An Invitation to Mathematical Physics
and Its History

Jont B. Allen

University of Illinois at Urbana-Champaign

http://jontalle.web.engr.illinois.edu/uploads/

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Preface:

Science evolved over thousands of years. It began out of curiosity about how the world around us works, as well as a need to know how to make things work better. From water management to space travel, science is essential for success.

The evolution of science is layered: Early science depended mostly on critical observation, thus the early scientist was considered a philosopher who told a story. Soon experiments were designed to test these theories. As soon as one layer is understood, it must be researched and resolved. We now know that there is a recursion of layers. Its like pealing an onion. As soon as one layer is removed, a new layer appears, and the process must be repeated. History has taught us that it would be a mistake to belive that once the layer is removed, the problem is solved. Some examples are helpful.

Moving from onions to physical examples is important because it prepares the student to understand that there may be no end to the layers. We don't know how many layers there are until we reach the end. There may be no end.

Newton was not the first person to find the layer he studied by suggesting that force and acceleration are proportional to mass, when he suggested that $F = ma$. He actually gained this insight from Galileo, who had the deeper insight, that falling objects, due to gravity, took the same time to hit the ground. If the object had a larger mass, or if it rolled down a inclined plane, the same law was responsible for the time of flight. So Galello's insight was the critical step that led Newton to introduce $F = ma$. At that time there was no understanding that gravity was much more subtle in its action.

Newton's new layer was very confusing. If two bodies were tied together by a spring, the joined body would fall at the very same time. Tying them together, made no difference in the result. This insight came from the realization that there was a very small force between the two bodies, due to their mutual gravity. While the effect was infinitesimal, it is not zero. It is zero only if their mutual force, thus acceleration, is the same. It follows that the spring must be the attractive mutual gravity. I assume Newton understood this. But where is the historical evidence for this assumption?

The next layer came when Einstein assumed that $E = mc_o^2$. The correct relaton is $E = \partial mv/\partial t$, as first described by Lorentz.[1]

Good science is observation and experimentation. Great science is the art of making models that explain experimental results. This always seems to result in a deeper question, suggesting new experiments. This cycle typically takes 10 to 50 years, thus progress much slower than a human lifetime.

To understand physics requires a knowledge of mathematics. The converse is not true. One can create new mathematics, but it may not be relevant to the new physics. Every layer requires new mathematics to explain the new physics. This process is successful when a new principle is established, leading to a deeper and reproducible experimental observation.

By definition, pure mathematics contains no physics. Yet historically, mathematics has a rich history filled with physical applications. Mathematics was developed by individuals who intend to make things work. As an engineer, I see these creators of early mathematics as budding engineers. This book is an attempt to tell the story of the development of mathematical physics, used by engineers. Each generation has its geniuses. One of the first philosophers, experimentalist, and mathematicians, was Galileo.

There are two distinct ways to learn mathematics: by learning definitions and relationships, or by associating each mathematical concept with a physical counterpart. Students of physics and engineering best learn mathematics based on underlying physical concepts. Students of pure mathematics are taught via definitions of abstract structures, not from the history of mathematical physics. These two teaching methods result in a very different understanding of the material.

There is a deep common thread between physics and mathematics: the chronological development, and the history of mathematics. This is because much of mathematics was developed to solve physical problems. Most early mathematics evolved from attempts to understand the world, with the goal of navigating the ocean. Pure mathematics followed as generalizations of these physical concepts.

This book is not a typical mathematics book; rather, it is about the relationship of math and physics, presented roughly in chronological order via their history. To teach mathematical physics in an orderly way, our treatment requires a step backward in terms of the mathematics, but a step forward in terms of the physics. Historically speaking, mathematics was created by individuals such as Galileo, who by modern standards, might be viewed as an engineer. Mathematics contains the basic information that well-informed engineers need to know.

Engineering and physics texts do not intend to be rigorous, in the mathematical sense. In some ways, mathematics cleans up the mess by proving theorems, which frequently start as speculations in physics and

[1] `https://en.wikipedia.org/wiki/Isidor_Rabi`

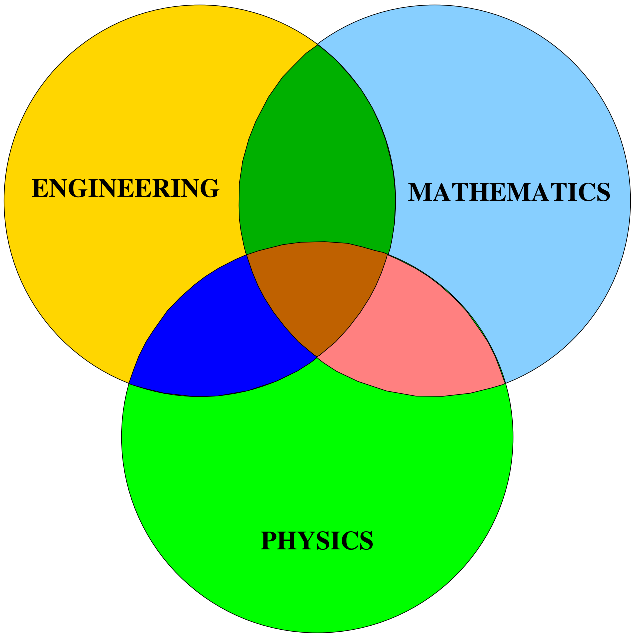

Figure 1: *There is a natural symbiotic relationship among mathematics, engineering, and physics (MEP), depicted in this Venn diagram.* Mathematics *provides the method and rigor.* Engineering *transforms the method into technology.* Physics *explores the boundaries. While these three disciplines work well together, there is poor communication, due to the vastly different vocabularies. But style may be more at issue. For example, mathematics rarely uses a system of units, whereas physics and engineering depend critically on them. Mathematics strives to abstract the ideas into proofs. Physics rarely uses a proof. When they attempt rigor, physicists and engineers typically get into difficulty. An important observation by Felix Klein about pure mathematicians, regarding the unavoidable inaccuracies in physics:* "It may be said that the idea [of inaccuracy] is usually so repulsive to [mathematicians] that its recognition sooner or later spoils their interest in natural science" *(Condon and Morse, 1929, p. 19).*

engineering. This cleanup is a slow tedious process. Just because something seems obvious, based on the known physical facts, does not render it as a fundamental theorem of mathematics.

Although there are similarities between this book and that of Graham et al. (1994), the differences are notable. First, Graham's *Concrete Mathematics* presents an impossible standard to be measured against. Second, it is clearly a math book, brilliantly written and targeted at computer science students. This book is not just a math book – it is a mathematical physics text, depending critically on mathematical concepts.

Organization: The book is organized into six sections: 1) Preface, 2) Number Systems, 3) Algebraic Equations, 4) Scalar Calculus 5) Laplace Transforms ($\mathcal{LT}$) and 6) Vector Calculus. The appendix address special topics: A) Eigen-analysis B) Pell's equation C) The Transmission matrix

New to the 2d edition is a detailed discussion on prime numbers, denoted here as ϱ_k. For example $\varrho_1 = 3, \varrho_2 = 5, \varrho_3 = 7, \varrho_4 = 11$, etc. The reciprocals of the primes are always periodic. The *degree of rationality* is defined by their period. Unlike primes, irrational numbers (e.g. $\sqrt{2}$) are never periodic, clearly distinguishing them from the primes. It then follows that $\varrho_{k+1}/\varrho_{k-1}$ is parsed into sub-groups.[2] [3]

Summary

This is foremost a math book, but not the typical math book, summarized as follows: 1) the goals of book and 2) the utility of math.

First, this book is for the engineering minded, for those who need to understand math, to do engineering, and to learn how things work. In this sense the book is more about physics and engineering than mathematics. Math skills are essential for making progress in building things, be it pyramids or computers, as clearly shown by the great civilizations of the Chinese, Egyptians, Mesopotamia's, Greeks, and Romans.

Second, this is a book about the math, created to explain physics, to enable people to engineer complex things. To sail around the world, one needs to know how to navigate. This requires a model of the planets and stars. You can know where you are on earth once you understand where earth is relative to the sun, planets, Milky Way, and distant stars. The answer to such a cosmic question depends on whom you ask. Who is qualified to answer such a question? We now know it was Tycho Brahe. It is best answered by those who study the mathematics of the universe. The utility and accuracy of the answer depend critically on the depth of understanding of the physics of the cosmic clocks.

The English astronomer Edmond Halley (1656–1742) was interested in understanding the physics of comets. So he asked Isaac Newton (1643–1727) about them. Apparently Newton's responded *"They are described by the ellipse,"* which stunned Halley (Stillwell, 2010, p. 176). Notably, both Galileo (ca. 1564-1642) and Kepler (ca. 1571-1630) had been studying the orbits of planets and their moons.[4] But the true source of this discovery, more than a century before Newton, was entirely due to the famous astronomer Tycho Brahe (Dec. 14, 1546 – Oct. 24, 1601). Tycho had a strange nose, or more precise, no nose.

[2]Google `https://blogs.ams.org/phdplus/2015/04/28/coffee-into-theorems/`

[3]For an interesting discussion, Google "What did Paul Erdős say about prime numbers?"

[4]Kepler (1571–1630) worked as a court mathematician to Emperors Rudolf II, Matthias, and Ferdinand II.

When Halley asked is older friend Issac Newton to explain, he responded “I calculated it.” But when asked to show the calculation, he was unable to reproduce it. The correct response should have been: “Tycho (Brahe) calulated it.”

This open challenge eventually led to Newton’s grand treatise *Philosophiae Naturalis Principia Mathematica* (July 5, 1687), hand written in latin script. I conjecture he did this so 1) he was on record, 2) using a languagae that could not be easily comprehend. The letter to Halley, explaining how to calculate the physics of the orbits of planets, had a humble beginning,

Likely Halley did not read latin. It seems to me that Newton was showing off. To do this, like Brahe, Newton needed both mathematics and latin, tools he had mastered. Today it is widely accepted that Newton and Gottfried Leibniz invented calculus. But the early record shows that perhaps Bhaskara II (1114–1185 CE) had mastered the art, well before Newton.[5] Newton was knighted in 1705.[6]

Third, the primary goal of this book is to teach mathematics to motivated engineers, in a way that it can be understood, mastered, and remembered. But how can this impossible goal be achieved? The answer is to fill in the gaps with *Who did what, and when?* Compared with math, the historical record is easily mastered.

To be an expert in a field, one must know its history. This includes who the people were, what they did, and the credibility of their story.

Who were those early explorers? They are names we all know: Archimedes, Pythagoras, Leonardo da Vinci, Galileo, Newton, and so on. All of these individuals had mastered some level of mathematics. The present book presents the tools taught to every engineer. Rather than memorizing complex formulas, we must make the relationships “obvious” by mastering the underlying concepts.

Fourth, when educators view this book, their reactions may be: *You have too much material crammed into one semester.* Tracking the students who have taken the course, looking at their grades, and interviewing them, supports my view that for the motivated student, the material presented here is appropriate for one semester. As stated on the Springer website, the first edition has sold more than 8,000 copies, in less than 5 years.

To write this book it was necessary to unify many technical terms. The goal becomes one of unifying the Mathematical and Physical terminology.

Author’s personal statement

An expert is someone who has made all possible mistakes in a small field. I certainly have made my share of mistakes. I openly state that I love making mistakes because I learn so much from them. One might call this the “expert’s corollary.”

