input specified the two equations are

\[
\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} j & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

To find the input we must invert the matrix (\( \Delta = -j - 1 \))

\[
\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \frac{1}{1 + j} \begin{bmatrix} 1 & 1 \\ 1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

Thus \( V_2 = 3/(1 + j) = 3(1 - j)/2 \), \( I_2 = (1 - 2j)/(1 + 1j) = -(1 + 3j)/2 \). The point of this exercise is that the two lines have a complex intersection point, not easily visualized.

### Integer equations: applications and solutions

Any equation for which we seek only integer solutions is called a Diophantine equation.

**Problem # 8:** *A practical example of using a Diophantine equation:*

“A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?”

- *Bachet de Béziriac (1623 CE)*

Here, weighing is performed using a balance scale having two pans, with weights being put on either pan. Thus, given weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with the 2-pound weight, and the 3-pound weigh in the other pan. Then, the scale will be balanced. A solution to the four weights for Bachet's problem is \( 1 + 3 + 9 + 27 = 40 \) pounds.

"Taken from: Joseph Rotman, “A first course in abstract algebra,” *Chapter 1, Number Theory* p. 50

– 8.1: Show how the combination of 1, 3, 9, & 27 pound weights may be used to weigh \( 1, 2, 3, \ldots, 8, 28, \) and \( 40 \) pounds of milk (or something else, such as flour). Assuming that the milk is in the left pan, provide the position of the weights using a negative sign ‘-’ to indicate the left pan and a positive sign ‘+’ to indicate the right pan. For example, if the left pan has 1 pound
of milk, then 1 pound of milk in the right pan, ‘+1’ will balance the scales.

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]^T$. The pan assignments matrix should only contain the values -1 (left pan), +1 (right pan), and 0 (leave out). You can indicate these using ‘-’, ‘+’, and blank spaces.

**Sol:** Any integer between 1 and 40 may be expanded using the weights 1, 3, 9, 27. Here is the problem stated in matrix form:

$$
\begin{bmatrix}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
\vdots \\
28 \\
\vdots \\
40
\end{bmatrix} = 
\begin{bmatrix}
+ & + & + & + \\
- & + & + & + \\
+ & + & + & + \\
- & - & + & + \\
+ & - & + & + \\
+ & + & - & + \\
- & + & - & + \\
+ & + & + & + \\
\vdots \\
+ & + & + & + \\
\vdots \\
+ & + & + & +
\end{bmatrix}
\begin{bmatrix}
1 \\
3 \\
9 \\
27
\end{bmatrix}
$$

The left column is the weight of the milk. The right-most column are the four weights. It should be clear that these four weights span the integers from 1-40 with binary weights. Each weight may be computed recursively from twice the sum of the previous weights +1, that is

$$W_{n+1} = 2W_n + 1 = 2^{n+1} \quad \text{since} \quad W_n = 2^n.$$  

For example to get 26 we place weights 9+3+1 in the pan with 26, and get 27-1. For example 27 = 2*(9+3+1)+1 is the next weight. Recursively, the weights are 3=2*1+1, 9=2*(3+1)+1, 27=2*(9+3+1)+1. The next weight (not shown) would be: 81=2*(27+9+3+1)+1 = 2*40+1. ■

**Ohm’s Law**

In general, impedance is defined as the ratio of a force over a flow. For electrical circuits, the voltage is the ‘force’ and the current is the ‘flow.’ Ohm’s law states that the voltage across and the current through a circuit element are related by the *impedance* of that element (which may be a function of frequency). For resistors, the voltage over the current is called the *resistance*, and is a constant (e.g. the simplest case, $V/I = R$). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g. $V/I = Z(s)$, where $s$ is the complex frequency $s \in \mathbb{C}$).

As described in Fig. 3.1 (p. 97) the impedance concept also holds in mechanics and acoustics. In mechanics, the ‘force’ is equal to the mechanical force on an element (e.g. a mass, dashpot, or spring), and the ‘flow’ is the velocity. In acoustics, the ‘force’ is pressure, and the ‘flow’ is the volume velocity or particle velocity of air molecules.

**Problem #9:** The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As it lights up, the resistance of the metal filament increases.

Ohm’s law says that the current

$$\frac{V}{I} = R(T).$$

where $T$ is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz].
3.2. PROBLEMS AE-2

9.1: Find the current when the light is first switched on.
Sol: Thus the current is
\[ I = \frac{120}{R} = \frac{120}{100} = 1.2 \text{ [Amps]} \]

As the bulb heats up, the current rapidly drops, and the resistance increases. This typically takes less than a millisecond [ms], which depends on the wattage of the light bulb. Such light bulbs are nonlinear.

These rules don’t apply to LED bulbs.

Problem # 10: The power in Watts is the product of the force and the flow.

10.1: What is the power of the light bulb of this example?
Sol: \[ P = V \cdot I = 120 \times 1.2 = 120 + 24 = 144 \text{ [W]} \]

Problem # 11: State the impedance \( Z(s) \) of each of the following circuit elements:

11.1: A resistor with resistance \( R \):
Sol: \( Z = R \)

11.2: An inductor with inductance \( L \):
Sol: \( Z = sL \) with \( s = \sigma + \omega j \). Note the flux \( \psi(t) = Li(t) \). The voltage \( v(t) \) is the time derivative of the flux
\[ v(t) = \frac{dv(t)}{dt} = L \frac{di(t)}{dt}. \]

11.3: A capacitor with capacitance \( C \):
Sol: \( Z = 1/sC \). Note the charge \( q(t) = Cv(t) \), thus the current \( i(t) \) is the time derivative of the charge
\[ i(t) = \frac{d}{dt}q(t) = C \frac{dv(t)}{dt}. \]

Problem # 12: Consider what happens at the triple-point of water. As water freezes or thaws, the temperature remains constant at 0 \( (\text{C}^\circ) \). Once all the water is frozen and more heat is removed, the temperature drops below 0 \( ^\circ \). As heat is added, water thaws, but the temperature remains at 0\( ^\circ \).

12.1: Once all the ice is melted, as more heat is added, find the temperature as more heat is added.
Model the triple point using a zener diode, a resistor and a capacitor. A zener holds the voltage constant independent of current. For the case of water’s triple-point, the voltage represents the temperature of water at the triple point, clamped at 0 \( [\text{C}^\circ] \). The current represents the heat flux. The latent heat of water at the triple point is 32 \( [\text{Cal/gm}] \). Thus as the temperature rises from below freezing, the water is clamped at 0\( ^\circ \) once the triple point is reached. Once there adding more head flux as no effect on the temperature until all the ice melts. Once melted, the temperature again begins to rise until it hits the boiling point, where it again stays at 100\( ^\circ \), until all the water has evaporated. Sol: Need a figure here showing how to model the triple point of water. The Heat capacity may be modeled by a capacitor, which is fixed at 0\( ^\circ \) as the capacitor discharges. Once it is empty, the temperature again begins to rise as the heat \( Q \) from the sun is added
\[ T^\circ = mcQ. \]

Thus the required circuit needs to emulate this temperature behavior due to the latent heat of melting ice and boiling water into steam.
Vector algebra in \( \mathbb{R}^3 \).

Definitions of the scalar (also called a dot product) \( \mathbf{A} \cdot \mathbf{B} \), cross \( \mathbf{A} \times \mathbf{B} \) and triple product \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \), may be found in Appendix A (p. 185), where \( \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^3 \subseteq \mathbb{C}^3 \), as shown in Fig. 3.4 page fig:VecDev. A fourth “double-cross” (\( \mathcal{A} \)) vector product is:

\[
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \alpha_\mathcal{A} \mathbf{B} - \beta_\mathcal{A} \mathbf{C}.
\]

where \( \alpha_\mathcal{A} = \mathbf{A} \cdot \mathbf{C} \) and \( \beta_\mathcal{A} = \mathbf{A} \cdot \mathbf{B} \) (Note: \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \)).

**Problem #13: Scalar product \( \mathbf{A} \cdot \mathbf{B} \)**

- 13.1: If \( \mathbf{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \) and \( \mathbf{B} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \), write out the definition of \( \mathbf{A} \cdot \mathbf{B} \).

Sol: See the definition in the above figure. \( \mathbf{A} \cdot \mathbf{B} = a_x b_x + a_y b_y + a_z b_z \). In general: \( \mathbf{A} \cdot \mathbf{B} = \sum_k A_k B_k . \)

- 13.2: The dot product is often defined as \( ||\mathbf{A}|| \ ||\mathbf{B}|| \cos(\theta) \), where \( ||\mathbf{A}|| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \) and \( \theta \) is the angle between \( \mathbf{A}, \mathbf{B} \). If \( ||\mathbf{A}|| = 1 \), describe how the dot product relates to the vector \( \mathbf{B} \).

Sol: See the definition in the above figure. The vector product is the portion of \( \mathbf{B} \) in the direction of \( \mathbf{A} \).

**Problem #14: Vector (cross) product \( \mathbf{A} \times \mathbf{B} \)**

- 14.1: If \( \mathbf{A} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \) and \( \mathbf{B} = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \), write out the definition of \( \mathbf{A} \times \mathbf{B} \).

Sol:

\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
a_x & a_y & a_z \\
b_x & b_y & b_z
\end{vmatrix} = \hat{x} \begin{vmatrix}
a_y & a_z \\
b_y & b_z
\end{vmatrix} - \hat{y} \begin{vmatrix}
a_x & a_z \\
b_x & b_z
\end{vmatrix} + \hat{z} \begin{vmatrix}
a_x & a_y \\
b_x & b_y
\end{vmatrix}.
\]

- 14.2: Show that the cross product is equal to the area of the parallelogram formed by \( \mathbf{A}, \mathbf{B} \), namely \( ||\mathbf{A}|| ||\mathbf{B}|| \sin(\theta) \), where \( ||\mathbf{A}|| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \) and \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \).

Sol: A parallelogram’s area is equal to its base times its height. Therefore, let’s say the base is length \( ||\mathbf{A}|| \), and the height \( ||\mathbf{B}|| \sin(\theta) \), which is the portion of \( \mathbf{B} \) that is perpendicular to \( \mathbf{A} \).

**Problem #15: Triple product \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \)**

Let \( \mathbf{A} = [a_1, a_2, a_3]^T \), \( \mathbf{B} = [b_1, b_2, b_3]^T \), \( \mathbf{C} = [c_1, c_2, c_3]^T \) be three vectors in \( \mathbb{R}^3 \).

- 15.1: Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{vmatrix} \).

Sol: Using the determinate-definition of the cross product,

\[
\mathbf{B} \times \mathbf{C} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
b_x & b_y & b_z \\
c_x & c_y & c_z
\end{vmatrix} = \hat{x} \begin{vmatrix}
b_y & b_z \\
c_y & c_z
\end{vmatrix} - \hat{y} \begin{vmatrix}
b_x & b_z \\
c_x & c_z
\end{vmatrix} + \hat{z} \begin{vmatrix}
b_x & b_y \\
c_x & c_y
\end{vmatrix}.
\]

Let \( \mathbf{D} = \mathbf{B} \times \mathbf{C} \) and compute \( \mathbf{A} \cdot \mathbf{D} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \). Finally compute the requested right-hand side, and compare the two. It should be clear that they are the same, because the dot product transfers the elements of vector \( \mathbf{A} \) to cross product and reduces the product to the scalar.

\(^4\)Greenberg p. 694, Eq. 8.
3.2. PROBLEMS AE-2

- **15.2: Describe why** $|A \cdot (B \times C)|$ **is the volume of parallelepiped generated by** $A, B$ **and** $C$.

**Sol:** Note that the norm of $B \times C$ is the area of the parallelogram generated by $C$ and $B$. Taking the dot product with $A$ results in the volume of the corresponding parallelepiped (prism). So the absolute value of triple product is volume of parallelepiped.  

- **15.3: Explain why three vectors** $A, B, C$ **are in one plane if and only if the triple product** $A \cdot (B \times C) = 0$.

**Sol:** (triple product is zero) if and only if: (volume is zero), if and only if: (they are in the same plane)  

**Problem # 16:** **Given two vectors** $A, B$ **in the** $\hat{x}, \hat{y}$ **plane shown in Fig. 3.3 (same as 3.4 on page 84), with** $B = \hat{y}$ (**i.e.,** $||B|| = 1$).

- **16.1: Show that** $A$ **may be split into two orthogonal parts, one in the direction of** $B$ **and the other perpendicular (⊥) to** $B$. **Hint:** Express the vector products of $A$ and $B$ (dot and cross) in polar coordinates (Greenberg, 1988).

\[
A = (A \cdot B)B + B \times (A \times B) = A_\parallel + A_\perp.
\]

**Sol:** Expressing the vector products in polar form makes this result transparent:

\[
A \cdot B = ||A|| \cos(\theta) \quad \text{and} \quad A \times B = ||A|| \sin(\theta)
\]

The first quantity is in the direction of $B$, while the second is in the direction $A \times B$, which is ⊥ to $B$. Thus

\[
A = ||A|| (B \cos(\theta) + A \times B \sin(\theta)) = A_\parallel + A_\perp.
\]
3.3 Problems AE-3

Topic of this homework:
Visualizing complex functions; Bilinear/Möbius transform; Riemann sphere.

2-port network analysis

Problem # 1: Perform an analysis of electrical 2-port networks, shown in Fig. 3.6 (page 93). This can be a mechanical system if the capacitors are taken to be springs, and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.

The definition of the ABCD transmission matrix (\(T\)) is

\[
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{AE-3.1}
\]

The impedance matrix, where the determinant \(\Delta_T = AD - BC\), is given by

\[
\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A & \Delta_T \\ 1 & D \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{AE-3.2}
\]

1.1: Derive the formula for the impedance matrix (Eq. AE-3.2) given the transmission matrix definition (Eq. AE-3.1).

Show your work. Sol: The formula may be easily derived by re-arranging the equations from the matrix (Eq. AE-3.2). Begin with

\[
V_1 = AV_2 - BI_2 \\
I_1 = CV_2 - DI_2
\]

From the second equation, we get

\[
V_2 = \frac{1}{C}I_1 + \frac{D}{C}I_2
\]

which gives (upon substitution)

\[
V_1 = \frac{A}{C}I_1 + \frac{AD}{C}I_2 - BI_2 = \frac{A}{C}I_1 + \left(\frac{AD}{C} - B\right)I_2
\]

which yields the matrix equation

\[
\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} A/C & (AD/C - B) \\ 1/C & D/C \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A & \Delta_T \\ 1 & D \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{AE-3.3}
\]

Problem # 2: Consider a single circuit element with impedance \(Z(s)\)
3.3. PROBLEMS AE-3

– 2.1: What is the ABCD matrix for this element if it is in ‘series’?
Sol: 
\[
\begin{bmatrix}
1 & Z(s) \\
0 & 1
\end{bmatrix}
\]

– 2.2: What is the ABCD matrix for this element if it is ‘shunt’?
Sol: 
\[
\begin{bmatrix}
1 & 0 \\
1/Z(s) & 1
\end{bmatrix}
\]

Problem # 3: Find the ABCD matrix for each of the circuits of Figure 3.6.
For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency \( s \in \mathbb{C} \), then (ii) substitute \( s = 1+j \) and calculate the total transmission matrix at this single frequency.

– 3.1: Left circuit (let \( R_1 = R_2 = 10 \) k\( \Omega \) ‘kilo-ohms’, and \( C = 10 \) nF ‘nano-farads’)
Sol: Write the system in chain matrix form:
\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
1 & Z_1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
Y_C & 1
\end{bmatrix} \begin{bmatrix}
1 & Z_3 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
\]

Now we substitute the given values:
\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
1 & 10^4 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
-j10^{-8} & 1
\end{bmatrix} \begin{bmatrix}
1 & 10^4 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix} = \begin{bmatrix}
1 + j10^{-4} & 2 \times 10^4 + j \\
-j10^{-8} & 1 + j10^{-4}
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
\]

– 3.2: Right circuit (use \( L \) and \( C \) values given in the figure), where the pressure \( P \) is analogous to the voltage \( V \), and the velocity \( U \) is analogous to the current \( I \).
Sol: Write the system in chain matrix form:
\[
\begin{bmatrix}
P_1 \\
U_1
\end{bmatrix} = \begin{bmatrix}
1 & sL_1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
sC_2 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 \frac{1}{sC_3} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
\frac{1}{sL_4} & 1
\end{bmatrix} \begin{bmatrix}
P_2 \\
-U_2
\end{bmatrix}
\]

Now we substitute the given values:
\[
\begin{bmatrix}
P_1 \\
U_1
\end{bmatrix} = \begin{bmatrix}
1 & j \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
2j & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 \frac{1}{3j} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
\frac{1}{3} & 1
\end{bmatrix} \begin{bmatrix}
P_2 \\
-U_2
\end{bmatrix} = \begin{bmatrix}
-\frac{2}{3} & \frac{4j}{3} \\
\frac{19}{12}j & \frac{5}{3}
\end{bmatrix} \begin{bmatrix}
P_2 \\
-U_2
\end{bmatrix}
\]

I used Matlab/Octave to evaluate this script:
\[
a=[1 \ j; 0 \ 1]; b=[1 \ 0; 2j \ 1]; c=[1 \ 1/3j; 0 \ 1]; d=[1 \ 0; 1/4j \ 1]; T=a*b*c*d.
\]

Finally I found \( T(2,1) \) to be 19/12 using the Matlab/Octave command: \( \text{rats}(1.5833, 6) \)

– 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Equation AE-3.2. Do this for the specific frequency \( s = 1+j \), as in the previous part (feel free to use Matlab/Octave for your computation).
Sol: Left circuit: Using the previous solution, and matlab:
\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \frac{1}{j10^{-8}} \begin{bmatrix}
1 + j10^{-4} & 1 \\
1 & 1 + j10^{-4}
\end{bmatrix} \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
\]

– 3.4: Right circuit: Using the previous solution, and matlab:
Sol:
\[
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix} = \frac{1}{1.5833j} \begin{bmatrix}
-\frac{2}{3} & 1 \\
\frac{5}{3} & 1
\end{bmatrix} \begin{bmatrix}
U_1 \\
U_2
\end{bmatrix}
\]
Algebra

Problem # 4: Fundamental theorem of algebra (FTA)

– 4.1: State the fundamental theorem of algebra (FTA).
Sol: There are multiple definitions of the FTA, which of course must be equivalent. Here are three (equivalent) answers from Wikipedia:

1. The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This may then be applied recursively till the degree is zero.

2. Every degree \( n \) polynomial with complex coefficients has, counted with multiplicity, exactly \( n \) roots. The equivalence of the two statements can be proven through the use of successive polynomial division.

