An Invitation to Mathematical Physics

4.2. PROBLEMS VC-2 85
Capacitor analysis

Problem # 10: Find the solution to the Laplace equation between two infinite parallel plates separated by a distance \( d \). Assume that the left plate at \( x = 0 \) is at voltage \( V \) and the right plate at \( x = d \) is at voltage \( 0 \). The electric field lines are perpendicular to the plates, running directly from one plate to the other. However, we solve for per-unit-area characteristics of the capacitor.

The horn equation may be written as

\[
\text{Webster horn equation}
\]

and its variables.

Design, cochlear mechanics, and any case that has wave propagation. Write the formula for classic applications of horns are in vocal tract acoustics, loudspeaker diaphragm design, and its wave equation in regions where the properties (i.e., area of the tube) vary along the axis.

Since \( \sigma \) is just a constant Thus using Gauss' law (\( \epsilon \) is the area of the horn at \( x \))

\[
\text{Integration is trivial since the solution must be of the form}
\]

To find the charge, we must first compute the electric field from the voltage using \( E \).

\[
\text{Sol: From the BC in your equation from question 10.2.}
\]

The units are farads per square-meter. Note that the sign must work out so that

\[
\text{Since}
\]

on the surface of each plate.

Problem # 11: Horns illustrate an important generalization of the solution of the one dimensional wave equation in one dimen-

\[
\text{Solve:}
\]

where \( \rho \) is the pressure and \( c \) is the wave speed.

Historically an important generalization of the solution of the one di-

\[
\text{Problem # 10.1: Write Laplace's equation in one dimension for a capacitor.}
\]

\[
\text{Problem # 10.2: Write the general solution to your differential equation for}
\]

\[
\text{Problem # 10.3: Apply the boundary conditions}
\]

\[
\text{Problem # 10.4: Find the charge density per unit area}
\]

\[
\text{Problem # 10.5: Determine the per-unit-area capacitance}
\]

\[
\text{Problem # 10.6: The capacitor is charged to a uniform potential per unit are}
\]

\[
\text{Problem # 10.7: The left plate is at a higher potential than the} \]
CHAPTER 4. VECTOR DIFFERENTIAL EQUATIONS

– 5.1: Find the divergence of the following equations 
\[ \nabla \cdot \mathbf{A} = \nabla \cdot \left( \frac{x}{y} \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z \right) \]
Sol: 
\[ \nabla \cdot \mathbf{A} = \frac{y}{y} + 1 + 1 = 3 \]

– 5.2: Evaluate \( \nabla \times \mathbf{A} \) for \( \mathbf{A} = xe^y \mathbf{e}_x + ye^x \mathbf{e}_y + ze^x \mathbf{e}_z \).
Sol: 
\[ \nabla \times \mathbf{A} = \left( \frac{\partial (ze^x)}{\partial y} - \frac{\partial (ye^x)}{\partial z} \right) \mathbf{e}_x + \left( \frac{\partial (xe^y)}{\partial z} - \frac{\partial (ze^x)}{\partial x} \right) \mathbf{e}_y + \left( \frac{\partial (ye^x)}{\partial x} - \frac{\partial (xe^y)}{\partial y} \right) \mathbf{e}_z \]

– 5.3: Evaluate \( \nabla \cdot (\nabla \times \mathbf{A}) \) and \( \nabla \times (\nabla \times \mathbf{A}) \) for \( \mathbf{A} = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z \).
Sol: 
\[ \nabla \cdot (\nabla \times \mathbf{A}) = \frac{\partial^2 (x)}{\partial x^2} + \frac{\partial^2 (y)}{\partial y^2} + \frac{\partial^2 (z)}{\partial z^2} = 3 \]
\[ \nabla \times (\nabla \times \mathbf{A}) = \left( \frac{\partial (y)}{\partial z} - \frac{\partial (z)}{\partial y} \right) \mathbf{e}_x + \left( \frac{\partial (z)}{\partial x} - \frac{\partial (x)}{\partial z} \right) \mathbf{e}_y + \left( \frac{\partial (x)}{\partial y} - \frac{\partial (y)}{\partial x} \right) \mathbf{e}_z \]

Second-order differentials

Problem # 9: This problem is about second-order vector differentials.

– 9.1: If \( \mathbf{v}(x, y, z) = \nabla \phi(x, y, z) \), then what is \( \nabla \cdot \mathbf{v}(x, y, z) \)?
Sol: 
\[ \nabla \cdot \mathbf{v} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi \]

– 9.2: Evaluate \( \nabla^2 \phi \) and \( \nabla \times \nabla \phi \) for \( \phi(x, y) = xe^y \).
Sol: 
\[ \nabla^2 \phi = \frac{\partial^2 (xe^y)}{\partial x^2} + \frac{\partial^2 (xe^y)}{\partial y^2} = e^y \]
\[ \nabla \times \nabla \phi = \left( \frac{\partial (xe^y)}{\partial y} - \frac{\partial (xe^y)}{\partial z} \right) \mathbf{e}_x + \left( \frac{\partial (xe^y)}{\partial z} - \frac{\partial (xe^y)}{\partial x} \right) \mathbf{e}_y + \left( \frac{\partial (xe^y)}{\partial x} - \frac{\partial (xe^y)}{\partial y} \right) \mathbf{e}_z = xe^y \mathbf{e}_x + ye^x \mathbf{e}_y - xe^y \mathbf{e}_z \]

– 9.3: Evaluate \( \nabla \times (\nabla \times \mathbf{v}) \) and \( \nabla \cdot (\nabla \times \mathbf{v}) \) for \( \mathbf{v} = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z \).
Sol: 
\[ \nabla \cdot (\nabla \times \mathbf{v}) = 3 \]
\[ \nabla \times (\nabla \times \mathbf{v}) = \left( \frac{\partial (y)}{\partial z} - \frac{\partial (z)}{\partial y} \right) \mathbf{e}_x + \left( \frac{\partial (z)}{\partial x} - \frac{\partial (x)}{\partial z} \right) \mathbf{e}_y + \left( \frac{\partial (x)}{\partial y} - \frac{\partial (y)}{\partial x} \right) \mathbf{e}_z = 0 \]

– 9.4: When \( V(x, y, z) = \nabla \left( \frac{1}{x} + 1/y + 1/z \right) \), what is \( \nabla \times \mathbf{V}(x, y, z) \)?
Sol: 
\[ \nabla \times \mathbf{V} = \left( \frac{\partial (1/y)}{\partial z} - \frac{\partial (1/z)}{\partial y} \right) \mathbf{e}_x + \left( \frac{\partial (1/z)}{\partial x} - \frac{\partial (1/x)}{\partial z} \right) \mathbf{e}_y + \left( \frac{\partial (1/x)}{\partial y} - \frac{\partial (1/y)}{\partial x} \right) \mathbf{e}_z = \frac{1}{z^2} \mathbf{e}_x - \frac{1}{y^2} \mathbf{e}_y - \frac{1}{x^2} \mathbf{e}_z \]

– 9.5: When was Maxwell born and when did he die? How long did he live (within ±10 years)?
Sol: 
He lived 48 years, from 1831 to 1879.
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Maxwell's Equations

The variables have the following names and defining equations (see Table 5.4, ... magnetic flux density B. After applying Gauss' Laws, the
surface integral becomes

\[ \Psi = - \int \int_S \mathbf{B} \cdot \hat{n} \; dS \quad (VC-2.11) \]

CONTENTS

Preface

Science has evolved over thousands of years. It began out of curiosity about how the world around us ... is effectively constant, since \( h \ll R_e \).

Newton's equation says the acceleration is constant,

\[ \frac{d^2 h(t)}{dt^2} = \frac{G m M}{R_e^2} \]

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\[ \frac{d^2 h(t)}{dt^2} = \frac{G m M}{R_e^2} \]

The evolution of science is layered: Early science depended mostly on critical observation, thus the early sci-

Note that \( J \) is independent of time. It follows that the force on a body is proportional to its acceleration

\[ F = ma \]

Gauss' law

\[ \oint \mathbf{E} \cdot d\mathbf{A} = 0 \]

The flux of \( \mathbf{E} \) through any closed surface is zero, implying \( \mathbf{E} \) is a conservative field. The fact that the flux of a conservative field through any closed surface is zero implies

\[ \mathbf{E} = \nabla \phi \]

The electric field is the gradient of a scalar potential.

\[ \mathbf{B} = \nabla \times \mathbf{A} \]

The magnetic field is the curl of a vector potential.

\[ \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} \]

The magnetic field strength is given by the magnetic flux density divided by the permeability.

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \]

The electric displacement is the electric field plus any permanent electric polarization.

\[ \mathbf{E} = \varepsilon_0 \frac{\partial \mathbf{D}}{\partial t} \]

The electric field is the time rate of change of the electric displacement divided by the permittivity.

\[ \mathbf{B} = \mu_0 \mathbf{H} \]

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Chapter 1

Instructors manual: Number systems

1.1 Problems NS-1

Topic of this homework:

Introduction to Matlab/Octave (see the Matlab or Octave tutorial for help)
Deliverables: Report with charts and answers to questions. Hint: Use LATEX.

Plotting complex quantities in Octave/Matlab

Problem # 12: Consider the functions $f(s) = s^2 + 6s + 25$ and $g(s) = s^2 + 6s + 5$.

Sol: The roots of $f(s)$ are $-3 \pm 4i$ (in Matlab: roots([1 6 25])). The roots of $g(s)$ are $-1$ and $-5$ (in Matlab: roots([1 6 5])). You will find the program that generates all these figures at https://jontalle.web.engr.illinois.edu/uploads/298.17/NS1.m.

Problem # 12: Show the roots of $f(s)$ as red circles and of $g(s)$ as blue plus signs.
The x-axis should display the real part of each root, and the y-axis should display the imaginary part. Use hold on and grid on when plotting the roots.

Sol: 

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4.1: Find the zeros of functions $f(s)$ and $g(s)$ using the command roots().

Sol: The roots of $f(s)$ are $-3 \pm 4i$ (in Matlab: roots([1 6 25])). The roots of $g(s)$ are $-1$ and $-5$ (in Matlab: roots([1 6 5])). You will find the program that generates all these figures at https://jontalle.web.engr.illinois.edu/uploads/298.17/NS1.m.

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1. The rotational (e.g., angular) part is defined by the vector potential $\mathbf{w}$, which requires that $\nabla \times \nabla \times \mathbf{w} \neq 0$.

A field is irrotational if $\nabla \times \mathbf{w} = 0$, meaning that the field is generated using only a scalar potential, $\mathbf{w} = \nabla \phi$ (note that this is how a conservative field is usually defined, by saying there exists some $\phi$ such that $\mathbf{w} = \nabla \phi$).

2. The compressible (e.g., radial) part of a field is defined by the scalar potential $\phi$, which requires that $\nabla \phi = \nabla^2 \phi$. A field is incompressible if $\nabla \cdot \mathbf{w} = 0$, meaning that the field is generated using only a vector potential, $\mathbf{w} = \nabla \times \mathbf{v}$.

The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities: (1) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ and (2) $\nabla \times (\nabla \phi) = 0$.

Problem # 3: Define the following:

- 3.1: A conservative vector field

Sol: A conservative vector field is defined as the gradient of a scalar potential $\mathbf{v} = \nabla \phi(x, y, z)$. Every conservative field is necessarily irrotational (the test for irrotational field is $\nabla \times \mathbf{v} = 0$).

- 3.2: An irrotational vector field

Sol: The vector field $\mathbf{v}$ is rotational if there exists a vector potential $\mathbf{w}$ such that $\mathbf{v} = \nabla \times \mathbf{w}(x, y, z)$. The field $\mathbf{v}$ is irrotational if $\nabla \times \mathbf{v} = 0$. A purely rotational field is not conservative.

- 3.3: An incompressible vector field

Sol: A field $\mathbf{v}$ is incompressible if $\nabla \cdot \mathbf{v} = 0$.

- 3.4: A solenoidal vector field

Sol: A rotational field is one having a divergence of zero, i.e., $\nabla \cdot \mathbf{v} = 0$, or alternatively, $\mathbf{v} = \nabla \times \mathbf{w}(x, y, z)$, since any field defined by a curl is rotational, since the divergence of the curl is always zero.

- 3.5: When is a conservative field irrotational?

Sol: Always.

- 3.6: When is an incompressible field irrotational?

Sol: A field is irrotational if $\nabla \times \mathbf{v} = 0$ and incompressible if $\nabla \cdot \mathbf{v} = 0$. So, almost never. The only case is the trivial solution $\mathbf{v} = 0$, or a constant field $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Problem # 4: For each of the following, (i) compute $\nabla \times \mathbf{v}$, (ii) compute $\nabla \times \mathbf{v}$, and (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.),

- 4.1: $\mathbf{v}(x, y, z) = -\nabla(3y^2 + y \log(y))$

Sol: The field is conservative (or irrotational) because it is defined by a gradient. To test for irrotational, show that $\nabla \times \mathbf{v} = 0$. Thus $\nabla \times \mathbf{v} = 0$ for any $\mathbf{v}(x, y, z)$. Thus you do not need to do any computation, just state the answer.

- 4.2: $\mathbf{v}(x, y, z) = xy\mathbf{k} - z\mathbf{j} + f(z)\mathbf{i}$

Sol: To test for a rotational field, take the curl, to see if it is zero:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ [0 & 0 & f(z)] \end{vmatrix} - z\mathbf{k},$$

which is not zero. We can also see by inspection that $\nabla \times \mathbf{v} \neq 0$. Thus the vector field is non-irrotational and non-compressible.

- 4.3: $\mathbf{v}(x, y, z) = \nabla \times (x\mathbf{k} - z\mathbf{j})$

Sol: $\mathbf{v}$ is $\mathbf{k}$. Therefore, $\nabla \times \mathbf{v} = 0$, and $\nabla \cdot \mathbf{v} = 0$. This field is technically incompressible and irrotational, but it is also very boring, since it is a constant.

- 4.4: $\mathbf{v}(x, y, z) = \nabla \times (\nabla \phi - z\mathbf{j})$

Sol: $\mathbf{v}$ is $\mathbf{k}$. Therefore, $\nabla \times \mathbf{v} = 0$, and $\nabla \cdot \mathbf{v} = 0$. This field is technically incompressible and irrotational, but it is also very boring, since it is a constant.

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1 A note about the relationship between the generating function and the test: You might imagine special cases where $\nabla \times \mathbf{w} = 0$ but $\nabla \times \nabla \times \mathbf{w} = 0$ (or $\mathbf{v} \neq 0$ but $\nabla \cdot \mathbf{v} = 0$). In these cases, the vector (or scalar) potential can be written as a scalar (or vector) potential. For example, consider a field $\mathbf{v} = \nabla \phi_0 + \mathbf{h}$, where $\mathbf{h} = \mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z$. Note that $\mathbf{h}$ can actually be generated by either a scalar potential ($\phi_0 = 1/2 [x^2 + y^2 + z^2]$), such that $\nabla \phi_0 = \mathbf{h}$, or a vector potential $(x_0 = 1/2 [x^2 + y^2 + z^2]$, such that $\nabla \times x_0 = \mathbf{h}$). We find that $\nabla \times \mathbf{v} = 0$, but $\mathbf{v}$ must be irrotational. We say this irrotational field is generated by $\mathbf{v} = \nabla (\phi_0 + \phi_1)$. The imaginary part. Use hold on and grid on when plotting the roots.

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### 4.2. PROBLEMS VC-2

- 2.2: Show that this equation is true:
  \[ \nabla^2 r \rho(r) \equiv \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \rho(r) = \frac{1}{r} \frac{\partial^2}{\partial r^2} \rho(r) \nabla^2 \nabla \cdot \begin{pmatrix} 1 \\ \mu \nabla \times \mathbf{B} \end{pmatrix} = 0 \]

- 12.3 Give your figure the title "Complex Roots of \( f(s) \) and \( g(s) \)." Label the x- and y-axes "Magnitude" and "Phase (radians)."

- 2.3: Use the results from Eq. VC-2.4 to show that the solution to the spherical wave equation

  \[ \begin{align*}
  \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^4} \frac{\partial^2 u}{\partial t^2} & = 0 \\
  \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \phi^2} + \frac{1}{r^4} \frac{\partial^2 \phi}{\partial t^2} & = 0
  \end{align*} \]

  for \( \phi \) satisfies the Helmholtz equation. Thus these orthogonal components are the solutions to the spherical wave equation for 1 dimensional waves. Thus

  \[ \begin{align*}
  \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^4} \frac{\partial^2 u}{\partial t^2} & = 0 \\
  \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \phi^2} + \frac{1}{r^4} \frac{\partial^2 \phi}{\partial t^2} & = 0
  \end{align*} \]

  This formula seems to be a natural extension of the algebraic products

  \[ \begin{align*}
  \psi_1 \psi_2 & = \psi_1 \psi_2 + \nabla_1 \cdot \nabla_2 \\
  \psi_1 \psi_2 & = \psi_1 \psi_2 + \nabla_1 \times \nabla_2
  \end{align*} \]

  where \( \psi_1 \psi_2 \) is the Heaviside step function.
1.1. PROBLEMS NS-1

14.3: Use the Matlab/Octave function primes to generate prime numbers between 1 and 10^6. Save them in a vector x. Plot this result using the command hist(x).

Sol:

14.4: Now try [n,bincenters] = hist(x). Use length(n) to find the number of bins.

Sol: length(n) is 10 * n

14.5: Set the number of bins to 100 by using an extra input argument to the function hist. Show the resulting figure, give it a title, and label the axes. Hint: help hist and doc hist.

Sol:

Problem # 15: Inf, NaN, and logarithms in Octave/Matlab.