This book has been written out of my love for the topic of mathematical physics, a topic that provides many insights, that lead to a deeper understanding of key physical concepts. I have developed a unified sense of physics and math. I understood that math can be physics, and physics math. I have discovered what I assume to be my conceptual holes that need filling. I sense deep relationships between math and physics that remain unresolved. What we presently teach is not wrong, but may miss some unifying relationships. I sense I may be missing an intuition for how math “works.” Good scientists “listen” to their data. In the same way, I want to listen to the language of mathematics. We need to let the mathematics guide us toward our engineering goals.

As summarized in Fig. 1, this marriage of math, engineering, and physics (MEP)[7] helps us make progress in understanding the physical world. We must turn to mathematics and physics when trying to understand the universe. My views follow from a lifelong attempt to understand human communication – that is, the perception and decoding of human speech sounds.

In this text it is my goal to clarify conceptual errors while telling the story of physics and mathematics. My views have been inspired by classic works, as documented in the Bibliography. There is a short summary of these books at the end of Chapter 6 **Suggested readings** §6.11.

This book was inspired by my reading Stillwell (2002) through Chapter 21. Somewhere in Chapter 22 I switched to the third edition (Stillwell, 2010), at which point I realized I had much more to master. It became clear that by teaching this material to undergraduate engineers, I could absorb the advanced material at a reasonable pace. This book soon followed.

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[5] `https://www-history.mcs.st-and.ac.uk/Projects/Pearce/Chapters/Ch8_5.html`

[6] Quoteing from Google’s “AI OVERVIEW:” Johannes Kepler and Galileo Galilei had a respectful scientific correspondents, but never met. While they corresponded on astronomical discoveries, and Kepler defended Galileo’s early findings, Galileo often ignored Kepler’s work, including his famous correct theory of elliptical orbits. For some reason their relationship ended in 1611. More historical research on their relationship would be helpful.

[7] MEP is a focused alternative to STEM.

Acknowledgments

Swati Meherishi of the Springer staff was perpetually positive about this project, and always steered me the right directions regarding the details of making a manuscript into a book.

Most important, I would like to thank my wife, Sheau Feng Jeng (Patricia Allen), for her unbelievable support and love. She delivered constant peace of mind, without which this project could never have been started, much less finished. Many others, including the many students who took courses based on this book, played important roles.

> C’est par la logique qu’on démontre, c’est par l’intuition qu’on invente.
> *It is by logic that we prove, but by intuition that we discover.*
>
> - Henri Poincare

Finally, I dedicate this book to my friend and colleague Ron Graham (1935-2020), who played a critical role in my mathematical development (Graham et al., 1989).

> “The main obstacle to progress is not ignorance but the illusion of knowledge.”
>
> -Ron Graham[8]

–Jont Allen, Mahomet, IL, December 17, 2025

[8] `https://today.ucsd.edu/story/ron-graham-obituary`

Contents

Chapter 1

Introduction

Mathematical Physics is a complex topic, designed to describe physical observations using mathematics. This book is an attempt to connect the two overlapping topics via their history, the glue provided by time. This is successful strategy.

Much of early mathematics, say before 1600 BCE, involved the love of art and music, the sensations of light and sound. Our psychological senses of color and pitch are determined by the frequencies (i.e., wavelengths). The Chinese, and later the Pythagorean's, are well known for their early contributions to music theory. We are largely ignorant of exactly what Chinese scholars knew. The best record of early mathematics comes from Euclid, who lived in the forth century BCE, after Pythagoras. Thus we can trace early mathematics back to the Pythagorean's in the sixth century (580–500 BCE), who focused on the Pythagorean theorem and early music theory.

Pythagoras firmly believed that "all is number," meaning that every number, and every mathematical and physical concept, could be explained by integer relationships, mostly based on either ratios or via the Pythagorean theorem. It is likely that his belief was based on Chinese mathematics from thousands of years earlier. It is also believed that his ideas about the importance of integers followed from music theory. With the discovery of irrational numbers, the concept of a number was significantly extended beyond these elementary views. As discussed by Stillwell (2010, p. 16), the Pythagorean view is still relevant today:

> *With the digital computer, digital audio, and digital video coding everything, at least approximately, into sequences of whole numbers, we are closer than ever to a world in which "all is number."*

The frequencies of the musical notes (pitches) obey integral ratio relationships based on the octave.[1] The western 12-tone scale breaks the octave into 12 equal irrational ratios called *semitones* (a factor of $f_2/f_1 = \sqrt[12]{2}$). Our innate sense of frequency ratios comes from the physiology of the the cochlea, with a fixed distance along the organ of Corti, the sensory organ of the inner ear [2]. Mozart's contribution was to slightly adjust the mathematical relation to make it more natural, today known as the *Well-Tempered Clavier*. His many contributions to music benefited from his mathematical skills.

1.1 Early science and mathematics

Although early Asian mathematics is lost, it defined its course for at least several millennia. The first recorded mathematics was from the Chinese (5000–1200 BCE) and the Egyptians (3300 BCE). Some of the best early records were left by the people of Mesopotamia (Iraq, 1800 BCE).

While the first 5000 years of math are poorly documented, the basic Time-line is in Fig. 1.1.

Thanks to Euclid, and later Diophantus (ca. 250 CE), we have some understanding of Chinese mathematics. For example, Euclid's formula (Eq. 1.1, p. 7). provides a method for computing Pythagorean triplets, a formula

[1]By definition, an octave is a doubling in frequency.

[2]Fahey and Allen (1985); Allen and Fahey (1993)

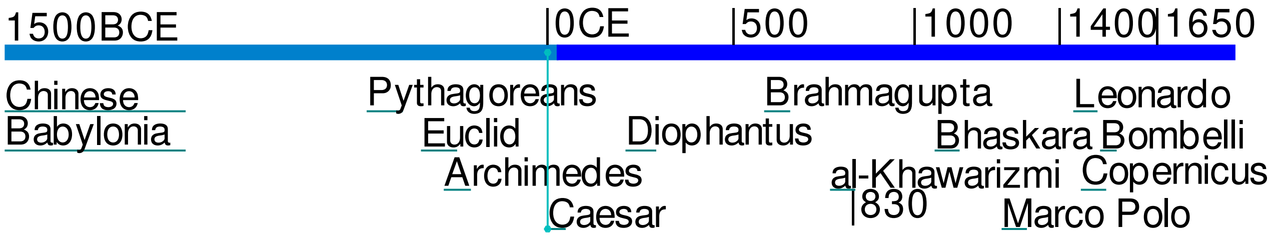

Figure 1.1: *Time-line between 1500* BCE *and 1700* CE. *The European renaissance is considered to have occurred between the 15th and 17th centuries CE. There is evidence that a Chinese "treasure ship" visited Italy in 1434, initiating the Italian renaissance (Menzies, 2008). This was not the first encounter between the Italians and the Chinese, as documented in* The Travels of Marco Polo *(ca. 1300* CE*).*

believed to be due to the Chinese.[3]

Chinese bells and stringed musical instruments were exquisitely developed with tonal quality, as documented by ancient physical artifacts (Fletcher and Rossing, 2008). In fact this development was so rich that one might ask why the Chinese failed to initiate the Industrial Revolution. Specifically, why did European innovation eventually dominate when it was the Chinese who were responsible for such extensive early invention?

An early insight into the scientific history of China came from Joseph Needham, an American chemist and scholar from Cambridge England. Needham learned Chinese from a colleague[4] and ended up researching early Chinese science and technology for the U.S. government (Winchester, 2009).

According to Lin (1995), the Needham question is:

> *Why did modern science, the mathematization of hypotheses about Nature, with all its implications for advanced technology, take its meteoric rise only in the West at the time of Galileo[, but] had not developed in Chinese civilization or Indian civilization?*

As discussed by Lin (1995) and Apte (2009), Needham cites the many developments in China:

> *Gunpowder, the magnetic compass, and paper and printing, which Francis Bacon considered as the three most important inventions facilitating the West's transformation from the Dark Ages to the modern world, were invented in China.*
>
> *Needham's works attribute significant weight to the impact of Confucianism and Taoism on the pace of Chinese scientific discovery, and emphasize what it describes as the "diffusionist" approach of Chinese science as opposed to a perceived independent inventiveness in the western world. Needham held that the notion that the Chinese script had inhibited scientific thought was "grossly overrated" (Grosswiler, 2004). Unlike most languages, Chinese characters are a "picture" representation.*

Lin (1995) focused on military applications, missing the significant importance of nonmilitary contributions. A large fraction of mathematics was developed to better understand the solar system, acoustics, musical instruments, and the theory of sound and light. Eventually the universe became a popular topic, as it remains today (Einstein (1905)).

Regarding the Needham question, I suspect the answer is now clear. In the end, China withdrew from its several earlier expansions because of internal politics (Menzies, 2004, 2008).

1.1.1 The Pythagorean theorem

Thanks to Euclid's *Elements* (written ca. 323 BCE), we have the historical record tracing the progress in geometry as established by the Pythagorean theorem, which states that *for any right triangle* having sides of

[3]One might reasonably view Euclid's role as that of a mathematical messenger, rather than as the originator of the mathematical language.

[4]They eventually conjoined (Winchester, 2009).

History of Mathematics to the 16th Century CE

20th	Chinese (primes; quadratic equation; Euclidean algorithm (GCD))	
18th	Babylonians (Mesopotamia/Iraq) (quadratic solution)	
6th	Thales of Miletus (first Greek geometry) (624)	
5th	Pythagoras and the Pythagorean "tribe"(570)	
4th	**Euclid**; Archimedes	
3rd	Eratosthenes (276–194)	BCE
1st	Diophantus (ca. 250)	CE
2ed	Library of Alexandria destroyed by fire (391)	
4th	Brahmagupta (negative numbers; quadratic equation) (598–670);	
7th	al–Khwarizmi (algebra) (830);	
8th	Hasan Ibn al–Haytham (Alhazen) (965–1040);	
10th	Bhaskara (calculus) (1114–1183)	
13th	Leonardo da Vinci (1452–1519); Michelangelo (1475–1564); Copernicus (1473–1543); Nicolo Tartaglia (1499) (cubic solution);	
14th	Bombelli (1526–1572); **Tycho Brahe** (1546–1601), Johannes Kepler (1571–1630), **Galileo Galilei** (1564–1642)	
15th	**Bernoulli's**, Newton (1683-1727)	
16th	**Euler** (1707–1783), Gauss (1777–1855), Cauchy (1789–1857)	
17th	Fermat (1601-1665); **Riemann** (1826–1866); **dAlembert** (1777-1783)	

lengths $(a, b, c) \in \mathbb{R}$ that are positive real numbers or, more interesting, integers $c > [a, b] \in \mathbb{N}$, such that $a + b > c$, since

$$c^2 = a^2 + b^2. \tag{1.1}$$

Early integer solutions were likely found by trial and error rather than by an algorithm.

If a, b, c are lengths, then a^2, b^2, c^2 are each the area of a square. Equation 1.1 says that the area a^2 plus the area b^2 equals the area c^2. Today a simple way to prove this is to compute the magnitude of the complex number $c = a + b\jmath$, which forces the right angle

$$|c|^2 = (a + b\jmath)(a - b\jmath) = a^2 + b^2. \tag{1.2}$$

However, complex arithmetic was not an option for the Greek mathematicians, since complex numbers and algebra had yet to be discovered.