3. The field of complex numbers is algebraically closed. Note: this one requires an understanding of the term algebraically closed.

Wikipedia warns:

In spite of its name, there is no purely algebraic proof of the theorem, since any proof must use the completeness of the reals (or some other equivalent formulation of completeness), which is not an algebraic concept.

Problem # 5: Order and complex numbers:
One can always say that \( 3 < 4 \), namely that real numbers have order. One way to view this is to take the difference, and compare to zero, as in \( 4 - 3 > 0 \). Here, we will explore how complex variables may be ordered.

– 5.1: Explain the meaning of \( |z_1| > |z_2| \).
Sol: \( |z| = \sqrt{x^2 + y^2} \) is the length of \( z \), so the above expression says that a disk of radius \( |z_1| \) contains a second disk of radius \( |z_2| \).

– 5.2: If \( x_1, x_2 \in \mathbb{R} \) (are real numbers), define the meaning of \( x_1 > x_2 \). 

Hint: Take the difference. Sol: This conditions is the same as \( x_1 - x_2 > 0 \). Order is meaningful on the real line, as a length.

– 5.3: Explain the meaning of \( z_1 > z_2 \).
Sol: It makes no sense to order complex numbers. A complex number has both a length and an angle (it is the same as a vector). The concept of an angle extends the sign of a real number, making order impossible. To show this, place to points on a plane, and ask which is larger than the other. The order of the \( x \) and \( y \) components, each have order. Thus order cannot be defined.

– 5.4: If time were complex how might the world be different?
Sol: As best we know, time is real, thus it has the order property: the is a past, present and future. If time were complex this would not be the case. Thus if time were complex, the past could be the future. (not graded)

Problem # 6: It is sometimes necessary to consider a function \( w(z) = u + iv \) in terms of the real functions, \( u(x, y) \) and \( v(x, y) \) (e.g. separate the real and imaginary parts). Similarly, we can consider the inverse \( z(w) = x + iy \) where \( x(u, v) \) and \( y(u, v) \) are real functions.
3.3. PROBLEMS AE-3

6.1: Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = 1/z \).

Sol: Multiply by the complex conjugate \( x - iy \)

\[
    w = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}
\]

Therefore \( u(x, y) = \frac{x}{x^2 + y^2} \) and \( v(x, y) = \frac{-y}{x^2 + y^2} \). ■

Problem # 7: Find \( u(x, y) \) and \( v(x, y) \) for \( w(z) = c^z \) with complex constant \( c \in \mathbb{C} \) for the following cases:

- 7.1: \( c = e \)

Sol: Since \( u + iv = e^z \),

\[
    u = e^x \cos y
\]

and

\[
    v = e^x \sin y.
\]

■

- 7.2: \( c = 1 \) (recall that \( 1 = e^{2\pi k} \) for \( k = 0, 1, 2, \ldots \))

Sol: From the general formula with \( c = 1 \)

\[
    1^z = e^{z \log 1} e^{ik2\pi y} = e^{0} e^{-yk2\pi} = e^{-yk2\pi}
\]

where \( k \) is any integer. Thus \( u = e^{-yk2\pi} \) and \( v = 0 \). ■

- 7.3: \( c = j \). Hint: \( j = e^{\pi i/2} \).

Sol: \( j^z = \left(e^{\pi i/2}\right)^z = e^{z\pi i/2} = e^{-\pi z/2} = 0.2079. \)

Thus \( j^z = \left(e^{\pi i/2}\right)^z = e^{z\pi i/2} = e^{j(x+y)\pi/2} = e^{-\pi y/2} (\cos(x\pi/2) + j \sin(x\pi/2)). \)

■

- 7.4: Find \( u(x, y) \) for \( w(z) = \sqrt{z} \). Hint: Begin with the inverse function \( z = w^2 \).

Sol: The simple solution is to work in polar coordinates, which gives \( w(z) = \sqrt{|z|} e^{j\phi/2} \).

Following the Hint, \( z = x + iy = u^2 - v^2 + 2uvj \). Breaking into real and imaginary parts we find \( x = u^2 - v^2 \) and \( y = 2uv \). Removing \( v = y/2u \) from the first equation gives

\[
    u^2 = x + (y/2u)^2
    \]

\[
    u^4 - xu^2 = (y/2)^2
\]

\[
    u^2 = \frac{x \pm (x^2 + y^2)}{2}
    \]

\[
    (u, v) = \left( \pm \sqrt{\frac{x \pm (x^2 + y^2)}{2}}, \frac{y}{2u(x, y)} \right)
\]

This should be the equation that shows as the contour lines for the Matlab command \( \text{W} = \text{sqrt}(Z) \). This is more informative if you scale \( Z \) by 5 or even 10. ■

- 7.5: Convolution of an impedance \( z(t) \) and its inverse \( y(t) \).

In the frequency domain a Brune impedance is defined as the ratio of an numerator polynomial \( N(s) \) over a denominator polynomial \( D(s) \).
– 7.6: Consider a brune impedance defined by the ratio of numerator and 
denominator polynomials \( Z(s) = \frac{N(s)}{D(s)} \). Since the admittance \( Y(s) \) is defined as the reciprocal of 
the impedance, the product must be 1. If \( z(t) \leftrightarrow Z(s) \) and \( y(t) \leftrightarrow Y(s) \), it follows that 
\( z(t) * y(t) = \delta(t) \). 
What property must \( n(t) \leftrightarrow N(s) \) and \( d(t) \leftrightarrow D(s) \) obey for this to be true? 
**Sol:** Since 
\[
1 = Z(s)Y(s) = \frac{N(s) D(s)}{D(s) N(s)}
\]
it follows that \( N(s)D(s) = D(s)N(s) \) thus \( n(t) * d(s) = d(t) * n(t) \). Namely the convolution of \( n(t) \) 
and \( d(t) \) commute (are independent of order). ■

– 7.7: The definition of a “minimum phase function” is that it must have a causal inverse. 
Show that every impedance is minimum phase. **Sol:** Since \( z(t) \) is causal and has a causal invers \( y(t) \), by 
definition every impedance must be minimum phase. ■

**Schwarz inequality**

**Problem #8:** Figure 3.4 page (same as Fig. 3.5 page 90) shows three vectors for an arbitrary 
value of \( \alpha \in \mathbb{R} \) and a specific value of \( \alpha = \alpha^* \).

– 8.1: Given Fig. 3.4, find the value of \( \alpha \in \mathbb{R} \) such that the length (norm) of \( E \) (i.e., 
\( \|E\| \geq 0 \)) is minimum? Show your derivation, not the answer (\( \alpha = \alpha^* \)). 
**Sol:** In Fig. 3.4 we see vectors \( V, U \), and for reference, \( V + \alpha^* U \). Also shown are scaled values of 
\( U, \alpha U \) and \( \alpha^* U \). The setup for the derivation is 
\[
\|E(\alpha)\|^2 = E \cdot E = (\vec{V} + \alpha \vec{U}) \cdot (\vec{V} + \alpha \vec{U}) \geq 0.
\] (AE-3.4)

Minimize with respect to \( \alpha \).

When \( U \) is scaled by \( \alpha^* \), length \( \|E(\alpha^*)\| \) is minimum, and \( (V - \alpha^* U) \perp U \), namely vector \( E(\alpha^*) \) 
is \( \perp \) to vector \( U \). This follows from 
\[
\frac{d}{d\alpha} \|E\|^2 = \frac{d}{d\alpha} ((\vec{V} + \alpha \vec{U}) \cdot (\vec{V} + \alpha \vec{U})) = 2(\vec{V} + \alpha \vec{U}) \cdot \vec{U} = 0.
\]
Thus 
\[
\alpha^* = -\frac{\vec{V} \cdot \vec{U}}{||\vec{U}||^2}
\]

– 8.2: Find the formula for \( \|E(\alpha^*)\|^2 \geq 0 \). Hint: Substitute \( \alpha^* \) into Eq. AE-3.4 and show 
that this results in the Schwarz inequality.
\[
|\vec{U} \cdot \vec{V}| \leq ||\vec{U}|| ||\vec{V}||.
\]
3.3. PROBLEMS AE-3

Sol: From Eq. AE-3.4
\[ ||V||^2 + 2\alpha^* V \cdot U + (\alpha^*)^2 ||U||^2 \geq 0 \]
Substituting \( \alpha^* \) gives
\[ ||V||^2 ||U||^2 - 2(V \cdot U)^2 + |U \cdot V|^2 \geq 0. \]
Simplifying
\[ ||V||^2 ||U||^2 \geq |U \cdot V|^2. \]
and taking the square root (and swap order), gives the Schwarz inequality
\[ |\vec{U} \cdot \vec{V}| \leq ||\vec{U}|| ||\vec{V}||. \]

Problem # 9: What is the geometrical meaning of the dot product of two vectors? Sol: The dot product of two vectors is the length of the \( \perp \) projection of one vector on the other. According to the Schwarz inequality, this projection length must be less than the product of the lengths of the two vectors.

– 9.1: Give the formula for the dot product between two vectors. Explain the meaning based on Fig. 3.4 (page 84).
Sol: \( \vec{V} \cdot \vec{U} = ||\vec{V}|| ||\vec{U}|| \cos \theta \). It represents the amount of one vector going in the direction of the other. In a drawing, it is a projection of the one on the other, found by dropping the \( \perp \) from the tip of one, on the other.

– 9.2: Write the formula for the “dot product” between two vectors: \( \vec{U} \cdot \vec{V} \) in \( \mathbb{R}^n \) in polar form (e.g., assume the angle between the vectors is equal to \( \theta \)).
Sol: \( \vec{U} \cdot \vec{V} = \sum_{i=1}^{n} a_i b_i (= ||\vec{U}|| ||\vec{V}|| \cos(\theta)) \). This last relationship defines the angle between two vectors.

– 9.3: How is the Schwarz inequality related to the Pythagorean theorem?
Sol: It says that for a right triangle, the case when \( a = a^* \), the lengths of the two vectors must be greater than the projection of one on the other, unless they are co-linear (i.e., the angle between them is zero).

– 9.4: Starting from \( ||U + V|| \) derive the triangle inequality
\[ ||\vec{U} + \vec{V}|| \leq ||\vec{U}|| + ||\vec{V}||. \]
Sol: \( ||\vec{U} + \vec{V}||^2 = (\vec{U} + \vec{V}) \cdot (\vec{U} + \vec{V}) = ||U||^2 + ||V||^2 + 2U \cdot V \leq ||U||^2 + ||V||^2 + 2||U \cdot V|| \) Using the Schwarz inequality we find \( ||\vec{U} + \vec{V}||^2 \leq ||U||^2 + ||V||^2 + 2||U|| \ ||V|| \). Completing the square on the right gives \( ||\vec{U} + \vec{V}||^2 \leq (||U|| + ||V||)^2 \). Final taking the square root gives the triangle inequality.

– 9.5: The triangular inequality \( ||\vec{U} + \vec{V}|| \leq ||\vec{U}|| + ||\vec{V}|| \) is true for 2 and 3 dimensions: Does it hold for 5 dimensional vectors?
Sol: It is true in any number of dimensions.

– 9.6: Show that the wedge product \( \vec{U} \wedge \vec{V} \perp \vec{U} \cdot \vec{V} \).
Sol: \( \vec{V} \wedge \vec{U} = ||\vec{V}|| ||\vec{U}|| \sin \theta \vec{\phi} \). See the discussion in the text on the wedge product. This is true in any number of dimensions.
Probability

Problem # 10: Basic terminology of experiments

<table>
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<tr>
<th>Problem</th>
<th>Question</th>
<th>Solution</th>
</tr>
</thead>
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<tr>
<td>10.1</td>
<td>What is the mean of a trial, and what is the average over all trials?</td>
<td>The mean and average are the same. What is different in these two questions is the size of the set being averaged.</td>
</tr>
<tr>
<td>10.2</td>
<td>What is the “expected value” of a random variable $X$?</td>
<td>This is a mathematical definition of how to compute the mean, as an inner product.</td>
</tr>
<tr>
<td>10.3</td>
<td>What is the “standard deviation” about the “mean”?</td>
<td>This is the expected value of the second moment of the random variable.</td>
</tr>
<tr>
<td>10.4</td>
<td>What is the definition of “information” of a random variable?</td>
<td>The information is $I = 1/P(X_k)$.</td>
</tr>
<tr>
<td>10.5</td>
<td>What is the “entropy” of a random variable?</td>
<td>The expected value of the log of the information: $I$.</td>
</tr>
<tr>
<td>10.6</td>
<td>What is the “sampling noise” of a random variable?</td>
<td>Sampling noise is the mean error due to a finite number of samples $N \in \mathbb{N}$. The magnitude of the mean sampling noise is $\sigma = 1/\sqrt{N}$.</td>
</tr>
<tr>
<td>10.7</td>
<td>How to you combine events? Hint: if the event is the flip of a biased coin, the event is $H = p, T = 1 - p$, so the event is ${ p, 1 - p }$.</td>
<td>To solve the problem, you must find the probabilities of two independent events. The convolution of two uncorrelated (independent) events then have probability $[p, 1 - p] \ast [p, 1 - p] = [p^2, 2p(1 - p), (1 - p^2)]$. Here $\ast$ represents convolution (Section 4.5.4, p. 136). Three events have four outcomes $[p, 1 - p] \ast [p, 1 - p] \ast [p, 1 - p] = [p, 1 - p]$. Pascal’s triangle is a related structure defined by recursive convolutions of $[1, 1]$, assuming $p = 1/2$.</td>
</tr>
<tr>
<td>10.8</td>
<td>What does the term “independent” mean in this context? Give an example.</td>
<td>This term means that one event (flip of a coin) has no influence on the next (or any other flip) of that same coin. An example of non-independent events might be that upon flipping the coin, it bent. thus changing the probability for any following flips.</td>
</tr>
<tr>
<td>10.9</td>
<td>Define the term “odds”.</td>
<td>The odds are the ratio of the two outcomes. Namely the odds are $p/(1 - p)$, or equivalently $(1 - p)/p = 1/p - 1$.</td>
</tr>
</tbody>
</table>
Chapter 4

Differential equations

4.1 Problems DE-1

Topic of this homework:
Complex numbers and functions (ordering and algebra); Complex power series; Fundamental theorem of calculus (real and complex); Cauchy-Riemann conditions; Multivalued functions (branch cuts and Riemann sheets)

Complex Power Series

Problem # 1: In each case derive (e.g. using Taylor’s formula) the power series of \( w(s) \) about \( s = 0 \) and state the ROC of your series. If the power series doesn’t exist, state why! Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at \( s = 0 \).

- 1.1: \( 1/(1 - s) \)
  Sol: \( 1/(1 - s) = \sum_{n=0}^{\infty} s^n \) which converges for \( |s| < 1 \) (e.g., the ROC is \( |s| < 1 \)).

- 1.2: \( 1/(1 - s^2) \)
  Sol: \( 1/(1 - s^2) = \sum_{n=0}^{\infty} s^{2n} \) which converges for \( |s^2| < 1 \). (e.g., the ROC is \( |s| < 1 \)). One can also factor the polynomial, thus write it as: \( \frac{1}{(1-s)(1+s)} \). There are two poles, at \( s = \pm 1 \), and each has an ROC of 1.

- 1.3: \( 1/(1 - s^2) \)
  Sol: To show this note that \( -\frac{d}{ds}(1 - s)^{-1} = \frac{1}{(1-s)^2} \). Expanding this gives \( \frac{1}{(1-s)^2} = -\frac{d}{ds} \sum_{n=0}^{\infty} s^n = \sum_{n=1}^{\infty} ns^{n-1} = \sum_{n=0}^{\infty} (n + 1)s^n \), which converges for \( |s| < 1 \). A second way is to factor \( 1 - s^2 \) and then convolve the coefficients of the \( \infty \) series of \( 1/(1 \pm is) \).

- 1.4: \( 1/(1 + s^2) \). Hint: This series could be ugly to derive if you try to take the derivatives \( \frac{d^n}{ds^n}[1/(1 + s^2)] \). Using the results of Problem 1 you should use the partial fraction expansion \( w(s) = -0.5i/(s - i) + 0.5i/(s + i) = \frac{i}{2} \frac{(s-i)-(s+i)}{s^2+1} \).
  Sol: The resulting series is \( 1/(1 + s^2) = 0.5 \sum_{n=0}^{\infty} s^n((-i)^n + (i)^n) \). The ROC is \( |s| < 1 \). We can see this by considering the poles of the function at \( s = \pm i \); both poles are 1 from \( s = 0 \), the point of expansion. An alternative is to write the function as \( 1/(1 - (is)^2) = \sum(is)^n \).
1.5: $1/s$
Sol: If you try to do a Taylor expansion at $s = 0$, the first term, $w(0) \to \infty$. Thus, the Taylor series expansion in $s$ does not exist. ■

1.6: $1/(1 - |s|^2)$
Sol: The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed. ■

**Problem # 2:** Consider the function $w(s) = 1/s$

- 2.1: Expand this function as a power series about $s = 1$. Hint: Let $1/s = 1/(1 - 1 + s) = 1/(1 - (1 - s))$.
Sol: The power series is
$$w(s) = \sum_{n=0}^{\infty} (-1)^n (s - 1)^n$$
which converges for $|s - 1| < 1$.