15.1: Try 1/0 and 0/0 in the Octave/Matlab command window. What are the results? What do these “numbers” mean in Octave/Matlab? Sol: 1/0 returns Inf (infinity) and 0/0 returns NaN (not a number).

15.2: Try log(0), log10(0), and log2(0) in the command window.

In Matlab/Octave, the natural logarithm log() is computed using the function log. Functions log10 and log2 are computed using log10 and log2. Sol: log(0) is -Inf. Working in any base results in a scale factor, so the value does not change in these different bases. For example if 10^x = 2^y then x = log10(2^y) = y log10(2) = 0.30103y.

15.3: Try log1(1) in the command window. What do you expect for log10(1) and log2(1)?

Sol: As with log(0), changing base of log(1)=0 gives the same result, because scaling 0 always gives 0.

4.2 Problems VC-2

Topics of this homework:
Partial differential equations; fundamental theorem of vector calculus (Helmholtz’s theorem); wave equation; Maxwell’s equations (ME) and variables (E, D, B, H); Second-order vector differentials, Webster horn equation.

Notation: The following notation is used in this homework:
1. s = σ + jω is the Laplace frequency, as used in the Laplace transform.
2. A Laplace transform pair is indicated by the symbol ↔: for example, f(t) ↔ F(s).
3. ϱ_k is the kth prime; for example, ϱ_5 ∈ P, ϱ_k = [2, 3, 5, 7, 11, 13, ...] for k = 1, ..., 6.

Partial differential equations (PDEs): Wave equation

Problem # 1: Solve the wave equation in one dimension by defining ξ = t + x/c.

1.1: Show that d’Alembert’s solution, g(x,t) = f(t-x/c) + g(t+x/c), is a solution to the acoustic pressure wave equation in one dimension:

\[ \frac{\partial^2 g(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 g(x,t)}{\partial t^2} \]

where f(ξ) and g(ξ) are arbitrary functions. Sol:

\[ \frac{\partial}{\partial t} g(x,t) = \frac{1}{c} f(t-x/c) + \frac{1}{c} g(t+x/c) = \frac{1}{c} f(t-x/c) + \frac{1}{c} g(t+x/c) \] (VC-2.1)

\[ \frac{\partial^2}{\partial x^2} g(x,t) = \frac{1}{c^2} f(t-x/c) + \frac{1}{c^2} g(t+x/c) = \frac{1}{c^2} f(t-x/c) + \frac{1}{c^2} g(t+x/c) \] (VC-2.2)

\[ \frac{\partial^2}{\partial x^2} g(x,t) = \frac{\partial}{\partial t} f(t-x/c) + \frac{\partial}{\partial t} g(t+x/c) = \frac{\partial}{\partial t} f(t-x/c) + \frac{\partial}{\partial t} g(t+x/c) \] (VC-2.3)

Problem # 2: Solving the wave equation in spherical coordinates (i.e., three dimensions)

2.1: Write the wave equation in spherical coordinates g(r,θ,φ, t). Consider only the radial term r (i.e., dependence on angles θ and φ is assumed to be zero). Hint: The form of the Laplacian as a function of the number of dimensions is given in Eq. 5.1.9 (page 175). Alternatively, look it up on the internet or in a calculus book.

Sol: Given the formula for the Laplacian in spherical coordinates, the wave equation is

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial g(r,t)}{\partial r} \right) \] (VC-2.4)
4.1. PROBLEMS VC-1 79

C/NC : causal (C)/noncausal (NC)

Sol: Responds only when driven for \( t \geq 0 \). Does not anticipate for negative \( t \).

\[ dy(t) \frac{dt}{dt} + (t-1) y^2(t) = i e^t \]


**Problem #** 1

**Problem #** 2

**Problem #** 3

**Problem #** 4

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**Notes:**

- **Problem #** 5: Try \( \log(-1) \) in the command window. What do you expect for \( \log(-1) \)?
- **Problem #** 6: There is no number that can be raised to the power of \( 2 \) to give \( -1 \), due to the fundamental theorem of algebra.
- **Problem #** 7: The time function is real (or complex).
- **Problem #** 8: The time function is real (or complex).

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**Problem #** 9

**Problem #** 10

**Problem #** 11

**Problem #** 12

**Problem #** 13

**Problem #** 14

**Problem #** 15

**Problem #** 16

**Problem #** 17

**Problem #** 18

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**Problem #** 19: We are interested in primes that are greater than \( 2 \).

**Problem #** 20: We are interested in primes that are greater than \( 2 \).

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**Problem #** 21: We are interested in primes that are greater than \( 2 \).

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1.1. PROBLEMS NS-1

Sol: Subtracting any line from the line following it, gives:

\[ (1 - 1) + 3 = 2^2 - 2^2 \]
\[ 5 = 3^2 - 3^2 \]
\[ 7 = 4^2 - 4^2 \]
\[ \vdots \]

\[ \sum_{n=0}^{N-1} 2n + 1 - \sum_{n=0}^{N-2} 2n + 1 = N^2 - (N - 1)^2 \]
\[ 2N - 1 = N^2 - (N^2 - 2N + 1) \]
\[ 2N - 1 = 2N - 1. \]

Thus the two sides are equal, as suggested by the above formula.

Can you find a simpler more constructive "proof?" Hint: assuming you know what integration by parts is, can you devise a concept called Summation by parts? ■

78 CHAPTER 4. VECTOR DIFFERENTIAL EQUATIONS

– 5.4: For any differentiable vector field \( V \), write two vector calculus identities that are equal to zero.

Sol: Curl of the gradient \( \nabla \times \nabla \Phi(x, y, z) = 0 \) and the divergence of the curl \( \nabla \cdot \nabla \times V(x, y, z) = 0 \) are both zero. (Page 780, Stillwell) ■

– 5.5: What is the most general form a vector field may be expressed in, in terms of scalar \( \Phi \) and vector \( A \) potentials?

Sol: \( V = \nabla \Phi(x, y, z) + \nabla \times A(x, y, z) \), where \( \Phi \) is the scalar potential and \( A \) is the vector potential. ■

Problem # 6: Perform the following calculations. If you can state the answer without doing the calculation, explain why.

– 6.1: Let \( \mathbf{v} = \sin(x) \hat{x} + y \hat{y} + z \hat{z} \). Find \( \nabla \cdot (\nabla \times \mathbf{v}) \).

Sol: 0 ■

– 6.2: Let \( \mathbf{v} = \sin(x) \hat{x} + y \hat{y} + z \hat{z} \). Find \( \nabla \times (\nabla \sqrt{\mathbf{v} \cdot \mathbf{v}}) \).

Sol: 0 ■

– 6.3: Let \( \mathbf{v}(x, y, z) = \nabla (x + y^2 + \sin(\log(z))) \). Find \( \nabla \times \mathbf{v}(x, y, z) \).

Sol: It is zero because \( \nabla \times \mathbf{v}(x, y, z) \) is always zero. ■

Integral theorems

Problem # 7: For each of the following problems, in a few words, identify either Gauss’s or Stokes’s law, define what it means, and explain the formula that follows the question.

– 7.1: What is the name of this formula?

\[ \int_S \hat{A} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{A} \, dV. \]

Sol: This is the integral form of Gauss’ law. The unit normal vector is \( \hat{A} \) to the surface \( S \) having area \( A \), \( A \equiv \int_S dA \). The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field \( \nabla \cdot \mathbf{A} \) over the volume contained by the surface, and defined as \( \Phi' \). ■

– 7.2: What is the name of this formula?

\[ \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_C \mathbf{A} \cdot d\mathbf{R}. \]

Give one important application. Sol: Stokes Theorem, which relates the differential to the integral form of Maxwell’s equations. ■

– 7.3: Describe a key application of the vector identity

\( \nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} \).

Sol: When we wish to reduce Maxwell's two curl equations to the vector wave equation, we must use this identity. ■

System Classification

Problem # 8: Complete this system classification problem about physical systems using the system postulates.

– 8.1: Provide a brief definition of these classifications: L/NL : linear (L)/nonlinear (NL) Sol: Superposition and scaling hold ■

TI/TV : time-invariant (TI)/time-varying (TV) Sol: The measurement time is irrelevant ■

P/A : passive (P)/active (A) Sol: An active system has a power source, a passive system does not. ■
4.1. PROBLEMS VC-1

– 2.4: Rewrite the pressure wave equation in the time domain using the time derivative property ...

– 5.3: \( v = \nabla \times A \), where \( A = y\hat{x} + x^2\hat{y} + z\hat{z} \)

Sol: \( \nabla \cdot (\nabla \times A) = 0 \), and \( \nabla \times (\nabla \times A) = -2\hat{y} \)

6. CHAPTER 1. INSTRUCTORS MANUAL: NUMBER SYSTEMS

1.2 PROBLEMS NS-2

Topic of this homework:
Prime numbers, greatest common divisor, the Euclidean algorithm

1. Find the Euclidean algorithm for the following pairs of numbers:

- \( (54, 18) \) - Answer: 18
- \( (72, 8) \) - Answer: 8
- \( (23, 476) \) - Answer: 1

2. Find the prime numbers between 1 and 100.

Sol: \( \left\lfloor \sqrt{100} \right\rfloor = 10 \) prime numbers.

3. If \( n \) is a composite number, what is \( \phi(n) \)?

Sol: \( \phi(n) = n \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} \cdot \frac{1}{6} \cdot \frac{1}{7} \cdot \frac{1}{8} \cdot \frac{1}{9} \cdot \frac{1}{10} \) for composite \( n \).

4. Prove that the Euclidean algorithm is an efficient way to find the greatest common divisor.

Problem 2: Find the opposite and the direction for the following vectors.

\( \vec{A} \): Vector operator

5. The sum of two vectors \( \vec{A} \) and \( \vec{B} \) can be written as a product of a scalar and a vector.

\( \vec{A} = \vec{B} \cdot \vec{C} \)

6. The gradient of the function \( f(x, y, z) \) is written as \( \nabla f \).

Sol: \( \nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} \)

7. The curl of the vector field \( \vec{A} \) is written as \( \nabla \times \vec{A} \).

Sol: \( \nabla \times \vec{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z} \)

8. The divergence of the vector field \( \vec{A} \) is written as \( \nabla \cdot \vec{A} \).

Sol: \( \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \)

9. The Laplacian of the vector field \( \vec{A} \) is written as \( \nabla^2 \vec{A} \).

Sol: \( \nabla^2 \vec{A} = \nabla \cdot (\nabla \times \vec{A}) \)

10. The gradient of the scalar field \( f(x, y, z) \) is written as \( \nabla f \).

Sol: \( \nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} \)

11. The curl of the scalar field \( f(x, y, z) \) is written as \( \nabla \times f \).

Sol: \( \nabla \times f = 0 \)

12. The divergence of the scalar field \( f(x, y, z) \) is written as \( \nabla \cdot f \).

Sol: \( \nabla \cdot f = 0 \)

13. The Laplacian of the scalar field \( f(x, y, z) \) is written as \( \nabla^2 f \).

Sol: \( \nabla^2 f = \nabla \cdot (\nabla f) \)
### Greatest common divisors

Consider using the Euclidean algorithm to find the greatest common divisor (GCD; the largest common prime factor) of two numbers. Note that this algorithm may be performed using one of two methods:

<table>
<thead>
<tr>
<th>Method</th>
<th>Division</th>
<th>Subtraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>On each iteration ...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$a_{i+1} = b_i$</td>
<td>$b_{i+1} = a_i$ - $a_{i+1} \times \text{floor}(a_i/b_i)$</td>
</tr>
<tr>
<td>Terminates when ...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>$b = 0$ (GCD = $a$)</td>
<td>$b = 0$ (GCD = $a$)</td>
</tr>
</tbody>
</table>

The division method (Eq. 2.1, Sec. 2.1.2, Ch. 2) is preferred because the subtraction method is much slower.

#### Problem # 4: Understanding the Euclidean algorithm (GCD)

- **4.1:** Use the Octave/Matlab command `factor` to find the prime factors of $a = 85$ and $b = 15$.

  **Solution:** From Octave’s `factor()` we find $85 = 17 \cdot 5$ and $15 = 3 \cdot 5$.

- **4.2:** What is the greatest common prime factor of $a = 85$ and $b = 157$?

  **Solution:** The largest common factor $	ext{gcd}(85, 157)$ is 5.

- **4.3:** By hand, perform the Euclidean algorithm for $a = 85$ and $b = 15$.

  **Solution:**

<table>
<thead>
<tr>
<th>$a_i$</th>
<th>$b_i$</th>
<th>$a_{i+1}$</th>
<th>$b_{i+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>85</td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- **4.4:** Write each of these numbers as a product of primes: $22, 30, 34, 43, 44, 48, 49$.

  **Solution:**

  - $22 = 2 \cdot 11$.
  - $30 = 2 \cdot 3 \cdot 5$.
  - $34 = 2 \cdot 17$.
  - $43$ is prime.
  - $44 = 2^2 \cdot 11$.
  - $48 = 2^4 \cdot 3$.
  - $49 = 7^2$.

- **4.5:** Find the largest prime $p_i \leq 100$. Do not use Matlab/Octave other than to check your answer.

  **Solution:** The prime numbers $2, 3, 5, 7, 11, \ldots, 97, 99$ are the largest primes less than or equal to 100.

- **4.6:** Find the largest prime $p_i \leq 1000$. Do not use Matlab/Octave other than to check your answer.

  **Solution:** The prime numbers $2, 3, 5, 7, 11, \ldots, 997, 999$ are the largest primes less than or equal to 1000.

- **4.7:** Explain why $\pi^c = 3^a \cdot 7^b \cdot 11^c \ldots$ before only the primes remain?

  **Solution:** To find the prime factors of a number, we can use the Euclidean algorithm. The algorithm involves dividing the larger number by the smaller number, then dividing the smaller number by the remainder, and so on, until the remainder is 0. The prime factors are the divisors used in each step.

- **4.8:** Generalize: For $n = 1, \ldots, N$, what is the largest number you need to consider before only the primes remain?

  **Solution:** For $n = 1, \ldots, N$, the largest number you need to consider is $N!$ (the factorial of $N$).

- **4.9:** Write the numbers starting with 100 and count backward: 100, 99, 98, 97, 96, 95, 94, 93, 92, 91, 90, 89, 88, 87, 86, 85, 84, 83, 82, 81, 80, 79, 78, 77, 76, 75, 74, 73, 72, 71, 70, 69, 68, 67, 66, 65, 64, 63, 62, 61, 60, 59, 58, 57, 56, 55, 54, 53, 52, 51, 50, 49, 48, 47, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1.

  **Solution:** The largest number is 1.

- **4.10:** Explain why $\pi^c = 3^a \cdot 7^b \cdot 11^c \ldots$ before only the primes remain?

  **Solution:** To find the prime factors of a number, we can use the Euclidean algorithm. The algorithm involves dividing the larger number by the smaller number, then dividing the smaller number by the remainder, and so on, until the remainder is 0. The prime factors are the divisors used in each step.

#### Problem # 2: Acoustic wave equation

**Note:** In this problem, we will work in the frequency domain.

- **2.1:** The basic equations of acoustics in one dimension are

  $$
  \frac{\partial}{\partial x} \rho_v \varphi = -\rho_s \varphi' \quad \text{and} \quad \frac{\partial}{\partial t} \varphi = \frac{\rho_s}{\eta_0 \rho_0} \varphi'.
  $$

  **Here** $\varphi(x, \omega)$ is the pressure (in the frequency domain), $\varphi'(x, \omega)$ is the volume velocity (the integral of the velocity over the wavefront with area $A$), $\varphi = \varphi + i \omega$, $\rho_s = 1.2$ is the specific density of air, $\eta_0 = 1.4$, and $P_0$ is the atmospheric pressure (i.e., $10^5$ Pa). Note that the pressure field $\varphi$ is a scalar (pressure does not have direction), while the volume velocity field $\varphi'$ is a vector (velocity has direction).

  We can generalize these equations to three dimensions using the $\nabla$ operator

  $$
  -\nabla \varphi = \rho_s \varphi' \quad \text{and} \quad -\nabla \cdot \varphi = \frac{\rho_s}{\eta_0 \rho_0} \varphi'.
  $$

- **2.2:** Starting from these two basic equations, derive the scalar wave equation in terms of the pressure $\varphi$.

  $$
  \sqrt{\rho_s} \nabla^2 \varphi = \frac{\rho_s}{\eta_0 P_0} \varphi',
  $$

  where $c_0$ is a constant representing the speed of sound.

  **Solution:** We wish to remove $\varphi'$ from the two equations, to obtain a single equation in pressure. If we take the partial wrt $x$ of the pressure equation, and then substitute the velocity equation, to remove the velocity:

  $$
  \sqrt{\rho_s} \nabla^2 \varphi = -\rho_s \varphi \cdot \nabla \varphi = \frac{\rho_s}{\eta_0 P_0} \varphi - \frac{\rho_s}{\eta_0 P_0} \varphi' = \frac{\rho_s}{\eta_0 P_0} \varphi^2 - \frac{\rho_s}{\eta_0 P_0} \varphi^2
  $$

  **2.3:** What is $c_0$ in terms of $\eta_0$, $\rho_0$, and $P_0$?

  **Solution:** Comparing the last two terms from the previous solution we see that

  $$
  c_0 = \sqrt{\rho_s P_0 / \eta_0}.
  $$
Vector differential equations

4.1 Problems VC-1

Topics of this homework:
Vector algebra and fields in \( \mathbb{R}^3 \), divergence and curl, Gauss's divergence and Stokes' theorems, system classification problems.