Almost 700 years after Euclid's *Elements*, the Library of Alexandria was destroyed by fire (391 CE), taking with it much of the accumulated Greek knowledge. As a result, one of the best technical records remaining is Euclid's *Elements*, along with some sparse mathematics due to Archimedes (ca. 300 BCE) on geometrical series, the volume of a sphere, the area of a parabola, and elementary hydrostatics. Arround 1572 a copy of Diophantus's *Arithmetic* was discovered by Bombelli in the Vatican library. (Burton, 1985; Stillwell, 2010, p. 51). His book became an inspiration for Galileo, Descartes, Fermat, Newton, and their followers.

Early number theory: Well before Pythagoras, the Babylonians (ca. 1,800 BCE) had tables of triplets of integers $[a, b, c]$ that obey Eq. 1.1, such as $[3, 4, 5]$. However, the triplets from the Babylonians were larger numbers, the largest being $a = 12,709$ and $c = 18,541$. A clay tablet (Plimpton-322) dating back to 1800 BCE was found with integers for $[a, c]$. Given such sets of two numbers, which determined a third positive integer $b = 13,500$ such that $b = \sqrt{c^2 - a^2}$, this table is more than convincing that the Babylonians were well aware of Pythagorean triplets (PTs), but less convincing that they had access to Euclid's formula, a formula for PTs, derived in §2.5.1.

It seems obvious that Euclid's *Elements* was largely the source of the fruitful era of the Greek mathematician Diophantus (215–285 CE) who developed the field of discrete mathematics, today known as either Number Theory or *Diophantine analysis*[5]. This term means that the solution, not the equation, is integral.

The work of Diophantus was followed by fundamental changes in mathematics, possibly leading to the development of algebra, but at least these four key discoveries:

[5]Assigning multiple names to something typically leads to chaos.

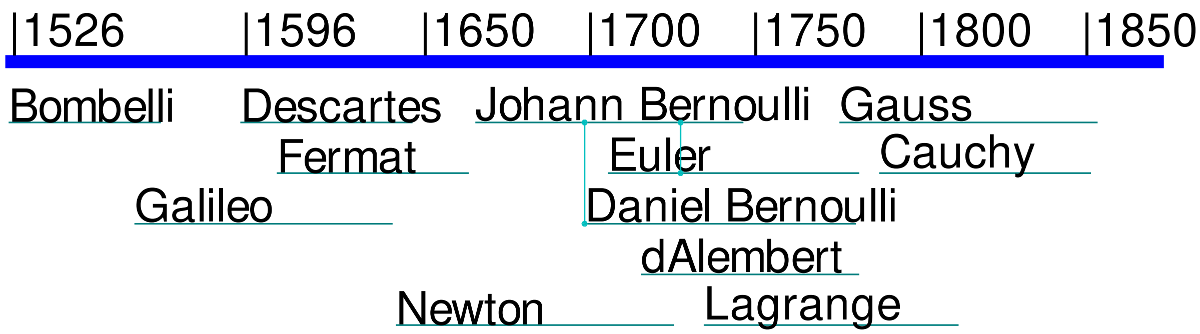

Figure 1.2: *Time-line from 1526 to 1850 includes the development of modern number theory, analytic geometry, calculus, differential equations, and linear algebra. Newton was born one year after Galileo died, thus was significantly influenced by his many discoveries. The vertical green lines indicate mentor-student relationships. Note the significant overlap among Newton, the Bernoulli's, and Euler, a nucleation point for modern mathematics. Lagrange played a key role in the development of linear algebra. Gauss had the advantage of input from Newton, Euler, d'Alembert, and Lagrange. Likely Cauchy had a significant contemporary influence on Gauss as well. Finally, note that Fig. 1.1 ends with Bombelli while this figure begins with him. He famously discovered a copy of Diophantus's book in the Vatican library. (1546-1601) and Kepler to this figure. I suspect they may be more important than This was the same book in which Fermat noted that the margin was too small to hold the "proof" of his "last theorem." I have updated the above time-line by highlighting the names of tagged by their century: 14th: Bombelli, Tycho Brahe, Kepler, Galileo; Descarte, Fermat; 15th: Netwon, Johann and Daniel Bernoulli; 16th: Euler, dAlembert, Lagrange, Gauss, Cauchy; 17th: Fermat, Riemann and . 18th: Einstein Itemize the individual's historical roles, along with its importance. Brahe's role deservies everyone's attention in the history of science, for drawing attention to the role of the sun as the center of our solar system. Kepler was the first to acknowledge Tycho's amazing contributions, by churning them into mathematics. In my present view, Tycho Brahe is the most important individual in the history of our solar system. Without Kepler's insightful analysis, Tycho's contribution could (would?!) have been lost, along with the work of Albert Einstein (ca. 1905).*

1. Negative numbers
2. Quadratic equations (Brahmagupta, 7th century)
3. Algebra (al-Khwarizmi, 9th century)
4. Complex arithmetic (Bombelli, 15th century)

These discoveries overlapped with the European Middle Ages (also known as the Dark Ages). Just because Europe went "dark" did not imply that European intellectuals ceased working.[6]

1.1.2 What is science?

Modern science is a process that quantifies hypotheses to build truths.It evolved from the early Greek philosophers Plato and Aristotle. Science works quite well, thanks to the language of mathematics. Using the scientific method we have discovered many deep secrets about the universe over the last 5000 years.

1.1.3 What is mathematics?

It seems strange when people say "I can't learn math"[7] but then claim to be good at languages. Before high school, students tend to confuse arithmetic with math. One does not need to be good at arithmetic to be good at math (but it doesn't hurt). Math is a language, with symbols taken from various languages, primarily Greek letters such as σ, Ω, ν, due to the popularity of Euclid's *Elements*. When you learn a new language you acquire a new skill set that expands your world view. The language of math may be the ideal example.

The evolution of the symbols of language is interesting (Mazur, 2014). These symbols are assembled into words and then sentences. In the Chinese language, spoken dialects can be quite different, yet the printed characters are the same for all dialects.[8] This means everyone can read the language, but the same people cannot speak to each other, unless they speak either Mandarin or English. Today the classic characters have been replace by modern fonts, to simplify the script and provide a universal simplified written language. This

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[6]It would be interesting to search the archives of the monasteries, where the records were kept, to determine exactly what happened during this religious blackout.

[7] "It looks like greek to me."

[8]Note: I can only write American English.

is similar to our universal written mathematics. Mathematical sentences can be read out loud in any language (dialect), but the symbols remain the same.

Learning a languages, such as math, is an advanced skill. However, the social outcomes of learning a language and learning math are very different. Learning a new language is fun because it opens doors to other cultures. Math is different due to the rigor of the grammar (rules of the language) as well as the way it is taught (i.e., not as a language). A third difference between math and language is that math evolved from physics, giving it technical context.

As with any language, the more mathematics you learn, the easier it is to understand, because mathematics is built from the bottom up. It's a continuous set of concepts, much like the construction of a house. If you try to learn calculus and differential equations but skip simple number theory, the lessons will be more difficult to understand. You will end up memorizing instead of understanding, and as a result you will likely soon forget it. When you truly understand something, it can never be forgotten. A beautiful example is the solution to a quadratic equation: If you learn how to complete the square you don't need the quadratic formula.

Mathematical *must* be learned sequentially. You can't build a house if you don't know about screws, nails and plaster. Likewise in mathematics, you can't learn to integrate a function if you have failed to understand the difference between integers, complex numbers, polynomials, and analytic functions.

A short list of topics in mathematics includes numbers ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{C}$), algebra, derivatives, anti-derivatives (i.e., integration), differential equations, vectors and the spaces they define, matrices, matrix algebra, eigenvalues and eigenvectors, solutions of systems of equations, and matrix differential equations and their eigenvectors. Learning is about understanding, not memorizing.

The rules of mathematics are formally defined by algebra. For example, the sentence $a = b$ means that the number a has the same value as the number b. The sentence is read as "a equals b." The numbers are nouns and the equal sign says they are equivalent; it plays the role of a verb, or action symbol. Following the rules of algebra, this sentence may be rewritten as $a - b = 0$. Here the symbols for minus and equal indicate two types of actions (verbs).

Sentences can become arbitrarily complex, such as the definition of the integral of a function or a differential equation. But in each case, the mathematical sentence is written down, may be read out loud, has a well-defined meaning, and may be manipulated into equivalent forms following the rules of algebra and calculus. This language of mathematics is powerful, with deep consequences, first known as algorithms but eventually as theorems.

The writer of an equation should always translate (explicitly summarize the meaning of the expression), so the reader will not miss the main point. This is a simple matter of clear writing.

Just as math is a language, so language may be thought of as mathematics. To properly write correct English it is necessary to understand the construction of the sentence. It is important to identify the subject, verb, object, and various types of modifying phrases. For example, look up the interesting distinction between *that* and *which*.[9] Thus, like math, language has rules. Most individuals use language that "sounds right," but if you're learning English as a second language, you must understand the rules, which are arguably easier to master than its foreign speech sounds.

A second answer to the question "What is Mathematics?" is the commonly held view that math is taught as a construction of definitions, based on axioms. Given a consistent application of the definitions, all the results of mathematics are assumed to follow. This definition seems simplistic.

1.2 Modern mathematics

1.2.1 Early physics as mathematics: Back to Pythagoras

We have established that math is the language of science. There is an additional answer to the question "What is mathematics?" The answer, the creation of algorithms and theorems, comes from studying its history, beginning with the earliest records. This chronological view starts, of course, with the study of numbers. First there is the taxonomy of numbers. It took thousands of years to realize that numbers are more than the counting numbers $\mathbb{N}$, to create a symbol for nothing (i.e., zero), and to invent negative numbers. With the invention of the abacus, a precise memory aid for manipulating complex sets of real integers, one could do very detailed analysis. But this required the discovery of algorithms (procedures) to add, subtract, multiply (many additions of the same number), and divide (many subtractions of the same number), such as the Euclidean algorithm for the greatest common divisor (GCD). Eventually it became clear to the experts (early mathematicians) that there were natural rules to be discovered; thus books (e.g., Euclid's *Elements*) were written to summarize these

[9] `https://en.oxforddictionaries.com/usage/that-or-which`

new concepts. Most of these books are now lost. This was not an act of carelessness by the authors. During what is know as the dark-ages, there was an anti-intellectual religious pillage, best exampled by the destruction of the library in Alexandria.

The role of mathematics is to summarize algorithms (i.e., sets of rules) and formalize an idea as a theorem. Pythagoras and his followers, the Pythagoreans, believed that there was a fundamental relationship between mathematics and the physical world. The Asian civilizations were the first to capitalize on the relationship between science and mathematics, to use mathematics to design things for profit.This may have been the beginning of capitalizing technology (i.e., engineering), based on the relationship between physics and math. This influenced commerce in many ways: i.e., map making, tools, implements of war (the wheel, gunpowder), art (music), water transport, sanitation, secure communication, food, namely all aspects of human existence. Of course it was the Asian cultures to first to master many of these early technologies.

The Pythagorean theorem did not begin with Euclid or Pythagoras; rather they appreciated its importance and documented its proof. Why is Eq. 1.1 called a *theorem*? Theorems require a proof. What exactly needs to be proved? We do not need to prove that (a, b, c) obeys this relationship, since this condition is observed. We do not need to prove that a^2 is the area of a square, as this is the definition of an area. What needs to be proved is that the relationship $c^2 = a^2 + b^2$ holds *if, and only if,* the angle between the two shorter sides is $90°$. Namely they are orthogonal.