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s,s,'ExpansionPoint',1)` which is equivalent to the shorthand `syms s; taylor(1/s,s,1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to $z = -1$ via the Möbius “translation” $s = z + 1$, and expand using the Taylor series
$$\frac{1}{s} = \frac{1}{1 + z} = \sum_{n=0}^{\infty} (-z)^n.$$ 
Next back-substitute $z = s - 1$ giving
$$\frac{1}{s} = \sum_{n=0}^{\infty} (-1)^n (s - 1)^n.$$ 
It follows that the RoC is $|z| = |s - 1| < 1$, as provided by Matlab/Octave. ■

- 2.2: What is the RoC?
Sol: $|s| < 1$ ■

- 2.3: Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$.
Sol: Let $z = s^{-1}$ and expand about $1$:
$$\frac{1}{1 - s^{-1}} = \frac{s}{s - 1} = -\frac{s}{1 - s} = s(1 + s + s^2 + s^3 \cdots) = s + s^2 + s^3 \cdots.$$ 
which has a zero at $0$ and a pole at $1$. ■

- 2.4: What is the ROC?
Sol: $|s| < 1$. ■

- 2.5: What is the residue of the pole?
Sol: -0. ■

**Problem # 3:** Consider the function $w(s) = 1/(2 - s)$

- 3.1: Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the ROC as a condition on $|s^{-1}|$.
Hint: Multiply top and bottom by $s^{-1}$.
Sol: $1/(2 - s) = -s^{-1}/(1 - 2s^{-1}) = -s^{-1} \sum 2^n s^{-n}$. The ROC is $|2/s| < 1$, or $|s| > 2$. ■
4.1. PROBLEMS DE-1

– 3.2: Find the inverse function \( s(w) \). Where are the poles and zeros of \( s(w) \), and where is it analytic?

**Sol:** Solving for \( s(w) \) we find \( 2 - s = 1/w \) and \( s = 2 - 1/w = (2w - 1)/w \). This has a pole at 0 and a zero at \( w = 1/2 \). The ROC is therefore from the expansion point out to, but not including \( w = 0 \).

**Problem # 4: Summing the series**

The Taylor series of functions have more than one region of convergence.

– 4.1: If \( a = 0.1 \), what is the value of 

\[
x = 1 + a + a^2 + a^3 \ldots?
\]

Show your work. **Sol:** To sum this series, use the fact that

\[
x - ax = (1 + a + a^2 + a^3 \ldots) - a(1 + a + a^2) = 1 + a(1 - 1) + a^2(1 - 1) + \ldots
\]

This gives \( x(1 - a) = 1 \), or \( x = 1/(1 - a) \). Now since \( a = .1 \), the sum is \( 1/(1 - 0.1) = 1.11 \).

– 4.2: If \( a = 10 \) what is the value of

\[
x = 1 + a + a^2 + a^3 \ldots?
\]

**Sol:** In this case the series clearly does not converge. To make it converge we need to write a formula for \( y = 1/x \) rather than for \( x \).

\[
y - y/a = (1 + 1/a + 1/a^2 + 1/a^3 \ldots) - 1/a(1 + 1/a + a^1/2) = 1 + (1 - 1)/a + (1 - 1)/a^2 + \ldots
\]

This gives \( y(1 - 1/a) = 1 \), or \( y = 1/(1 - 1/a) \). Now since \( a = 10 \), the sum is \( y = 1/(1 - 0.1) = 9 \). We might conclude that since \( x = 1/y \), \( x = 1/9 \). Does this make sense?

**Quadratic forms**

A matrix that has positive eigenvalues is said to be positive-definite. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy since the power is the voltage times the current. Given an impedance matrix

\[
V = ZI,
\]

the power \( P \) is

\[
P = I \cdot V = I \cdot ZI,
\]

which must be positive definite for the system to obey conservation of energy.

**Problem # 5: For the following problems, consider the 2 × 2 impedance matrix**

\[
Z = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.
\]

– 5.1: Solve for the power \( P(i_1, i_2) \) by multiplying out the matrix equation below (which is in quadratic form) \( (I \equiv \begin{bmatrix} i_1 & i_2 \end{bmatrix}^T) \)

\[
P(i_1, i_2) = I^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} I.
\]

**Sol:**

\[
P(i_1, i_2) = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2i_1 + i_2 \\ i_1 + 4i_2 \end{bmatrix} = 2i_1^2 + 2i_1i_2 + 4i_2^2.
\]
5.2: Is the impedance matrix positive definite? Show your work by finding the eigenvalues of the matrix $Z$.

**SOL:** Yes, as it is positive definite if the eigenvalues are both positive. You need to show that the eigenvalues are positive (not zero or negative). They are, so it is. How to do all this is worked on in Example 3, page 593.

\[
\begin{vmatrix}
 2 - \lambda & 1 \\
 1 & 4 - \lambda
\end{vmatrix} = 0 \Rightarrow \lambda = 3 \pm \sqrt{2} > 0
\]

5.3: Should an impedance matrix always be positive definite? Explain.

**SOL:** Yes.

---

**Cauchy-Riemann Equations**

For the following problem: $i = \sqrt{-1}$, $s = \sigma + i\omega$, and $F(s) = u(\sigma, \omega) + iv(\sigma, \omega)$.

**Problem #6:** According to the Fundamental theorem of complex calculus the integration of a complex analytic function is independent of the path. It follows that the derivative of $F(s)$ is defined as

\[
\frac{dF}{ds} = \frac{d}{ds} [u(\sigma, \omega) + iv(\sigma, \omega)].
\]  

(DE-1.1)

If the integral is independent of the path, then the derivative must also be independent of direction

\[
\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \omega}.
\]  

(DE-1.2)

The Cauchy-Riemann (CR) conditions

\[
\frac{\partial u(\sigma, \omega)}{\partial \sigma} = \frac{\partial v(\sigma, \omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma, \omega)}{\partial \omega} = -\frac{\partial v(\sigma, \omega)}{\partial \sigma}
\]

may be used to show where Equation DE-1.2 holds.

6.1: Assuming Equation DE-1.2 is true, use it to derive the CR equations.

**SOL:** First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations.

6.2: Merge the CR equations to show that $u$ and $v$ obey Laplace’s equation

\[
\nabla^2 u(\sigma, \omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma, \omega) = 0.
\]

**SOL:** Take partial derivatives with respect to $\sigma$ and $\omega$ and solve for one equation in each of $u$ and $v$. What can you conclude?

**SOL:** One may conclude that the real and imaginary parts of complex analytic functions must obey these conditions.

**Problem #7:** Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g. where the function $F(s)$ is or is not analytic). Hint: Review where CR-1 and CR-2 hold.

7.1: $F(s) = e^s$

**SOL:** CR conditions hold everywhere.

7.2: $F(s) = 1/s$

**SOL:** CR conditions are violated at $s = 0$. The function is analytic everywhere except $s = 0$. 

4.1. PROBLEMS DE-1

Branch cuts and Riemann sheets

Problem #8: Consider the function \( w^2(z) = z \). This function can also be written as \( w(z) = \sqrt{z} \).
Define \( z = re^{\phi j} \) and \( w(z) = re^{\phi j/2} \).

- 8.1: How many Riemann sheets do you need in the domain (\( z \)) and the range (\( w \)) to fully represent this function as single valued?
Sol: There are two sheets for \( z \) and one sheet for \( w = \sqrt{z} \). When the point in domain \( z \) (being mapped to \( w(z) \)) crosses the \( z \) branch cut, the range (\( w \)) switches from the \( z_+ \) sheet to the \( z_- \) sheet. \( w(z) \) remains analytic on the cut, since it is analytic everywhere. The function \( w(z) = \sqrt{z} \) is analytic everywhere, even at \( z = 0 \). ■

- 8.2: Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.
Sol: Above we show the mapping for the square root function \( w(z) = \sqrt{z} = \sqrt{re^{j\phi/2}} \).

- 8.3: Use \( zviz.m \) to plot the positive and negative square roots \( +\sqrt{z} \) and \( -\sqrt{z} \). Describe what you see.
Sol: The sheet for the positive root is shown in Fig 3.2 (page 106 of the Oct 24 version of the class notes.) Two view the two sheets use Matlab command \( zviz \ sqrt(w) -sqrt(w) \).

- 8.4: Where does \( zviz.m \) place the branch cut for this function?
Sol: Typically the cut is placed along the negative real \( z \) axis \( \phi = \pm \pi \). This is Matlab’s/Octave’s default location. In the figure above, it has been placed along the positive real axis, \( \phi = 0 = 2\pi \). ■

- 8.5: Must it necessarily be in this location?
Sol: No, it can be moved, at will. It must start from \( z = 0 \) and end at \( |z| \to \infty \). The cut may be move when using \( zviz.m \) by multiplying \( z \) by \( e^{j\rho} \). For example, \( zviz \ W = sqrt(j*Z) \) rotates the cut by \( \pi/2 \). The colors of \( w(z) \) (angle maps to color) always ‘jump’ at the branch cut, as you make the transition across the cut. ■

Problem #9: Consider the function \( w(z) = \log(z) \). As before define \( z = re^{\phi j} \) and \( w(z) = re^{\phi j} \).

- 9.1: Describe with a sketch, and then discuss the branch cut for \( f(z) \).
Sol: From the plot of \( zviz \ w(z) = \log(z) \) of Lecture 18, we see a branch cut going from \( w = 0 \) to \( w = -\infty \). If we express \( z \) in polar coordinates \( z = re^{j\phi} \), then
\[
w(z) = \log(r) + \phi j = u(x, y) + v(x, y) j,
\]
where \( r(x, y) = |z| = \sqrt{x^2 + y^2} \) and \( \phi = \angle z = \phi(x, y) \). Thus a zero in \( w(z) \) appears at \( z = 1 + 0j \), and only appears on the principle sheet of \( z \) (between \( -\pi < \angle z = \phi < \pi \)), because this is the only place where \( \phi = 0 \). As the angle \( \phi \) increases, the imaginary part of \( w = \angle z \), which increases without bound. Thus \( w \) is like a spiral stair case, or cork-screw. If \( r = 1 \) and \( \phi \neq 0 \), \( w(r) = \log(1) + \phi j \) is not zero, since the angle is not zero. ■

- 9.2: What is the inverse of this function, \( z(f) \)? Does this function have a branch cut (if so, where is it)?
Sol: \( z(w) = e^w \) is a single valued function, so a branch cut is not appropriate. Only multi-valued functions require a branch cut. ■

- 9.3: Using \( zviz.m \), show that
\[
\tan^{-1}(z) = -\frac{j}{2} \log \frac{j - z}{j + z}. \tag{DE-1.3}
\]
In Fig. 4.1 (p. 116) these two functions are shown to be identical. Sol: Use the Matlab commands \( \text{atan}(Z) \) and \( -(j/2) \times \log((j+Z)/(j-Z)) \). ■
9.4: Algebraically justify Eq. 4.1.0.2. Hint: Let \( w(z) = \tan^{-1}(z), \) \( z(w) = \tan w = \sin w / \cos w, \) then solve for \( e^{wz}. \)

Sol: Following the hint gives

\[
    z(w) = -\tan^{-1}(z) = \frac{1 + zj}{1 - zj} = \frac{j - z}{j + z}.
\]

Solving this bilinear equation for \( e^{2wz} \) gives

\[
    e^{2wz} = \frac{1 + zj}{1 - zj} = \frac{j - z}{j + z}.
\]

Taking the log and using our definition of \( w(z) \) we find

\[
    w(z) = \tan^{-1}(z) = -\frac{j}{2} \log \frac{j - z}{j + z}.
\]

A Cauer synthesis of any Brune impedance

Problem # 10: One may synthesize a transmission line (ladder network) from a positive real impedance \( Z(s) \) by using the continued fraction method. To obtain the series and shunt impedance values, one may use residue expansion. Here we shall further explore this method.

10.1: Starting from the Brune impedance \( Z(s) = \frac{1}{s + 1}, \) find the impedance network as a ladder network.

Sol: Taking the reciprocal we find the sum of two shunt admittances, and capacitor and resistor

\[
    Y(s) = s + 1.
\]

The the impedance is \( Z(s) = 1/(s + 1). \)

10.2: Use a residue expansion to mimic the CFA floor function (§2.3.2, p. 45) for polynomial expansions. Find the residue expansion of \( H(s) = s^2/(s + 1) \) and express it as a ladder network.

Sol: Verify that

\[
    Z(s) = s^2/(s + 1) = s - 1 + 1/(s + 1). \tag{DE-1.4}
\]

Thus the Cauer synthesis is a series combination \( s - 1 \) (an inductor \( L = 1 \) and a resistor \( R = -1 \) ohms) and a shunt \( 1\|s \) (i.e., \( Y(s) = 1 + s \), a resistor of \( R = 1 \) in parallel with a capacitor \( C = 1 \).) It would appear that \( Z(s) \) is not a positive real impedance.

10.3: Discuss how the series impedance \( Z(s, x) \) and shunt admittance \( Y(s, x) \) determine the wave velocity \( \kappa(s, x) \) and the characteristic impedance \( z_o(s, x) \) when

1. \( Z(s) \) and \( Y(s) \) are both independent of \( x \)

Sol: In the most general case

\[
    z_o(s, x) = \sqrt{Z(s, x)/Y(s, x)}
\]

and

\[
    \kappa(s, x) = \sqrt{Z(s, x)Y(s, x)}.
\]

The general equations for \( z_o(s, s) \) and \( \kappa(s, x) \) are given in Mason (1927), and discussed in Appendix G (p. 233).
2. $Z(s, x)$ and $Y(s)$ ($Y(s)$ is independent of $x$, $Z(s, x)$ depends on $x$)

3. $Z(s)$ and $Y(s, x)$ ($Z(s)$ is independent of $x$, $Y(s, x)$ depends on $x$)

4. $Z(s, x)$ and $Y(s, x)$ (both $Y(s, x)$, $Z(s, x)$ depend on $x$)

This shows that a Cauer synthesis may be implemented with the residue expansion replacing the floor function in the CFA. This seems to solve Burne’s network synthesis problem.
4.2 Problems DE-2

Topic of this assignment:
Integration of complex functions; Cauchy’s theorem, integral formula, residue theorem; power series; Riemann sheets and branch cuts; inverse Laplace transforms

Two fundamental theorems of calculus

Fundamental Theorem of Calculus (Leibniz):
According to the Fundamental Theorem of (Real) Calculus (FTC)
\[ f(x) = f(a) + \int_a^x F(\xi)d\xi, \]  
(DE-2.1)
where \( x, a, \xi, F, f \in \mathbb{R} \). This is an indefinite integral (since the upper limit is unspecified). It follows that
\[ \frac{df(x)}{dx} = \frac{d}{dx} \int_a^x F(x)dx = F(x). \]
This justifies also calling the indefinite integral the anti-derivative.

For a closed interval \([a, b]\), the FTC is
\[ \int_a^b F(x)dx = f(b) - f(a), \]  
(DE-2.2)
thus the integral is independent of the path from \( x = a \) to \( x = b \).

Fundamental Theorem of Complex Calculus:
According to the Fundamental Theorem of Complex Calculus (FTCC)
\[ f(z) = f(z_0) + \int_{z_0}^z F(\zeta)d\zeta, \]  
(DE-2.3)
where \( z_0, z, \zeta, f, F \in \mathbb{C} \). It follows that
\[ \frac{df(z)}{dz} = \frac{d}{dz} \int_{z_0}^z F(\zeta)d\zeta = F(z). \]

For a closed interval \([s, s_0]\), the FTCC is
\[ \int_{s_0}^s F(\zeta)d\zeta = f(s) - f(s_0), \]  
(DE-2.5)
thus the integral is independent of the path from \( x = a \) to \( x = b \).

Problem # 1
4.2. PROBLEMS DE-2

– 1.1: Consider Equation DE-2.1. What is the condition on $F(x)$ for which this formula is true?

**Sol:** The sufficient condition is that the integrand $F(x)$ is be analytic, namely $F(x) = \sum_{n=0}^{\infty} a_n x^n$. This assures that $F(x)$ is single valued and may be integrated, since it may integrated term by term. It follows that as long as $x < \text{ROC}$, this integral exists. Thus the integral equals $F(x) - F(a)$. Note that if the integrand has a Taylor series, all of its derivatives exist within the ROC, because the coefficients depend on derivatives of $F(x)$. ■

– 1.2: Consider Equation DE-2.3. What is the condition on $F(z)$ for which this formula is true?

**Sol:** The sufficient condition is that the integrand $F(z)$ must be complex analytic, namely $F(z) = \sum_{n=0}^{\infty} c_n z^n$, with $c_n \in \mathbb{C}$. ■

**Problem # 2:** In the following problems, solve the integral

$I = \int_C F(z) dz$

for a given path $C$.

– 2.1: Perform the following integrals ($z = x + iy \in \mathbb{C}$):

1. $I = \int_{0}^{1+j} z \, dz$

**Sol:**

$I = \left. \frac{1}{2} z^2 \right|_0^{1+j} = \frac{1}{2} (1 + j)^2 = \frac{1}{2} (1 - 1 + 2j) = j$ ■

2. $I = \int_{0}^{1+j} z \, dz$, but this time make the path explicit: from 0 to 1, with $y=0$, and then to $y=1$, with $x=1$.