Example: The heat flux is proportional to the change in temperature times the thermal conductivity \( \kappa \) of the medium. The heat flux is given by:

\[
\text{heat flux} = \kappa \cdot \text{change in temperature}.
\]

The algorithm iteratively converges on the GCD by subtracting out multiples of the GCD until only the GCD is left. The algorithm can be described in general as:

- \( \gcd(a, b) \) where \( a \geq b > 0 \)
- Subtract \( b \) from \( a \) if \( a \) is not a multiple of \( b \)
- Repeat until \( a = 0 \) or \( b = 0 \)

- If \( a = 0 \), then \( \gcd(b, 0) = b \)
- If \( b = 0 \), then \( \gcd(a, b) = a \)

The algorithm is used to find the greatest common divisor (GCD) of two numbers. It can be implemented using either the subtraction method or the division method.

- **Subtraction method:**
  \[
  a_1 = a - b \cdot \left\lfloor \frac{a}{b} \right\rfloor,
  b_1 = b
  \]
  Repeat until \( b = 0 \)

- **Division method:**
  \[
  a_1 = a / b,
  b_1 = b
  \]
  Repeat until \( b = 0 \)

Note: The algorithm can be used to find the prime number factors of a number. For example, to find the prime factors of 77:

- 77 = 7 × 11

The answer is 7 × 11, which is the prime factorization of 77.
Problem # 5: Coprimes

5.1: Define the term coprime.

Sol: When two integers have no common factors, they are said to be coprime.

5.2: How can the Euclidean algorithm be used to identify coprimes?

Sol: If \( \gcd(a,b) = 1 \) they only have 1 as a common factor, thus they are coprime.

5.3: Give at least one application of the Euclidean algorithm.

Sol: Given two integers \( n,d \in \mathbb{Z} \), if we wish to reduce the fraction \( \frac{n}{d} \), we must cancel the common factors.

Example: If \( n = 9 \), \( d = 6 \) then \( \frac{9}{6} = \frac{3 \cdot 3}{2 \cdot 3} = \frac{3}{2} \). The GCD, 3, may be identified using the Euclidean algorithm. While this fraction may be easily simplified via inspection, the GCD algorithm could be very helpful for larger numbers \( n,d \).

5.4: Write a Matlab function, \( \text{function } x = \text{my}\_gcd(a,b) \) that uses the Euclidean algorithm to find the GCD of any two inputs \( a \) and \( b \).

Test your function on the \((a,b)\) combinations from the previous problem. Include a printout (or hand-write) your algorithm to turn in.

Hints and advice:

- Don’t give your variables the same names as Matlab functions! Since \( \text{gcd} \) is an existing Matlab/Octave function, if you use it as a variable or function name, you won’t be able to use \( \text{gcd}() \) to check your \( \text{gcd()} \) function. Try \( \text{clear all} \) to recover from this problem.
- Try using a “while” loop for this exercise (see Matlab documentation for help).

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- Try using a “while” loop for this exercise (see Matlab documentation for help).

Algebraic generalization of the GCD (Euclidean) algorithm

Problem # 6: In this problem we are looking for integer solutions \((m,n) \in \mathbb{Z}\) to the equations \( ma + nb = \gcd(a,b) \) and \( ma - nb = 0 \) given positive integers \((a,b) \in \mathbb{Z}^+\). Note that this requires that either \( m \) or \( n \) be negative. These solutions may be found using the Euclidean algorithm only if \((a,b)\) are coprime \((a \perp b)\). Note that integer (whole number) polynomial relations such as these are known as Diophantine equations. Such equations (e.g., \( ma - nb = 0 \)) are linear Diophantine equations, possibly the simplest form of such relations.

Example: \( \gcd(2,3) = 1 \): For \((a,b) = (2,3)\), the result is

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
0 & 1 \\
1 & -2
\end{bmatrix} \cdot \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix} \cdot \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
-1 & 1 \\
3 & -2
\end{bmatrix}
\]

Thus from the above equation we find the solution \((m,n)\) to the integer equation

\[2m + 3n = \gcd(2,3) = 1;\]
3.3. PROBLEMS

– 4.6: What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?

Sol:

$$
F_1 \frac{V_1}{V_1} = T...
$$

cells, the delay linearly increases with $N$, since each eigenvalue represents the delay of one cell, and delay adds. ■

Problem #

– 6.3: Does the equation $(m,n) = (−1,1)$ (i.e., $−2 + 3 = 1$). There is also a second representation will be infinitely long. We can represent the CFA coefficients $\alpha$ as a vector of integers $n_k$, $k = 1, 2, ... , \infty$:

$$
...'
$$

where

$$
\pm e^m \pm bi
$$

is the one way delay.

Division method

Sol:

$$
\frac{1}{2} = \left( \begin{array}{c}
1 \\
1
\end{array} \right)
$$

The result of this method is one of $a$, $b$, $c$, or $d$ in the set

$$
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
$$

The final result of the Division method is $a_n = x = a_1$.

The results of the Division method are $c_n = a_1$ and $d_n = b_1$.

Suppose we choose to divide $\alpha$ into the $n_k$, $k = 1, 2, ... , \infty$. We can represent the CFA coefficients $\alpha = (\alpha_1, \alpha_2, ... , \alpha_n)$ as a vector of integers $n_k$, $k = 1, 2, ... , \infty$.

The results of the Division method are $c_n = a_1$ and $d_n = b_1$. Therefore, the Division method is more complicated than the Division method, because it allows us to compute the coefficients for each $n_k$, $k = 1, 2, ... , \infty$. The Division method is more complex because it requires more calculations than the Division method.

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α = [n₁; n₂, n₃, n₄,...] = n₁ + 1/n₂ + 1/n₃ + 1/n₄ + ... .

As discussed in Sec. 2.4.3 (p. 28), the CFA is recursive, with three steps per iteration. For α₁ = π, n₁ = 3, r₁ = π – 3, and α₂ = 1/r₁:

α₂ = 1/0.1416 = 7.0625...

As discussed in Sec. 1.2.11, the CFA is recursive, with three steps per iteration. For α₁ = π, n₁ = 3, r₁ = π – 3, and α₂ = 1/r₁:

α₂ = 1/0.1416 = 7.0625...

α₁ = n₁ + 1/n₂ + 1/n₃ + 1/n₄ + ... = n₁.

In terms of a Matlab/Octave script:

```matlab
alpha0 = pi;
K = 10;
n = zeros(1, K); alpha = zeros(1, K);
alpha[1] = alpha0;
for k = 2:K, % k=1 to K
    n(k) = round(alpha(k-1));
    alpha(k) = 1/[alpha(k-1) - n(k)];
    disp([n(k), alpha(k)]);
end
```

8.1: By hand (you may use Matlab/Octave as a calculator), find the first three values of r₃ for α = e.The CFA for this is: eₙ = 23.1407 ... [23.7, 9.4, ... .]

8.2: For the preceding question, what is the error (remainder) when you truncate the continued fraction after n₁, ..., n₄? Give the absolute value of the error and the percentage error relative to the original α.
The remainder is εₚ = (23 + 1/(7 + 1/9)) which gives an error of ε = |εₚ| = (23 + 1/(7 + 1/9))/εₚ = 2.92 × 10⁻⁸ = 0.00008%.

8.3: Use the Matlab/Octave program provided to find the first 10 values of αₙ for α = e and verify your result using the Matlab/Octave command π/α.αₙ = 23.1407 ... [23.7, 9.4, 2.591, 2.591, 2.591, 2.591, 2.591, 2.591, 2.591, 2.591, ... .]

8.4: Discuss the similarities and differences between the Euclidean algorithm and the CFA.

1. Both are recursive, meaning that the steps are repeated one after another.
2. The EA starts from two numbers (a, b). The output of the gca(a, b) is the GCD. The CFA starts with a single number and the output is a sequence of integers. If the sequence terminates the number was rational. If the sequence does not terminate, the number is irrational.
3. The EA works with the difference between the minimum and maximum of the two numbers whereas the CFA works with the rounding function and the reciprocal of the error.
4. It would seem that the goals of the two algorithms, the starting point, and the results are totally different. Both are very useful and powerful. Both generalize to more difficult situations than working with simple numbers.

72. CHAPTER 3. DIFFERENTIAL EQUATIONS

Express H₁₂ in terms of a residue expansion. Sol: From Eq. DE-3.4a, V₁ = sCF₂ – (s²MC/2 + 1)V₂. Since F₂ = 0,

\[ V₁ \frac{V₂}{s²MC/2 + 1} = c₁e^{x₁} + c₂e^{x₂}, \]

having eigenfrequencies \( \lambda₁ = ±\sqrt{\frac{MC}{2}} \), and associated eigenvalues \( c₁ = ±\sqrt{\frac{MC}{2}} \). ❑

4.2: Find h₂₃(t) \( \leftrightarrow \) H₂₃(s).

Sol:

\[ h(t) = \frac{e^{t/\tau_0}}{s²MC/2 + 1} \cdot \frac{e^{-t/\tau_2}}{s²MC/2 + 1} \cdot \frac{e^{t/\tau_1}}{1 + \frac{sMC}{M}}, \]

The integral follows from the Cauchy Residue theorem (CRT). ❑

4.3: What is the input impedance \( Z₂ \rightarrow V₂/\bar{V}_₂ \), assuming \( \bar{V}_2 = -r_0V₂ \)?

Sol: From Eq. DE-3.4a find \( Z₂ \):

\[ Z₂(s) = \left( T \left[ \frac{V}{V_2} \right] = \frac{V}{V_2} = \frac{1 + s²MC/2}{s²MC/2 + 1} \frac{sMC}{s²MC/2 + 1} \right) \]

4.4: Simplify the expression for \( Z₂ \) as follows:

1. Assume the characteristic impedance \( r_0 = \sqrt{MC} \).
2. Terminate the system in \( r_0: \bar{V}_2 = -r_0V₂ \) (e., \( V₂ \) cancels).
3. Assume higher-order frequency terms are less than 1 (|s₁/s₂| < 1).
4. Let the number of cells \( N \rightarrow \infty \). Thus |s₁/s₂| = 0.

When a transmission line is terminated in its characteristic impedance \( r_0 \), the input impedance \( Z₂(s) = r_0 \). Thus, when we simplify the expression for \( Z₂(s) \), it should be equal to \( r_0 \). Show that this is true for this setup.

Sol: Applying the Nyquist approximation (i.e., ignore second order frequency terms |s₁/s₂| = 0)

\[ Z₂(s) = \frac{r₀(1 + s²MC/2)}{r₀ + s²MC/2} = \frac{sMC}{s²MC/2} = \frac{M}{r₀}, \]

We conclude that below the Nyquist cutoff frequency, as \( N \rightarrow \infty \) the system equals a transmission line terminated by its characteristic impedance thus \( Z₂(s) = r₀ \). ❑

4.5: State the ABCD matrix relationship between the first and Nth nodes in terms of the cell matrix. Write out the transfer function for one cell, \( H_{23} \).

Sol:

\[ T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \]

Now use the formulae for the eigenvalues and vectors to obtain \( T \) for \( N = 1 \):

\[ T = E\Lambda E^{-1} = E \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} E^{-1}. \]
The solution is a repeat of what is summarized above: the system in Fig. 3.3 represents a problem, find the transfer function \( H(s) \equiv \frac{V_2}{V_1} \). From the results of the \( T \) matrix, find \( H_{21}(s) = \frac{V_2}{V_1} \left|_{T_2=0} \right. \)

**Problem:** Find the vector transformation
\[
\begin{bmatrix} 1 \sqrt{e^{i2\pi/5}} \end{bmatrix}
\]
where \( \sqrt{e^{i2\pi/5}} \) is a primitive 5th root of unity.

**Solution:**
\[
\sqrt{e^{i2\pi/5}} = \cos \left( \frac{2\pi}{5} \right) + i \sin \left( \frac{2\pi}{5} \right)
\]

The characteristic impedance defined is
\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]

The matrix equation for the CFA is derived in Sec. F, p. 249. We conclude that taking Eq. 2.4.2 to the other way around. In one case we can determine a band-limited approximation of
\[
\frac{V_2}{V_1} \equiv H(s)
\]
and swapping rows, results in a CFA matrix. However I believe this must be iterated. It follows that the GCD and
\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]

Using the formula for the wavelength in terms of the wave velocity and frequency we find
\[
\lambda = \frac{c}{f}
\]

The approximation is highly accurate below the Nyquist cutoff frequency
\[
f < \frac{c}{2\lambda}
\]

The approximation is also useful above the Nyquist frequency
\[
f > \frac{c}{2\lambda}
\]

The corresponding unnormalized eigenvectors are
\[
\begin{bmatrix} 1 \sqrt{e^{i2\pi/5}} \end{bmatrix} \text{ and } \begin{bmatrix} 1 \sqrt{e^{i4\pi/5}} \end{bmatrix}
\]

We can either let
\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]

Thus the Nyquist condition represents a computational bound, not a physical limitation.

The length of the CFA is
\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]

Each cell, composed of 2 cells, or \( [1., 2] \). Matlab gives
\[
\text{rat}(23/7) = [2; 2,2,2,2,2,\cdots]
\]

As functions of \( c \) and \( T \) is not a problem and an appropriate

**Problem 3:** Find the CFA and expression of
\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]

\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]

\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]

\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]

\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]

\[
\frac{3}{7} = \frac{1}{2} + \frac{1}{2 + 1/(2 + 1/(2 + \cdots))}
\]
9.7: Show that
\[
\frac{1}{1 - \sqrt{a}} = a + a^2 + a^4 + a^6 + \cdots
\]
This is a Taylor expansion of 1 expressed in terms of removable singularities. See Cotes Theorem (1716) (Stillwell, 2010, p. 289). ■

Transmission-line analysis

Problem # 2: Train-mission-line We wish to model the dynamics of a freight train that has \( N \) such cars and study the velocity transfer function under various load conditions.

As shown in Fig. 3.3, the train model consists of masses connected by springs.

Use the ABCD method (see the discussion in Appendix B.3, p. 230) to find the matrix representation of the system of Fig. 3.3. Define the force on the soft train car \( f_2(t) \leftrightarrow F_2(\omega) \) and the velocity \( v_2(t) \leftrightarrow V_2(\omega) \).

Break the model into cells consisting of three elements: a series inductor representing half the mass \((L = M/2)\), a shunt capacitor representing the spring \((C = 1/K)\), and another series inductor representing half the mass \((L = M/2)\), transforming the model into a cascade of symmetric \((A = 0)\) identical cell matrices \( T(s) \).

- 2.1: Find the elements of the ABCD matrix \( T \) for the single cell that relate the input node 1 to output node 2

\[
\begin{bmatrix}
F \\
V_1
\end{bmatrix} = T \begin{bmatrix}
F(\omega) \\
V(\omega)
\end{bmatrix},
\]

where

\[
T = \begin{bmatrix}
\frac{sM}{2} & 1 \\
0 & \frac{1}{sC}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & \frac{1}{sC} \\
\frac{sM}{2} & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 + s^2MC/2 \\
1 + s^2MC/2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & sM/2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & sM/2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & sM/2 \\
0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 + s^2MC/2 \\
1 + s^2MC/2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 + s^2MC/2 \\
1 + s^2MC/2
\end{bmatrix}
\]

\[
\text{DE-3.3}
\]

\[
\text{DE-3.4a}
\]

- 2.2: Express each element of \( T(s) \) in terms of the complex Nyquist ratio \( s/s_n < 1 \) \((s = 2\pi f, s_n = 2\pi f_n)\). The Nyquist wavelength sampling condition is \( \lambda_c > 2\Delta \). It says the critical wavelength \( \lambda_c \) must be greater than the Nyquist wavelength \( 2\Delta \).

Namely it is defined in terms the minimum number of cells \( 2\Delta \) per minimum wavelength \( \lambda_c \).

The Nyquist wavelength sampling theorem says that there are at least two cars per wavelength.

Proof: From the figure, the distance between cars \( \Delta = c_s T_s [\text{m}] \), where

\[
c_s = \frac{1}{\sqrt{MC}} [\text{m/s}]
\]

The cutoff frequency obeys \( f_s = c_s \). The Nyquist critical wavelength is \( \lambda_c = c_s / f_c > 2\Delta \). Therefore the Nyquist sampling condition is

\[
f < f_c \equiv \frac{c_s}{\lambda_c} = \frac{1}{2\Delta \sqrt{MC}} [\text{Hz}].
\]

\[
\text{DE-3.5}
\]

Finally, \( s_n = 2\pi f_n \).
### 3.3 PROBLEMS DE-3

#### – 1.2: Take the Laplace transform (LT) of Eq. DE-3.2 and find the total impedance $Z(s)$ of the system. Apply the Cauchy-Riemann (CRT) theorem to find the residue at $s = 0$.

$$z(t) = c_+ e^{st} + c_- e^{-st},$$

where $s_\pm = -1/2 \pm \sqrt{3}/2$ and $c_\pm = 1/2 \pm \sqrt{3}/2$. ■

#### – 1.3 Problems NS-3

**Topic of this homework:** Pythagorean triplets, Pell's equation.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>119</td>
<td>169</td>
</tr>
<tr>
<td>3367</td>
<td>4825</td>
</tr>
<tr>
<td>4601</td>
<td>6649</td>
</tr>
<tr>
<td>12709</td>
<td>18541</td>
</tr>
</tbody>
</table>

Thus, $a \in \mathbb{Z}$ and $c \in \mathbb{Z}$.

**Problem #1**: Find the roots of the numerator of $p(s) = s^2 + 1$ and the roots of the denominator of $q(s) = s^2 - 1$.

**Sol:**

Let $s_1$ and $s_2$ be the roots of $p(s)$ and $q(s)$, respectively.