Eventually the Pythagoreans were murdered, likely the result of mixing a powerful technology with politics:

> *[It was] said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in (Stillwell, 2010, p. 16) nearby Mesopotamia in 497 BCE.*

Modern mathematics (what we practice today) was born in the 15th and 16th centuries, in the minds of Leonardo da Vinci, Bombelli, Galileo, Descartes, and Fermat (Burton, 1985). Since mathematics is the product of human effort, the first way to study the topic is to determine the great minds who studied math. This is not possible since their are no records. There was a period where books were destroyed by those who understood the threat of scientific knowledge. Doing the best we can with what little we know today, the story starts with the Pythagoreans, a group of mathematicians who were extremely secretive about how they solved problems. It seems obvious that they kept their profile low so they wouldn't be destroyed. However this eventual changed, starting with Galileo, Mersenne, Descartes, and Newton, which allowed mathematics to blossom. Developments during this time may seem hectic and disconnected, but this is a wrong impression. Due to Galileo's studies of the pendulum and the telescope, along with his more accurate time and frequency measurements, and a better understanding of the relationship $f\lambda = c_o$ among frequency f, wavelength λ, and wave speed c_o, progress was made. The developments were dependent of new technologies, such as the telescope (optics).

Following centuries of analysis by many authors, allowed the development of *categories* of mathematics. I assume that the first of these was *Number theory*, the analysis of numbers. As the methods evolved, new areas were discovered and named. Examples include calculus and analytic function theory. This progress continues to this day, and will likely never end with a last area of mathematics, just like areas of science. The fields evolve in a never-ending inward spiral.

1.2.2 Science meets mathematics

Early studies of vision and hearing: Since light and sound (music) played a key role in the development of the early science. It was important to fully understand the mechanism of our perception of light and sound. There are many outstanding examples where physiology impacted mathematics. Leonardo da Vinci (1452–1519) is well known for his early studies of human anatomy, the knowledge of which was key when it came to drawing and painting the human form. Depicting the human body as art, had a massive impact on science. Art and science became one.

Tycho Brahe: Tycho Brahe has the great distinction of being the first astronomer to determine that the earth revolves around the sun rather than the popular view, at that time (1550) that the planets revolved around the earth, with the earth being the center of the universe. How he determined this is a complex topic Ferguson (2020).

Johannes Kepler: Johannes joined Brahe about 30 years into Brahe's research. His huge accompulishment was to formulate a mathematical theory around Brahe's data on the orbits of the planets around the sun. Both of these contributions were a new critical change in our understanding of the solar system.

Galileo: In 1589 Galileo Galilei (1564–1642) famously conceptualized an experiment in which he suggested dropping two different masses from the Leaning Tower of Pisa. He suggested that both must take the same time to hit the ground.

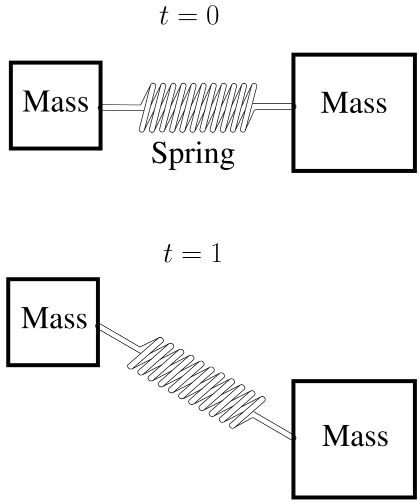

Figure 1.3: *Depiction of Galileo's argument (from his unpublished book of 1638) as to why objects of different masses (i.e., weights) must fall with the same velocity, contrary to what Archimedes had proposed in about 250 BCE. Mass and weight are orthogonal concepts. While this two-body problem is stable, a three-body problem is potentially chaotic, an issue of some concern, even to this day.*

Conceptually this is a mathematically sophisticated experiment, driven by a mathematical argument in which Galileo considered the two masses to be connected by an elastic cord (a spring) or rolling down a friction less inclined plane (see Fig. 1.3). His studies resulted in the concept of conservation of energy, one of the cornerstones of physical theory since that time.

Being joined with an elastic cord, regardless of the tension, the masses become one. If the velocity were proportional to the mass, as Archimedes believed, the sum of the two masses would necessarily fall even faster. This results in a logical fallacy: How can two masses fall faster than either mass alone? This also violates the concept of conservation of energy, as the total energy of the two masses would be greater than that of the parts. I suspect Galileo's argument may have been the first time that the principle of conservation of energy was clearly stated.

It seems likely that Galileo was attracted to this model of two masses connected by a spring because he was also interested in planetary motion, which consists of masses (sun, earth, moon, etc), mutually attracted by gravity (i.e., springs).

Galileo performed related experiments on pendulums, where he varied the length l, mass m, and angle θ of the swing. By measuring the period[10] he was able to formulate precise relationships between the variables. This experiment also measured the force exerted by gravity, so the experiments were related, but in very different ways. The pendulum served as the ideal clock, as it needed very little energy to keep it going, due to its very low friction (energy loss).

In a related experiment, Galileo measured the length of a day by counting the number of swings of the pendulum in 24 hours, thus measuring precisely the daily period of a star as it crossed the tip of a church steeple. The number of seconds in a day is the highly composite number $24{\cdot}60{\cdot}60 = 86,400 = 2^7 3^3 5^2$ [s/day]. Since 86,400 is the product of the first three primes, it is highly composite and thus may be expressed in many equivalent ways. For example, the day can be divided evenly into 2, 3, 4, 5, 6, 8, 9, ... (but not 7) parts, leaving invariant the number of seconds per day. I would love to know who was responsible for this highly composite number of seconds per day, which is actually wrong, due to leap years. My guess is either Galileo or one of his colleagues.

Factoring the number of days in a year $(365 = 5 \cdot 73)$ is a poor choice, since it cannot be decomposed into small primes. If the year were taken as $364 = 2^2 \cdot 7 \cdot 13$ days, it would make for shorter years (by 1 day), 13 months per year, perfect quarters, $28 = 4 \cdot 13$ day months, and $52 = 4 \cdot 13$ weeks. Every holiday would always fall on the same day, every year. It would be a calendar that humans could actually understand. We can only guess as to why the number of days per year was so poorly chosen.

Galileo also studied the relationship between the wavelength and frequency of a sound wave in musical instruments. He greatly improved the telescope, which he needed for his observations of the planets and their moons, such as Io, leading to the first accurate determination of the speed of light in 1676 by Ole Rømer.

[10]The term *period* refers to the duration in units of time of a periodic function. For example, the periods of the moon and the sun are 28 days and one year, respectively.

Galileo's original experiments on pendulums and rolling masses down slopes were flawed by inaccurate data. It is likely that he didn't have accurate clocks. He soon solved this problem. We don't know whether Mersenne repeated Galileo's experiments and then appreciated his theory, or whether he communicated with Galileo. But the final resolution was that once the experimental method was improved, the results were improved. Galileo claimed that the distance to reach the ground is proportional to the square of the time. This expression is equivalent to $F = m_o a$ assuming constant mass m_o and acceleration a.

Many of Galileo's contributions resulted in new mathematics, leading to Newton's discovery of the wave equation (1687), followed 60 years later by its one-dimensional general solution by d'Alembert (1747), and not to be forgotten, Maxwell's equation (1865), or Einstein's 1905 work (See Time-line Fig. 1.1). Ideally multiple resonators, such as the planets, need to be synchronized. If not, they can interact in a random, or even chaotically.

Mersenne: Marin Mersenne (1588–1648) contributed to our understanding of the relationship between the wavelength and the dimensions of musical instruments and is said to be the first to measure the speed of sound. At first Mersenne strongly rejected Galileo's views, partially due to errors in Galileo's reports of his results. But once Mersenne saw the significance of Galileo's conclusion, he became Galileo's strongest advocate, helping to spread the word (Palmerino, 1999).

Mersenne was also a decent mathematician, inventing in 1644 Mersenne primes (MP) ϱ_m of the form

$$\varrho_m = 2^{\varrho_k} - 1,$$

where ϱ_k ($k < m$) denotes the kth prime.

As of December 2018, 51 MPs are known.[11] The first MP is $3 = \varrho_1 = 2^2 - 1$, and the largest known MP prime is $\varrho_{12251} = 2^{\varrho_7} - 1$.[12]

Newton: With the closure of Cambridge University due to the plague of 1665, Isaac Newton (1642–1726) returned home to Woolsthorpe-by-Colsterworth (95 miles north of London) to work by himself for over a year.[13] It was during this solitary time that he did his most creative work.

Exploring our physiological senses requires a scientific understanding of the physical processes of vision and hearing, first considered by Newton, but researched later in much greater detail by Helmholtz (Stillwell, 2010, p. 261). While Newton may be best known for his studies on light and gravity, he was the first to predict the speed of sound. However, his theory was in error by $\sqrt{c_p/c_v} = \sqrt{1.4} \approx 1.1832$.[14] This famous error would not be resolved for 129 years, awaiting the formulation of thermodynamics and the equi-partition theorem by Laplace in 1816 (Britannica, 2004).

In 1676, just 11 years prior to Newton's 1687 *Principia*, there was a basic understanding that sound and light traveled at very different speeds, due to the experiments of Ole Rømer.

> *Ole Rømer first demonstrated in 1676 that light travels at a finite speed (as opposed to instantaneously) by studying the apparent motion of Jupiter's moon Io. In 1865, James Clerk Maxwell proposed that light was an electromagnetic wave, and therefore traveled at the speed c_o appearing in his theory of electromagnetism. (Wikipedia: Speed of Light, 2019)*

The idea behind Rømer's discovery was that due to the large distance between Earth and Io, there was a difference between the period of the moon when Jupiter was closest and furthest from Earth. The moon Io acts like a clock due to its highly regular orbit around Jupiter. This difference in distance and the constant speed of light, caused a fixed delay in the observed eclipse of Io as it went behind Jupiter, delayed by the difference in time due to the speed of light and the difference in distance. This is equivalent to watching a video of a clock's motion. When the video is delayed or slowed down, the time will be earlier.

The amazing Bernoulli family: The first individual who seems to have openly recognized the importance of mathematics, enough to actually teach it, was Jacob Bernoulli. Jacob worked on what is now viewed as the standard package of analytic "circular" (i.e., periodic) functions: $\sin(x)$, $\cos(x)$, $\exp(x\jmath)$, $\log(x)$.

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[11] `https://mathworld.wolfram.com/MersennePrime.html`

[12] $\varrho_7 = 17$ which gives $131,071 = 2^{17} - 1$.

[13] Because the calendar was modified during Newton's lifetime, his birth date is no longer given as Christmas 1642 (Stillwell, 2010, p. 175).

[14] The square root of the ratio of the specific heat capacity at constant pressure c_p to that at constant volume c_v.

Figure 1.4: *Above left: Jacob (1655–1705) and right: Johann (1667–1748) Bernoulli, both painted by their portrait painter brother, Nicolaus. Below left: Leonhard Euler (1707–1783) and right: Jean le Rond d'Alembert (1717–1783). Euler was blind in his right eye, hence the left portrait view.*

From Fig. 1.1 we may conclude that Jacob (1654–1705), the elder brother, would have been strongly influenced by Newton.[15] Newton would have been influenced by Fermat, Descartes, and Galileo, who died one year before Newton was born[16] (White, 1999).[17]

Jacob Bernoulli, like all successful mathematicians of the day, was largely self-taught. Jacob was in a new category of mathematicians because he was an effective teacher. Jacob taught his sibling Johann, who then taught his sibling Daniel. But most important, Johann taught Euler, the most prolific (thus influential) of all mathematicians, including Gauss. This teaching resulted in an explosion of new ideas and understanding. It is most significant that all four mathematicians published their methods and findings. Much later, Jacob studied with students of Descartes (Stillwell, 2010, pp. 268–69).