**Sol:**

$I = \int_{x=0}^{1} (x + 0j) \, dx + \int_{y=0}^{1} (1 + yj) \, dy \, j$

$I = \left. \frac{1}{2} x^2 \right|_0^1 + \left. (1 + yj) \right|_0^1$

$I = \frac{1}{2} + (yj - \frac{1}{2} y^2) \bigg|_0^1$

$I = \frac{1}{2} + j - \frac{1}{2} = j$

We conclude that the integration of $z$ is independent of the path. This is true for any integrand $z^n$ with $n \in \mathbb{Z}$. ■

3. Discuss whether your results agree with Eq. DE-2.4?

**Sol:** Yes the two integrals must agree, because the function is analytic, and the integral must be the same, independent of the path. ■

– 2.2: Perform the following integrals on the closed path $C$, which we define to be the unit circle. You should substitute $z = e^{i\theta}$ and $dz = ie^{i\theta} \, d\theta$, and integrate from $\{-\pi, \pi\}$ to go once around the unit circle.
1. \( \int_C zdz \)
\[
\int_C zdz = \int_{-\pi}^{\pi} e^{i\theta} d\theta = \int_{-\pi}^{\pi} e^{i2\theta} id\theta = e^{i2\theta}|_{-\pi} = 0.
\]

2. \( \int_C \frac{1}{z} dz \)
\[
\int_C \frac{1}{z} dz = 2\pi i.
\]

3. Discuss whether your results agree with Eq. DE-2.4?
\[
\text{Sol: (a) obeys the FTCC because } f(z) = z \text{ is analytic everywhere, (b) does not obey the FTCC because } f(z) = 1/z \text{ is not analytic at } z=0 \text{ (inside } C). \]

**Problem # 3: FTCC and integration in the complex plane**

Let the function \( F(z) = c^z \), where \( c \in \mathbb{C} \) is given for each problem below. *Hint: Can you apply the FTCC?*

- **3.1:** For the function \( f(z) = c^z \), where \( c \in \mathbb{C} \) is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that \( f(z) \) is analytic for all \( z \in \mathbb{C} \).

\[
\text{Sol: We may rewrite this function as } f(z) = e^{\ln(c)z}, \text{ where } z = x + iy \text{ and } f = u + iv. \text{ Thus }
\]
\[
\begin{align*}
\frac{\partial u}{\partial x} &= \ln(c) e^{\ln(c)x} \cos(\ln(c)y), \\
\frac{\partial v}{\partial y} &= \ln(c) e^{\ln(c)x} \sin(\ln(c)y) \\
\frac{\partial u}{\partial y} &= -\ln(c) e^{\ln(c)x} \sin(\ln(c)y), \\
\frac{\partial v}{\partial x} &= -\ln(c) e^{\ln(c)x} \cos(\ln(c)y)
\end{align*}
\]

Thus the CR conditions are satisfied everywhere and the function is analytic for all \( z \in \mathbb{C} \). ■

- **3.2:** Find the anti-derivative of \( F(z) \).

\[
\text{Sol: Since } c^z = e^{\ln(c)z}, \text{ the indefinite integral (anti-derivative) is }
\]
\[
I(z) = \frac{1}{\ln c} e^{\ln(c)z} \text{ since } \frac{d}{dz} I(z) = \frac{d}{dz} \frac{1}{\ln c} e^{\ln(c)z} = \frac{e^{\ln(c)z}}{\ln c} = F(z).
\]

■

- **3.3:** \( c = 1/e = 1/2.7183 \ldots \) where \( C \) is \( \zeta = 0 \to i \to z \)

\[
\text{Sol: The integrand is } F(z) = e^{-z}, \text{ which is entire. Thus the the integral is independent of the path (i.e., } C \text{ is not relevant to the final answer).}
\]
\[
I(z) = \int_0^1 e^{-\zeta} d\zeta + \int_i^z e^{-\zeta} d\zeta = F(z) - F(i) + F(i) - F(0)
\]
\[
= \int_0^z e^{-\zeta} d\zeta = -e^z \bigg|_0^z = -(e^{-z} - 1)
\]

■

- **3.4:** \( c = 2 \) where \( C \) is \( \zeta = 0 \to (1 + i) \to z \)

\[
\text{Sol: The integrand is } F(z) = 2^z, \text{ where } 2 = e^{\ln 2}. \text{ The path } C \text{ is not relevant to the final answer.}
\]
\[
I(z) = \int_0^z 2^\zeta d\zeta = \int_0^z e^{\ln 2 \zeta} d\zeta = \frac{e^{\ln 2 \zeta} \bigg|_0^z}{\ln 2} = (e^{z \ln 2} - 1)/\ln 2 \approx 1.443(e^{0.693z} - 1)
\]

■
4.2. PROBLEMS DE-2

- 3.5: \( c = i \) where the path \( C \) is an inward spiral described by \( z(t) = 0.99^t e^{iz2\pi t} \) for \( t = 0 \rightarrow t_0 \rightarrow \infty \).

**Sol:** \( i = e^{i\pi/2} e^{i2\pi n} \). We have already proved that the path doesn’t matter for any \( F(z) = e^z \), so we just need to evaluated \( z(t) \) for \( t = 0 \) and \( t \rightarrow \infty \). This gives \( z(0) = 1 \) and \( z(t \rightarrow \infty) = 0 \).

\[
I = \int_{z(0)}^{z(t \rightarrow \infty)} i^z dz = \int_{z(0)}^{z(t \rightarrow \infty)} e^{iz2\pi/2} dz = \left. \frac{2e^{iz\pi/2}}{i\pi} \right|_1^0 = \frac{2}{i\pi} (1 - e^{i\pi/2}) = \frac{-2(i + 1)}{\pi}
\]

- 3.6: \( c = e^{t-\tau_0} \) where \( \tau_0 > 0 \) is a real number, and \( C \) is \( z = (1 - i\infty) \rightarrow (1 + i\infty) \). Hint: Do you recognize this integral? If you do not recognize the integral, please do not spend a lot of time trying to solve it via the ‘brute force’ method.

**Sol:** This is basically the inverse Laplace transform of \( e^{-\tau_0 z} \), we are just missing the scale factor \( \frac{1}{2\pi i} \).

\[
I(t) = \int_{1-i\infty}^{1+i\infty} e^{(t-\tau_0)z} dz = \int_{1-i\infty}^{1+i\infty} e^{-\tau_0 z} e^{zt} dz = 2\pi i \delta(t - \tau_0)
\]

**Problem #4: Cauchy’s theorems for integration in the complex plane**

There are three basic definitions related to Cauchy’s integral formula. They are all related, and can greatly simplify integration in the complex plane. When a function depends on a complex variable we shall use uppercase notation, consistent with the engineering literature for the Laplace transform.

1. **Cauchy’s (Integral) Theorem CT-1** (Stillwell, p. 319; Boas, p. 45)

   \[
   \oint_{C} F(z) dz = 0,
   \]

   if and only if \( F(z) \) is complex-analytic inside of \( C \).

   This is related to the Fundamental Theorem of Complex Calculus (FTCC)

   \[
   f(z) = f(a) + \int_{a}^{z} F(z) dz,
   \]

   where \( f(z) \) is the anti-derivative of \( F(z) \), namely \( F(z) = df/dz \). The FTCC requires \( F(z) \) to be complex-analytic for all \( z \in \mathbb{C} \). By closing the path (contour \( C \)), Cauchy’s theorem (and the following theorems) allows us to integrate functions that may not be complex-analytic for all \( z \in \mathbb{C} \).

2. **Cauchy’s Integral Formula CT-2** (Boas, p. 51; Stillwell, p. 220)

   \[
   \frac{1}{2\pi i} \oint_{C} \frac{F(z)}{z - z_0} dz = \begin{cases} \int F(z_0), z_0 \in C \text{ (inside)} \\ 0, z_0 \notin C \text{ (outside)} \end{cases}
   \]

   Here \( F(z) \) is required to be analytic everywhere within (and on) the contour \( C \). \( F(z_0) \) is called the residue of the pole.

3. **(Cauchy’s) Residue Theorem CT-3** (Boas, p. 72)

   \[
   \oint_{C} F(z) dz = 2\pi j \sum_{k=1}^{K} \text{Res}_k,
   \]

   where \( \text{Res}_k \) are the residues of all poles of \( F(z) \) enclosed by the contour \( C \).
How to calculate the residues: The residues can be rigorously defined as

$$\text{Res}_k = \lim_{z \to z_k} [(z - z_k)f(z)].$$

This can be related to Cauchy’s integral formula: Consider the function $F(z) = w(z)/(z - z_k)$, where we have factored $F(z)$ to isolate the first-order pole at $z = z_k$. If the remaining factor $w(z)$ is analytic at $z_k$, then the residue of the pole at $z = z_k$ is $w(z_k)$.

– 4.1: Describe the relationships between the three theorems:

1. (1) and (2) Sol: When $z_0$ falls outside of $\mathcal{C}$, (2) reduces to (1).

2. (1) and (3) Sol: When there are no poles inside $\mathcal{C}$, all the residues are zero, and (3) reduces to (1).

3. (2) and (3) Sol: Case (2) has only one induced pole at $z = z_0$, having residue $F(z_0)$. Thus (3) is the same as (2) when $K = 1$, the pole at $z_0$ is within contour $\mathcal{C}$, and the single residue is $F(z_0)$.

– 4.2: Consider the function with poles at $z = \pm j$

$$F(z) = \frac{1}{1 + z^2} = \frac{1}{(z - j)(z + j)}$$

Find the residue expansion.

Sol:

$$F(z) = \frac{j}{2} \left( \frac{1}{z + j} - \frac{1}{z - j} \right).$$

– 4.3: Apply Cauchy’s theorems to solve the following integrals. State which theorem(s) you used, and show your work.

1. $\oint_{\mathcal{C}} F(z)dz$ where $\mathcal{C}$ is a circle centered at $z = 0$ with a radius of $\frac{1}{2}$.

   Sol: Because the contour $\mathcal{C}$ does not include the poles, $F(z)$ is analytic everywhere inside $\mathcal{C}$. Using Cauchy’s integral theorem, the integral is 0.

2. $\oint_{\mathcal{C}} F(z)dz$ where $\mathcal{C}$ is a circle centered at $z = j$ with a radius of 1.

   Sol: Since we only enclose the pole at $z = j$, use the integral formula with $F(z) = 1/(z + j)$:

   $$\oint_{\mathcal{C}} \frac{F(z)}{z - j}dz = 2\pi j \text{Res}_j = 2\pi j \left( \frac{1}{z + j} \right)_{z=j} = 2\pi j \frac{1}{2j} = \pi$$

3. $\oint_{\mathcal{C}} F(z)dz$ where $\mathcal{C}$ is a circle centered at $z = 0$ with a radius of 2.

   Sol: Since we enclose both poles, using the residue theorem:

   $$\oint_{\mathcal{C}} F(z)dz = 2\pi j (\text{Res}_j + \text{Res}_{-j}) = 2\pi j \left( \frac{1}{2j} - \frac{1}{2j} \right) = 0$$

As a side note, the inverse Laplace transform for $F(z)$ is $\sin(t)$, which is zero for $t = 0$, consistent with this result.
Problem # 5: Integration in the complex plane
In the following questions, you’ll be asked to integrate \( F(s) = u(\sigma, \omega) + iv(\sigma, \omega) \) around the contour \( C \) for complex \( s = \sigma + i\omega \),

\[
\oint_C F(s)ds.
\]

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations (but you should if you can’t answer the question ‘by inspection’).

- 5.1: \( F(s) = \sin(s) \)

**Sol:** Analytic everywhere.

\[
\cos(s) = \int_{0}^{2\pi} \sin(s)ds = 0.
\]

This function is entire (i.e., has no poles) so the integral must be zero.

- 5.2: Given function \( F(s) = \frac{1}{s} \)

1. State where the function is and is not analytic. **Sol:** Analytic everywhere except at \( s = 0 \), where it has a pole.

2. Explicitly evaluate the integral when \( C \) is the unit circle, defined as \( s = e^{i\theta}, 0 \leq \theta \leq 2\pi \). **Sol:**

\[
\oint_C F(s)ds = \int_{0}^{2\pi} \frac{1}{es}i e^{i\theta}d\theta = \int_{0}^{2\pi} i d\theta = 2\pi i
\]

3. Evaluate the same integral using Cauchy’s theorem and/or the residue theorem. **Sol:** The residue is 1 so the integral is \( 2\pi i \).

- 5.3: \( F(s) = \frac{1}{s^2} \)

1. State where the function is and is not analytic. **Sol:** Analytic everywhere except at \( s = 0 \), where it has a 2\(^{nd} \) order pole.

2. Explicitly evaluate the integral when \( C \) is the unit circle, defined as \( s = e^{i\theta}, 0 \leq \theta \leq 2\pi \). **Sol:**

\[
\oint_C F(s)ds = \int_{0}^{2\pi} \frac{1}{e^{i2\theta}}i e^{i\theta}d\theta = \int_{0}^{2\pi} i e^{-i\theta}d\theta = i \frac{1}{-i} e^{-i\theta} \bigg|_{0}^{2\pi} = 1(e^{-i2\pi} - e^{0}) = 0
\]

3. What does your result imply about the residue of the 2\(^{nd} \) order pole at \( s = 0 \)? **Sol:** The residue is 0.

- 5.4: \( F(s) = e^{st} \)

1. State where the function is and is not analytic. **Sol:** Analytic everywhere.

2. Explicitly evaluate the integral when \( C \) is the square \( (\sigma, \omega) = (1, 1) \rightarrow (-1, 1) \rightarrow (-1, -1) \rightarrow (1, -1) \rightarrow (1, 1) \). **Sol:** When you perform this integral piece-wise, you will find that all terms cancel out and the result is 0.

3. Evaluate the same integral using Cauchy’s theorem and/or the residue theorem. **Sol:** The function is analytic everywhere, so the integral is 0 by Cauchy’s theorem.
1. State where the function is and is not analytic. Sol: Analytic everywhere except at \( s = -2 \), where it has a pole.

2. Let \( C \) be the unit circle, defined as \( s = e^{i\theta}, 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

   Sol: The function is analytic everywhere inside \( C \), so the integral is 0 by Cauchy’s theorem.

3. Let \( C \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

   Sol: This contour contains the pole. The residue is 1, therefore the integral is equal to \( 2\pi i \).

\[ -5.6: \quad F(s) = \frac{1}{2\pi i} \frac{e^{st}}{(s+4)} \]

1. State where the function is and is not analytic. Sol: Analytic everywhere except at \( s = -4 \), where it has a pole.

2. Let \( C \) be a circle of radius 3, defined as \( s = 3e^{i\theta}, 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

   Sol: This contour contains the pole. The residue is \( \frac{1}{2\pi i} e^{-2t} \), therefore the integral is equal to \( e^{-2t} \).

3. Let \( C \) contain the entire left-half \( s \)-plane. Evaluate the integral using Cauchy’s theorem and/or the residue theorem. Do you recognize this integral? Sol: This contour contains the pole. The residue is \( \frac{1}{2\pi i} e^{-2t} \), therefore the integral is equal to \( e^{-2t} \). This contour is the inverse Laplace transform.

\[ -5.7: \quad F(s) = \pm \frac{1}{\sqrt{s}} \quad (e.g. \quad F^2 = \frac{1}{s}) \]

1. State where the function is and is not analytic. Sol: Analytic everywhere except \( s = 0 \), where there is a pole.

2. This function is multivalued. How many Riemann sheets do you need in the domain \( (s) \) and the range \( (f) \) to fully represent this function? Indicate (e.g. using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range. Sol: There are 2 sheets in the domain (for the \( \pm \) square root) which map to 1 sheet in the range.

3. Explicitly evaluate the integral

   \[ \int_C \frac{1}{\sqrt{z}} \, dz \]

   when \( C \) is the unit circle, defined as \( s = e^{i\theta}, 0 \leq \theta \leq 2\pi \). Is this contour ‘closed’? State why or why not. Sol: The solution is

   \[ 2\sqrt{z} \bigg|_{\theta=0}^{2\pi} = 2e^{j\theta/2} \bigg|_{0}^{2\pi} = 2(e^{j\pi} - e^{0}) = -4. \]
4.2. PROBLEMS DE-2

In polar coordinates

\[
\int_0^{2\pi} \frac{ds}{\sqrt{s}} = \int_0^{2\pi} \frac{d\theta}{e^{i\theta/2}} = i \int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta/2}} d\theta = i \int_0^{2\pi} e^{i\theta/2} d\theta = \left. 2 e^{i\theta/2} \right|_0^{2\pi} = 2 \left[ e^{i\pi} - 1 \right] = 2(-2) = -4.
\]

This contour is not closed. One way to determine this is to see if going once around the unit circle returns \( F(s) \) to its original value.

\[
F(e^{i0}) = 1 \neq F(e^{i2\pi}) = e^{-i\pi} = -1.
\]

\[\square\]

4. Explicitly evaluate the integral

\[
\int_C \frac{1}{\sqrt{z}} \, dz
\]

when \( C \) is twice around the unit circle, defined as \( s = e^{i\theta}, 0 \leq \theta \leq 4\pi \). Is this contour ‘closed’? State why or why not. Hint: Note that

\[
\sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi}e^{i\theta}} = e^{i\pi} \sqrt{e^{i\theta}} = -1 \sqrt{e^{i\theta}}
\]

Sol:

\[
\int_0^{4\pi} \frac{ds}{\sqrt{s}} = \int_0^{4\pi} \frac{d\theta}{e^{i\theta/2}} = i \int_0^{4\pi} \frac{e^{i\theta}}{e^{i\theta/2}} d\theta = i \int_0^{4\pi} e^{i\theta/2} d\theta = \left. 2 e^{i\theta/2} \right|_0^{4\pi} = 2 \left[ e^{i2\pi} - 1 \right] = 2(0) = 0.
\]

This contour is closed. One way to determine this is to see if going twice around the unit circle returns \( F(s) \) to its original value. \( F(e^{i0}) = 1 = F(e^{i4\pi}) = e^{-i2\pi} = 1 \).

\[\square\]

5. What does your result imply about the residue of the (twice-around \( \frac{1}{2} \) order) pole at \( s = 0 \)?

Sol: The residue is 0.

\[\square\]

6. Show that the residue is zero. Hint: apply the definition of the residue. Sol: \( c_{-1} = \lim_{z \to z_k} z/\sqrt{z} = \lim_{z \to z_k} \sqrt{z} = 0 \).

\[\square\]

**Problem # 6: A two-port network application for the Laplace transform**
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CHAPTER 4. DIFFERENTIAL EQUATIONS

Figure 4.1: This three-element electrical circuit is a system that acts to low-pass filter the signal voltage \( V_1(\omega) \), to produce signal \( V_2(\omega) \).