**Problem #2**: Assume that $M(s) = s^2 + 1 \neq 0$ and $N(s) = s^2 - 1 \neq 0$.

**Problem #3**: Assume that $M(s) = s^2 + 1 = 0$ and $N(s) = s^2 - 1 = 0$.

**Problem #4**: Assume that $M(s) = s^2 + 1 \neq 0$ and $N(s) = s^2 - 1 = 0$.

**Problem #5**: Assume that $M(s) = s^2 + 1 = 0$ and $N(s) = s^2 - 1 \neq 0$.

**Problem #6**: Assume that $M(s) = s^2 + 1 \neq 0$ and $N(s) = s^2 - 1 \neq 0$.

**Problem #7**: Assume that $M(s) = s^2 + 1 = 0$ and $N(s) = s^2 - 1 = 0$.

**Problem #8**: Assume that $M(s) = s^2 + 1 = 0$ and $N(s) = s^2 - 1 \neq 0$.

**Problem #9**: Assume that $M(s) = s^2 + 1 \neq 0$ and $N(s) = s^2 - 1 \neq 0$.

**Problem #10**: Assume that $M(s) = s^2 + 1 \neq 0$ and $N(s) = s^2 - 1 = 0$.

**Problem #11**: Assume that $M(s) = s^2 + 1 = 0$ and $N(s) = s^2 - 1 \neq 0$.

**Problem #12**: Assume that $M(s) = s^2 + 1 \neq 0$ and $N(s) = s^2 - 1 = 0$.

**Problem #13**: Assume that $M(s) = s^2 + 1 = 0$ and $N(s) = s^2 - 1 = 0$.
Find a formula for $a$ in terms of $p$ and $q$.

Sol:

\[(a, c) = (119, 169) \quad (p, q) = \pm (12, 5) \]

\[(a, c) = \ldots \text{ and eigenvectors.} \]

The eigenvalues $\lambda_k$ and eigenvectors $\vec{e}_k$ of a square matrix $A$ are related by

\[A\vec{e}_k = \lambda_k\vec{e}_k, \quad (NS-3.1)\]

---

3.3 Problems DE-3

Topics of this homework: Brune impedance

lattice transmission line analysis

Brune Impedance

Problem #1: Residue form

A Brune impedance is defined as the ratio of the force $F(s)$ to the flow $V(s)$ and may be expressed in residue form as

\[Z(s) = c_0 + \sum_{k=1}^{N} \frac{c_k}{s - s_k} - \frac{N(s)}{D(s)} \quad (DE-3.1)\]

with

\[D(s) = \prod_{k=1}^{N}(s - s_k) \quad \text{and} \quad c_2 = \lim_{s \to s_k} (s - s_k)D(s) = \prod_{k=1}^{N}(s - s_k).\]

The prime on the index $n'$ means that $n = k$ is not included in the product.

---

3.1: Find the Laplace transform ($LT$) of (a) a (1) spring, (2) dashpot, and (3) mass.

Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: (1) Hooke’s law $f(t) = Kx(t)$, (2) dashpot resistance $f(t) = Rv(t)$, and (3) Newton’s law for mass $f(t) = Md(t)/dt$.

1. Hooke’s law $f(t) = Kx(t)$. Taking the $LT$ gives

\[F(s) = KX(s) = KV(s)/s \leftrightarrow f(t) = Ku(t) \leftrightarrow V(t) = K \int f(t) dt,\]

since

\[v(t) = \frac{1}{s}F(t) \leftrightarrow V(s) = sX(s).\]

Thus the impedance of the spring is

\[Z_s(s) = \frac{K}{s} \leftrightarrow Z(t) = Ku(t),\]

which is analogous to the impedance of an electrical capacitor. The relationship may be made tighter by specifying the compliance of the spring as $C = 1/K$.

2. Dashpot resistance $f(t) = Rv(t)$. From the $LT$ this becomes

\[F(s) = RV(s)\]

and the impedance of the dashpot is then

\[Z_v = R \leftrightarrow R(t),\]

analogous to that of an electrical resistor.

3. Newton’s law for mass $f(t) = Md(t)/dt$. Taking the $LT$ gives

\[f(t) = MD(t)/dt \leftrightarrow F(s) = sMv(s),\]

thus

\[Z_m(s) = sM \leftrightarrow M\frac{d}{dt},\]

analogous to an electrical inductor.

---

Pell’s equation:

Problem 3: Pell’s equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \notin \mathbb{Q}$. We seek integer solutions of

\[x^2 - Ny^2 = 1.\]

As shown on page 42, the solutions $x_n, y_n$ for the case of $N = 2$ are given by the linear $2 \times 2$ matrix recursion

\[
\begin{bmatrix}
  x_{n+1} \\
  y_{n+1}
\end{bmatrix} =
\begin{bmatrix}
  1 & 2 \\
  1 & 1
\end{bmatrix}
\begin{bmatrix}
  x_n \\
  y_n
\end{bmatrix}
\]

with $[x_0, y_0]^T = [1, 0]^T$ and $1 + \sqrt{2} = e^{\pi/4}$. It follows that the general solution to Pell’s equation for $N = 2$ is

\[
\begin{bmatrix}
  x_n \\
  y_n
\end{bmatrix} =
\begin{bmatrix}
  e^{\pi/4} & 1 \\
  1 & 1
\end{bmatrix}^n
\begin{bmatrix}
  x_0 \\
  y_0
\end{bmatrix},
\]

To calculate solutions to Pell’s equation using the matrix equation above, we must calculate

\[A^n = e^{n\pi/4} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},\]

which becomes tedious for $n > 2$.

---

Diagonalization of a matrix (eigenvalue/eigenvector decomposition):

As derived in Appendix B, the most efficient way to compute $A^n$ is to diagonalize the matrix $A$ by finding its eigenvalues and eigenvectors.

The eigenvalues $\lambda_k$ and eigenvectors $\vec{e}_k$ of a square matrix $A$ are related by

\[A\vec{e}_k = \lambda_k\vec{e}_k, \quad (NS-3.1)\]
3.2. PROBLEMS

Every impedance $Z(s)$ has a corresponding reflectance function given by a M"obius transformation, which may be expressed as:

$$0 = (y - \lambda) \left[ \begin{array}{cc} 1 & 1 \\ \tau & \tau \end{array} \right] \left[ \begin{array}{c} \lambda \\ 1 \end{array} \right] = (y - \lambda) \left( \frac{\lambda - \tau}{\lambda - \tau} \right) \Rightarrow \lambda = \frac{3 \pm \sqrt{2}}{2} > 0$$

Problem 10.3: Should an impedance matrix always be positive-definite? Explain.

Sol: Yes.

In the time domain, the matrices that are diagonal and off-diagonal elements $\delta$ and $\tau$ are determined from Eq. NS-3.1, by factoring out $(\lambda - \tau)$ from the matrix equation:

$$A \vec{e} = \vec{0}$$

Thus, from Eq. NS-3.2, we can conclude if $\vec{e}$ is not zero, yet when operated on by $A$, the roots of which are the eigenvalues of the matrix $A$.

These concepts may be easily extended to higher dimensions.

Problem 10.2: Is the impedance matrix positive-definite? Show your work by finding the eigenvalues $\lambda$ of the matrix $A$.

Sol: The determinant equation results in a second-degree polynomial in $\lambda$:

$$\det(A - \lambda I) = \det \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] = ad - bc = 0$$

Thus, the matrix equation $A \vec{e} = \vec{0}$ is a linear system of equations, the solutions of which are the eigenvalues $\lambda$ of the matrix $A$.

Note: This problem considers the relationship between the impedance matrix and its eigenvalues.

\begin{align*}
\tau & = 1 \\
\Lambda & = \begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\end{align*}

The power $P$ in the network, considering the impedance matrix $Z$ and the current $I$, is given by:

$$P = V \cdot I$$

where $V$ is the voltage across the impedance matrix and $I$ is the current flowing through the network.
1.3. PROBLEMS NS-3

Finding the eigenvectors:

An eigenvector \( \vec{e}_k \) can be found for each eigenvalue \( \lambda_k \) from Eq. NS-3.1,

\[
(A - \lambda_k I)\vec{e}_k = \vec{0}.
\]

The left side of the above equation becomes a column vector, where each element is an equation in the elements of \( \vec{e}_k \), set equal to 0 on the right side. These equations are always degenerate, since the determinant is zero. Thus the two equations have the same slope.

Solving for the eigenvectors is often confusing because they have arbitrary magnitudes, \( ||\vec{e}_k|| = \sqrt{e_{k1}^2 + e_{k2}^2} = d \). From Eq. NS-3.1, we can determine only the relative magnitudes and signs of the elements of \( \vec{e}_k \), so we have to choose a magnitude \( d \). It is common practice to normalize each eigenvector to have unit magnitude (\( d = 1 \)).

– 3.1: Find the companion matrix and thus the matrix \( A \) that has the same eigenvalues as Pell’s equation. Hint: Use Matlab’s function \([E, \text{Lambda}] = \text{eig}(\lambda)\) to check your results!

Sol: The companion matrix is

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\]

– 3.2: Solutions to Pell’s equation were used by the Pythagoreans to explore the value of \( \sqrt{2} \). Explain why Pell’s equation is relevant to \( \sqrt{2} \).

Sol: As discussed Sec. 2.5.2 Chapter 2, as the iteration increases, the ratio of the \( x_n/y_n \) approaches \( \sqrt{2} \).

– 3.3: Find the first three values of \((x_n, y_n)^T\) by hand and show that they satisfy Pell’s equation for \( n = 2 \). See class notes (slide 9.4.2) for this calculation.

Sol: By hand, find the eigenvectors \( \lambda_k \), of the 2 \( \times \) 2 Pell’s equation matrix

\[
A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}
\]

The eigenvalues are given by the roots of the equation \((1 - \lambda_k)^2 = 2\). Thus \( \lambda_k = 1 \pm \sqrt{2} = 2.4142, -4142 \).

– 3.4: By hand, show that the matrix of eigenvectors, \( E \), is

\[
E = \begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}
\]

Sol: The eigenvectors \( \vec{e}_k \) may be found by solving

\[
\begin{align*}
A \begin{bmatrix} e_1 \\
  e_2 \end{bmatrix} &= \lambda_k \begin{bmatrix} e_1 \\
  e_2 \end{bmatrix} \\
(A - \lambda_k I) \begin{bmatrix} e_1 \\
  e_2 \end{bmatrix} &= \vec{0}
\end{align*}
\]

For \( \lambda_k \), this gives

\[
\begin{align*}
0 &= \begin{bmatrix} 1 - (1 + \sqrt{2}) & 2 \\
  1 & 1 - (1 + \sqrt{2}) \end{bmatrix} \begin{bmatrix} e_1 \\
  e_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 2 \\
  1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1 \\
  e_2 \end{bmatrix}
\end{align*}
\]

which gives the relation between the elements of \( \vec{e}_k \), \( e_1, e_2 \), as \( e_1 = -\sqrt{2} e_2 \).

The eigenvectors are defined to be unit length and orthogonal, namely

1. \( ||\vec{e}_k||^2 = \vec{e}_k \cdot \vec{e}_k = 1 \)
2. \( \vec{e}_k \cdot \vec{e}_k = 0 \).

Once we normalize \( \vec{e}_k \) to have unit length, we obtain the first eigenvector

\[
\vec{e}_k = \frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{2} \\
  1 \end{bmatrix}
\]

\[
\alpha q(t) = \alpha q(t - T_p) + v(t)
\]

\[
i(t) = q(t) - (1/\alpha)(q(t - T_p))
\]

Figure 3.2: This feedback network, described by a time-domain difference equation with delay \( T_p \), has an all-pole transfer function \( \psi(\alpha) \equiv \rho(s)/\rho'(s) \) given by Eq. DE-2.12, which physically corresponds to a stub of a transmission line, with the input at one end and the output at the other. To describe the \( \psi(\alpha) \) function we must take \( \alpha = -1 \). A transfer function \( \psi(s) \equiv \rho(s)/\rho'(s) \) has the same poles as \( \psi(\alpha) \), but with zeros as given by Eq. DE-2.14, in the input admittance \( Y(s) = I(s)/V(s) \) of the transmission line, defined as the ratio of the Laplace transform of the current \( i(t) \) to the voltage \( v(t) \).

Explain what this means in physical terms. Start with two terms (e.g., \( z_1(t) \times z_2(t) \)). Hint: The input admittance of this cascade may be interpreted as the analytic continuation of \( \psi(\alpha) \) by defining a cascade of eigenvectors with eigenvalues derived from the primes. For a discussion of this idea see Sec. 3.2.3 and C.1.1.

Sol: In terms of the physics, these transmission line equations are telling us that \( \psi(\alpha) \) may be decomposed into an infinite cascade of transmission lines, each having a delay given by \( T_p = \ln \rho_p \).

Physical interpretation: Such functions may be generated in the time domain, as shown in Fig. 3.2 (p. 68), using a feedback delay of \( T_p \) seconds, described by the two equations in the Fig. 3.2 with a unity feedback gain \( \alpha = -1 \). Taking the Laplace transform of the system equation, we see that the transfer function between the state variable \( q(t) \) and the output \( i(t) \) is given by \( \zeta_1 \), which is an all-pole function, since

\[
Q(s) = e^{-\alpha T_p}Q(s) + V(s), \quad \zeta_1(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-\alpha T_p}},
\]

(C.2.12)

Closing the feed-forward path gives a second transfer function \( Y(s) = I(s)/V(s) \)—namely,

\[
Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1}{1 - e^{-\alpha T_p}} + e^{-\alpha T_p}.
\]

(C.2.13)

If we take \( i(t) \) as the current and \( v(t) \) as the voltage at the input to the transmission line, then \( y(t) \equiv \zeta_1(s) \) represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the \( j\omega \) axis. By a slight modification, \( \zeta_1(s) \) may alternatively be written as

\[
Y_1(s) = e^{-\alpha T_p} + e^{-\alpha T_p} = j \tan(\frac{\alpha T_p}{2}).
\]

(C.2.14)

Here we use a shorthand double-parentheses notation to define the infinite (one-sided) sum \( f(t) = \sum_{t=0}^{\infty} f(t - kT) \).
3.2. PROBLEMS

– 7.4: Try the following Matlab/Octave commands, and then comment on your findings.
% Take the inverse LT of 1/sqrt(1+s^2)
 Sol:
 J_0(t) = \frac{\sin(t)}{t}

Problem 8: Here we seek the inverse transform for the partial differential Eq. 2.8-3.6,

\[ \frac{\partial^2 \zeta_p(s)}{\partial s} = \int_0^\infty e^{-st} \zeta(s) \, dt \]

and then proceed on as for the partial case.

In the case of the geometric series representation

\[ \sum_{k=0}^{\infty} e^{-skT_p} = \frac{1}{1 - e^{-sT_p}} \]

for which you can look up the LT⁻¹ of each term.

Repeating this for \( \lambda \), gives

\[ \vec{e}^{-t} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ & \ 1 \\ & \ 1 \end{pmatrix} \]

Thus, the matrix equation above may be generated by

\[ \begin{cases} x_n = x_{n-1} + x_{n-2} \\ y_n = y_{n-1} + y_{n-2} \end{cases} \quad n \geq 0 \]

for which you can look up the LT⁻¹ equation (the problem is for extra credit).