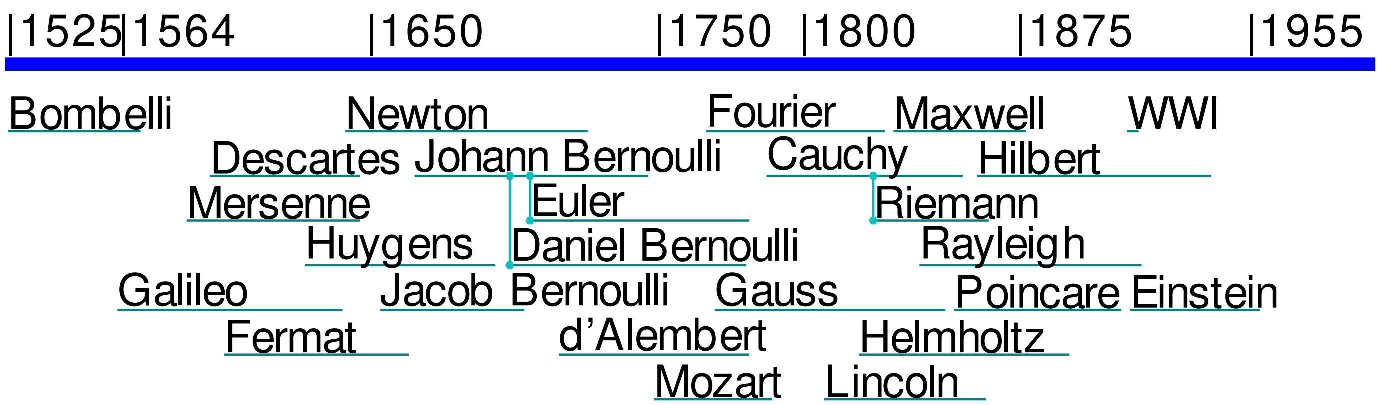

Figure 1.5: *Timeline for the 16th through 20th centuries covering Bombelli to Einstein. It seems likely that Bombelli's discovery of Diophantus's book* Arithmetic *in the Vatican library triggered many of the ideas presented by Galileo, Descartes, and Fermat. Thus Bombelli's discovery might be considered as a magic moment in mathematics. Napoleon hired Fourier, Lagrange, and Laplace to help with bloody military campaigns. The vertical green lines indicate mentor–student relationships. Mozart and Lincoln are indicated along the bottom for orientation.*

Euler: Leonhard Euler's mathematical talent went far beyond that of the Bernoulli family (Burton, 1985). Another special strength of Euler was his large number of publications. First he would master a topic, and then he would publish. Once the tools of mathematics were openly published, largely by Euler, mathematics grew exponentially.[18] His papers continued to appear long after his death (Calinger, 2015). It is also interesting that Euler and Gauss coincided with Mozart, Fourier, Riemann, Maxwell and Lincoln, who had a deep personal interest mathematics. It seems obvious that Einstein would also have been influenced by this cascade of math and science, pouring from from these many individuals, Fourier, Gauss, etc., ending with Hilbert. During this time, Einstein stayed in Berlin but eventually escaped to Prague (1911-1912), where he developed the general theory of relativity (Gordin, 2020).

d'Alembert: Another individual of that time who also published extensively was Jean la Rond d'Alembert (Fig. 1.4). Some of the most innovative ideas were first proposed by d'Alembert. Unfortunately, and perhaps unfairly, his rigor was criticized by Euler, and later by Gauss (Stillwell, 2010).

Gauss: Figure 1.4 shows Time-lines of the most famous mathematicians. This was one of the most creative times in mathematics. Carl Friedrich Gauss was born at the end of Euler's long and productive life. I suspect that Gauss owed a great debt to Euler; surely he must have been a scholar of Euler. One of Gauss's most significant achievements may have been his contribution to solving the open question about the density of prime numbers, and his use of least-squares.[19]

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[15]For a similar Time-line see `https://www.famousscientists.org/joseph-louis-lagrange/`

[16]`https://www-history.mcs.st-andrews.ac.uk/Biographies/Newton.html`

[17]`https://en.wikipedia.org/wiki/Early_life_of_Isaac_Newton`

[18]There are at least three useful exponential scales: factors of 2, factors of $e \approx 2.7$, and factors of 10. The octave and decibel use factors of 2 (6 [dB]) and 10 (20 [dB]). Information theory uses factors of 2 (1 [bit]). Circuit theory uses all three scales.

[19]`https://www-history.mcs.st-andrews.ac.uk/Biographies/Gauss.html`

Cauchy: Augustin-Louis Cauchy (1789–1857) was the son of a well-to-do family, but had the misfortune of being born during the time of of the French Revolution, which likely had its origins in the Seven Years' War, around 1756. To this day the French celebrate Bastille Day (July 14, 1789), which they viewed as a celebration of the revolution. The French Revolution left Cauchy with a lifelong scorn for French politics that deeply influenced his life. Cauchy had an unmatched intellect for mathematics. His most obvious achievement was complex analysis, for which he proved several basic theorems, named in his honor.

Helmholtz: Hermann von Helmholtz (1821–1894) was educated and experienced as a military surgeon. He also mastered classical music, acoustics, physiology, vision, hearing (Helmholtz, 1863b), and, most important of all, mathematics. He was the first person to measure the speed of a neural spike in a short patch of frog neuron, which he correctly found to be 27 [m/s]. Gustav Kirchhoff frequently expanded on Helmholtz's acoustic contributions. It is reported that Lord Rayleigh learned German so he could read Helmholtz's great works, which he amplified in his famous treatise on acoustics (Rayleigh, 1896).

Helmholtz's studies and theories of music and perception of sound are fundamental scientific contributions (Helmholtz, 1863a). His best-known mathematical contribution is "The fundamental theorem of vector calculus," or simply "Helmholtz theorem".

Lord Kelvin: Lord Kelvin (William Thompson, 1824–1907) was one of the first true engineer-scientists, equally acknowledged as a mathematical physicist, well known for his interdisciplinary research, and knighted by Queen Victoria in 1866.[20] Lord Kelvin coined the term *thermodynamics*, a science more fully developed by Maxwell (the same Maxwell of electrodynamics).[21]

The history during this time is complex. For example, in 1850 Lord Kelvin wrote a letter to George Stokes, suggesting that Stokes try to prove what is today known as "Stokes's theorem." As a result, Stokes posted a reward (Smith's Prize), searching for a proof of "Lord Kelvin's theorem," which was finally achieved by Hermann Hankel (1839–1873).[22] Many new concepts were being proved and appreciated over this productive period. Maxwell had published his famous equations using modern vector notation developed by Oliver Heaviside, J. Willard Gibbs, and Heinrich Hertz. Fig. 1.1 should put to rest the myth that one's best work is done in the early years. Many of these scientists were fully productive into old age. Those who were not, died early, due to poor health or accidents.

James Clerk Maxwell (1831–1879) In 1869 a Cambridge senate committee was formed to create the Cambridge Physics Laboratory and "the founding of a special Professorship." The Chancellor of Cambridge was the seventh Duke of Devonshire and a distant relative of Henry Cavendish, his family name. Thus the new laboratory became known as "The Cavendish."

> There was, naturally, much speculation about the choice of the new Professor of Experimental Physics. [Lord] Kelvin was the most likely candidate, but on being approached in private, refused in order to stay in Glasgow. Another likely candidate was Lord Rayleigh, a brilliant mathematician and physicist who had left Cambridge to work in his private laboratory at his country seat in Essex. When the appointment was eventually announced, the reaction was, if anything, one of disappointment. The new Professor, James Clerk Maxwell, was relatively unknown.[23]
> He was a much respected mathematician, but he had not since made any great name for himself — his major and astounding books on Electricity and Kinetic Theory had yet to be published. Moreover, the six years before his appointment had been spent in isolation at his Scottish home. His appointment was announced on March 8th 1871, and in spite of the initial disappointment, his inaugural lecture was looked forward to by his likely students as much as by the rest of the Cambridge scientists.
>
> When, a few days later, Maxwell began his first course with a lecture on Heat, his students had the delight of seeing the lecture room packed with their tutors, lecturers, professors and all the important personages of the University. Thinking that this was his first public appearance they sat, in their formal academic dress, while Maxwell, "with a perceptible twinkle in his eye," gravely expounded the difference between Fahrenheit and Centigrade, and the principle of the air thermometer.

[20]Lord Kelvin was one of a half dozen interdisciplinary mathematical physicists, all working about the same time, who made a fundamental change in our scientific understanding. Others include Helmholtz, Stokes, Green, Heaviside, Rayleigh, and Maxwell.

[21]Thermodynamics is another topic that warrants an analysis along historical lines (Kuhn, 1978).

[22]`https://en.wikipedia.org/wiki/Hermann\_Hankel`

[23]`https://www.youtube.com/watch?v=aFYKKSoXC5Y`

It was felt afterwards that Maxwell had done it on purpose, perhaps out of modesty, perhaps out of his later well-known sense of humor, or perhaps because he knew of the still considerable opposition his new laboratory had to face. As he had written to his friend Lord Rayleigh, "if we succeed too well, and corrupt the minds of youth till they observe vibrations and deflections and become Senior Ops. instead of Wranglers, we may bring the whole University and all the parents about our ears.
However, Maxwell made only a casual announcement of his inaugural lecture which was not to be in the Senate House, as expected, but in an out-of-the-way lecture room. Consequently only his students got to hear of it and it was to them, rather than a general gathering, that he delivered an exciting and interesting lecture, mapping out his plans for the future of Cambridge physics. ...."
[24]

Lord Rayleigh (William Strutt): Lord Rayleigh (1842–1919) wrote a classic text (1896) that is widely read even today by those who study acoustics. In 1904 he received the Nobel Prize in Physics for his investigations of the densities of the most important gases and for his discovery of argon in connection with these studies.

Poincare Henri Poincare (1854-1912) was a polymath from a young age, with many creative observations to his credit, including formulating the *Lorentz transformation* $\sqrt{1-(v/c_o)^2}$, which is critical for characterizing symmetries of Maxwell's equations, and Einstein's formulation of $E = mc_0^2$ (using the Lorentzian), from this venerable interdisciplinary view of science by splitting the disciplines into into smaller parts whenever we perceived a natural educational boundary. Reforging these natural connections into the curriculum is essential for the proper training of STEM students.[25] [26]

1.2.3 Three Streams from the Pythagorean theorem

From the outset of his presentation, Stillwell (2010, p. 1) defines "three great *streams* of mathematical thought: *Numbers, Geometry and Infinity*" that flow from the Pythagorean theorem, as summarized in Fig. 1.6. This is a useful concept, based on reasoning not as obvious as one might think. Many factors are in play here. One of these is the strongly held opinion of Pythagoras that all mathematics should be based on integers. The rest are tied up in the long, necessarily complex history of mathematics, as best summarized by the fundamental theorems Fig.1.6 (Page 16), each of which is discussed in detail in a later chapter.