– 6.1: Find the 2x2 ABCD matrix representation of Fig. 4.1.

Sol:

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
(1 + R_1Cs) & R_1 \\
R_2 & 1
\end{bmatrix} \begin{bmatrix}
1 & R_2 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}
\]

– 6.2: Find the eigenvalues of the 2x2 ABCD matrix. Hint: See Appendix B.3, page 201.

Sol: The eigenvalues of every 2x2 matrix are given in Appendix B.3 (page 201)

\[
\lambda_{\pm} = \frac{1}{2} \left( (A + D) - \sqrt{(A - D)^2 + 4BC} \right), \quad \text{(DE-2.6)}
\]

When \( R \equiv R_1 = R_2 \) this simplifies to

\[
\lambda_{\pm} = \left( 1 + RCs - \sqrt{BC} \right) \left( 1 + RCs + \sqrt{BC} \right). \quad \text{(DE-2.7)}
\]

where \( \sqrt{BC} = \sqrt{sC(2R + R^2Cs)} \approx 2RCs \) below the cutoff frequency and \( \sqrt{BC} = \sqrt{sC(2R + R^2Cs)} \approx RCs \) above. It is helpful to note that the low-pass cutoff frequency is \( \omega_c = 1/RC \) where \( \tau = RC \) is the time-constant.

– 6.3: Assuming that \( I_2 = 0 \), find the transfer function \( H(s) \equiv V_2/V_1 \). From the results of the ABCD matrix you determined above, show that

\[
H(s) = \frac{1}{1 + R_1Cs}.
\]

Sol: Since \( I_2 = 0 \) the upper row of the ABCD matrix gives the relationship between \( V_1 \) and \( V_2 \) as

\[
V_1 = (1 + R_1C)V_2
\]

Thus the ratio is as desired.

– 6.4: The transfer function \( H(s) \) has one pole. Where is the pole? Find the residue of this pole.

Sol: If we rewrite \( H(s) \) in the standard form, the pole \( s_p \) and residue \( A \) may be easily identified:

\[
H(s) = \frac{A}{s - s_p} = \frac{1}{1 + R_1Cs} = \frac{1/(R_1C)}{s + 1/(R_1C)}
\]

Thus the pole is \( s_p = -1/R_1C \) and the residue is \( A = 1/R_1C \).
4.2. PROBLEMS DE-2

– 6.5: Find \( h(t) \), the inverse Laplace transform of \( H(s) \).

Sol:

\[
h(t) = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{1 + R_1 Cs} \frac{ds}{2\pi j} = \frac{1}{R_1 C} e^{-t/RC} u(t)
\]

The integral follows from the **Residue Theorem**. The pole is at \( s_p = -1/RC \) and the residue is \( 1/R_1 C \).

– 6.6: Assuming that \( V_2 = 0 \) find \( Y_{12}(s) \equiv I_2/V_1 \).

Sol: Setting \( V_2 = 0 \) we may easily read off the requested function as

\[
V_1 = -(R_1 + R_2 + R_1 R_2 Cs) I_2
\]

thus

\[
Y_{12}(s) = -\frac{1}{R_1 + R_2 + R_1 R_2 Cs} = \frac{A}{s - s_p}
\]

with residue \( A = 1/R_1 R_2 C \) and pole \( s_p = -(R_1 + R_2)/(R_1 R_2 C) \).

– 6.7: Find the input impedance to the right-hand side of the system, \( Z_{22}(s) \equiv V_2/I_2 \) for two cases:

1. \( I_1 = 0 \)
2. \( V_1 = 0 \)

Sol: There are two cases. When \( I_1 = 0 \), \( Z_{22} = R_2 + 1/sC \). When \( V_1 = 0 \)

\[
Z_{22}(s) = R_2 + \frac{1}{sC} \frac{R_1}{R_1 + 1/sC} = R_2 + \frac{R_1}{1 + R_1 Cs}
\]

\[
= \frac{R_1 + R_2 + R_1 R_2 Cs}{1 + R_1 Cs}
\]

Reading this second case off of our matrix solution gives

\[
0 = (1 + R_1 C)V_2 - (R_1 + R_2 + R_1 R_2 Cs) I_2
\]

or solving for \( Z_{22} \) gives the brute-force result.

– 6.8: Compute the determinant of the \( ABCD \) matrix. Hint: It is always \( \pm 1 \).

Sol:

\[
\begin{vmatrix}
1 + R_1 Cs & R_1 + R_2 + R_1 R_2 Cs \\
R_1 + R_2 + R_1 R_2 Cs & 1 + R_2 Cs
\end{vmatrix}
= 1 + (R_1 + R_2)Cs + R_1 R_2(Cs)^2 - (R_1 + R_2)Cs - R_1 R_2(Cs)^2
= 1
\]

■
– 6.9: Compute the derivative of \( H(s) = \frac{V_2}{V_1} I_1 = 0 \).

**Sol:** From the result of the previous problem 2

\[
H(s) = \frac{1}{1 + R_1 C s}.
\]

Thus we wish to find

\[
\frac{d}{ds} H(s) = \frac{d}{ds} (1 + R_1 C s)^{-1} = \frac{-R_1 C}{(1 + R_1 C s)^2}.
\]

Here is a slightly easier way, using the log function:

\[
\frac{1}{H(s)} \frac{dH(s)}{ds} = \frac{d}{ds} \ln H(s) = -\frac{d}{ds} \ln(1 + R_1 C s) = \frac{-1}{1 + R_1 C s} \frac{d}{ds} (1 + R_1 C s) = \frac{-R_1 C}{(1 + R_1 C s)^2}.
\]

Thus

\[
\frac{dH(s)}{ds} = H(s) \frac{-R_1 C}{(1 + R_1 C s)^2} = \frac{-R_1 C}{(1 + R_1 C s)^2}
\]

**Problem # 7:** With the help of a computer

In the following we shall look at a few important concepts using Matlab/Octave using `syms` commands or Wolfram Alpha’s symbolic math toolbox.

**Example:** To find the Taylor series expansion about \( s = 0 \) of

\[
F(s) = -\log(1 - s),
\]

first consider the derivative and its Taylor series (about \( s = 0 \))

\[
F'(s) = \frac{1}{1 - s} = \sum_{n=0}^{\infty} s^n.
\]

Then, integrate this series term by term

\[
F(s) = -\log(1 - s) = \int_0^s F'(s) ds = \sum_{n=0}^{\infty} \frac{s^n}{n}.
\]

Alternatively you may use Matlab/Octave commands:

```matlab
syms s
taylor(-log(1-s),’order’,7)
```

– 7.1: Use Octave’s `taylor(-log(1-s))` to 7th order, as in the example above.

1. Try the above Matlab/Octave commands. Give the first 7 terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above). **Sol:**

\[
F(s) = \cdots + \frac{s^7}{7} + \frac{s^6}{6} + \frac{s^5}{5} + \frac{s^4}{4} + \frac{s^3}{3} + \frac{s^2}{2} + s
\]

2. What is the inverse Laplace transform of this series? Consider the series term by term.

**Sol:** \( f(t) = \sum \delta^{(n)}/n \)
4.2 PROBLEMS DE-2

71 – 7.2: The function \(1/\sqrt{z}\) has a branch point at \(z = 0\), thus it is singular there.

1. Can you apply Cauchy’s integral theorem when integrating around the unit circle? Sol: No, one cannot apply the Cauchy Theorem since it is not analytic at \(z = 0\). But the integral may be evaluated.

2. Below is a Matlab/Octave code that computes \(\int_0^{4\pi} \frac{dz}{\sqrt{z}}\) using Matlab’s/Octave’s symbolic analysis package:

```
syms z
I=int(1/sqrt(z))
J = int(1/sqrt(z),exp(-j*pi),exp(j*pi))
eval(J)
```

Run this script. What answers do you get for \(I\) and \(J\)?

Sol: This script returns the answers \(I = 2\sqrt{z}\) and \(J = 2.4493e-16\), which is numerically the same as zero.

3. Modify this code to integrate \(f(z) = 1/z^2\) once around the unit circle. What answers do you get for \(I\) and \(J\)? Sol: This function has a 2d order pole at \(s = 0\). Thus from the CIT, the integral evaluates to zero.

Proof:

\[
I = \oint ds = -\frac{1}{s}\left|_0^{2\pi}\right. = -e^{-i\theta}\left|_0^{2\pi}\right. = -(1 - 1) = 0
\]

More generally \(I = \oint \frac{ds}{s^n} = 0\) for \(n \neq 1\). As best I know, this holds for any \(n \in \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{R}, \mathbb{C}\). For \(n = 1\) it has a value of \(2\pi j\).

– 7.3: Bessel functions can describe waves in a cylindrical geometry

The Bessel function has a Laplace transform with a branch cut

\[J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}.\]

Draw a hand sketch showing the nature of the branch cut. Hint: Use \(z\text{viz}\). Sol: The roots are given by \(s_\pm = \pm j\). The branch cut connects the two roots, or can go from each root to \(\infty\). Either choice is valid.

**Problem # 8: Matlab/Octave exercises:**

– 8.1: Comment on the following Matlab/Octave exercises

1. Try the following Matlab/Octave commands, and then comment on your findings.

   ```matlab
   %Take the inverse LT of 1/sqrt(1+s^2)
syms s
   I=ilaplace(1/(sqrt(1+s^2)));
disp(I)
   Sol: I = J_0(t)u(t).
   ```

   ```matlab
   %Find the Taylor series of the LT
   T = taylor(1/sqrt(1+s^2),10); disp(T);
   Sol:
   T = \cdots + \frac{3s^4}{8} - \frac{s^2}{2} + 1
   ```

   ```matlab
   %Verify this
   syms t
   J=laplace(besselj(0,t));
disp(J);
   Sol: I = \frac{1}{\sqrt{1+s^2}}.
   ```

   ```matlab
   %plot the Bessel function
   t=0:0.1:10*pi;
b=besselj(0,t);
plot(t/pi,b);
grid on;
Sol: Plot of J_0(t)u(t).
```
– 8.2: When did Friedrich Bessel live?
Sol: 1784-1846, in Königsberg Germany.

– 8.3: What did he use Bessel functions for?
Sol: Solving the Bessel equation, which is the wave equation in 2D. Bessel functions were first introduced by the Daniel Bernoulli.

– 8.4: Using \texttt{zviz}, for each of the following functions

1. Describe the plot generated by \texttt{zviz \texttt{S=Z}}. Sol: It is a polar plot of the function, with intensity coding the magnitude and color coding the phase. Red is a positive real number while and blue is a negative real number.

2. Are the functions defined below legal Brune impedances? (i.e., Do they function obey $\Re Z(\sigma > 0) \geq 0$)? \texttt{Hint: Consider the phase (color). Plot \texttt{zviz Z} for a reminder of the colormap.}

\begin{enumerate}
\item \texttt{zviz 1./sqrt(1+S.ˆ2)}
Sol: No. The RHP has blue near the branch cut, in the RHP.
\item \texttt{zviz 1./sqrt(1-S.ˆ2)}
Sol: NO, there is a branch cut in the RHP.
\item \texttt{zviz 1./(1+sqrt(S))}
Sol: Yes, its red almost everywhere even though it has a branch cut from $[-\infty < \sigma \leq -10]$. Since $1/\sqrt{s}$ has an \texttt{LT}^{-1}, this function must as well. Matlab found
\[
\frac{1}{\sqrt{1+s}} \leftrightarrow \frac{e^{-t}}{\sqrt{\pi} \sqrt{t}} u(t),
\]
however Octave failed to find the inverse transform, (but was able to find the forward transform).
\end{enumerate}

\textbf{Problem # 9:} This problem is for extra credit.
Find the \texttt{LT}^{-1} of one factor of the Riemann zeta function $\zeta_p(s)$ where
\[
\zeta_p(s) \leftrightarrow z_p(t).
\]
Describe your results in words. \texttt{Hint: see Eq. AE-1.10, p. 33}
\texttt{Hint: Consider the geometric series representation}
\[
\zeta_p(s) = \frac{1}{1-e^{-st_p}} = \sum_{k=0}^{\infty} e^{-skT_p},
\]
\texttt{(DE-2.8)}

for which you can look up the \texttt{LT}^{-1} transform of each term.
Sol: Since each term in the series is a pure delay\textsuperscript{2}
\[
z_p(t) = \delta(t)T_p \equiv \sum_{k=0}^{\infty} \delta(t - kT_p) \leftrightarrow \frac{1}{1-e^{-st_p}}.
\]
\texttt{(DE-2.9)}

\textbf{Problem # 10: Inverse transform of products:}
The time domain version of Eq. DE-2.8 may be written as the convolution of all the $z_k(t)$ factors
\[
z(t) \equiv z_2(t) \ast z_3(t) \ast z_5(t) \ast z_7(t) \cdots \ast z_p(t) \ast \cdots ,
\]
\texttt{(DE-2.10)}

\textsuperscript{2}Here we use a shorthand double-parentheses notation to define the infinite (one-sided) sum $f(t)_T \equiv \sum_{k=0}^{\infty} f(t - kT).$
4.2. PROBLEMS DE-2

If we take a function \( Y = \Gamma(\zeta) \) with the input at one end and the output at the other. To describe the transfer function since impedance is also related to the round-trip delay \( T \), we can use the equation:

\[
q(t) = \alpha q(n - T_p) + v(t)
\]

\[
i(t) = q(t) - (1/\alpha)q(t - T_p)
\]

**Figure 4.2:** This feedback network is described by a time-domain difference equation with delay \( T_p \), has an all-pole transfer function \( \zeta_p(s) \equiv Q(s)/I(s) \) given by Eq. DE-2.11, which physically corresponds to a stub of a transmission line, with the input at one end and the output at the other. To describe the \( \zeta(s) \) function we must take \( \alpha = -1 \). A transfer function \( Y(s) = V(s)/I(s) \) that has the same poles as \( \zeta_p(s) \), but with zeros as given by Eq. DE-2.12, is the input admittance \( Y(s) = I(s)/V(s) \) of the transmission line, defined at the ratio of the Laplace transform of the current \( i(t) \leftrightarrow I(s) \) over the voltage \( v(t) \leftrightarrow V(s) \).

**Physical interpretation:** Such functions may be generated in the time domain as shown in Fig. 4.2 (p. 73), using a feedback delay of \( T_p \) seconds described by the two equations in the figure with a unity feedback gain \( \alpha = -1 \). Taking the Laplace transform of the system equation we see that the transfer function between the state variable \( q(t) \) and the input \( x(t) \) is given by \( \zeta_p(s) \), which is and all-pole function, since

\[
Q(s) = e^{-sT_p}Q(s) + V(s), \quad \text{or} \quad \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}.
\]

(Closing the feed-forward path gives a second transfer function \( Y(s) = I(s)/V(s) \), namely

\[
Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}.
\]

If we take \( i(t) \) as the current and \( v(t) \) as the voltage at the input to the transmission line, then \( y_p(t) \leftrightarrow \zeta_p(s) \) represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the \( j\omega \) axis. By a slight modification \( \zeta_p(s) \) may alternatively be written as

\[
Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j \tan(sT_p/2).
\]

Every impedance \( Z(s) \) has a corresponding reflectance function given by a Möbius transformation, which may be read off of Eq. DE-2.12 as

\[
\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}
\]

since impedance is also related to the round-trip delay \( T_p \) on the line. The inverse Laplace transform of \( \Gamma(s) \) is the round trip delay \( T_p \) on the line

\[
\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}.
\]
Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations, that are not obvious in the frequency domain. Transforming to the time domain is a form of analytic continuation of \( \zeta(s) \), that depends on the assumption that \( z(t) \) is one-sided in time (causal).
4.3 Problems DE-3

**Topic of assignment:**
Brune impedance, lattice transmission line analysis.

**Brune Impedance**

**Problem # 1: Residue form**

A Brune impedance is defined as the ratio of the force \( F(s) \) over the flow \( V(s) \), and may be expressed in residue form as

\[
Z(s) = c_0 + \sum_{k=1}^{K} \frac{c_k}{s - s_k} = \frac{N(s)}{D(s)} \quad (DE-3.1)
\]

with

\[
D(s) = \prod_{k=1}^{K} (s - s_k) \quad \text{and} \quad c_k = \lim_{s \to s_k} (s - s_k)D(s) = \prod_{n'=1}^{K-1} (s - s_{n'}). 
\]

The prime on index \( n' \) means that \( n = k \) is not included in the product.