Inverse of Riemann–Liouville fractional integral

\[ I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t^\alpha - s^\alpha) f(s) \, ds \quad \alpha > 0 \quad \alpha 
eq 1 \]

Solving the Bessel equation, which is the wave equation in 2D. Bessel functions were first introduced by the

Inverse of Riemann–Liouville fractional derivative

\[ D_a^\alpha f(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \nu)} \int_0^t (t^\nu - s^\nu) f(s) \, ds \quad \nu > 0 \quad \nu 
eq 1 \]

Inverse of Riemann–Liouville fractional derivative

\[ D_a^\alpha f(t) = \frac{\Gamma(\alpha-
u)}{\Gamma(\alpha)} \int_0^t (t^\nu - s^\nu) f(s) \, ds \quad \nu > 0 \quad \nu < 1 \]

Inverse of Riemann–Liouville fractional derivative

\[ D_a^\alpha f(t) = \frac{\Gamma(\alpha-
u)}{\Gamma(\alpha)} \int_0^t (t^\nu - s^\nu) f(s) \, ds \quad \nu < 0 \quad \nu 
eq -1 \]

Inverse of Riemann–Liouville fractional derivative

\[ D_a^\alpha f(t) = \frac{\Gamma(\alpha-
u)}{\Gamma(\alpha)} \int_0^t (t^\nu - s^\nu) f(s) \, ds \quad \nu < 0 \quad \nu = -1 \]

Inverse of Riemann–Liouville fractional derivative

\[ D_a^\alpha f(t) = \frac{\Gamma(\alpha-
u)}{\Gamma(\alpha)} \int_0^t (t^\nu - s^\nu) f(s) \, ds \quad \nu < 0 \quad \nu = -1 \]
1.3. PROBLEMS NS-3

– 4.2: What is the relationship between \( y_n \) and \( x_n \)?
Sol: This equation says that \( x_n = x_{n-1} + y_{n-1} \). The latter equation may be rewritten as \( y_n = x_{n-1} \). Thus
\[
x_n = x_{n-1} + x_{n-2}.
\]

which is Eq. NS-3.5. ■

– 4.3: Write a Matlab/Octave program to compute \( x_n \), using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is \( x_{20} \)? Note: Consider using the eigenanalysis of \( A \), described by Eq. NS-3.3 (p. 18).
Sol: You can try something like:

```matlab
function x = fib(n)
    A = [1 1; 1 0]; % [E, D] = eig(A); % x = E*D; % x = fib1();
    % A = xy(1);
    Given the initial conditions we defined, \( x_{20} = 165,580,141.1 \). ■
```

– 4.4: Using the eigenanalysis of the matrix \( A \) (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence
\[
x_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}.
\]
\[(\text{NS-3.8)}\]

– 4.5: What are the eigenvalues \( \lambda \) of the matrix \( A \)?
Sol: The eigenvalues of the Fibonacci matrix are given by
\[
\lambda = \frac{1 \pm \sqrt{5}}{2}.
\]
then \( \lambda_1 = \frac{1 + \sqrt{5}}{2} \) and \( \lambda_2 = \frac{1 - \sqrt{5}}{2} \). ■

– 4.6: How is the formula for \( x_n \) related to these eigenvalues? Hint: Find the eigenvectors.
Sol: The eigenvectors (determined from the equation \( (A - \lambda_1 I)\vec{e}_1 = \vec{0} \), and normalized to 1) are given by
\[
\vec{e}_1 = \left[ \frac{\lambda_1}{\sqrt{\lambda_1^2 - \lambda_2^2}} \right] \quad \vec{e}_2 = \left[ \frac{\lambda_2}{\sqrt{\lambda_2^2 - \lambda_1^2}} \right] \quad E = \left[ \vec{e}_1 \quad \vec{e}_2 \right].
\]

From the eigenanalysis, we find that
\[
\left[\begin{array}{c} x_n \\ y_n \end{array}\right] = E \left[ \begin{array}{c} \lambda_1^n \\ \lambda_2^n \end{array} \right] E^{-1} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = \left[ \begin{array}{cc} c_{11} & c_{12} \\ c_{21} & c_{22} \end{array} \right] \left[ \begin{array}{c} \lambda_1^n \\ \lambda_2^n \end{array} \right] = \left[ \begin{array}{c} c_{21} - c_{11} \\ c_{22} - c_{12} \end{array} \right].
\]

Solving for \( x_n \) and \( y_n \), we find that
\[
x_n = \frac{1}{c_{11}c_{22} - c_{12}c_{21}} \left( \lambda_1^n c_{11}c_{22} - \lambda_2^n c_{12}c_{21} \right) - \frac{1}{\sqrt{\lambda_1^2 - \lambda_2^2}} \left( \frac{\lambda_1^n}{\sqrt{\lambda_1^2 + 1}} - \frac{\lambda_2^n}{\sqrt{\lambda_2^2 + 1}} \right).
\]
\[
= \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}.
\]
\[\text{■}

– 4.7: What happens to each of the two terms
\[
\left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} \quad \text{and} \quad \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}.
\]
Sol: \( \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} \rightarrow 0 \) and \( \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \rightarrow \infty \) ■

we first consider the derivative and its Taylor series (about \( s = 0 \))
\[
F'(s) = \frac{1}{s} = \sum_{n=0}^{\infty} s^n.
\]

Then, we integrate this series term by term:
\[
F(s) = - \log(1-s) = \int F(s)ds = \sum_{n=0}^{\infty} \frac{s^n}{n}.
\]

Alternatively we can use Matlab/Octave commands:
```
syms s
syms x
Taylor = taylor(-log(1-s),'order',7)
solve(Taylor == 0)
```

1. Draw a hand sketch showing the nature of the branch cut. Hint: Use \( x^2 + y^2 = 0 \). The branch cut connects the two roots, or can go from each root to \( \infty \). Either choice is valid. ■

– 7.1: Use Octave’s taylor (-log(1-s)) to the seventh order, as in the example above.

1. Try the above Matlab/Octave commands. Give the first seven terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above). Sol:
\[
F(s) = \sum_{n=1}^{\infty} \left( \frac{s^n}{n} \right).
\]

2. What is the inverse Laplace transform of this series? Consider the series term by term.
Sol: \( f(t) = \sum \delta(t)/n \)

– 7.2: The function 1/\( \sqrt{z} \) has a branch point at \( z = 0 \); thus it is singular there.

1. Can you apply Cauchy’s integral theorem when integrating around the unit circle? Sol: No, one cannot apply the Cauchy Theorem since it is not analytic at \( z = 0 \). But the integral may be evaluated. ■

2. This Matlab/Octave code computes \( \int_0^{2\pi} \frac{dz}{z} \) using Matlab/Octave’s symbolic analysis package:
```
syms z
I = int(1/sqrt(z))
```

Run this script. What answers do you get for \( I \) and \( J \)?
Sol: This script returns the answers \( I = 2 \sqrt{2} \) and \( J = 2.4495e-16 \), which is numerically the same as zero. ■

3. Modify this code to integrate \( f(z) = 1/z^2 \) once around the unit circle. What answers do you get for \( I \) and \( J \)? Sol: This function has a 2d order pole at \( s = 0 \). Thus from the CIT, the integral evaluates to zero.

Proof:
\[
I = \int \frac{ds}{s^2} = \frac{1}{s^2} \Big|_0^\infty - e^{-s^2} \bigg|_0^\infty = -(1-1) = 0
\]

More generally \( I = \int \frac{dz}{z^n} = 0 \) for \( n \neq 1 \). As best I know, this holds for any \( n \in \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{R}, \mathbb{C} \). For \( n = 1 \) it has a value of \( 2e \). ■

– 7.3: Bessel functions can describe waves in a cylindrical geometry.

The Bessel function has a Laplace transform with a branch cut
\[
\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}.
\]

Draw a hand sketch showing the nature of the branch cut. Hint: Use \( \mathcal{L}\{x^2\} \). The roots are given by \( s_k = \pm \lambda \) for each root to \( \infty \). Either choice is valid. ■
3.2. PROBLEMS DE-2

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– 6.6: Assuming that \( V_2 = 0 \), find \( Y_{12}(s) \equiv I_2/V_1 \).

Sol: Setting \( V_2 = 0 \) we may read off the ...  to find the Taylor series expansion about \( s = 0 \) of
\[ F(s) = -\log(1 - s), \]

1https://www.wolframalpha.com/

CHAPTER 1. INSTRUCTORS MANUAL: NUMBER SYSTEMS

– 4.8: What happens to the ratio \( x_{n+1}/x_n \)?

Sol: \( x_{n+1}/x_n \rightarrow (1 + \sqrt{5})/2 \), ... = \( f_N f_{N+1} - f_N f_{N-1} \),
\[ f_N = f_{N+1} - f_{N-1} \]

Thus the relation only holds for the Fibonacci recursion formula. ■

Problem 9: Consider the expression

\[ \frac{5(x + s)}{\sqrt{5}} \]

– 5.1: What matrix is used to calculate this sequence?

Problem 10: What matrix is used to calculate the new sequence? What happens at \( n \rightarrow \infty \)?

Problem 11: Consider the expression

\[ \frac{1}{1 + x} \]

– 5.5: Modify your computer program to calculate the new sequence? Is \( \lambda_1 = 0 \)?

Problem 12: Consider the expression

\[ I \]

– 5.6: What are the eigenvalues of your new sequence? Is \( \lambda_2 = 0 \)?

Problem 13: Consider the expression

\[ \frac{1}{x} \]

– 5.7: Find the input impedance to the right-hand side of the system.

Problem 14: Consider the expression

\[ \frac{V}{R} \]

– 5.8: What are the eigenvalues such that the value \( \lambda_1 = 0 \)?

Problem 15: Consider the expression

\[ \frac{1}{x} \]

– 5.9: What properties are the eigenvalues?

Problem 16: Consider the expression

\[ I \]

– 5.10: What matrix is used to calculate this sequence?

Problem 17: Consider the expression

\[ \frac{V}{R} \]

– 5.11: What properties are the eigenvalues?

Problem 18: Consider the expression

\[ I \]

– 5.12: What matrix is used to calculate this sequence?

Problem 19: Consider the expression

\[ \frac{V}{R} \]

– 5.13: What properties are the eigenvalues?

Problem 20: Consider the expression

\[ I \]

– 5.14: What matrix is used to calculate this sequence?
1.3. PROBLEMS NS-3

CFA as a matrix recursion

Problem # 7: The CFA may be written as a matrix recursion. For this we adopt a special notation, unlike other matrix notations,\(^1\) with \(k \in \mathbb{N}:\)

\[
\begin{bmatrix}
[0] \\
[0] \\
[k]
\end{bmatrix}
= \begin{bmatrix}
[x_k] \\
[x_k]
\end{bmatrix}
\]

This equation says that \(n_{k+1} = [x_k]\) and \(x_{k+1} = 1/(x_k - [x_k]).\) It does not mean that \(n_{k+1} = [x_k] x_k,\) as would be implied by standard matrix notation. The lower equation says that \(x_k - [x_k]\) is the remainder—namely, \(x_k = [x_k] + r_k\) (Octave/Matlab's `floor(x)` function), also known as \(\text{mod}(x,y,\ldots).\)

\(- 7.1: \) Start with \(n_0 = 0 \in \mathbb{N}, x_0 \in \mathbb{R}, n_1 = [x_0] \in \mathbb{N}, r_1 = x - [x] \in \mathbb{L},\) and \(x_1 = 1/r_1 \in \mathbb{R},\) \(k \neq 0.\) For \(k = 1\) this generates on the left the next CFA parameter \(n_2 = [x_1]\) and \(x_2 = 1/r_2 = 1/(x_0 - [x_0])\) from \(n_0\) and \(x_0.\) Find \([n, x]_{k=1}^n\) for \(k = 2, 3, 4, 5.\)

Sol: If \(x_0 = \pi,\) then \(n_1 = [\pi] = 3, r_1 = \pi - n_1 = 0.14159\ldots\), and \(x_1 = 1/r_1 \approx 7.06: \)

\[
\begin{bmatrix}
3 \\
[7.06251]
\end{bmatrix}
\]

and for \(n = 2\)

\[
\begin{bmatrix}
7 \\
[7.06252]
\end{bmatrix}
= \begin{bmatrix}
[0] \\
[0]
\end{bmatrix}
\begin{bmatrix}
[7] \\
[7.0625]
\end{bmatrix}
\begin{bmatrix}
3 \\
[7.06251]
\end{bmatrix}
\]

For \(n = 3,\) \(n_1 = n_2 = n_3 = [3.7, 15.].\) Continuing \(n_4 = [1.003418] = 1\) and \(n_5 = 292.\)

\(- 6.2: \) Find the eigenvalues of the \(2 \times 2\) ABCD matrix. Hint: See Appendix B.3, page 230.

Sol: The eigenvalues of every \(2 \times 2\) matrix are given in Appendix B.3 (page 230)

\[
\lambda_x = \frac{1}{2} \left( 3 + \sqrt{3^2 - 4 \times 2 \times 2} \right)
\]

This simplifies to

\[
\lambda_x = \frac{1}{2} \left( 3 + \sqrt{3^2 - 4 \times 2 \times 2} \right)
\]

where \(\sqrt{3^2 - 4 \times 2 \times 2} = \sqrt{9 - 8} = \sqrt{1} = 1\)

When \(|s| < 1, \) \(\sqrt{\frac{1}{1 - s}} = \sqrt{\frac{1}{1 - s}}.\)

\(- 6.3: \) Assuming that \(I_2 = 0,\) find the transfer function \(H(s) \equiv V_2/V_1.\) From the results of the ABCD matrix you determined in questions 6.1 and 6.2, show that

\[
H(s) = \frac{s}{s + 1}
\]

Sol: Since \(I_2 = 0\) the upper row of the ABCD matrix gives the relationship between \(V_1\) and \(V_2\)

\[
V_1 = (1 + s)\times V_2
\]

Thus the ratio is as desired.

\(- 6.4: \) The transfer function \(H(s)\) has one pole. Where is the pole and residue? Sol: If we rewrite \(H(s)\) in the standard form, the pole \(s_p\) and residue \(A\) may be identified:

\[
H(s) = \frac{A}{s - s_p} = \frac{s}{s + 1}
\]

Thus the pole is \(s_p = -1\) and the residue is \(A = 1.\)

\(- 6.5: \) Find \(h(t),\) the inverse Laplace transform of \(H(s).\)

Sol:

\[
h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s + 1}\right\} = e^{-t}u(t)
\]

The integral follows from the Residue Theorem. The pole is at \(s_p = -1\) and the residue is \(s_p.\)
In polar coordinates
\[ \int_{0}^{2\pi} ds \sqrt{s} = \int_{0}^{2\pi} d\theta_i e^{i\theta/2} = i \int_{0}^{2\pi} e^{i\theta} e^{i\theta/2} d\theta = \cdots \]
results in terms of the dimensionless ratio \( s/\omega_c \), where \( \omega_c = 1/\tau \) is the cutoff frequency and \( \tau \) is the time constant.

**Problem 6**: A two-port network application for the Laplace transform

1. Find the \( 2 \times 2 \) ABCD matrix representation of Fig. 3.1. Express the results in terms of the dimensionless ratio \( s/\omega_c \), where \( \omega_c = 1/\tau \) is the cutoff frequency and \( \tau = RC \) is the time constant.
Chapter 2

Algebraic Equations

2.1 Problems AE-1


Note: The term analytic is used in two different ways. (1) An analytic function is a function that may be expressed as a locally convergent power series; (2) analytic geometry refers to geometry using a coordinate system.

Polynomials and the fundamental theorem of algebra (FTA)

Problem #1: A polynomial of degree \( N \) is defined as

\[
P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.
\]

1.1: How many coefficients \( a_n \) does a polynomial of degree \( N \) have?
Sol: \( N + 1 \)

1.2: How many roots does \( P_N(x) \) have?
Sol: \( N \)

Problem #2: The fundamental theorem of algebra (FTA)

2.1: State and then explain the FTA.
Sol: The FTA says that every polynomial has at least one root \( x = x_0 \).

2.2: Using the FTA, prove your answer to question 1.2. Hint: Apply the FTA to prove how many roots a polynomial \( P_N(x) \) of order \( N \) has.
Sol: When a root is determined, it may be factored out, leaving a new polynomial of degree one less than the first. Specifically,

\[
P_{N-1}(x) = \frac{P_N(x)}{x - x_0}
\]

Thus it follows that by a recursive application of this theorem, a polynomial has a number of roots equal to its degree. All the roots must be counted, including repeated and complex roots and roots at \( \infty \).

Problem #3: Consider the polynomial function \( P_2(x) = 1 + x^2 \) of degree \( N = 2 \) and the related function \( F(x) = 1/P_2(x) \). What are the roots (e.g., zeros) \( x_n \) of \( P_2(x) \)? Hint: Complete the square on the polynomial \( P_2(x) = 1 + x^2 \) of degree 2, and find the roots.
Sol: For the roots by setting \( P_2(x) = 0 \) gives \( x_0 = -1 \), leading to \( x = \pm 1 \).

- 5.4: \( F(x) = e^{\pi i x} \)

1. State where the function is and is not analytic. Sol: Analytic everywhere.

2. Explicitly evaluate the integral when \( \zeta \) is the square \( s = 1, 1 \rightarrow (-1, 1) \rightarrow (-1, 1) \rightarrow (1, 1) \). Sol: When you perform this integral piece-wise, you will find that all terms cancel out and the result is \( 0 \).

3. Evaluate the same integral using Cauchy’s theorem and/or the residue theorem. Sol: The function is analytic everywhere, so the integral is \( 0 \) by Cauchy’s theorem.

- 5.5: \( F(x) = \frac{1}{x^2 + \pi} \)

1. State where the function is and is not analytic. Sol: Analytic everywhere except at \( x = -2 \), where it has a pole.

2. Let \( \zeta \) be the unit circle, defined as \( s = e^{\pi i} \), \( 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

Sol: The function is analytic everywhere inside \( \zeta \), so the integral is \( 0 \) by Cauchy’s theorem.

3. Let \( \zeta \) be a circle of radius 3, defined as \( s = 3e^{\pi i} \), \( 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

Sol: This contour contains the pole. The residue is \( 1 \), therefore the integral is equal to \( 2\pi i \).

- 5.6: \( F(x) = \frac{1}{x^2 + e^{\pi i}} \)

1. State where the function is and is not analytic. Sol: Analytic everywhere except at \( x = -2 \), where it has a pole.

2. Let \( \zeta \) be a circle of radius 3, defined as \( s = 3e^{\pi i} \), \( 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

Sol: The function is analytic everywhere inside \( \zeta \), so the integral is \( 0 \) by Cauchy’s theorem.

3. Let \( \zeta \) be a circle of radius 3, defined as \( s = 3e^{\pi i} \), \( 0 \leq \theta \leq 2\pi \). Evaluate the integral using Cauchy’s theorem and/or the residue theorem.

Sol: This contour contains the pole. The residue is \( 1 \), therefore the integral is equal to \( 2\pi i \).

- 5.7: \( F(s) = \pm \sqrt{\frac{1}{s + \pi}} \) (e.g., \( F^2 = \frac{1}{1 + \pi} \))

1. State where the function is and is not analytic. Sol: Analytic everywhere except \( s = 0 \), where there is a pole.

2. This function is multivalued. How many Riemann sheets do you need in the domain \( s \) and the range \( t \) to fully represent this function? Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range. Sol: There are 2 sheets in the domain (for the \( \pm \) square root) which map to 1 sheet in the range.

3. Explicitly evaluate the integral

\[
\int C \frac{1}{\sqrt{2 + \pi}} dt
\]

when \( \zeta \) is the unit circle, defined as \( s = e^{\pi i} \), \( 0 \leq \theta \leq 2\pi \). Is this contour closed? State why or why not.

Sol: The solution is

\[
2\sqrt{2 + \pi} \sqrt{\frac{1}{2 + \pi}} = 2(e^{\pi i} - e^{\pi i}) = -4.
\]
3.2. PROBLEMS DE-2

2.
∮
C F(z)dz, where C is a circle centered at z = j with a radius of 1
Sol: Since we only consider the circle around z = j, the integral becomes
0 = \int_{|z| = 1} F(z)dz, where F(z) = \frac{1}{z - j}
Σ

3. What does your result imply about the residue of the second-order pole at s = 0? Sol: The residue is 0. ■

CHAPTER 2. ALGEBRAIC EQUATIONS
Problem # 4: F(x) may be expressed as (A, B, x ± ∈C)
F(x) = A
x−x+
+ B
x−x−
, ... the function 1/(1 −x), it is defined as the series 1 + x+ x2 + x3 + ···, such that the ratio of
consecutive terms is x.