1. **Numbers**
   - $\mathbb{N}$ *counting numbers,* $\mathbb{Q}$ *rationals,* $\mathbb{P}$ *primes (6th century BCE)*
   - $\mathbb{Z}$ *common integers,* $\mathbb{I}$ *irrationals (5th century BCE)*
   - *zero* $\in \mathbb{Z}$ *(7th century CE)*
2. **Geometry** *(e.g., lines, circles, spheres, toroids, other shapes)*
   - *Composition of polynomials (Descartes, Fermat),*
   - *Euclid's geometry and algebra* $\Rightarrow$ *analytic geometry (17th century CE)*
   - *Fundamental theorem of algebra (18th century CE)*
3. **Infinity** *($\infty \rightarrow$ sets)*
   - *Taylor series, functions, calculus (Newton, Leibniz) (17th and 18th century CE)*
   - $\mathbb{R}$ *real,* $\mathbb{C}$ *complex (19th century CE)*
   - *Set theory (20th century CE)*

Figure 1.6: *Three streams that follow from the Pythagorean theorem: numbers, geometry, and infinity.*

[24] `https://www.phy.cam.ac.uk/history/years/firstten`, Moralee (1995)
[25] Perhaps it is time to put the STEM Humpty Dumpty back together.
[26] It seems we have retreated Why? Because subtraction that created numbers outside the defined set, would become impossible.

As shown in Fig. 1.6, Stairwell's concept of three streams, following from the Pythagorean theorem, is the organizing principle behind this book.

Ch. 1: The *Introduction* is a historical survey of pre-college mathematical physics, presented in terms of the three main Pythagorean streams (stream 1–stream 3), leading to the book's five chapters. Stream 3 is split into 3A and 3B.

Ch. 2: *Number systems* presents some important ideas from number theory, starting with prime numbers, complex numbers, vectors, and matrices. Five classic number theory problems are discussed: the Euclidean algorithm (GCD), continued fractions (CFA), Euclid's formula, Pell's equation, and the Fibonacci difference equation. The general solution of these problems leads to the concept of the eigen-function analysis, which is introduced in §1.2.3.

Ch. 3: *Complex Algebraic equations* discusses complex Algebra and its development, as we know it today. The chapter presents the theory of algebraic equations as functions of real and complex variables. Newton's method for finding complex roots of polynomials, poles vs. zeros, and the Gauss-Lucas theorem (bounds on the root locations of the derivative of a polynomial) Allen (2025). Complex impedance $Z(s)$ of complex frequency $s = \sigma + \omega \jmath$ is covered with some care, developing the topic that is needed for engineering mathematics.

While the algebra of real and complex functions is identical, the calculus is fundamentally different. This leads to the concepts of complex–analytic functions, complex Taylor series, the Cauchy-Riemann conditions, branch points, branch cuts, and Riemann sheets. All of these ideas are fundamental to impedance functions that describe the linear relationships between force and flow in the complex frequency domain (i.e., impedance $\in \mathbb{C}$).

Ch. 4: *Scalar calculus* (Stream 3A) covers ordinary differential equations and integral theorems of simple physical systems (mass-springs, inductors-capacitors, heat dynamics), solutions to scalar differential equations that have constant coefficients, colorized mappings of complex–analytic functions, multi-valued functions, Cauchy's theorems, and inverse Laplace transforms.

Ch. 5: *Vector calculus* (Stream 3B) introduces vector partial differential equations, as well as gradient, divergence, and curl differential operators, Stokes's and Green's theorems, and Maxwell's equations.

Chapter 2

Stream 1: Number Systems

Number theory (the study of numbers) was a starting point for many key ideas. In Euclid's early geometrical constructions, the Pythagorean theorem for real numbers $[a, b, c]$ was accepted as true. As we shall see, the derivation of the formula for Pythagorean triplets is the first of a rich body of mathematical constructions, including the solution of Pell's equation [1] [2] (p. 19), and recursive difference equations, such as solutions of the Fibonacci recursion formula $f_{n+1} = f_n + f_{n-1}$, that go back at least to the Chinese (2000 BCE). These are early pre-limit forms of calculus, best analyzed using an eigen-function (e.g., eigenmatrix) expansion, which was a geometrical concept from linear algebra, as an orthogonal set of normalized unit-length vectors (see Appendix A.3).

It unimaginable that anyone who uses an abacus would fail to appreciate the concept of zero and negative numbers. It does not take much imagination to go from counting numbers $\mathbb{N}$ to the set of all integers $\mathbb{Z}$, including zero. On an abacus, subtraction is obviously the inverse of addition. Subtraction to obtain zero abacus beads is no different from subtraction from zero, which requires *negative* beads. To assume that the Romans, who first developed counting sticks, or the Chinese, who then deployed the concept using beads, did not understand negative numbers, is unimaginable. However, understanding the concept of zero (and negative numbers) is not the same as having a symbolic notation. The Roman number system has no such symbols. The first recorded use of a symbol for zero is said to be by Brahmagupta[3] in 628 CE.[4] [5] However, this is likely wrong, given the notation developed by the Mayan civilization, which existed from 2000 BCE to 900 CE.[6] There is speculation that the Mayans cut down so much of the Amazon jungle that it eventually resulted in global warming, possibly leading to their demise.

The definition of zero depends on the concept of subtraction, which formally requires the creation of algebra (ca. 830 CE; see Fig. 1.1). But apparently it took more than 600 years from the time Roman numerals were put into use, without any symbol for zero, to the time the symbol for zero is first documented. Likely this delay was more about the political situation, such as government rulings, than mathematics.

The concept that caused much more difficulty was the "sleeping 8" (∞) known as infinity, first proposed by Bernhard Riemann in 1851 with the development of the extended plane, which mapped the plane to a sphere (see Fig. 3.14). His construction made it clear that the point at ∞ is simply another point on the open complex plane, since rotating the sphere (extended plane) moves the point at ∞ to a finite point on the plane, thereby closing the complex plane.

2.1 The taxonomy of numbers: $\mathbb{P}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{I}, \mathbb{R}, \mathbb{C}$

2.1.1 What is a number?

There are many realizations of math that do not use numbers. A good example is the mathematics of *logic*, which deals with *boolean logic*, the algebra of true/faults T,F, AND/OR, IF THEN, EXCEPT and NOT.

Once symbols for zero and negative numbers were accepted, progress could be made. This led to the theory of numbers, today known as *Number theory*. It is interesting that complex numbers such as $1 + \sqrt{-1}$, and irrational numbers, such as π and $e^{\sqrt{-1}}$, were soon discovered.

[1] Heisenberg, an inventor of the matrix algebra form of quantum mechanics, learned mathematics by studying Pell's equation.

[2] `https://www.aip.org/history-programs/niels-bohr-library/oral-histories/4661-1`

[3] `https://https://en.wikipedia.org/wiki/Brahmagupta`

[4] `https://news.nationalgeographic.com/2017/09/origin-zero-bakhshali-manuscript-video-spd`

[5] `https://www.nytimes.com/2017/10/07/opinion/sunday/who-invented-zero.html`

[6] `https://www.storyofmathematics.com/mayan.html`

To fully understand numbers, a transparent notation is essential. First one must identify the different classes (genus) of numbers, providing a notation that clearly defines each of these classes, along with their relationships. It is logical to start with the most basic counting numbers, which we indicate with the double-bold symbol $\mathbb{N}$.

An transparent notation is also required for the many generalizations of math operations, such as the many non-numeric operations $\{\cdot\}, \cup, \cap, \in, \not\subset, \perp$. For easy access, mathematical notation is summarized in Appendix 2.1.1. Most important is the definition of a *number*, for example $y = AB^C$, where A, B, C are typically signed integers. Equally important are a subset of very special functions, typically taught before high school, such as $\sin, \cos, \tan$ and $\log$. The $\log$ function is critically important, since by definition $\log(y) = \log(A) + C \log B$. Here B is called the *base*, or less frequently, the *radix*. The most common radix use by humans is base 10. Computers use base 2, known as *binary*.

Once calculus is taught, all of these elementary cases are greatly expanded, and complex number and function become common-place.

Counting numbers $\mathbb{N}$: These are known as the *natural numbers* $\mathbb{N} = \{1, 2, 3, \dots\}$ and denoted by the double-bold symbol $\mathbb{N}$. For clarity we shall refer to the natural numbers as *counting numbers*, since *natural*, which means *integer*, is vague. Note that 0 is not in $\mathbb{N}$.

The mathematical sentence "$2 \in \mathbb{N}$" is read as "2 is a member of the set of counting numbers." The word *set* is defined as the collection of any objects that share a specific property. Typically a set may be defined either by a sentence or by example.

Primes $\mathbb{P}$: A number is prime ($\pi_n \in \mathbb{P}$) if its only factors are 1 and itself. An interesting exception are complex conjugate factors, such as

$$\pi_3 = 5 = \left(2 + \sqrt{-1}\right)\left(2 - \sqrt{-1}\right), \tag{2.1}$$

which voids the traditional definition of the reciprocal of a prime as a rational number, since

$$\frac{1}{5} = \frac{1}{\left(2 + \sqrt{-1}\right)\left(2 - \sqrt{-1}\right)} = \frac{\alpha}{2 - \jmath} + \frac{\beta}{2 + \jmath} = \frac{\alpha(2 + \jmath) + \beta(2 - \jmath)}{5} = 2\frac{\alpha + \beta}{5}. \tag{2.2}$$

Thus $\alpha = \beta$ and $\alpha + \beta = 1/2$.

The primes $\mathbb{P}$ are a subset of the counting numbers ($\mathbb{P} \subset \mathbb{N}$).[7] The *logarithmic integral function* $li(x)$ is the best approximation to the limit of $x/ln(x)$. Example: 2, 3, 5 are $\in \mathbb{P}$ but $4 = 2^2 \notin \mathbb{P}$. *Based on the video of footnote 4, the gaps between primes are around 8% of the numbers for* $N = 1e4$ *numbers.* An important properties of primes is that their reciprocals are always rational. [8] [9]

A fun but perhaps misleading plot of the primes is discussed at: https://www.3blue1brown.com/lessons/prime-spirals.

One may generate the first 2 million primes with the Octave/Matlab command $P = primes(N)$, and verify their primality $factor(P(n))$. Here N is the number of primes you wish to tabulate. $N = 10^9$ takes 4.5 ± 0.2 [s] on my IBM laptop. Many textbooks use $p(n)$ rather than π_n for the primes.

A was well known to the earliest mathematicians, that every integer may be written as a unique product of primes. A second key fact is the density of primes $\rho_\pi(N) \sim 1/\log(N)$; that is, $\rho_\pi(N)$ is inversely proportional to the log of N, an observation first quantified by Gauss (Goldstein, 1973). A third is that there is a prime between every integer $N \geq 2$ and $2N$. In Fig. 2.2 we demonstrate that this is a massive under estimate of the density of primes per-Octave.[10]

A fourth is that $1/\pi_k$ is always cyclic.[11] For example $1/7 = 0.142857, 142857, 142857, \cdots$. Furthermore, maximum period of the cycle is $\pi_k - 1$. A fifth strange but interesting observation, is that $142 + 857$=999.

In the exercises below we denote π_n as the prime numbers indexed by $n \in \mathbb{N}$. The first 12 primes are $\{n|1 \leq n \leq 12\} = \{\pi_n|2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$. Primes less than 100 are relatively easy to identify. Primes always end in the odd numbers 1, 3, 7 and 9. All that end in 2 are even, thus cannot be prime. There are other oddities, as we shall see in Fig. 2.3 (p. 34) .