---

1. **1.1: Find the Laplace transform (LT) of a 1) spring, 2) dashpot and 3) mass.**

Express these in terms of the force \( F(s) \) and the velocity \( V(s) \), along with the electrical equivalent impedance:

1. **Hooke’s Law** \( f(t) = Kx(t) \). **Sol:** Taking the LT gives

\[
F(s) = KX(s) = KV(s)/s \leftrightarrow f(t) = Ku(t) \ast v(t) = K \int_{0}^{t} v(t),
\]

since

\[
v(t) = \frac{d}{dt} x(t) \leftrightarrow V(s) = sX(s).
\]

Thus the impedance of the spring is

\[
Z_s(s) = \frac{K}{s} \leftrightarrow z(t) = Ku(t),
\]

which is analogous to the impedance of an electrical capacitor. The relationship may be made tighter by specifying the compliance of the spring as \( C = 1/K \). ■

2. **Dash-pot resistance** \( f(t) = Rv(t) \). **Sol:** From the LT this becomes

\[
F(s) = RV(s)
\]

and the impedance of the dash-pot is then

\[
Z_r = R \leftrightarrow R\delta(t),
\]

analogous to that of an electrical resistor. ■
3. Newton’s Law for Mass \( f(t) = M \frac{dv(t)}{dt} \). **Sol:** Taking the \( \mathcal{L} \) gives

\[
f(t) = M \frac{d}{dt} v(t) \Leftrightarrow F(s) = M sV(s),
\]

thus

\[
Z_m(s) = sM \leftrightarrow M \frac{d}{dt},
\]

analogous to an electrical inductor.

– 1.2: Take the Laplace transform (\( \mathcal{L} \)) of Eq. DE-3.2 and find the total impedance \( Z(s) \) of the mechanical circuit.

\[
M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + K x(t) = f(t) \Leftrightarrow (Ms^2 + Rs + K)X(s) = F(s). \tag{DE-3.2}
\]

**Sol:** From the properties of the \( \mathcal{L} \) that \( \frac{dx}{dt} \leftrightarrow sX(s) \), we find

\[
f(t) \leftrightarrow F(s) = Ms^2X(s) + RsX(s) + KX(s).
\]

In terms of velocity this is \( (Ms + R + K/s)V(s) = F(s) \). Thus the circuit impedance is

\[
z(t) \leftrightarrow Z(s) = \frac{F}{V} = \frac{K + Rs + Ms^2}{s}.
\]

– 1.3: What are \( N(s) \) and \( D(s) \) (e.g. Eq. DE-3.1)?

**Sol:** \( D(s) = s \) and \( N(s) = K + Rs + Ms^2 \).

– 1.4: Assume that \( M = R = K = 1 \), find the residue form of the admittance \( Y(s) = 1/Z(s) \) (e.g. Eq. DE-3.1) in terms of the roots \( s_{\pm} \). Hint: Check your answer with the Octave/Matlab residue command.

**Sol:** First find the roots of the numerator of \( Z(s) \) (the denominator of \( Y(s) \)):

\[
s_{\pm}^2 + s_{\pm} + 1 = (s_{\pm} + 1/2)^2 + 3/4 = 0,
\]

which is

\[
s_{\pm} = \frac{-1 \pm j\sqrt{3}}{2}.
\]

Second form a partial fraction expansion

\[
s \frac{1}{1 + s + s^2} = c_0 + \frac{c_+}{s - s_+} + \frac{c_-}{s - s_-} = \frac{s(c_+ + c_-) - (c_+ s_- + c_- s_+)}{1 + s + s^2}.
\]

Comparing the two sides shows that \( c_0 = 0 \). We also have two equations for the residues \( c_+ + c_- = 1 \) and \( c_+ s_- + c_- s_+ = 0 \). The best way to solve this is to set up a matrix relation and take the inverse

\[
\begin{bmatrix} 1 & 1 \\ s_- & s_+ \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{thus:} \quad \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{s_+ - s_-} \begin{bmatrix} s_+ & -1 \\ -s_- & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

which gives \( c_+ = \pm \frac{s_+}{s_+ - s_-} \). The denominator is \( s_+ - s_- = j\sqrt{3} \) and the numerator is \( \pm 1 + j\sqrt{3} \). Thus

\[
c_\pm = \pm \frac{s_\pm}{s_+ - s_-} = \frac{1}{2} \left( \pm \frac{j}{\sqrt{3}} \right).
\]

As always, finding the coefficients is always the most difficult part. Using 2x2 matrix algebra automates the process. Always check your final result as correct.
4.3. PROBLEMS DE-3

- 1.5: By applying the CRT, find the inverse Laplace transform ($L^{-1}$). Use the residue form of the expression that you derived in the previous exercise.

\[
z(t) = \frac{1}{2\pi j} \int_C Z(s)e^{st}ds.
\]

where $C$ is the Laplace contour which encloses the entire left-half $s$ plane. Applying the CRT

\[
z(t) = c_+ e^{s_+ t} + c_- e^{s_- t}.
\]

where $s_{\pm} = -1/2 \pm j\sqrt{3}/2$ and $c_{\pm} = 1/2 \pm j/(2\sqrt{3})$. ■

**Problem #2: Train-mission-line** We wish to model the dynamics of a freight-train having $N$ such cars, and study the velocity transfer function under various load conditions.

As shown in Fig. 4.3, the train model consists of masses connected by springs.

**Physical description:**

![Diagram of a train model consisting of masses connected by springs.](image)

Figure 4.3: Depiction of a train consisting of cars, treated as a mass $M$ and linkages, treated as springs of stiffness $K$ or compliance $C = 1/K$. Below it is the electrical equivalent circuit, for comparison. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is $\Delta$ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL. This method, called a Cauer synthesis, is based on the ABCD transmission line method of Secs. 3.5 (p. 92).

Use the ABCD method (see discussion in Appendix B.3, p. 201) to find the matrix representation of the system of Fig. 4.3. Define the force on the $n$th train car $f_n(t) \leftrightarrow F_n(\omega)$, and velocity $v_n(t) \leftrightarrow V_n(\omega)$.

Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($A = D$) identical cell matrix $T(s)$.

- 2.1: Find the elements of the ABCD matrix $T$ for the single cell that relate the input node 1 to output node 2

\[
\begin{bmatrix}
F \\
V
\end{bmatrix}_1 = T \begin{bmatrix}
F(\omega) \\
-V(\omega)
\end{bmatrix}_2.
\]

\[(DE-3.3)\]

Sol:

\[
T =\begin{bmatrix}
1 & sM/2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & sM/2
\end{bmatrix}
=\begin{bmatrix}
1 + s^2MC/2 & (sM)(1 + s^2MC/4) \\
MC & 1 + s^2MC/2
\end{bmatrix}
\]

\[(DE-3.4a)\]

■
2.2: Express each element of $T(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi j f$, $s_c = 2\pi j f_c$). The Nyquist sampling cutoff frequency $f_c$ is defined in terms the minimum number of cells (i.e., 2) of length $\Delta$ per wavelength:

The Nyquest sampling theorem says that there are at least two cars per wavelength (more than two time samples at the highest frequency). From the figure, the distance between cars $\Delta = c_o T_o [m]$ where $c_o = 1/\sqrt{M C}$.

The cutoff frequency obeys $f_c \lambda = c_o$ where the Nyquist wavelength is $\lambda = 2\Delta$. Therefore the Nyquist sampling condition is

$$\omega < \omega_c = 2\pi f_c \equiv \frac{2\pi c_o}{2\Delta} = \frac{\pi}{\Delta \sqrt{MC}}.$$

(DE-3.5)

Sol: Reiterating what was said above: the system in Fig. 4.3 represents a transmission line having a wave speed of $c_o = 1/\sqrt{MC}$ and characteristic impedance $r_o = \sqrt{M/C}$. Each cell, composed of 2 masses $M$ connected by one spring $K$, has length $\Delta$.

We wish to define the Nyquist frequency $f_c$ such that the wavelength $\lambda > 2\Delta$, where $\Delta$ is the cell length. Using the formula for the wavelength in terms of the wave velocity and frequency we find

$$\lambda = c_o/f_c = 2\Delta,$$

thus we conclude that

$$f < f_c = \frac{c_o}{2\Delta} = \frac{1}{2\Delta \sqrt{MC}}.$$

(DE-3.6)

If we wish to have the system be accurate for a given frequency we may make the cell length $\Delta$ smaller, while keeping the velocity constant ($MC$ is held constant). Thus the characteristic resistance [ohms/unit length] $r_o$ must change as $f_c \to \infty$ and $\Delta \to 0$. We can either let $M \to \infty$ and $C \to 0$ (their product remains constant), or the other way around. In one case $r_o \to \infty$ and in the other case it goes to 0.

2.3: Use the property of the Nyquist sampling frequency $f < f_c$ (Eq. DE-3.6) to remove higher order powers of frequency

$$1 + \left(\frac{s}{s_c}\right)^2 \approx 1$$

(DE-3.7)

to determine a band-limited approximation of $T(s)$. Sol:

$$T = \begin{bmatrix} 1 + 2(s/s_c)^2 & sM(1 + (s/s_c)^2) \\ sC & 1 + 2(s/s_c)^2 \end{bmatrix} \begin{bmatrix} 1 & sM \\ sC & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 \\ sC & 1 \end{bmatrix}$$

The approximation is highly accurate below the Nyquist cutoff frequency $s < s_c$. Given any desired frequency $f$, we can always make the cell size $\Delta$ smaller by decreasing $M$ and $C$, while keeping $f < f_c$ and the cell velocity constant ($c_o = 1/\sqrt{MC}$). Thus the Nyquist condition represents a computational bound, not a physical limitation.

Problem #3

Now consider the cascade of $N$ such $T(s)$ matrices, and perform an eigen analysis.

3.1: Find the eigenvalues and eigenvectors of $T(s)$, as functions of $s/s_c$.

Sol: Matrix $T(s)$ has eigenvalues

$$\lambda_{\pm} = 1 \mp 2s/s_c \approx e^{\pm 2s/s_c} = e^{\mp sT_c}.$$
From this we can interpret the eigenvalues as the cell delay $T_c = 2/s_c$.

The corresponding unnormalized eigenvectors are

$$E_\pm = \begin{bmatrix} \pm \sqrt{M/C} \\ 1 \end{bmatrix},$$

where the characteristic impedance defined is $r_o = \sqrt{M/C}$. ■

**Problem # 4**

Finally, find the velocity transfer function:

- 4.1: Assuming that $N = 2$ and that $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv \frac{V_2}{V_1}$. From the results of the $T$ matrix, find

$$H_{21}(s) = \frac{V_2}{V_1} \mid_{F_2=0}$$

Express $H_{12}$ in terms of a residue expansion. Sol: From Eq. DE-3.4a, $V_1 = sCF_2 - (s^2MC/2 + 1)V_2$. Since $F_2 = 0$

$$\frac{V_2}{V_1} = \frac{-1}{s^2MC/2 + 1} = \left( \frac{c_+}{s - s_+} + \frac{c_-}{s - s_-} \right)$$

having eigenfrequencies $s_\pm = \pm \sqrt{\frac{2}{MC}} = \pm s_c$ and residues $c_\pm = \pm j\sqrt{2MC} = \pm s_c$. ■

- 4.2: Find $h_{21}(t) \leftrightarrow H_{21}(s)$.

Sol:

$$h(t) = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \frac{e^{st}}{s^2MC/2 + 1} \frac{ds}{2\pi j} = c_+ e^{-s_+ t} u(t) + c_- e^{-s_- t} u(t).$$

The integral follows from the Cauchy Residue theorem (CRT). ■

- 4.3: What is the input impedance $Z_1 = F_1/V_1$, assuming $F_2 = -r_0V_2$?

Sol: Starting from Eq. DE-3.4a find $Z_2$

$$Z_2(s) = \frac{F_2}{V_2} = T \begin{bmatrix} F \\ -V \end{bmatrix}_2 = \frac{-(1 + s^2CM/2)r_0V_2 - sM(1 + s^2CM/4)V_2}{-sCr_0V_2 - (1 + s^2CM/2)V_2}$$

■

- 4.4: Simplify the expression for $Z_2$ assuming:

1. Assuming the characteristic impedance $r_o = \sqrt{M/C}$
2. terminate the system in $r_o$: $F_2 = -r_0V_2$ (i.e., $-V_2$ cancels)
3. Assume higher order frequency terms are less than 1 ($|s/s_c| < 1$)
4. Let the number of cells $N \to \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteristic impedance $r_o$, the input impedance $Z_1(s) = r_o$. Thus when we simplify the expression for $T(s)$ it should be equal to $r_o$. Show that this is true for this setup.

Sol: Applying the Nyquist approximation (i.e., ignore second order frequency terms $(s/s_c)^2 \approx 0$)

$$Z_1(s) = \frac{r_o(1 + s^2CM/2)^0 + sM(1 + s^2CM/4)^0}{r_0sC + (1 + s^2CM/2)^0}$$

$$\approx \frac{r_o + sM}{1 + r_0sC} \cdot \frac{r_o + sM}{1 + r_0sC} = \frac{M}{1} \cdot \frac{M}{1} \cdot \frac{r_o + sM}{r_0sC + sM} = \frac{r_2}{M + r_0s/s_c}$$

$$\approx \frac{r_o^2}{M + r_0s/s_c} = \frac{r_o^2}{M}$$

$$= r_o.$$
We conclude that below the Nyquist cutoff frequency, as $N \to \infty$ the system equals a transmission line terminated by its characteristic impedance thus $Z_1(s) = r_o$. ■

4.5: State the ABCD matrix relationship between the first and $N$th node in terms of the cell matrix. Write out the transfer function for one cell: $H_{21}$?

Sol:

$$\mathcal{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Now use the formulae for the eigenvalues and vectors to obtain $\mathcal{T}$ for $N = 1$:

$$\mathcal{T} = E \Lambda E^{-1} = E \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} E^{-1}.$$  ■

4.6: What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?

Sol:

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \mathcal{T}^N \begin{bmatrix} F_N(\omega) \\ -V_N(\omega) \end{bmatrix}$$

along with the eigenvalue expansion

$$\mathcal{T}^N = EA^N E^{-1} = E \begin{bmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{bmatrix} E^{-1}.$$  

where $\lambda_{\pm}^N = e^{\pm s NT_o}$. Recall that $NT_o$ is the one way delay.

We conclude that as we add more cells, the delay linearly increases with $N$, since each eigenvalue represents the delay of one cell, and delay adds. ■
Chapter 5

Vector differential equations

5.1 Problems VC-1

Topic of this assignment:
Vector algebra and fields in $\mathbb{R}^3$; Gradient and scalar Laplacian operator; Definitions of Divergence and Curl; Gauss’s (divergence) & Stokes’ (Curl) Law
Schwarz inequality; Quadratic forms; System postulates

Scalar fields and the $\nabla$ operator

Problem # 1: Let $T(x, y) = x^2 + y$ be an analytic scalar temperature field in 2 dimensions (single-valued $\in \mathbb{R}^2$).

– 1.1: Find the gradient of $T(x)$ and make a sketch of $T$ and the gradient.
Sol: $\nabla (x^2 + y) = 2x \hat{x} + \hat{y}$. The temperature is quadratic in $x$ and linear in $y$, which has the shape of a trough in $x$, linearly increasing in $y$. In the $y$ ($\hat{y}$) direction the gradient is constant, and in the $\hat{x}$ direction, it is linear, and goes through zero at $x = 0$, with $T(0) = 0$. Skiing in the $y$ direction would be a constant ride of slope 1. If the snow had no friction, you would accelerate, but the terminal velocity would be due to the friction of the snow on the skis. Along the $x$ direction, you would accelerate, at first, coming down, and at $x = 0$ you would stop accelerating, and begin slow down. This would be a more interesting problem if you treated it in terms of the forces on the skis and included friction as well as gravity.

– 1.2: Compute $\nabla^2 T(x)$, to determine if $T(x)$ satisfies Laplace’s equation.
Sol: Forming this operation we find that
$$\frac{\partial^2}{\partial x^2} x^2 + \frac{\partial^2}{\partial y^2} y = 2.$$ So $T(x)$ does not satisfy laplace’s equation, rather it satisfies the Poisson equation $\nabla^2 T(x) = 2$.

– 1.3: Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.
Sol: The iso-potential contours are the concave parabolas $y = T_0 - x^2$. 

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– 1.4: The heat flux\(^1\) is defined as \( J(x,y) = -\kappa(x,y)\nabla T \) where \( \kappa(x,y) \) is a constant denoting thermal conductivity at the point \( (x,y) \). Assuming \( \kappa = 1 \) everywhere (the medium is homogenous), plot the vector \( J(x,y) = -\nabla T \) at \( x = 2, y = 1 \). Be clear about the origin, direction and length of your result.

**Sol:** \( J = \nabla T = -2\hat{x} - \hat{y} \) thus \(-\kappa \nabla T(2,1) = J = -(4\hat{x} + \hat{y})\), which has a length of \( \sqrt{17} \) and is pointed 1 unit down and \( 4/\sqrt{17} \) units to the left.

– 1.5: Find the vector \( \perp \) to \( \nabla T(x,y) \), namely tangent to the iso-temperature contours. Hint: Sketch it for one \((x,y)\) point (e.g., \( 2,1 \)) and then generalize.

**Sol:** We may invoke the third dimension \( \hat{z} \) to generate this vector: \( \pm \hat{z} \times \nabla T = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \pm 1 \\ 2x & 1 & 0 \end{bmatrix} = \mp (1\hat{x} - 2x\hat{y} + 0\hat{z}) \). Alternatively, rotate \( \nabla T \) by \( \pm \pi/2 \) in the \((x,y)\) plane.

– 1.6: The thermal resistance \( R_T \) is defined as the potential drop \( \Delta T \) over the magnitude of the heat flux \( |J| \). At a single point the thermal resistance is

\[
R_T(x,y) = -\nabla T/|J|.
\]

How is \( R_T(x,y) \) related to the thermal conductivity \( \kappa(x,y) \)?

**Sol:** \( R_T(x,y) = 1/\kappa(x,y) \). In general, resistance is the reciprocal of conductivity (conductance). This is true for electrical and acoustic systems as well.