Overview:
Analytic functions are defined by infinite (power) series. The function
Analytic functions
− ∞
+ ∞
| x
1
− x
2
| = 1 + x+ x2 + x3 + ···, such that the ratio of
consecutive terms is x.

Problem # 5: Integration in the complex plane
1. State where the function is and is not analytic.
Sol: The function is analytic everywhere except at x = 1.
2. Explicitly evaluate the integral when
P
| x
1
− x
2
| = 1 + x+ x2 + x3 + ···, such that the ratio of
consecutive terms is x.

3. Evaluate the same integral using Cauchy's theorem and/or the residue theorem.
Problem # 5: The geometric series

5.1. What is the region of convergence (RoC) for the power series Eq. AE-1.2 of \(1/(1-x)\) given above—for example, where does the power series \(P(x)\) converge to the function value \(f(x)\)? State your answer as a condition on \(x\). Hint: What happens to the power series when \(x > 1\)?

Sol: \(|x| < 1\) because for \(|x| \geq 1\), the power series diverges to infinity.

5.2. In terms of the pole, what is the RoC for the geometric series in Eq. AE-1.2?

Sol: The nearest pole relative to the expansion point, at \(x = 0\) if \(x\) is for the geometric series in Eq. AE-1.2, is the pole that allows us to integrate functions that may not be complex analytic for all \(z \in C\).

5.3. How does the RoC relate to the location of the pole of \(1/(1-x)\)?

Sol: The nearest pole relative to the expansion point, at \(x = 0\) is at \(x = 1\). Thus the RoC is 1.

5.4: Where are the zeros, if any, in Eq. AE-1.2?

Sol: There is a single zero at \(x = -1\).

5.5. Assuming \(z\) is in the RoC, prove that the geometric series correctly represents \(1/(1-x)\) by multiplying both sides of Eq. AE-1.2 by \(1-x\).

Sol:

\[
1 = \frac{1}{1-x} = 1 + x + x^2 + \cdots
\]

for all \(x \neq 1\).

The introduction of the pole introduces an added zero since \(P_0(x)_{|x=1} = N\).

If one lets \(z = 1/x\) the relation becomes

\[
1 = \frac{1 - z}{1 - z}
\]

which is valid for \(x \neq 1\), which when expanded the RoC is \(|z| < 1\), or \(x > 1\).

Problem # 6: Use the geometric series to study the degree \(N\) polynomial. It is very important to note that all the coefficients \(c_n\) of this polynomial are 1.

\[
P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{N} x^n.
\]

6.1. Prove that

\[
P_N(x) = \frac{1-x^{N+1}}{1-x}.
\]

---

CHAPTER 3. DIFFERENTIAL EQUATIONS

if and only if \(F(z)\) is complex analytic inside of \(\mathcal{C}\). This is related to the FTCCC.

\[f(z) = f(a) + \int_{\mathcal{C}} F(z)dz,
\]

where \(f(z)\) is the antiderivative of \(F(z)\)—namely, \(F(z) = df/dz\). The FTCCC requires \(F(z)\) to be complex analytic for all \(z \in C\). By closing the path \(\mathcal{C}\), Cauchy’s theorem (and the following theorems) allows us to integrate functions that may not be complex analytic for all \(z \in C\).

2. Cauchy’s Integral Formula CT-2 (Boas 1987, p. 51; Stillwell, 2010, p. 220)

\[
\int_{\mathcal{C}} \frac{F(z)}{z - a}dz = 2\pi i \sum_{k=1}^{K} \text{Res}_k,
\]

where \(\text{Res}_k\) are the residues of all poles of \(F(z)\) enclosed by the contour \(\mathcal{C}\).

How to calculate the residues: The residues can be rigorously defined as

\[
\text{Res}_k = \lim_{z \to a_k} \frac{1}{z - a_k}F(z).
\]

This can be related to Cauchy’s integral formula: Consider the function \(F(z) = w(z)/(z - a_k)\), where we have factored \(F(z)\) to isolate the first-order pole at \(z = a_k\). If the remaining factor \(w(z)\) is analytic at \(a_k\), then the residue of the pole at \(z = a_k\) is \(w(a_k)\).

4.1: Describe the relationships between the theorems:

1. CT-1 and CT-2: Sol: When \(a_k\) falls outside of \(\mathcal{C}\), CT-2 reduces to CT-1.

2. CT-1 and CT-3: Sol: When there are no poles inside \(\mathcal{C}\), all the residues are zero, and CT-3 reduces to CT-1.

3. CT-2 and CT-3: Sol: Case CT-2 has only one induced pole at \(z = a_k\), having residue \(F(a_k)\). Thus CT-3 is the same as CT-2 when \(K = 1\), the pole at \(a_0\) is within contour \(\mathcal{C}\), and the single residue is \(F(a_0)\).

4.2: Consider the function with poles at \(z = \pm j\).

\[
F(z) = \frac{1}{1 + z^2} = \frac{1}{(z - j)(z + j)}.
\]

Find the residue expansion.

Sol: \(F(z) = \frac{1}{2j} \left(\frac{1}{z - j} - \frac{1}{z + j}\right)\).

4.3: Apply Cauchy’s theorem to solve the following integrals. State which theorem(s) you used and show your work.

1. \(\int_{\mathcal{C}} F(z)dz\), where \(\mathcal{C}\) is a circle centered at \(z = 0\) with a radius of \(\frac{1}{2}\).

Sol: Because the contour \(\mathcal{C}\) does not include the poles, \(F(z)\) is analytic everywhere inside \(\mathcal{C}\). Using Cauchy’s integral theorem, the integral is 0.
3.2: Problems DE-2

3.2: Find the antiderivative of $F(z)$.
Sol: Since $c\, z = e^{\ln(c)\, z}$, the indefinite integral is:

$$\int c\, z \, dz = \frac{e^{\ln(c)\, z}}{\ln(c)} + C = \frac{e^{\ln(c)\, z}}{\ln(c)} + C$$


$$\int_{C} F(z) \, dz = 0$$

---

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Sol:

$$P_N(x) = 1 + x + x^2 + \cdots + x^N = \sum_{n=0}^{\infty} x^n - \sum_{n=N+1}^{\infty} x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$.

The integrand is

$$\mathcal{I}(t) = \int e^{-\pi \tau i} \, d\tau$$

where the path $C$ is the unit circle.

The path $C$ is an integer.

The path $C$ is the unit circle.

The path $C$ is the unit circle.

Cauchy's theorems CT-1, CT-2, CT-3

$\blacksquare$

---

Hint: Can it be represented by a different power series outside this RoC? Problem 5?

---

26 CHAPTER 2. ALGEBRAIC EQUATIONS

Sol:

$$P_{100}(0) = 1$$

and

$$P_{100}(0.9) = 9.999760947410014$$

with a difference of $-3.55271 \times 10^{-15}$ (i.e., $-16\times\varepsilon$).

$\blacksquare$

---

Problem 5?
2.1. PROBLEMS AE-1

Problem # 7 The exponential series

– 7.1: What is the RoC for the exponential series Eq. 3.2.11?
Sol: The exponential is convergent everywhere on the open real line. ■

– 7.2: Let \( z = y \) in Eq. 3.2.11, and write out the series expansion of \( e^y \) in terms of its real and imaginary parts.
Sol: 
\[
e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \\
= 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots
\]

– 7.3: Let \( z = y \) in Eq. 3.2.11, and write out the series expansion of \( e^y \) in terms of its real and imaginary parts. How does your result relate to Euler’s identity \((e^{iy} = \cos(y) + i\sin(y))\)?
Sol: 
\[
e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \\
= 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots
\]

Inverse analytic functions and composition

Overview: It may be surprising, but every analytic function has an inverse function. Starting from the function \((x, y) \in \mathbb{C}\)
\[g(x) = \frac{1}{y}\]
the inverse is
\[x = y - \frac{1}{y} = 1 - \frac{1}{y}\]

Problem # 8: Consider the inverse function described above

– 8.1: Where are the poles and zeros of \(z(y)\)?
Sol: The pole is at \( y = 0 \), and the zero is at \( y = 1 \). There are no poles or zeros at \( \infty \) because \( \lim_{y \to \infty}(y-1)/y = 1 \). ■

– 8.2: Where (for what condition on \( y \)) is \( z(y) \) analytic?
Sol: It is analytic anywhere but the pole, at \( y = 0 \). ■

Problem # 9 Consider the exponential function \( z(x) = e^x \) \((x, z \in \mathbb{C})\).

2.2. PROBLEMS AE-1

Problem # 9 Consider the exponential function \( z(x) = e^x \) \((x, z \in \mathbb{C})\).

– 9.1: What is the RoC for the exponential series Eq. 3.2.11?
Sol: The exponential is convergent everywhere on the open real line. ■

– 9.2: Let \( z = y \) in Eq. 3.2.11, and write out the series expansion of \( e^y \) in terms of its real and imaginary parts.
Sol: 
\[
e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \\
= 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots
\]

– 9.3: Let \( z = y \) in Eq. 3.2.11, and write out the series expansion of \( e^y \) in terms of its real and imaginary parts. How does your result relate to Euler’s identity \((e^{iy} = \cos(y) + i\sin(y))\)?
Sol: 
\[
e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \\
= 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots
\]

– 9.4: Where are the poles and zeros of \(z(y)\)?
Sol: The pole is at \( y = 0 \), and the zero is at \( y = 1 \). There are no poles or zeros at \( \infty \) because \( \lim_{y \to \infty}(y-1)/y = 1 \). ■

– 9.5: Where (for what condition on \( y \)) is \( z(y) \) analytic?
Sol: It is analytic anywhere but the pole, at \( y = 0 \). ■

Discussion whether your results agree with Eq. DE-2.4?

– 1. \( \int Cdz\)
Sol: \( \int_C dz = \int_C e^y dy = \int_C e^y i\theta \frac{d\theta}{i} = \int_C e^y de \quad (\text{C is a circle})
\]

This example does not obey FTCC because \( f(z) = 1/z \) is not analytic at \( z = 0 \) (inside C), instead it satisfies CT-2. ■

– 2. \( \int_C dz\)
Sol: \( \int_C zdz = \int_C e^y i\theta \frac{d\theta}{i} = \int_C e^y de \quad (\text{C is a circle})
\]

This example does not obey FTCC because \( f(z) = 1/z \) is not analytic at \( z = 0 \) (inside C), instead it satisfies CT-2. ■

– 3. \( \int_C dz\)
Sol: \( \int_C zdz = \int_C e^y i\theta \frac{d\theta}{i} = \int_C e^y de \quad (\text{C is a circle})
\]

This example obeys the FTCC because \( f(z) = 1/z \) is analytic everywhere; ■

– 4. \( \int_C zdz\)
Recall that the path of integration is the unit circle, starting and ending at -1.
Sol: Let \( \zeta = z = 2 \), then the limits become \([-1 + 2i, 1 + 2i] = (2\text{r} + e^{-i\pi/2}, 2\text{r} + e^{i\pi/2})\).
\[
I = \int_C \frac{dz}{z} = \int_{-1}^{1} e^{-i\theta} d\theta = \int_{-1}^{1} e^{-i\theta} i\theta \frac{d\theta}{i} = \int_{-1}^{1} e^{-i\theta} + e^{i\theta} - 1 - e^{-i\theta} = 0.
\]

Thus the example reduces to the case of (3), and therefore must have the same conclusion as (3). But in this case the reasoning is different because the second order pole (singular point) is outside the unit circle, thus the function is analytic inside \( \zeta \), so CT-1 applies. ■

Problem # 3: FTCC and integration in the complex plane

Let the function \( F(z) = e^z \) \((z \in \mathbb{C})\) is given for each question. Hint: Can you apply the FTCC?

– 3.1: For the function \( f(z) = e^z \) \((z \in \mathbb{C})\) is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that \( f(z) \) is analytic for all \( z \in \mathbb{C} \).
Sol: We may rewrite this function as \( f(z) = e^{x+iy} \) where \( z = x + iy \) and \( f = u + iv \). Thus
\[
u(x, y) = e^{x+iy} \cos(\text{ln}(c))
\]
\[
u(x, y) = e^{x+iy} \sin(\text{ln}(c))
\]
\[
\frac{\partial u}{\partial x} = \ln(e) e^{x+iy} \cos(\text{ln}(c))
\]
\[
\frac{\partial u}{\partial y} = \ln(e) e^{x+iy} \sin(\text{ln}(c))
\]
\[
\frac{\partial v}{\partial x} = -\ln(e) e^{x+iy} \sin(\text{ln}(c))
\]
\[
\frac{\partial v}{\partial y} = -\ln(e) e^{x+iy} \cos(\text{ln}(c))
\]

Thus the CR conditions are satisfied everywhere and the function is analytic for all \( z \in \mathbb{C} \). ■
Problems 3.2

1.2: Consider Equation DE-2.3. What is the condition on \( F(z) \) for which this formula is valid? 

\[
I = \frac{1}{2} \int_{1}^{0} \frac{(z-y)}{y} \, dy = \frac{1}{2} \int_{0}^{1} \frac{(y-z)}{y^2} \, dy = \frac{1}{2} + \frac{j}{2} = j
\]

9.1: Find the inverse \( x(z) \).

Sol: Taking the natural log (\( \ln \)) of both sides gives \( x = \ln(1/z) \).

11.2: What is \( h(x) \)?

Sol: \( h(x) = x^6 + 5x^5 + 10x^4 + 12x^3 + 11x^2 + 7x + 2 \).

Convolution

The convolution of two functions \( f(t) \) and \( g(t) \) is given by:

\[
(f * g)(t) = \int_{0}^{t} f(\tau) g(t-\tau) \, d\tau
\]

Problem 2. In the following problems, solve the integral with itself.

\[
\int_{1}^{1} (1 + e^x) \, dx = 2
\]

Problem 3. In the following problems, solve the integral with itself.

\[
\int_{1}^{1} \frac{1}{z} \, dz = 2
\]

Convolution of sequences:

\[
C(x, y) = \sum_{k=0}^{\infty} x^{k} y^{n-k}
\]

Problem 1. Convolve the sequence \( x(z) \).

Sol: \( h(x) = x^{6} + 5x^{5} + 10x^{4} + 12x^{3} + 11x^{2} + 7x + 2 \).

Problem 1: Integrate the following problems over the interval

\[
\int_{1}^{1} (x + e^z) \, dx = 2
\]

Chapter 2: Algebaic Equations

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Newton's root-finding method

Problem # 12: Use Newton's iteration to find the roots of the polynomial

\[ P(x) = 1 - x^3 \]

- 12.1: Draw a graph describing the first step of the iteration starting with \( x_0 = (1/2, 0) \).

Sol: Start with an \((x,y)\) coordinate system and put points at and the vertex of \( P(x) \).

- 12.2: Calculate \( x_1 \) and \( x_2 \). What number is the algorithm approaching?

Sol: First we must find \( P'_x(x) = -3x^2 \). Thus the equation we must iterate is Eq. 3.1.14 (p. 56):

\[ x_{n+1} = x_n - \frac{P'_x(x_n)}{P''_x(x_n)} \]

Given a first guess for the root \( x_0 \), the next are \( x_1 = x_0 + \frac{-1}{3} \) and \( x_2 = x_1 + \frac{-1}{3} \). Note that if \( x = 0 \) is the root, then \( x_1 = x_0 \), and we are done. However, if \( x_0 = 0 \), then \( x_1 = \infty \), since \( x_0 = 0 \) is a root of \( P_1(x) \). Thus we must not start at the roots of \( P'_1(x) = 0 \).

- 12.3: Here is an Octave/Matlab script for the \( P_2(x) \) case. Modify it to find \( P_3(x) \):

```matlab
x=[1/2, 1/10]; x(2)=0.2; x(1)=-10
y(1)=x(1); for n=2:10
    y(n) = x(n-1) + (1-3*(n-1)^2)/(2*(3*(n-1)));
end
semilogy(abs(x)-1); hold on
semilogy(abs(x[1]-1),"or"); hold off
```

Sol:

- 12.4: For \( n = 4 \), what is the absolute difference between the root and the estimate, \( |x_r - x_4| \)?

Sol: 4.6E-8 (very small!)

- 12.5: Does Newton's method work for \( P_2(x) = 1 + x^2 \) ? If so, why? Hint: What are the roots in this case?

Sol: Here \( P'_2(x) = 2x \), thus the iteration gives

\[ x_{n+1} = x_n - \frac{1 + x^2}{2x} \]

In this case the roots are \( x_0 = \pm 1 \)—namely, purely imaginary. The solution will converge for complex roots as long as the starting point is complex. If we start with a real number for \( x_{00} \) and use real arithmetic, Newton's method fails because there is no way for the answer to become complex. Real in = Real out.

- 12.6: What if we let \( x_0 = (1 + j)/2 \) for the case of \( P_2(x) = 1 + x^2 \)?

Sol: By starting with a complex initial value, we fix the Real in = Real out problem.

Riemann zeta function \( \zeta(s) \)

Definitions and preliminary analysis:

The zeta function \( \zeta(s) \) is defined by the complex analytic power series

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \]

This series converges, and thus is valid, only in the ROC given by \( \Re(s) < 1 \), since there \( |n^{-s}| \leq 1 \). To determine its formula in other regions of the s plane, one must extend the series via analytic continuation (see p. 69).
Problem #9: One may synthesize a transmission line (ladder network) from a positive real impedance values, we can use a residue expansion. Here we shall explore this method.