[7] Google "The prime number theorem": `https://www.youtube.com/watch?v=qoJacpk_OXo`

[8] A rational number may be expressed as the ratio of two integers Thus $\mathbb{R} = \mathbb{I}\mathbb{Q}$.

[9] `https://en.wikipedia.org/wiki/Palindromic_prime`

[10] `https://en.wikipedia.org/wiki/Reciprocals_of_primes`

[11] `https://en.wikipedia.org/wiki/Reciprocals_of_primes#`

Exercise #*1*

Write the first 20 integers in prime-factored form.

Exercise #*2*

Write the integers 2 to 20 in terms of π_n. Here is a table to assist you:

n	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
π_n	2	3	5	7	11	13	17	19	23	29	31	$\cdots$

Coprimes are two numbers that have no common factors. For example, $21 = 3 \cdot 7$ and $10 = 2 \cdot 5$ are coprime, whereas $4 = 2 \cdot 2$ and $6 = 2 \cdot 3$, which have 2 as a common factor, are not coprime. By definition all unique pairs of primes are coprime. We shall use the notation $m \perp n$ to indicate that m and n are coprime. The ratio of two coprimes is *reduced*, meaning it contains no common primes.

The ratio of two numbers that are not coprime may always be reduced by canceling the common factors. *Gaussian primes* are of the form $\pi_k + \sqrt{-1}\pi_l$ for $k, l \in \mathbb{N}$, with $k \neq l$.

When doing numerical work, it is always beneficial to work with coprimes, to minimize the effort. Generalizing this idea, we define *triprimes* as three prime numbers with no common prime factors, such as $\{\pi_3, \pi_9\pi_5, \pi_2\pi_7\}$, or numerically $\{5, 143, 51\}$ = $\{5, 13 \cdot 11, 3 \cdot 17\}$.

The *fundamental theorem of arithmetic* states that every integer may be uniquely expressed as a unique product of primes. The *prime number theorem* states the mean density of primes $\rho_\pi(n)$ as a function of $n \in \mathbb{N}$. One of the more interesting properties of primes are that their reciprocals are rational ($\in \mathbb{Q}$), as in the case of 7, which has a cycle period of 6.

I speculate that primes might also play a role in natural language processing (NLP). Every word, and thus word-sequences, could be mapped onto primes numbers, using frequency of the combinations to assign the mapping. This then converts natural language (text) into a unique efficient sequence of primes and their products. Given this unique representation (language coded as primes), it seems possible, but challenging, that one could extract meaning from the textual input. This concept seems related to Claude Shannon's definition of information which he defined as *information* (Allen, 1994).

Integers $\mathbb{Z}$: The integers include positive and negative counting numbers and zero. Notionally we indicate as $\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$, which is read as: "The integers are composed of the natural numbers $-\mathbb{N}$, zero, and $\mathbb{N}$."

Real numbers $\mathbb{R}$: Reals are the union of rational and irrational numbers—namely, $\mathbb{R} = \mathbb{Q} \cup \mathbb{I} = \mathbb{Z} \cup \mathbb{F} \cup \mathbb{I}$. Lengths in Euclidean geometry are reals. Many people assume that IEEE-754 floating-point numbers (ca. 1985) are real (i.e., $\in \mathbb{R}$) In fact, they are rational ($\mathbb{Q} = \{\mathbb{F} \cup \mathbb{Z}\}$) approximations to real numbers, designed to have a very large dynamic range. The hallmark of fractional numbers ($\mathbb{F}$) is their power in making highly accurate approximations of any real number. Due to IEEE-754, it is not possible to define an irrational number on a computer. The famous example is the popular representation of $\pi \approx 22/7$. The use of IEEE-754 gives the best possible approximation for π, given the hardware limitations, close to 4 orders of magnitude better than 22/7.

Using Euclid's compass and ruler methods, one can make the length of a line proportionally shorter or longer or (approximately) the same. A line may be made be twice as long, or an angle can be bisected. However, the concept of an integer length in Euclid's geometry was not defined.[12]

Real numbers were first fully accepted only after set theory was developed by Georg Cantor in 1874 (Stillwell, 2010, p. 461). This seems amazing, given how widely accepted real numbers are today.

Rational numbers $\mathbb{Q}$: Given two real numbers $x, y \neq 0 \in \mathbb{R}$, we define $\pm x/y \in \mathbb{Q}$. The rational number 3.0/1.0 $\in \mathbb{R}$. The word rational is the concatenation "ratio + nal". The utility of rational numbers is they can efficiently approximate any real irrational number to any desired finite precision, the classic example being $\pi \approx 22/7$. Specific examples include:

```
octave:> round(pi*2^8)/(2.0001^8)
ans = 3.140336299224576

octave:> round(pi*2^15)/(2.0001^15)
ans = 3.141601562500000
```

[12] As best I know.

The above demonstrates the effect of rounding after multiplying by 2^8 and 2^{15}, following a rational approximation of π on my ThinkPad X1 laptop computer.

The reciprocal of every prime is a repeating fraction. Examples include $1/3 = 0.3333\cdots$ and $1/5 = 0.20000\cdots$. If we cross-multiply ($3/3 = 5/5 = 5 \cdot 0.200 = 1$) the result is always 1. If we assuming that the classification is independent of the base, then repeating 0 are rational. Thus we assume that repeating decimals, either 3 or 0, should be taken as repeating. For most scientific computing this is more than adequate.

Irrational numbers $\mathbb{I}$: Irrational numbers include the reciprocals of primes $1/\pi_k$, e, and the square roots of primes. If an irrational number is truncated after 100 digits, the result must be rational.

The following question seems interesting: *Does the sum of reciprocals of primes converge? How about the sum of the reciprocals of irrational numbers?* This relationship seems analogous to that of the integers $\mathbb{Z}$ and fractionals $\mathbb{F}$, which split the rationals ($\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$, $\mathbb{Z} \perp \mathbb{F}$).

Irrational numbers $\mathbb{I}$ were famously problematic for the Pythagoreans, who incorrectly theorized that all numbers were rational. Like ∞, irrational numbers required mastering a new and difficult concept before they could even be defined: It was essential to understand the factorization of counting numbers into primes (i.e., the *fundamental theorem of arithmetic*) before the concept of irrationals could be sorted out. Clearly irrational numbers cannot be factored. Recall that $\pi \approx 22/7 = 3 + 1/7$, cannot be factored.

As shown in Fig. 2.7, fractionals can approximate any irrational number with arbitrary accuracy. Integers are also important, but for a very different reason. All numerical computing today is done with $\mathbb{Q} = \mathbb{F} \cup \mathbb{Z}$. Indexing uses integers $\mathbb{Z}$, while the rest of computing (flow dynamics, differential equations, etc.) is done with fractionals $\mathbb{F}$ (i.e., IEEE-754). Computer scientists are trained on these topics, and computer engineers are expected to be at least conversant with them.

A subset of irrational numbers (e.g., $n \in \mathbb{I}$) cannot be classified, since to do so would require a computer having an ∞ of bits, which is physically impossible. This argument is supported by Fig. 2.2.

It is well known that certain numbers, such as π and e, are irrational. For example, the periods of reciprocals of prime numbers are rational. Yet is it difficult to *prove* that a number is irrational, since the required property is that it does not have a repeating decimal. This is impossible to prove since the repeat period can be larger than total the memory of the computer.

It is only possible to identify a rational number by expanding it as a continued fraction, and then showing that the sequence repeats. Every rational number is not irrational ($\mathbb{R} \in \mathbb{I} \perp \mathbb{Q}$). Only if the number is rational, can it be proved to not be irrational.

Complex numbers $\mathbb{C}$: Today the accepted way to write a complex number is using the notation $z = a + b\jmath \in \mathbb{C}$, where $a, b \in \mathbb{R}$. Here $1\jmath = \sqrt{-1}$. We also define $1\imath = -1\jmath$ to account for the two possible signs of the square root. Accordingly $1\jmath^2 = 1\imath^2 = -1$.

The ratio of two integers is always rational. The word *complex*, as used here, does *not* mean that the numbers are complicated or difficult. In the following we may use the symbol $\jmath$ as an alias for $\sqrt{-1}$ and $\imath$ for $-\sqrt{-1}$. If $b \in \mathbb{R}$, then $b\jmath$ is known as the *imaginary* part a complex number. This does not mean the number disappears.

Complex numbers are heavily used in engineering mathematics, for example, as roots of polynomials. The most obvious example is the quadratic formula for the roots of polynomials of degree 2. All real numbers have a natural order on the real line. Complex numbers do not have a natural order. For example, $\jmath > 1$ has no meaning.

Cartesian multiplication of complex numbers follows the basic rules of real algebra, for example, using the rules for multiplying a monimal (a monomial is a polynomial of degree one) times a polynomial. Multiplication of two gives

$$(a + bx)(c + dx) = ac + (ad + bc)x + bdx^2.$$

If we substitute $1\jmath$ for x and use the definition $1\jmath^2 = -1$, we obtain the Cartesian product of two complex numbers:

$$(a + b\jmath)(c + d\jmath) = ac - bd + (ad + bc)\jmath.$$

Thus multiplication and division of complex numbers obey the common rules of algebra.

However, there is a critical extension: Cartesian multiplication holds only when the angles sum to less than $\pm\pi$–namely, the range of the complex plane ($\pm 180°$). When the angles add to more than $\pm\pi$, one must use polar coordinates (Boas, 1987, p. 8). This is particularly striking for the Laplace transform of a delay.

Complex numbers can be challenging and may provide unexpected results. For example, it is not obvious that $\sqrt{3 + 4\jmath} = \pm(2 + \jmath)$.

Fractional numbers $\mathbb{F}$: Fractional numbers play a special role in modern computing since they cannot be stored on a computer. Every fractional number $\mathbb{F}$ is defined as the ratio of signed coprimes. If $n, d \neq 0 \in \pm\mathbb{P}$, then $n/d \in \mathbb{F}$. It is well know that the reciprocal of primes are rational ($1/\mathbb{P} \in \mathbb{R}$). Given this definition, $\mathbb{F} \subset \mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$. For example, $1/3 = 0.3333\cdots$, denoted $0.((3))_1$. Likewise, 1/7 is $0.((142857))_6$.

Because of the powerful approximating power of rational numbers, fractional integers have special utility, since they cannot be reduced. This allows for a reduced memory footprint of irrational numbers on a computer. For example, $\pi \approx 3+\frac{1}{7} = \frac{22}{7} \in \mathbb{F}$ with a relative error of 0.04% (same for $1/\pi$). Also $e \approx 19/7$ with an error of 0.4%, and $\sqrt{2} \approx 7/5$, accurate to 1.4%. For greater precision, when using Octave/Matlab, one may use either $rat()$ or $rats()$ along with Octave's FORMATLONG command. For example, try $rat(\pi, 1e-9)$ or $rats(pi, 5)$. A sort of amazing option is $rats(pi, 15) = 1146408/364913 = 2^3 \cdot 3 \cdot 37 \cdot 1291/364913$.

However rats(pi,20) = 817696623/260280919 has zero error when compared with π. How can this be explained? It cannot be correct since π is irrational while rats(pi,20) is rational. This example demonstrates the "magic" of the IEEE-754 standard.

The reciprocals of prime number π_k have a period of a factor of $\pi_k - 1$. For example, let the prime be 37, the period is either 36, 6, 3 or 2. Since every prime number is odd, $\pi_k - 1$ must be even, thus has a factor 2 since all even numbers may be divided by 2.

The most frequent alternative seems to have a period of $\pi_k - 1$, which by definition is even, since all primes other than 2 are odd.