**Problem # 2: Acoustic wave equation:** Note: In the following problem, we will work in the frequency domain.

The basic equations of acoustics in 1 dimension are

\[
-\frac{\partial}{\partial x} P = \rho_o s \psi \quad \text{and} \quad -\frac{\partial}{\partial x} \psi = \frac{s}{\eta_o P_o} P.
\]

Here \( P(x,\omega) \) is the pressure (in the frequency domain), \( \psi(x,\omega) \) is the volume velocity (integral of the velocity over the wave-front having area \( A \)), \( s = \sigma + \omega j \), \( \rho_o = 1.2 \) is the specific density of air, \( \eta_o = 1.4 \) and \( P_o \) is the atmospheric pressure (i.e., \( 10^5 \) [Pa]). Note that the pressure field \( P \) is a scalar (pressure does not have direction), while the volume velocity field \( \psi \) is a vector (velocity has direction).

We can generalize these equations to 3 dimensions using the \( \nabla \) operator

\[
-\nabla P = \rho_o s \psi \quad \text{and} \quad -\nabla \cdot \psi = \frac{s}{\eta_o P_o} P.
\]

– 2.1: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure \( P \),

\[
\nabla^2 P = \frac{s^2}{c_0^2} P
\]

where \( c_0 \) is a constant representing the speed of sound. **Sol:** We wish to remove \( \psi \) from the two equations, to obtain a single equation in pressure. If we take the partial wrt \( x \) of the pressure equation, and then substitute the velocity equation, to remove the velocity:

\[
\nabla^2 P = -\rho_o s \nabla \cdot \psi = \frac{s^2 \rho_o}{\eta_o P_o} P = \frac{s^2}{c_0^2} P
\]

\(^1\)The heat flux is proportional to the change in temperature times the thermal conductivity \( \kappa \) of the medium.
5.1. PROBLEMS VC-1

- 2.2: What is \( c_0 \) in terms of \( \eta_0, \rho_0, \) and \( P_0 \)?

Sol: Comparing the last two terms from the previous solution we see that

\[
c_0 = \sqrt{\frac{\eta_0 P_0}{\rho_0}}.
\]

- 2.3: Rewrite the pressure wave equation in the time domain, using the time derivative property of the Laplace transform (e.g. \( \frac{dx}{dt} \leftrightarrow sX(s) \)). For your notation, define the time–domain signal using a lowercase letter, \( p(x,y,z,t) \leftrightarrow P \).

Sol:

\[
\nabla^2 p(x,y,z,t) = \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} p(x,y,z,t)
\]

Vector fields and the \( \nabla \) operator

Vector Algebra

Problem # 3: Let \( \mathbf{R}(x,y,z) \equiv x(t) \mathbf{\hat{x}} + y(t) \mathbf{\hat{y}} + z(t) \mathbf{\hat{z}} \):

- 3.1: If \( a, b, c \) are constants, what is \( \mathbf{R}(x,y,z) \cdot \mathbf{R}(a,b,c) \)?

Sol: Using the formula for a scalar dot-product:

\[
\mathbf{R}(x,y,z) \cdot \mathbf{R}(a,b,c) \equiv [x(t) \mathbf{\hat{x}} + y(t) \mathbf{\hat{y}} + z(t) \mathbf{\hat{z}}] \cdot [a \mathbf{\hat{x}} + b \mathbf{\hat{y}} + c \mathbf{\hat{z}}]
\]

\[
= x(t)a + y(t)b + z(t)c.
\]

- 3.2: If \( a, b, c \) are constants, what is \( \frac{d}{dt} (\mathbf{R}(x,y,z) \cdot \mathbf{R}(a,b,c)) \)?

Sol: \( \left( a \frac{d}{dt} x(t) + b \frac{d}{dt} y(t) + c \frac{d}{dt} z(t) \right) \).

Problem # 4: Find the divergence and curl of the following vector fields:

- 4.1: \( \mathbf{v} = \mathbf{\hat{x}} + \mathbf{\hat{y}} + 2 \mathbf{\hat{z}} \)

Sol: \( \nabla \cdot \mathbf{v} = 0, \nabla \times \mathbf{v} = 0 \)

- 4.2: \( \mathbf{v}(x,y,z) = x \mathbf{\hat{x}} + xy \mathbf{\hat{y}} + z^2 \mathbf{\hat{z}} \)

Sol: \( \nabla \cdot \mathbf{v} \equiv \partial_x x + \partial_y y + \partial_z z^2 = 1 + x + 2z \nabla \times \mathbf{v} \equiv \left| \begin{array}{ccc} \mathbf{\hat{x}} & \mathbf{\hat{y}} & \mathbf{\hat{z}} \\ \partial_x & \partial_y & \partial_z \\ x & y & z^2 \end{array} \right| = (0-0) \mathbf{\hat{x}} + (0-0) \mathbf{\hat{y}} + (y-0) \mathbf{\hat{z}} = y \mathbf{\hat{z}} \)

- 4.3: \( \mathbf{v}(x,y,z) = x \mathbf{\hat{x}} + xy \mathbf{\hat{y}} + \log(z) \mathbf{\hat{z}} \)

Sol: Divergence: \( \partial_x x + \partial_y xy + \partial_z \log(z) = 1 + x + 1/z \), Curl: \( \mathbf{\hat{x}} \left( \partial_y \log(z) - \partial_z y \right) + \mathbf{\hat{y}} \left( \partial_x x - \partial_z \log(z) \right) + \mathbf{\hat{z}} \left( \partial_x y - \partial_y x \right) = \mathbf{\hat{y}} \)

- 4.4: \( \mathbf{v}(x,y,z) = \nabla (1/x + 1/y + 1/z) \)

Sol: First find \( \mathbf{v} = -\left( \mathbf{\hat{x}}/x^2 + \mathbf{\hat{y}}/y^2 + \mathbf{\hat{z}}/z^2 \right) \). Divergence of \( \mathbf{v} \): \( -(\partial_x 1/x^2 + \partial_y 1/y^2 + \partial_z 1/z^2) = 2(1/x^3 + 1/y^3 + 1/z^3) \), Curl of \( \mathbf{v} \): 0, because the curl of the gradient is always zero.
Vector & scalar field identities

Problem # 5: Find the divergence and curl of the following vector fields:

– 5.1: \( \mathbf{v} = \nabla \phi \), where \( \phi(x, y) = xe^y \)
Sol: \( \nabla \times \nabla \phi = 0 \) and \( \nabla^2 \phi = xe^y \) ■

– 5.2: \( \mathbf{v} = \nabla \times \mathbf{A} \), where \( \mathbf{A} = x \hat{x} + y \hat{y} + z \hat{z} \)
Sol: \( \nabla \cdot (\nabla \times \mathbf{A}) = 0 \) and \( \nabla \times (\nabla \times \mathbf{A}) = 0 \) ■

– 5.3: \( \mathbf{v} = \nabla \times \mathbf{A} \), where \( \mathbf{A} = y \hat{x} + x^2 \hat{y} + z \hat{z} \)
Sol: \( \nabla \cdot (\nabla \times \mathbf{A}) = 0 \) and \( \nabla \times (\nabla \times \mathbf{A}) = -2 \hat{y} \) ■

– 5.4: For any differentiable vector field \( \mathbf{V} \), write down two vector-calculus identities that are equal to zero.
Sol: Curl of the gradient \( \nabla \times \nabla \Phi(x, y, z) = 0 \) and the divergence of the curl \( \nabla \cdot \nabla \times \mathbf{V}(x, y, z) = 0 \) are both zero. (Page 780, Stillwell) ■

– 5.5: What is the most general form of a vector field may be expressed in, in terms of scalar \( \Phi \) and vector \( \mathbf{A} \) potentials?
Sol: \( \mathbf{V} = \nabla \Phi(x, y, z) + \nabla \times \mathbf{A}(x, y, z) \), where \( \Phi \) is the scalar potential and \( \mathbf{A} \) is the vector potential. ■

Problem # 6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.

– 6.1: Let \( \mathbf{v} = \sin(x) \hat{x} + y \hat{y} + z \hat{z} \). Find \( \nabla \cdot (\nabla \times \mathbf{v}) \).
Sol: 0 ■

– 6.2: Let \( \mathbf{v} = \sin(x) \hat{x} + y \hat{y} + z \hat{z} \). Find \( \nabla \times (\nabla \sqrt{\mathbf{v} \cdot \mathbf{v}}) \)
Sol: 0 ■

– 6.3: Let \( \mathbf{v}(x, y, z) = \nabla(x + y^2 + \sin(\log(z))) \). Find \( \nabla \times \mathbf{v}(x, y, z) \).
Sol: It is zero because \( \nabla \times \nabla f(x, y, z) \) is always zero. ■

Integral theorems

Problem # 7: For each of the following problems, in a few words, identify either Gauss’ or Stokes’ law, define what it means, and explain the formula that follows the question.

– 7.1: What is the name of this formula?
\[
\int_{S} \mathbf{n} \cdot \mathbf{v} \, dA = \int_{\mathcal{V}'} \nabla \cdot \mathbf{v} \, dV.
\]
Sol: This is the integral form of Gauss’ law. The unit normal vector is \( \perp \) to the surface \( S \) having area \( A \equiv \int_{S} dA \) The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field \( \nabla \cdot \mathbf{v} \) over the volume contained by the surface, and defined as \( \mathcal{V}' \). ■

– 7.2: What is the name of this formula?
\[
\int_{S} (\nabla \times \mathbf{V}) \cdot d\mathbf{S} = \oint_{C} \mathbf{V} \cdot d\mathbf{R}.
\]
Give one important application. Sol: Stokes Theorem, which relates the differential to the integral form of Maxwell’s equations. ■
5.1. PROBLEMS VC-1

– 7.3: Describe a key application of the vector identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}.$$ 

**Sol:** When we wish to reduce Maxwell’s two curl equations to the vector wave equation, we must use this identity.

**System Classification**

**Problem # 8:** Answer the following system classification questions about physical systems, in terms of the system postulates.

– 8.1: Provide a brief definition of the following properties: 

**L/NL :** linear(L)/nonlinear(NL): 
**Sol:** Superposition and scaling hold

**TI/TV :** time-invariant(TI)/time varying(TV): 
**Sol:** The measurement time is irrelevant

**P/A :** passive(P)/active(A): 
**Sol:** An active system has a power source, a passive system does not.

**C/NC :** causal(C)/non-causal(NC): 
**Sol:** Responds only when driven for \( t \geq 0 \). Does not anticipate for negative time.

**Re/Clx :** real(Re)/complex(Clx): 
**Sol:** The time function is real (or complex).

– 8.2: Along the rows of the table, classify the following systems: In terms of a table having 5 columns, labeled with the abbreviations: L/NL, TI/TV, P/A, C/NC, Re/Clx:

<table>
<thead>
<tr>
<th>#</th>
<th>Case:</th>
<th>Definition</th>
<th>L/NL</th>
<th>TI/TV</th>
<th>P/A</th>
<th>C/NC</th>
<th>Re/Clx</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Resistor</td>
<td>( v(t) = r_0 \ i(t) )</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>2</td>
<td>Inductor</td>
<td>( v(t) = L \frac{d}{dt} i(t) )</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: P</td>
<td>Sol: C</td>
</tr>
<tr>
<td>3</td>
<td>Switch</td>
<td>( v(t) = \begin{cases} 0 &amp; t \leq 0 \ V_0 &amp; t &gt; 0 \end{cases} )</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: P</td>
<td>Sol: P</td>
<td>Sol: C</td>
</tr>
<tr>
<td>5</td>
<td>Transistor</td>
<td>( I_{out} = g_m(V_{in}) )</td>
<td>Sol: NL</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>7</td>
<td>Resistor</td>
<td>( v(t) = r_0 \ i(t + 3) )</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: P</td>
<td>Sol: NC</td>
</tr>
<tr>
<td>8</td>
<td>modulator</td>
<td>( f(t) = e^{i2\pi t} g(t) )</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: P</td>
<td>Sol: P</td>
<td>Sol: C</td>
</tr>
</tbody>
</table>

**Sol:** Notes:
1. is a nonlinear system and is active system only when it is connected to a battery, similar to a diode.
2. The current is non-causal since it has a \( 3 \) [s] negative time delay, specified in the time domain.
3. is 1 Hz complex-modulation, so it is both complex and time-varying (TV)

– 8.3: Using the same classification scheme, characterize the following equations:

<table>
<thead>
<tr>
<th>#</th>
<th>Case:</th>
<th>L/NL</th>
<th>TI/TV</th>
<th>P/A</th>
<th>C/NC</th>
<th>Re/Clx</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( A(x) \frac{dy(t)}{dx} + D(t) y(x, t) = 0 )</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: P</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{dy(t)}{dt} + \sqrt{t} y(t) = \sin(t) )</td>
<td>Sol: L</td>
<td>Sol: TV</td>
<td>Sol: ?</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>3</td>
<td>( y^2(t) + y(t) = \sin(t) )</td>
<td>Sol: NL</td>
<td>Sol: TI</td>
<td>Sol: ?</td>
<td>Sol: C</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{dy}{dt} + xy(t + 1) + x^2 y = 0 )</td>
<td>Sol: L</td>
<td>Sol: TI</td>
<td>Sol: P</td>
<td>Sol: NC</td>
<td>Sol: Re</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{dy}{dt} + (t - 1) y^2(t) = ie^t )</td>
<td>Sol: NL</td>
<td>Sol: TV</td>
<td>Sol: A?</td>
<td>Sol: C</td>
<td>Sol: Clx</td>
</tr>
</tbody>
</table>
5.2 Problems VC-2

Topic of this assignment:
Maxwell’s equations (ME) and variables (E, D; B, H); Compressible and rotational properties of vector fields; fundamental theorem of vector calculus (Helmholtz’ Theorem); Riemann zeta function; Wave equation.

Notation: The following notation is used in this assignment:
1. \( s = \sigma + j \omega \) is the Laplace frequency, as used in the Laplace transform.
2. A Laplace transform pair are indicated by the symbol \( \leftrightarrow \): e.g., \( f(t) \leftrightarrow F(s) \).
3. \( \pi_k \) is the \( k \)th prime (i.e., \( \pi_k \in \mathbb{P} \), e.g., \( \pi_k = [2, 3, 5, 7, 11, 13, \cdots] \) for \( k = 1..6 \)).

Partial differential equations (PDEs): Wave equation

**Problem #1: Solve the wave equation in one dimension by defining \( \xi = t - x/c \).**

\[-1.1: \text{Show that d’Alembert’s solution, } \varrho(x,t) = f(t - x/c) + g(t + x/c), \text{ is a solution to the acoustic pressure wave equation, in 1-dimension:}
\]

\[
\frac{\partial^2 \varrho(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \varrho(x,t)}{\partial t^2},
\]

where \( f(\xi) \) and \( g(\xi) \) are arbitrary functions. **Sol:**

\[
\frac{\partial}{\partial x} \varrho(x,t) = \frac{\partial}{\partial x} f(t - x/c) + \frac{\partial}{\partial x} g(t + x/c) = \frac{-1}{c} f'(t - x/c) + \frac{1}{c} g'(t + x/c) \quad \text{(VC-2.1)}
\]

\[
\frac{\partial^2}{\partial x^2} \varrho(x,t) = \frac{\partial^2}{\partial x^2} f(t - x/c) + \frac{\partial^2}{\partial x^2} g(t + x/c) = \frac{1}{c^2} f''(t - x/c) + \frac{1}{c^2} g''(t + x/c) \quad \text{(VC-2.2)}
\]

\[
\frac{\partial^2}{\partial t^2} \varrho(x,t) = \frac{\partial^2}{\partial t^2} f(t - x/c) + \frac{\partial^2}{\partial t^2} g(t + x/c) = f''(t - x/c) + g''(t + x/c) \quad \text{(VC-2.3)}
\]

**Problem #2: Solution to the wave equation in spherical coordinates (i.e, 3-dimensions):**

\[-2.1: \text{Write out the wave equation in spherical coordinates } \varrho(r,\theta,\phi,t). \text{ Only consider the radial term } r (i.e., \text{dependence on angles } \theta, \phi \text{ is assumed to be zero}). \text{ Hint: The form of the Laplacian as a function of the number of dimensions is given in Eq. 5.1.3.10 (page 145). Alternatively, look it up on the internet or in a calculus book.}
\]

**Sol:** Given the formula for the Laplacian in spherical coordinates, the wave equation is

\[
\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \varrho(r,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varrho(r,t)
\]
5.2. PROBLEMS VC-2

- 2.2: Show that the following is true:

$$\nabla^2 r \varrho(r) \equiv \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \varrho(r) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \varrho(r).$$  \hfill (VC-2.4)

*Hint: Expand both sides of the equation.*  
*Sol:* Both sides of the equation expand to

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r} \frac{\partial R}{\partial r}.$$

- 2.3: Use the results from Eq. VC-2.4 to show that the solution to the spherical wave equation is

$$\nabla^2 r \varrho(r,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varrho(r,t)$$  \hfill (VC-2.5)

$$\varrho(r,t) = \frac{f(t-r/c)}{r} + \frac{g(t+r/c)}{r}.$$  \hfill (VC-2.6)

*Sol:* This proceed exactly as in the rectangular case (see above) except one must first recognize that the Laplacian in spherical coordinates may be written as

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} r \varrho(r).$$  \hfill (VC-2.7)

One then may proceed to use the solution for the rectangular case, but for $r \varrho(r)$, and then divide that solution by $r$.

- 2.4: With $f(\xi) = \sin(\xi) u(\xi)$ and $g(\xi) = e^{\xi} u(\xi)$, write down the solutions to the spherical wave equation, where $u(\xi)$ is the Heaviside step function.