The above defines each factor \( \zeta_j(s) \) as a product of the form:

\[
\prod_j \left( 1 - \frac{s}{\zeta_j(s)} \right)
\]

with each step the RoC is larger, resulting in an analytic function that has its RoC approaching \( -\infty \) as \( s \to \infty \). Thus the Cauer synthesis is a series combination of \( s \)-dependent \( \zeta_j(s) \) factors and the residue expansion replacing the floor function in the CFA. This seems to solve Brune's network synthesis problem. ■
2.1. PROBLEMS AE-1

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\[ w = \frac{1}{1 - \exp(-s\pi)} \]

\[ w \]

\[ \sigma \]

\[ j\omega \]

\[ -2 \quad -1 \quad 0 \quad 1 \quad 2 \]

\[ 2 \]

\[ 1 \]

\[ 0 \]

\[ -1 \]

\[ -2 \]

\[ \text{Figure 2.1: Plot of } w(s) = \ldots \text{ of } \zeta \text{p}(s) \text{ (Eq. AE-1.8). Here } w_k(s) \text{ has poles where } 1 = e^{s\pi k}, \text{ namely at } s_n = n2\pi j, \text{ as may be seen from the colorized plot.} \]

\[ \text{Problem # 8: Consider the function } w(z) = \log(z). \text{ As in Problem 7, let } z = re^{i\theta} \text{ and } w(z) = pe^{i\theta}. \]

\[ - 8.1: \text{ Describe with a sketch and then discuss the branch cut for } f(z). \]

\[ \text{Sol: From the plot of } zviz \ w(z) = \log(z) \text{ of Lecture 18, we see a branch cut going from } w = 0 \text{ to } w = -\infty. \text{ If we express } z \text{ in polar coordinates } (z = re^{i\theta}), \text{ then} \]

\[ w(z) = \log(r) + i\phi = w(x,y) + i\#, \]

where \( r(x,y) = |z| = \sqrt{x^2 + y^2} \) and \( \phi = \angle z = \phi(x,y) \). Thus a zero in \( w(z) \) appears at \( z = 1 + 0j \), and only appears on the principle sheet of \( z \) (between \( -\pi < \angle z < \pi \)), because this is the only place where \( \phi = 0 \). As the angle \( \phi \) increases, the imaginary part of \( w = \angle z \) which increases without bound. Thus \( w \) is like a spiral staircase, or cork-screw. If \( \rho = 1 \) and \( \phi \neq 0 \), \( w(r) = \log(1 + i\#) \) is not zero, since the angle is not zero. ■

\[ - 8.2: \text{ What is the inverse of the function } z(f) \text{? Does this function have a branch cut? If so, where is it?} \]

\[ \text{Sol: } z(w) = e^w \text{ is a single valued function, so a branch cut is not appropriate. Only multi-valued functions require a branch cut.} \]

\[ - 8.3: \text{ Using } zviz \text{, show that} \]

\[ \tan^{-1}(z) = -\frac{1}{2} \log \frac{z - 2}{z + 2}, \]

\[ \text{(DE-1.3)} \]

\[ \text{In Fig. 4.1 (p. 134) these two functions are shown to be identical.} \]

\[ \text{Sol: Use the Matlab commands } \text{zviz atan(z)} \text{ and } \text{zviz } -(1/2)\log((1+jz)/(1-jz)). \]

\[ - 8.4: \text{ Algebraically justify Eq. DE-1.3. Hint: Let } w(z) = \tan^{-1}(z) \text{ and } w(z) = \tan w = \sin w/\cos w; \text{ then solve for } e^{2w}. \]

\[ \text{Sol: Following the hint gives} \]

\[ z(w) = -j e^{iw} - e^{-iw} = -\frac{e^{2iw} - 1}{e^{2iw} + 1} \]

\[ \text{Solving this bilinear equation for } e^{2iw} \text{ gives} \]

\[ e^{2iw} = \frac{1 + zj}{1 - zj} \]

\[ \frac{1 - zj}{1 + zj} \]

\[ \text{Taking the log and using our definition of } w(z) \text{ we find} \]

\[ w(z) = \tan^{-1}(z) = -\frac{1}{2} \log \frac{f + jz}{f - jz}. \]

\[ \text{■} \]
3.1. PROBLEMS DE-1

Cauchy-Riemann Equations

Problem # 5: For this problem \( \sqrt{-1} = j \), \( s = \sigma + j\omega \), and \( F(s) = u(\sigma,\omega) + ... \) map to the sheet(s) in the range.

Sol: Above we show the mapping for the square root function \( w(z) = \sqrt{z} = \sqrt{re^{j\phi}/2} \). ■

2.2 Problems AE-2

Topics of this homework:

- Linear vs nonlinear systems of equations
- Euclid’s formula
- Gaussian elimination
- Matrix permutations
- Ohm’s law

Identify the elementary row operations that this matrix performs.

\[
G_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix}
\]

What can you conclude?

Sol:

The first GE matrix is given by

\[
G_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix}
\]

Multiply them to show this.

The Crumpton-Riemann equation is

\[
\frac{\partial z}{\partial x} = \frac{e^{j\omega}}{2}
\]

Deliverables: Answers to problems
2.2. PROBLEMS AE-2

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which scales the second row by -1 and adds it to the third row. Thus we have

\[
G_2 G_1 |A|b| = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & -3 & 1 & 0 & 3 & 1 & 1 \\
0 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & -1 & 1 \\
0 & -2 & 4 \\
0 & 0 & 1
\end{bmatrix}
\]

2.3 Find a third GE matrix \( G_3 \) that scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.

Sol: \( G_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \)

which scales the second row by -1/2. Thus we have

\[
G_3 G_2 G_1 |A|b| = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1/2 & 0 & 0 & 1 & 0 & -3 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 3 & 1 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & -1 & 1 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix}
\]

2.4: Find the last GE matrix, \( G_p \), which subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix \( (G_p G_3 G_2 G_1 A |b|) = I \).

Sol: \( G_p = \begin{bmatrix}
1 & 1 & -1 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix} \)

Thus we find \( G_p G_3 G_2 G_1 |A|b| \) is

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

2.5: Solve for \( \{x_1, x_2, x_3\}^T \) using the augmented matrix format \( G_p G_3 G_2 G_1 |A|b| \) (where \( \{A|b\} \) is the augmented matrix). Note that if you’ve performed the preceding steps correctly, \( x = G_p G_3 G_2 G_1 b \).

Sol: From the preceding problems, we see that \( \{x_1, x_2, x_3\}^T = \{3, -1, 1\} \)

2.6: Find the pivot matrix \( G \) that rescales the second row of the augmented matrix \( A|b| \) by 1/3.

Sol: \( G_1 = \begin{bmatrix}
1 & 1/3 \\
1/3 & 1
\end{bmatrix} \)

Proceeding

\[
G_1 A = \begin{bmatrix}
1 & 1 & -1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1/3 \\
1/3 & 1 \\
1/3 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & -1 \\
1 & 3 & 1/3 \\
1 & 1 & 4
\end{bmatrix}
\]

CHAPTER 3. DIFFERENTIAL EQUATIONS

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Sol: The power series is

\[ w(s) = \sum \frac{(-1)^n}{n!} (s - 1)^n, \]

which converges for \( |s - 1| < 1 \).

To convince you this is correct, use the Matlab/Octave command \texttt{syms s; Taylor((1/s, s, 'ExpansionPoint', 1), 1)}, which is equivalent to the shorthand \texttt{syms s; Taylor(1/s, s, 1)}. What is missing is the logic behind this expansion, given as follows: First move the pole to \( z = -1 \) via the Möbius “translation” \( s = z + 1 \), and expand using the Taylor series

\[ \frac{1}{z} = \frac{1}{1 + s} = \sum_{n=0} \frac{(-1)^n}{n!} (s - 1)^n, \]

next back-substitute \( z = s - 1 \) giving

\[ \frac{1}{s} = \sum_{n=0} (-1)^n (s - 1)^n. \]

It follows that the ROC is \( |z| = |s - 1| < 1 \), as provided by Matlab/Octave.

2.2: What is the ROC?

Sol: \( |s| < 1 \)

2.3: Expand \( w(s) = 1/s \) as a power series in \( s^{-1} = 1/s \) about \( s^{-1} = 1 \).

Sol: Let \( s = s^{-1} \) and expand about 1:

\[ \frac{1}{1 - s^{-1}} = \frac{1}{s} = \sum_{n=0} \frac{1}{n+1} s^n = s + s^2 + s^3 \ldots, \]

which has a zero at 0 and a pole at 1.

2.4: What is the ROC?

Sol: \( |s| < 1 \)

2.5: What is the residue of the pole?

Sol: \( 0 \)

Problem # 3: Consider the function \( w(s) = 1/(2 - s) \)

3.1: Expand \( w(s) \) as a power series in \( s^{-1} = 1/s \). State the ROC as a condition on \( |s^{-1}| \).

Hint: Multiply top and bottom by \( s^{-1} \).

Sol: \( 1/(2-s) = -s^{-1}((1-2s^{-1}) = -s^{-1} \sum \frac{2^n s^{-n}}{n!} \). The ROC is \( 2/|s| < 1 \), or \( |s| > 2 \).

3.2: Find the inverse function \( s(w) \). Where are the poles and zeros of \( s(w) \), and where is it analytic?

Sol: Solving for \( s(w) \) we find 2 - 2s - w = 0 or \( s = 1/(1+w) \). This has a pole at 0 and a zero at \( w = 1/2 \). The ROC is therefore from the expansion point out to, but not including \( w = 0 \).

Problem # 4: Summing the series

The Taylor series of functions have more than one region of convergence.

4.1: Given some function \( f(x) \), if \( a = 0 \), what is the value of

\[ f(a) = 1 + a + a^2 + a^3 + \cdots? \]

Show your work. Sol: To sum this series, we may use the fact that

\[ f(a) - af(a) = (1 + a + a^2 + a^3 + \cdots) - a(1 + a + a^2 + a^3 + \cdots) = (1 - a) f(a) = 1, \text{ or } f(a) = 1/(1-a). \]

This gives \((1-a)f(a) = 1\), or \( f(a) = 1/(1-a) \). Now since \( a = 1 \), the sum is \( 1/(1-0.1) = 1.11 \).

4.2: Let \( a = 10 \). What is the value of

\[ f(a) = 1 + a + a^2 + a^3 + \cdots? \]

Sol: In this case the series clearly does not converge. To make it converge we need to write a formula for \( y = 1/x \) rather than for \( x \).

\[ f(1/y) - f(1/y)/a = (1 + 1/a + 1/a^2 + 1/a^3 + \cdots) - a(1 + 1/a + 1/a^2 + \cdots) = 1/(1-1/a) + (1-1)/a^2 + \cdots \]

This gives \( f(1/a) = -a^{-1}/(1-a^{-1}) \). Now since \( a = 10 \), the series sums to \( f(10) = -0.1/(1-0.1) = -1/9 \).
Chapter 3
Differential equations
3.1 Problems DE-1

Topics of this homework:
Complex numbers and functions (ordering ... w(s) = 1/s
– 2.1: Expand this function as a power series about s= 1. Hint: Let 1/s= 1/(1−1+ s) = 1/(1 −(1 −s)).

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CHAPTER 2. ALGEBRAIC EQUATIONS
Two linear equations
Problem # 3 In this problem we transition from a general pair of equations...

The determinant is non-zero, so the determinant of the matrix A is non-zero. Thus the determinant cannot...

Problem # 3.1 What does it mean, graphically, if these two linear equations have:
1. A unique solution
2. No solution
3. An infinite number of solutions

If the slopes are the same but have different intercepts (are parallel to each other) there is no solution.

If they have the same slope but different intercepts (are parallel to each other) there is no solution.

Thus, the solution is: x = ±(s−1)/s.

Problem # 2: Consider the function w(s) = 1/s. Expand this function as a power series about s = 1.

Hint: Let 1/s = 1/(1−1+ s) = 1/(1 −(1 −s)).

There are three possibilities:

1. When they have different slopes, they meet at one (x,y) point, which is the solution.
2. If the two lines are identical, any point on the line is a solution.
3. If they have the same slope but different intercepts (are parallel to each other) there is no solution.

Problem # 2.1: Expand this function as a power series about s = 1.

Hint: Let 1/s = 1/(1−1+ s) = 1/(1 −(1 −s)).
Problem # 4: The application of linear functional relationships between two variables
We use 2 × 2 matrices to describe two-port networks, as discussed in Sec. 3.8 (p. 106). Transmission lines are a
great example: Both voltage and current must be tracked as they travel along the line. Figure 3.10 (p. 110) shows
an example segment of a transmission line.
Suppose you are given the following pair of linear relationships between the input (source) variables V₁ and
I₁ and the output (load) variables V₂ and I₂ of the transmission line:
\[
\begin{bmatrix}
V₁ \\ I₁
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
V₂ \\ I₂
\end{bmatrix}
\]

− 4.1: Let the output (the load) be V₂ = 2 and I₂ = 2 (i.e., V₂/I₂ = 1/2). Find the input voltage and current, V₁ and I₁.

Sol: This case corresponds to
\[
\begin{bmatrix}
V₁ \\ I₁
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
2 \\ 2
\end{bmatrix} = \begin{bmatrix}
2 + 2 \\ 2 - 2
\end{bmatrix} = \begin{bmatrix}
1 \\ 0
\end{bmatrix}
\]
Thus V₁ = 1 and I₁ = 0.

− 4.2: Let the input (source) be V₁ = 1 and I₁ = 2. Find the output voltage and current, V₂ and I₂.

Sol: With the input specified the two equations are
\[
\begin{bmatrix}
V₁ \\ I₁
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
V₂ \\ I₂
\end{bmatrix}
\]
To find the input we must invert the matrix (Δ = − j – 1)
\[
\begin{bmatrix}
V₁ \\ I₁
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
V₂ \\ I₂
\end{bmatrix}
\]
Thus V₂ = 3/(1 + j) = 3(1 – j)/2, I₂ = (1 – 2j)/(1 + j) = −(1 + j)/2. The point of this exercise is that the
two lines have a complex intersection point, not easily visualized.

Integer equations: applications and solutions
Any equation for which we seek only integer solutions is called a Diophantine equation.

Problem # 5: A practical example of using a Diophantine equation:

A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found
that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight
between 1 and 40 pounds. What were the weights of the pieces? – 16. Bachet de Béziers (1623)4

Here, weighing is performed using a balance scale that has two pans, with weights on either pan. Thus, given
weights of 1 and 3 pounds, one can weigh a 2-pound weight by putting the 1-pound weight in the same pan with
the 2-pound weight, and the 3-pound weight in the other pan. Then the scale will be balanced. A solution to the
four weights for Bachet’s problem is 1 + 3 + 9 + 27 = 40 pounds.

Solution: 2 = 1 + 3

− 5.1: Show how the combination of 1, 3, 9, and 27-pound weights can be used to weigh
1, 2, 3, . . . , 8, 25, and 40 pounds of milk (or something else, such as flour). Assuming that the
milk is in the left pan, provide the position of the weights using a negative sign – to indicate

```
2

Solution: 2 = 1 + 3
```

Taken from Rotman (1996, p. 30)
2.3. PROBLEMS AE-3

Problem 2.4: Vector (cross) product $A \times B$

- Problem 2.4: Scalar product $A \cdot B$

Problem 2.4: Vector algebra in $\mathbb{R}^3$

Schwarz inequality

- Problem 6: The dot product is often defined as $\langle A, B \rangle = A \cdot B$

- Problem 10.2: Give the formula for the dot product of two vectors. Explain the meaning based on the definition of the dot product.

Problem 16: The vector (cross) product $A \times B$ is defined as $A \times B = ||A|| ||B|| \sin \theta$, where $\theta$ is the angle between $A$ and $B$.

Problem 17: The triple product $A \cdot (B \times C)$ is defined as $A \cdot (B \times C) = ||A|| ||B|| ||C|| \cos \gamma$, where $\gamma$ is the angle between $A$ and the vector $B \times C$.

Problem 18: The scalar product $A \cdot B$ is defined as $A \cdot B = ||A|| ||B|| \cos \theta$, where $\theta$ is the angle between $A$ and $B$.

Problem 19: The vector (cross) product $A \times B$ is defined as $A \times B = ||A|| ||B|| \sin \theta$, where $\theta$ is the angle between $A$ and $B$.

Problem 20: The triple product $A \cdot (B \times C)$ is defined as $A \cdot (B \times C) = ||A|| ||B|| ||C|| \cos \gamma$, where $\gamma$ is the angle between $A$ and the vector $B \times C$.

Problem 21: The scalar product $A \cdot B$ is defined as $A \cdot B = ||A|| ||B|| \cos \theta$, where $\theta$ is the angle between $A$ and $B$.

Problem 22: The vector (cross) product $A \times B$ is defined as $A \times B = ||A|| ||B|| \sin \theta$, where $\theta$ is the angle between $A$ and $B$.

Problem 23: The triple product $A \cdot (B \times C)$ is defined as $A \cdot (B \times C) = ||A|| ||B|| ||C|| \cos \gamma$, where $\gamma$ is the angle between $A$ and the vector $B \times C$.

Problem 24: The scalar product $A \cdot B$ is defined as $A \cdot B = ||A|| ||B|| \cos \theta$, where $\theta$ is the angle between $A$ and $B$.
2.2. PROBLEMS AE-2 37

- 7.2: Show that the cross product is equal to the area of the parallelogram formed by A, B, namely \(|A||B| \sin(\theta)|, where \(|A| = \sqrt{A \cdot A}\) and \(\theta\) is the angle between A and B.
   Sol: A parallelogram’s area is equal to its base times its height. Therefore, let’s say the base is length \(|A|\) and the height \(|B| \sin(\theta)|\), which is the portion of B that is perpendicular to A.

Problem 8: Triple product A \(\times (B \times C)\)

Let A = \(a_1i + a_2j + a_3k\), B = \((b_1, b_2, b_3)^T\), C = \((c_1, c_2, c_3)^T\) be three vectors in \(\mathbb{R}^3\).

- 8.1: Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: \(A \cdot (B \times C) = (A \cdot B)B + B \times (A \times B) = A \parallel + A \perp\).
   Sol: Using the determinate-definition of the cross product,
   \[
   B \times C = \begin{vmatrix}
   \hat{x} & \hat{y} & \hat{z} \\
   b_x & b_y & b_z \\
   c_x & c_y & c_z
   \end{vmatrix}
   \]
   and let D = B \times C and compute A \(= A \times (B \times C)\). Finally compute the requested right-hand-side, and compare the two. It should be clear that they are the same, because the dot product transfers the elements of vector A to cross product and reduces the product to the scalar.

Figure 2.2: This figure is identical to Fig. 3.4 (p. 87), Sec. 3.5. Definitions of vectors A, B, C (vectors in \(\mathbb{R}^3\)) used in the definition of A \(\cdot B \times C \times A\). There are two algebraic vector products, the scalar (dot) product A \(\cdot B \in \mathbb{R}\) and the vector (cross) product A \(\times B \in \mathbb{R}^3\). Note that the result of the dot product is a scalar, while the vector product yields a vector, which is \(\perp\) to the plane containing A and B.

- 8.2: Describe why \(|A \cdot (B \times C)|\) is the volume of parallelepiped generated by A, B, and C.
   Sol: Note that the norm of B \(\times C\) is the area of the parallelogram generated by B and C. Taking the dot product with A results in the volume of the corresponding parallelepiped (prism). So the absolute volume of triple product is volume of parallelepiped.

Problem 9: Given two vectors \(A, B\) in the \(\hat{x}, \hat{y}\) plane shown in Fig. 2.2 (same as 3.4 on page 87), with \(B = \hat{y}\) (i.e., \(|B| = 1\)).

- 9.1: Show that A may be split into two orthogonal parts, one in the direction of B and the other perpendicular (\(\perp\)) to B. Hint: Express the vector products of A and B (dot and cross) in polar coordinates (Greenberg, 1988).
   \[
   A = (A \cdot B)B + B \times (A \times B) = A \parallel + A \perp.
   \]

Chapter 2: Algebraic Equations

- 7.2: c = 1 (recall that 1 = \(e^{i2\pi k}\) for \(k = 0, 1, 2, \ldots\))
   Sol: From the general formula with c = 1
   
   \[
   t^k = e^{i\log_1 e^{i2\pi k}} = e^{i2\pi k} = e^{i2\pi (+k\text{mod}2)}
   \]
   where \(k\) is any integer. Thus u = \(e^{-ik\pi} \cos k2\pi x\) and v = \(e^{-ik\pi} \sin k2\pi x\).

- 7.3: c = \(\pi\), Hint: \(j = e^{-i\pi/2} \mathbb{Z}\) \(\cap m \in \mathbb{Z}\).
   Sol: \(y^m = (e^{i\pi/2})^m = e^{i\pi/2} = 0\) if \(m = 2n\) \(\in \mathbb{Z}\).
   Thus for \(m = 0\), \(y^m = (e^{i\pi/2})^0 = 1\) \(\in \mathbb{Z}\).

- 7.4: Find \(w(x, y)\) for \(w(z) = |z|\). Hint: Begin with the inverse function \(z = w^2\).
   Sol: The simplest solution is to work in polar coordinates, which gives \(w(z) = \sqrt{|z|^2} / 2\).

Problem 8: Convolution of an impedance z(t) and its inverse \(y(t)\)

In the frequency domain a Brune impedance is defined as the ratio of a numerator polynomial \(N(s)\) to a denominator polynomial \(D(s)\).

- 8.1: Consider a Brune impedance defined by the ratio of numerator and denominator polynomials, \(Z(s) = N(s)/D(s)\). Since the admittance \(Y(s)\) is defined as the reciprocal of the impedance, the product must be 1. If \(z(t) \leftrightarrow Z(s)\) and \(y(t) \leftrightarrow Y(s)\), it follows that \(z(t) * y(t) = \delta(t)\). What property must \(n(t) \leftrightarrow N(s)\) and \(d(t) \leftrightarrow D(s)\) obey for this to be true?
   Sol: Since
   \[
   1 = Z(s)Y(s) = \frac{N(s)D(s)}{D(s)N(s)}
   \]
   it follows that \(N(s)D(s) = D(s)N(s)\) thus \(n(t) * d(t) = \delta(t) * n(t)\). Namely the convolution of \(n(t)\) and \(d(t)\) commute (are independent of order).

- 8.2: The definition of a minimum phase function is that it must have a causal inverse. Show that every impedance is minimum phase.
   Sol: Since \(z(t)\) is causal and has a causal inverse \(y(t)\), by definition every impedance must be minimum phase.

Schwarz inequality

Problem 9: The above figure shows three vectors for an arbitrary value of \(\alpha \in \mathbb{R}\) and a specific value of \(\alpha = \alpha^*\).

- 9.1: Find the value of \(\alpha \in \mathbb{R}\) such that the length (norm) of \(\hat{E}\) (i.e., \(|\hat{E}| \geq 0\) is minimum. Show your derivation, not the answer (\(\alpha = \alpha^*\)).
2.3. PROBLEMS

2. Every degree polynomial with complex coefficients has, counted with multiplicity, exactly \(7.1, 7.2, \) and \(7.3:\)

– 7.1: \(c = e\)

Sol: Since \(u + iv = e^z = e^{x+yi} = e^x \cos y + \sin y,\)

\(u = ex \cos y\)

and

\(v = ex \sin y.\)

■

CHAPTER 2. ALGEBRAIC EQUATIONS

Sol: Expressing the vector products in polar form makes this result transparent:

\[ A \cdot B = |A||B| \cos \theta. \]

Problem 4: It is sometimes necessary to consider a function whose complete derivative.

Problem 5: Order and complex numbers:

Problem 6: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 7: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 8: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 9: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 10: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 11: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 12: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 13: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 14: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 15: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 16: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 17: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 18: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 19: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]

Problem 20: It is sometimes necessary to consider a function with complex

\[ \frac{dz}{dt} = \frac{dc}{dt} = \frac{d}{dt} \left( e^{rt} \right) = re^{rt}. \]
independent of current. For the case of water’s triple point, the voltage represents the temperature of water at the triple point, clamped at 0°C. The current represents the heat flux. The latent heat of water at the triple point is 32 Cal/gm. Thus as the temperature rises from below freezing, the water is clamped at 0°C once the triple point is reached. At that point, adding more heat flux has no effect on the temperature until all the ice melts. Once the ice has melted, the temperature again begins to rise until it hits the boiling point, where it again stays at 100°C until all the water has evaporated. Sol: Need a figure here showing how to model the triple point of water. The Heat capacity may be modeled by a capacitor, which is fixed at 0 as the capacitor discharges. Once it is empty, the temperature again begins to rise as the heat Q from the sun is added

\[ T^n = n \pi Q \]

Thus the required circuit needs to emulate this temperature behavior due to the latent heat of melting ice and boiling water into steam.

**Nonlinear (quadratic) to linear equations**

In the following problems we deal with algebraic equations in more than one variable that are not linear equations. For example, the circle \( x^2 + y^2 = 1 \) may be solved for \( y(x) = \pm \sqrt{1-x^2} \). If we let \( z = x + y \) and \( x + y = \sqrt{1-x^2} \), we obtain the equation for half a circle \( y > 0 \). The entire circle is described by the magnitude of \( z \) as \( |z|^2 = (x+y)(x-y) = 1 \).

**Problem #14: Give the curve defined by the equation:**

\[ x^2 + xy + y^2 = 1 \]

- 14.1: Find the function \( y(x) \).

Sol: Completing the square in \( y \) and solve for \( y(x) \):

\[
\begin{align*}
(y + x/2)^2 - x^2/4 + x^2 &= 1 \\
(y + x/2)^2 &= 1 - \frac{x^2}{4} \\
y + x/2 &= \sqrt{1 - \frac{3x^2}{4}} \\
y &= -\frac{x}{2} \pm \sqrt{\frac{4}{3} - x^2} \\
\end{align*}
\]

Thus we find the equation is a rotated ellipse.

- 14.2: Using Matlab/Octave, plot \( y(x) \) and describe the graph.

Sol: 

![Graph of y(x)](image)

Thus we find the equation is a rotated ellipse.

- 14.3: What is the name of this curve?

Sol: It is an ellipse, rotated by 45 degrees.

**Problem #3: Find the ABCD matrix for each of the circuits of Fig. 3.8.**

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency \( s \in \mathbb{C} \), then (ii) substitute \( s = 1 \) and calculate the total transmission matrix at this single frequency.

- 3.1: Left circuit (let \( R1 = R2 = 10 \) kilo-ohms and \( C = 10 \) nano-farads)

Sol: Write the system in chain matrix form:

\[
\begin{bmatrix}
V_1 \\
I_1 \\
\end{bmatrix} =
\begin{bmatrix}
1 & Z_C & 0 & 0 \\
0 & 1 & Z_1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2 \\
\end{bmatrix}
\]

Now we substitute the given values:

\[
\begin{bmatrix}
1 & 10^4 & 0 & 0 \\
0 & 1 & 10^4 & 0 \\
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2 \\
\end{bmatrix} =
\begin{bmatrix}
1 + j10^{-4} & 2 \times 10^{-4} & 0 \\
0 & 1 + j10^{-4} & 0 \\
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2 \\
\end{bmatrix}
\]

- 3.2: Right circuit (use \( L \) and \( C \) values given in the figure), where the pressure \( P \) is analogous to the voltage \( V \), and the velocity \( U \) is analogous to the current \( I \).

Sol: Write the system in chain matrix form:

\[
\begin{bmatrix}
P_1 \\
U_1 \\
\end{bmatrix} =
\begin{bmatrix}
1 & sL & 1 & 0 \\
0 & 1 & 1/j2 \pi & 1 \\
\end{bmatrix}
\begin{bmatrix}
P_2 \\
U_2 \\
\end{bmatrix}
\]

Now we substitute the given values:

\[
\begin{bmatrix}
1 & 1/j2 \pi \\
0 & 1/j2 \pi \\
\end{bmatrix}
\begin{bmatrix}
P_2 \\
U_2 \\
\end{bmatrix} =
\begin{bmatrix}
1 + j10^{-4} & 1 + j10^{-4} \\
0 & 1 + j10^{-4} \\
\end{bmatrix}
\begin{bmatrix}
P_1 \\
U_1 \\
\end{bmatrix}
\]

I used Matlab/Octave to evaluate this script:

\[
a=[1 \ 3; 0 \ 1]; b=[1 \ 3 \ 1 \ 4]; c=[1 \ 1/3; 0 \ 1]; d=[1 \ 0; 1/4 \ 1]; T=a+b*c+d.
\]

Finally I found \( T(2,1) \) to be 19/12 using the Matlab/Octave command: \( rats(1.5833, 6) \).

- 3.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. AE-3.2. Do this for the specific frequency \( s = 1 \) as in the previous part (feel free to use Matlab/Octave for your computation).

Sol: Left circuit: Using the previous solution, and Matlab:

\[
\begin{bmatrix}
V_1 \\
V_2 \\
\end{bmatrix} =
\begin{bmatrix}
1 + j10^{-4} & 1/2 \pi \\
0 & 1 + j10^{-4} \\
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
\end{bmatrix}
\]

- 3.4: Right circuit: Repeat the analysis as in question 3.3.

Sol: 

\[
\begin{bmatrix}
P_1 \\
P_2 \\
\end{bmatrix} =
\begin{bmatrix}
1 + j10^{-4} & 1/2 \pi \\
0 & 1 + j10^{-4} \\
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
\end{bmatrix}
\]

**Algebra**

**Problem #4: Fundamental theorem of algebra (FTA).**

- 4.1: State the fundamental theorem of algebra (FTA).

Sol: There are multiple definitions of the FTA, which of course must be equivalent. Here are three (equivalent) answers from Wikipedia

1. The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This may then be applied recursively till the degree is zero.

**Nonlinear (quadratic) to linear equations**

- 14.3: What is the name of this curve?

Sol: It is an ellipse, rotated by 45 degrees.
2.3 Problems AE-3

Topics of this homework:
Visualizing complex functions, bilinear/Möbius transformation, two-port network analysis.

Problem #2: Consider a single circuit element with impedance $Z(s) = 1 + 2s + 3s^2$.

(a) Find the quadratic function in $x$:

$$0 = 1 + 2x + 3x^2$$

(b) Solve the equation for $x$.

Solution:
1. From the first equation, we get:
   $$1 = Z(s)$$
   $$1 = 1 + 2s + 3s^2$$
   $$0 = 2s + 3s^2$$
   $$s(2 + 3s) = 0$$
   $$s = 0, -\frac{2}{3}$$

2. From the second equation, we get:
   $$0 = 1 + 2x + 3x^2$$
   $$x^2 + \frac{2}{3}x + \frac{1}{3} = 0$$
   $$x = \frac{-\frac{2}{3} \pm \sqrt{\left(-\frac{2}{3}\right)^2 - 4\left(\frac{1}{3}\right)}}{2\left(\frac{1}{3}\right)}$$
   $$x = -\frac{1}{3}, -1$$

The roots are $x = 0, -\frac{2}{3}, -\frac{1}{3}, -1$.

Problem #3: Consider a two-port network with impedance $Z(s)$.

(a) Find the transfer function $H(s)$.

Solution:
1. From the first equation, we get:
   $$1 = Z(s)$$
   $$1 = 1 + 2s + 3s^2$$
   $$0 = 2s + 3s^2$$
   $$s(2 + 3s) = 0$$
   $$s = 0, -\frac{2}{3}$$

2. From the second equation, we get:
   $$0 = 1 + 2x + 3x^2$$
   $$x^2 + \frac{2}{3}x + \frac{1}{3} = 0$$
   $$x = \frac{-\frac{2}{3} \pm \sqrt{\left(-\frac{2}{3}\right)^2 - 4\left(\frac{1}{3}\right)}}{2\left(\frac{1}{3}\right)}$$
   $$x = -\frac{1}{3}, -1$$

The roots are $x = 0, -\frac{2}{3}, -\frac{1}{3}, -1$.
This is a quartic, but quadratic in $x^2$. Thus it may be solved for $x^2$ by the completion of squares

$$x^4 + \frac{4}{3}x^2 = \frac{1}{3} \implies \left( x^2 + \frac{2}{\sqrt{3}} \right)^2 = \frac{1}{3} \left( \frac{4}{3} - 1 \right) \implies x^2 = \frac{2}{3} \pm \frac{1}{\sqrt{3}}\sqrt{3} = \pm \frac{1}{\sqrt{3}} \text{ and } \pm \frac{2}{\sqrt{3}}$$

resulting in four roots.

### Nonlinear intersection in analytic geometry

Euclid’s formula for Pythagorean triplets (Eq. 2.5.6, p. 41) can be derived by intersecting a circle and a secant line: Consider the nonlinear equation of a unit circle having radius 1, centered at $(x, y) = (0, 0)$,

$$x^2 + y^2 = 1$$

and the secant line through $(-1, 0)$,

$$y = t(x + 1)$$

a linear equation having slope $t$ and intercept $x = -1$. If the slope $0 < t < 1$, the line intersects the circle at a second point $(a, b)$ in the positive $x, y$ quadrant. The goal is to find $a, b \in \mathbb{N}$ and then show that $c^2 = a^2 + b^2$. Since the construction gives a right triangle with short sides $a, b \in \mathbb{N}$, then it follows that $c \in \mathbb{N}$.

#### Euclidean Proof:

1) $2\phi + \eta = \pi$
2) $\eta + \Theta = \pi$
3) $\phi = \Theta / 2$

#### Diophantus’s Proof:

1) $c^2 = a^2 + b^2$
2) $k(a) = \text{trace}(a + c)$
3) $\zeta(t) = a + jb = \frac{\sqrt{t^2 + 1}}{t}$
4) $\zeta = |c|e^{i\xi} = |c| \left[ \cos(\phi) + i \sin(\phi) \right]$

#### Pythagorean triplets:

1) $t = p/q \in \mathbb{Q}$
2) $a = p^2 - q^2$
3) $b = 2pq$
4) $c = p^2 + q^2$

![Figure 2.3: Derivation of Euclid’s formula for the Pythagorean triplets (PT) $[a, b, c]$, based on a composition of a line, having a rational slope $t = p/q \in \mathbb{Q}$, and a circle $x^2 + y^2 = a^2 + b^2$, where $a, b, c \in \mathbb{N}$. This analysis is attributed to Diophantus (Diophantus’ time) (280 CE), and today such equations are called Diophantine (Diophantus’ time) equations. PTs have applications in architecture and scheduling, and many other practical problems. Most interesting is their relation to Ramanujan’s formula for the eigenvalues of the hydrogen atom (Appendix B).]

### Problem #15: Derive Euclid’s formula

- 15.1: Draw the circle and the line, given a positive slope $0 < t < 1$.
  Sol: See in given in Fig. 2.3

### Problem #16: Substitute $y = t(x + 1)$ (the line equation) into the equation for the circle, and solve for $x(t)$. Hint: Because the line intersects the circle at two points, you will get two solutions for $x$. One of these solutions is the trivial solution $x = -1$.
  Sol: $x(t) = \left( 1 - t^2 \right) / (1 + t^2)$