*Sol:* In each case replace $\xi = t - x/c$ to obtain the solution to the wave equation for 1 dimensional waves. Thus

$$\varrho(r,t) = \frac{f(t-r/c)}{r} + \frac{g(t+r/c)}{r} = \frac{\sin(t-r/c)}{t-r/c} u(t-r/c) + \frac{e^{(t+r/c)} u(t+r/c)}{t+r/c}.$$

- 2.5: Sketch this last case for several times (e.g., 0, 1 2 seconds), and describe the behavior of the pressure $\varrho(r,t)$ as a function of time $t$ and radius $r$.  

*Sol:* Plot the functions at several times (e.g., 0, 1 2 seconds), as a function of $x$. The first function becomes smaller as the radius grows. The second function becomes larger as the inbound waves approaches $r = 0$.

- 2.6: What happens when the inbound wave reaches the center at $r = 0$?  

*Sol:* Stand back. It blows up. The equations fail when the solution becomes so large that the linearity assumption fails. I’m not sure what actually happens, in practice. This seems to be how they detonate nuclear weapons.

**Helmholtz formula**

Every differentiable vector field may be written as the sum of a *scalar potential* $\phi$ and *vector potential* $w$. This relationship is best known as *The Fundamental theorem of vector calculus (Helmholtz’ formula).*
where $\phi$ is the scalar potential and $w$ is the vector potential. This formula seems a natural extension of the\(^2\) algebraic $A \cdot B \perp A \times B$, since $A \cdot B \propto \|A\|\|B\| \cos(\theta)$ and $A \times B \propto \|A\|\|B\| \sin(\theta)$ as developed in Appendix A.3.1, page 190. Thus these orthogonal components have magnitude 1 when we take the norm, due to Euler’s identity ($\cos^2(\theta) + \sin^2(\theta) = 1$).

As described in Table 5.1 (p. 142), Helmholtz’ formula separates a vector field (i.e., $v(x)$) into compressible and rotational parts:

1. The rotational (e.g. angular) part of a field is defined by the vector potential $w$, requiring $\nabla \times \nabla \times w \neq 0$. A field is irrotational (conservative) when $\nabla \times v = 0$, meaning that the field $v$ can be generated using only \(^2\) a scalar potential, $v = \nabla \phi$ (note this is how a conservative field is usually defined, by saying there exists some $\phi$ such that $v = \nabla \phi$).

2. The compressible (e.g. radial) part of a field is defined by the scalar potential $\phi$, requiring $\nabla \cdot \nabla \phi = \nabla^2 \phi \neq 0$. A field is incompressible (solenoidal) when $\nabla \cdot v = 0$, meaning that the field $v$ can be generated using only a vector potential, $v = \nabla \times w$.

The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities:

1. $\nabla \cdot (\nabla \times w) = 0$
2. $\nabla \times (\nabla \phi) = 0$

**Problem #3: Define the following:**

---

**– 3.1: A conservative vector field**

**Sol:** A conservative vector field is defined as the gradient of a scalar potential $v = \nabla \phi(x, y, z)$. Every conservative field is necessarily irrotational (the test for an irrotational field is $\nabla \times v = 0$). ■

**– 3.2: A irrotational vector field**

**Sol:** The vector field $v$ is rotational if there exists a vector potential $w$ such that $v = \nabla \times w(x, y, z)$. The for irrotational is $\nabla \times v = 0$. A purely rotational field is not conservative. ■

**– 3.3: An incompressible vector field**

**Sol:** A field $v$ is incompressible if $\nabla \cdot v = 0$. ■

**– 3.4: A solenoidal vector field**

**Sol:** A rotational field is one having a divergence of zero, i.e., $\nabla \cdot v = 0$, or alternatively, $v \equiv \nabla \times w(x, y, z)$, since any field defined by a curl is rotational, since the divergence of the curl is always zero. ■

**– 3.5: When is a conservative field irrotational?**

**Sol:** Always! ■

**– 3.6: When is a incompressible field irrotational?**

**Sol:** A field is incompressible if $\nabla \cdot v = 0$ and irrotational if $\nabla \times v = 0$. So, almost never. The only case is the trivial solution $v = 0$, or a constant field $v = x_0 \hat{x} + y_0 \hat{y} + z_0 \hat{z}$. ■
Problem #4: For each of the following, (i) compute \( \nabla \cdot \mathbf{v} \), (ii) compute \( \nabla \times \mathbf{v} \), (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.):

- 4.1: \( \mathbf{v}(x, y, z) = -\nabla(3yx^3 + y \log(xy)) \)

**Sol:** The field is conservative (or irrotational) because it is defined by a gradient. To test for irrotational, show that the curl is zero. But \( \nabla \times \nabla \phi(x, y, z) = 0 \) for any \( \phi(x, y, z) \). Thus you do not need to do any computation, just state the answer. ■

- 4.2: \( \mathbf{v}(x, y, z) = xy\hat{x} - z\hat{y} + f(z)\hat{z} \)

**Sol:** To test for a irrotational field, take the curl, to see if it is zero:

\[
\nabla \times \mathbf{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ xy & -z & f(z) \end{vmatrix} = \hat{x} - x\hat{z},
\]

(VC-2.9)

which is not zero. We can also see by inspection that \( \nabla \cdot \mathbf{v} \neq 0 \). Thus the vector field is rotational and compressible. ■

- 4.3: \( \mathbf{v}(x, y, z) = \nabla \times (x\hat{x} - z\hat{y}) \)

**Sol:** \( \mathbf{v} = \hat{x} \). Therefore, \( \nabla \times \mathbf{v} = 0 \), and \( \nabla \cdot \mathbf{v} = 0 \). This field is technically incompressible and irrotational, but it is also very boring, since it is a constant. ■

Maxwell’s Equations

The variables have the following names and defining equations (Table 5.4, p. 171):

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Equation</th>
<th>Name</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>( \nabla \times E = -\dot{\mathbf{B}} )</td>
<td>Electric Field strength</td>
<td>[Volts/m]</td>
</tr>
<tr>
<td>( D = \varepsilon \varepsilon_0 E )</td>
<td>( \nabla \cdot D = \rho )</td>
<td>Electric Displacement (flux density)</td>
<td>[Col/m²]</td>
</tr>
<tr>
<td>( H )</td>
<td>( \nabla \times H = \mathbf{J} + \dot{\mathbf{D}} )</td>
<td>Magnetic Field strength</td>
<td>[Amps/m]</td>
</tr>
<tr>
<td>( B = \mu \mu_0 H )</td>
<td>( \nabla \cdot B = 0 )</td>
<td>Magnetic Induction (flux density)</td>
<td>[Webers/m²]</td>
</tr>
</tbody>
</table>

Note that \( \mathbf{J} = \sigma E \) is the current density (which has units of [Amps/m²]). Furthermore the speed of light in vacuo is \( c_0 = 3 \times 10^8 \) [m/s], and the characteristic resistance of light \( r_0 = 377 = \sqrt{\mu_0/\varepsilon_0} \) [Ω (i.e., ohms)].

Speed of light

**Problem #5:** The speed of light in vacuo is \( c_0 = 1/\sqrt{\mu_0 \varepsilon_0} \approx 3 \times 10^8 \) [m/s]. The characteristic resistance in in-vacuo is \( r_0 = \sqrt{\mu_0/\varepsilon_0} \approx 377 \) [Ω].

- 5.1: Find a formula for the in-vacuo permittivity \( \varepsilon_0 \) and permeability in terms of \( c_0 \) and \( r_0 \).

**Sol:** \( \varepsilon_0 = 1/r_0 c_0 \) and \( \mu_0 = r_0 c_0 \). ■ Based on your formula, what are the numeric values of \( \varepsilon_0 \) and \( \mu_0 \)?

**Sol:** \( \varepsilon_0 \approx 10^{-8}/3 \cdot 377 = 8.84 \cdot 10^{-12} \) and \( \mu_0 \approx 377/3 \cdot 10^8 = 1.26 \cdot 10^{-6} \). ■

- 5.2: In a few words, identify the law, define what it means, and explain the following formula:

\[
\int_S \hat{n} \cdot \mathbf{v} \, dA = \int_V \nabla \cdot \mathbf{v} \, dV
\]

**Sol:** This is the integral form of Gauss’ law. The unit normal vector is \( \perp \) to the surface \( S \) having area \( A \equiv \int_A \, dA \). The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field \( \nabla \cdot \mathbf{v} \) over the volume contained by the surface, and defined as \( V' \). ■
CHAPTER 5. VECTOR DIFFERENTIAL EQUATIONS

Application of ME

Problem # 6: The electric Maxwell equation is $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, where $\mathbf{E}$ is the electric field strength and $\mathbf{B}$ is the time rate of change of the magnetic induction field, or simply the magnetic flux density. Consider this equation integrated over a two-dimensional surface $S$, where $\hat{n}$ is a unit vector normal to the surface (you may also find it useful to define the closed path $C$ around the surface):

$$\iint_S (\nabla \times \mathbf{E}) \cdot \hat{n} dS = -\frac{\partial}{\partial t} \oint_C \mathbf{B} \cdot \hat{n} dS$$

– 6.1: Apply Stokes’ theorem to the left-hand side of the equation.
Sol: The surface $S$ must be open, with its edge $C$ defining the path for the line integral. $\emf \equiv \iint_S (\nabla \times \mathbf{E}) \cdot \hat{n} dS = \int_C \mathbf{E} \cdot dR$. (VC-2.10)

From Stokes’ theorem: the electromotive force (emf) is the line integral of $\mathbf{E}$ around the rim of the open surface. Think of the flux change as the Thevenin source driving the voltage. ■

– 6.2: Consider the right-hand side of the equation. How is it related to the magnetic flux $\Psi$ through the surface $S$?
Sol: It is equal to the negative time rate of change of the flux, $-\dot{\Psi}$. From Gauss’ Law the total magnetic flux $\Psi$ is the surface integral over the normal component of the magnetic flux density $\mathbf{B}$. After applying Gauss’ Laws, the surface integral becomes

$$\Psi = -\iint_S \mathbf{B} \cdot \hat{n} dS$$

(VC-2.11) ■

– 6.3: Assume the right-hand side of the equation is zero. Can you relate your answer to part (a) to one of Kirchhoff’s laws?
Sol: This result is well know as Kirchhoff’s first (voltage) law (KVL), $\emf = \sum_k V_k = -\dot{\Psi}$. When the flux induced into the loop may be ignored (e.g., it is very small), the sum of the voltages around the loop is zero. In rectangular coordinates with a plane surface this is simply $\Phi = B_n A$, where $A$ is the area and $B_n$ the normal component of $\mathbf{B}$ (⊥ to the surface $S$). ■

Problem # 7: The magnetic Maxwell equation is $\nabla \times \mathbf{H} = C \equiv \mathbf{J} + \dot{\mathbf{D}}$, where $\mathbf{H}$ is the magnetic field strength, $\mathbf{J} = \sigma \mathbf{E}$ is the conductive (resistive) current density and the displacement current $\dot{\mathbf{D}}$ is the time rate of change of the electric flux density $\mathbf{D}$. Here we defined a new variable $C$ as the total current density.

– 7.1: First consider the equation over a two dimensional surface $S$,

$$\iint_S (\nabla \times \mathbf{H}) \cdot \hat{n} dS = \iint_S [\mathbf{J} + \dot{\mathbf{D}}] \cdot \hat{n} dS = \iint_S C \cdot \hat{n} dS$$

Apply Stokes’ theorem to the left-hand side of this equation. In a sentence or two, explain the meaning of the resulting equation. Hint: What is the right-hand side of the equation? Sol: The surface $S$ must be open, with its edge $C$ prescribing the line integral, and its surface of $C$ defines the total current $I(t)$. The normal component of the surface integral over the total current $C$ gives total current $I(t)$. By Stokes theorem:

$$\mmf \equiv \iint_S \nabla \times \mathbf{H} \cdot \hat{n} dS = \int_C \mathbf{H} \cdot d\mathbf{R} = \iint_S C \cdot \hat{n} dS = I(t)$$

This is Ampere’s Law. ■
Problem # 8: Now consider this equation in three dimensions. Take the divergence of both sides, and integrate over a volume \( V \) (closed surface \( S \)).

\[
\iiint_V \nabla \cdot (\nabla \times \mathbf{H}) dV = \iiint_V \nabla \cdot \mathbf{C} dV
\]

- 8.1: What happens to the left-hand side of this equation? Hint: Can you apply a vector identity?
Sol: It is 0.

Apply the divergence theorem (sometimes known as Gauss’s theorem) to the right-hand side of the equation, and interpret your result. Hint: Can you relate your result to one of Kirchhoff’s laws?
Sol: We get

\[
\iiint_V \nabla \cdot \mathbf{C} dV = \iint_S \mathbf{C} \cdot \hat{n} dS = 0
\]

This result is Kirchhoff’s second (current) law (KCL), \( \sum I_k = \int \mathbf{D}(t) \cdot dS \). When the stray capacitance (\( \dot{D} \)) can be ignored the sum of the currents into the ‘node’ is zero. Generalizing, a ‘node’ to a volume \( V \), the total current \( I(t) \) flowing in/out of the volume is the integral of the normal component of the current density over the cross-sectional closed surface area, which equals 0.

Second-order differentials

Problem # 9: In this section we ask questions about second order vector differentials.

- 9.1: If \( \mathbf{v}(x, y, z) = \nabla \phi(x, y, z) \), then what is \( \nabla \cdot \mathbf{v}(x, y, z) \)?
Sol: Since \( \nabla \cdot \nabla = \nabla^2 \) this is \( \nabla^2 \phi(x, y, z) \).

- 9.2: Evaluate \( \nabla^2 \phi \) and \( \nabla \times \nabla \phi \) for \( \phi(x, y) = xe^y \).
Sol: CoG = 0 \( \nabla \times \nabla \phi = 0 \), \( \nabla^2 \phi = xe^y \).

- 9.3: Evaluate \( \nabla \cdot (\nabla \times \mathbf{v}) \) and \( \nabla \times (\nabla \times \mathbf{v}) \) for \( \mathbf{v} = x\hat{x} + y\hat{y} + z\hat{z} \).
Sol: \( \nabla \cdot (\nabla \times \mathbf{v}) = 0 \), \( \nabla \times (\nabla \times \mathbf{v}) = 0 \).

- 9.4: When \( \mathbf{V}(x, y, z) = \nabla (1/x + 1/y + 1/z) \) what is \( \nabla \times \mathbf{V}(x, y, z) \)?
Sol: This is always zero.

- 9.5: When was Maxwell born (and die)? How long did he live (within \( \pm 10 \) years)?
Sol: He lived 48 years, from 1831-1879.

Capacitor analysis

Problem # 10: Find the solution to the Laplace equation between two infinite \(^3\) parallel plates, separated by a distance of \( d \). Assume that the left plate, at \( x = 0 \), is at a voltage of \( V(0) = 0 \), and the right plate, at \( x = d \), is at a voltage of \( V_d = V(d) \).

- 10.1: Write down Laplace’s equation in one dimension for \( V(x) \).
Sol: This is the Laplace equation for rectangular coordinates

\[
\frac{\partial^2 V(x)}{\partial x^2} = 0
\]

\(^3\)We study plates that are infinite because this means the electric field lines will be perpendicular to the plates, running directly from one plate to the other. However, we will solve for per-unit-area characteristics of the capacitor.
– 10.2: Write down the general solution to your differential equation for \( V(x) \).
Sol: Integration is trivial since the solution must be of the form \( V(x) = A + Bx \).

– 10.3: Apply the boundary conditions \( V(0) = 0 \) and \( V(d) = V_d \) to solve for the constants in your equation from the previous part.
Sol: From the BC \( A = 0 \) and \( B = \frac{V_d}{d} \). Thus \( V(x) = \frac{V_d}{d} x \).

– 10.4: Find the charge density per unit area (\( \sigma = Q/A \), where \( Q \) is charge and \( A \) is area) on the surface of each plate. Hint: \( E = -\nabla V \), and Gauss’s Law states that \( \int_S \mathbf{D} \cdot \mathbf{n} dS = \text{Q enclosed} \).
Sol: To find the charge, we must first compute the electric field from the voltage using \( E = -\nabla V(x) \)

\[-E \equiv \nabla V(r) = \hat{x} \frac{\partial}{\partial x} V(x) = \hat{x} V_d \]

Since \( \mathbf{D} = \varepsilon_0 \mathbf{E} \) we find the normal component of the \( \mathbf{D} \) field

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} = -\varepsilon_0 \nabla V \]

is just a constant Thus using Gauss’ law \( (\sigma = -\frac{1}{A} \int_S D_x dA = D_r) \), the surface charge density \( \sigma \) in farads per square-meter is

\[ \sigma = \frac{\varepsilon_0}{d} V_d \]

– 10.5: Determine the per-unit-area capacitance \( C \) of the system.
Sol: Since \( \sigma = CV_d \), the capacity \( C \) per unit area is

\[ C = \frac{\varepsilon_0}{d} \text{[F/m}^2\text{]} \]

The units are farads per square-meter. Note that the sign must work out so that \( C > 0 \).


Webster Horn Equation

**Problem #11:** Horns provide an important generalization of the solution of the 1D wave equation, in regions where the properties (i.e., area of the tube) vary along the axis of wave propagation. Classic applications of horns are vocal tract acoustics, loudspeaker design, cochlear mechanics, any case having wave propagation.

– 11.1: Write out the formula for the Webster horn equation, and explain the variables.
Sol: The horn equation may be written as

\[
\frac{1}{A(x)} \frac{\partial}{\partial x} \left( A(x) \frac{\partial \varrho}{\partial x} \right) = \frac{1}{c^2} \frac{\partial^2 \varrho}{\partial t^2}. 
\]  

\text{(VC-2.12)}

where \( A(x) \) is the area of the horn at \( x \) (range variable). \( \varrho(x, t) \) is the pressure and \( c \) is the wave speed.