Another example is $27 = 3^3$, which has a period of 3. While $10/(27-1)$ has a period $384615))_6$ of 6, $\frac{100}{370} = 1/270))_3$. This last case seems consistent since $37^2 = 370/0.270))_3$. Namely consistent with $10/27 = 0.370, 370, 370, 370, 370\ldots$.

It seems obvious that assuming a long-hand recursive calculation, it would be possible to demonstrate a repeating sequence, which would determine the period of any repeating sequence. The flaw with this approach is when the calculation is terminated before the repeat is found. One would need an upper bound of the period, to know when to terminate the calculation. When the period is large, verification becomes impossible. Factoring such large numbers may be the only option for proving they are primes.

By definition, every real number $\mathbb{R}$ is either rational $\mathbb{Q}$ or irrational ($\mathbb{I}$). These are decimal numbers, requiring infinite precision in their representation. The rationals $\mathbb{Q}$ and irrationals $\mathbb{I}$ split the reals ($\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, $\mathbb{Q} \perp \mathbb{I}$). It follows that $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{I} \subset \mathbb{R}$. Such numbers cannot be represented on a computer, as they would require an infinite number of bits (precision). All prime numbers are rational. However their reciprocals are repeating decimals, thus are irrational. This violates some sort of natural intuition. One must conclude that fractionals, the ratio of integers, are rational, but if the denominator is prime, it is irrational. Logically this seems wrong. Yet $1/3 = 0.33333\cdots$, repeats forever and 3/3=1. If we truncate $0.3333333000\cdots$, its product with 3 is not 1. One cannot define themselves out of this conclusion. The reciprocal of every prime is irrational!

I fail see any way out of the conclusion that "The reciprocals of every integer that is not periodic, is irrational." While this is not satisfying, it seems that logically there is no alternative. The same logic applies to time, there can be no time that is equal to zero. I see this as a strange conclusion. Clearly there is a past and a future. But there is no 'now.' I was not prepared for this conclusion.

Exercise #*3*

Verify that $\sqrt{3+4\jmath} = \pm(2+\jmath)$. **Solution:** Squaring the left side gives $\sqrt{3+4\jmath}^2 = 3+4\jmath$. Squaring the right side gives $(2+\jmath)^2 = 4+\jmath^2+4\jmath = 3+4\jmath$. Thus the two are equal.

Exercise #*4*

What is special about the above example?

Solution: Note this is a $\{3, 4, 5\}$ triangle. Can you find another example like this one? Namely, how does one find integers that obey Eq. 1.1

An alternative to Cartesian multiplication of complex numbers is to work in polar coordinates. The polar form of the complex number $z = a + b\jmath$ is written in terms of its magnitude $\rho = \sqrt{a^2+b^2}$ and angle $\theta = \angle z = \tan^{-1} z = \arctan z$ as

$$z = \rho e^{\theta\jmath} = \rho(\cos\theta + \jmath\sin\theta).$$

From the definition of the complex natural log function, we have

$$\ln z = \ln \rho e^{\theta\jmath} = \ln \rho + \theta\jmath,$$

which is important, even critical, in engineering calculations. When the angles of two complex numbers are greater that $\pm\pi$, one must use polar coordinates. It follows that for computing the phase, the log function is different from the single- and double-argument $\angle\theta = \arctan(z)$ function.

The polar representation makes clear the utility of a complex number: Its magnitude scales while its angle θ rotates. The property of scaling and rotating is what makes complex numbers useful in engineering calculations. This is especially obvious when dealing with impedance's, which have complex roots with very special properties.

Matrix representation: The alternative way to represent complex numbers is in terms of 2×2 matrices.

$$a + b\jmath \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad 1 \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad 1\jmath \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e^{\theta\jmath} \leftrightarrow \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{2.3}$$

The *conjugate* of $a + b\jmath$ is then defined as $a - b\jmath \leftrightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. By taking the inverse of the 2×2 matrix (assuming $|a + b\jmath| \neq 0$), we can define the ratio of one complex number by another. It may not seem obvious, or even possible, until you try it.

This representation proves that $1\jmath$ is not necessary when defining a complex number. What $1\jmath$ can do is to conceptually simplify the algebra. It is worth your time to become familiar with the matrix representation, to clarify any possible confusions you might have about multiplication and division of complex numbers. This matrix representation can save you time, heartache, and messy algebra. Once you have learned how to multiply two matrices, it's a lot simpler than doing the complex algebra. In many cases we will leave the results of the analysis in matrix form, to avoid the algebra altogether.[13] Thus both representations are important.

Exercise #5

Using Matlab/Octave, verify that

$$\frac{a + b\jmath}{c + d\jmath} = \frac{ac + bd + (bc - ad)\jmath}{c^2 + d^2} \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \frac{1}{c^2 + d^2}. \tag{2.4}$$

Solution: The typical solution may use numerical examples. A more precise way to show this is to use the Matlab/Octave symbolic code:

```
syms a b c d A B
A=[a -b;b a];
B=[c -d;d c];
C=A*inv(B)
```

History of complex numbers: It is amazing that complex numbers were not accepted until 1851 even though Bombelli introduced them in the 16th century. One might have thought that the solution of the quadratic polynomial, known to the Chinese, would have settled this question. It seems that complex integers (Gaussian integers) were accepted before complex numbers. Perhaps this was because real numbers were not accepted (i.e., proved to exist, thus mathematically defined), until the development of real analysis in the late 19th century, thus providing a proper definition of real numbers.

2.1.2 Numerical taxonomy

The taxonomy structure may be summarized with a valuable single compact sentence, starting with the prime numbers π_k and ending with the closed set of complex numbers $\mathbb{C}$.

Summarizing

1. The set of prime numbers $\{\pi_k\}$ is the set: $\{2, 3, 5, 7, 11, \cdots, \infty\} \in \mathbb{P}$
   defined as the numbers that have no factors other than 1 and π_k

2. where $\mathbb{P}$ is a subset of the set of counting numbers $\mathbb{N}$: $\{1, 2, 3, 4, 5, \cdots\}$

[13] Sometimes we let the computer do the final algebra, numerically, as 2×2 matrix multiplications.

3. which is a subset of the set of signed integers $\mathbb{Z} \in -\mathbb{N}, \mathbb{N}$: $\{\cdots, -1, 0, 1, \cdots\}$
4. which is a subset fractionals $\mathbb{F}$: $1/\mathbb{Z} \cup \mathbb{Z}\ \{-7, -\frac{1}{3}, 0, 0.25, \frac{22}{7}, 5, \cdots\}$,
5. which is a subset of the set of reals $\mathbb{R} \in \mathbb{Q} \cup \mathbb{I}$: $\{-\sqrt{2}, 0, e^\pi, \}$
6. which is a subset of the set of real irrationals $\mathbb{I} \in \mathbb{Z} \cup \mathbb{F} : \{-\sqrt{2}, 0, \frac{1}{\pi}, e^\pi, \} \in \mathbb{R}$,
7. which is a subset of the set of complex numbers $\mathbb{C}$: $\{\frac{-1}{\sqrt{2\pi}} + \jmath, 10^{-6\jmath} + \pi\jmath, \cdots\} \in \mathbb{C}$.

It seems obvious, that by definition, every rational $\mathbb{Q}$ is a unique mapping from each member $\mathbb{P}$, thus repeating decimals are a basic property of rationals. For the reciprocal prime, $1/3 = 0.3333\cdots$), the period is 1. The repeat factor for $\pi_4 = 7$ is six (Example: $1/7 = 0.142857, 142857, \cdots))_6$. The rationals $\mathbb{Q}$ may be further decomposed into the fractionals $\mathbb{F}$ and the integers $\mathbb{Z}$ ($\mathbb{Q} = \mathbb{F} \cup \mathbb{Z}$) $(1 + 2/3 = 1.66666\cdots)$. The reals $\mathbb{R}$ map may be rationals (1/7) $\mathbb{Q}$ or irrational π, thus $\mathbb{I}$ ($\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$) The final classification contains complex numbers. The numbers in the braces are a few typical examples.

As discussed in Appendix 2.1.1, all numbers may be viewed as complex; that is, every real number is complex if we let the imaginary part be zero (Boas, 1987). For example, $2 \in \mathbb{P} \subset \mathbb{C}$. Likewise, every purely imaginary number (e.g., $0 + 1\jmath$) is complex with zero real part. From this point of view, the classifications seem arbitrary, but still a useful way of sorting the different classes of numbers.

Note that complex numbers $\mathbb{C}$ do not have order property, which means that one complex number cannot be larger or smaller than another. It makes no sense to say that $\jmath > 1$. The real and imaginary parts, and the magnitude and phase, have order. Order is restricted to $\mathbb{R}$.

If time t were complex, there could be no yesterday and tomorrow.[14]

2.1.3 Definition of a number

This URL is helpful: *https://en.wikipedia.org/wiki/Square_root*

- Rule 1: The product of two numbers is always a number.
- Rule 2: if $y = x^2$ or $y = 1/x$ and x is a number, then y is a number unless $x = 0$. 0 is a number.

  There are other rules of algebra (math) that are important:
- Rule 3: if $x = 0$ then 1/x is not a number. rather it is ∞.
- Rule 4: Another important exception: if $x = \sqrt{y}$ then x is a number even when $y < 0$. or $y = 0$ or $y = i = \sqrt{-1}$. Example: if $y = -1$ then x is called i, which is considered in math to be a number. This is a fact easily proved (it is not a conjecture).

  Newton call $\imath$ an *imaginary number* because he believe it didn't exist. It turned out he was wrong, but it took over 100 years to clear up the uncertainty.
- Rule 5: The sum or difference or product including $1/x * x = 1$ of two numbers is a number. The only exception is $1/x$ when $x = 0$.

  Example: Another VERY interesting example is: $(2 + i)(2 - i) = 4 + 1 + 2i - 2i = 5$.

  Since 5 may be factored as the product of two Gaussian integers, *it is not a prime.* Initially such logic seems extreme. Newton would agree, yet he called i an *imaginary number*.

God's rule: If you don't understand complex numbers then you cannot be successful as an engineer.

Math is a special language with important rules, that follow from logic and analysis.

Summary: It is easy to remember all these rules: Algebra of numbers is always a number as long as you avoid 1/0, which is not a number.

[14]One can meaningfully define $\xi = x + \jmath c_o t$ to be complex $(x, t \in \mathbb{R}, \xi \in \mathbb{C})$, with x in meters [m], t is in seconds [s], and the speed of light c_o [m/s], but I'm not sure it is useful.

2.1.4 Applications of integers

Why are integers important? First, we count with them so that we can keep track of "how many." But there is much more to numbers than counting: We use integers for any application where absolute accuracy is essential, such as in banking transactions (making change), the precise computing of dates (Stillwell, 2010, p. 70) and locations ("I'll meet you at 34th and Vine at noon on Jan. 1, 2034"), and the construction of roads and pyramids out of bricks (objects built from a fixed unit size). The field known as *Number theory*, is more likely to be studied by mathematicians than by Engineering students. However due to its fundamental nature, it defines methods that can be useful in many scientific applications.

Second, Integers are important because they are precise. To navigate we need to know how to accurately predict the tides and the locations of the moon and sun. But such an integral representation of our position or time is not possible since time is not an integer. Unlike space, time ($t \in \mathbb{R}$) is a one dimensional causal continuous and one-sided. The future is unknown, even unknowable, determined only by the past.

The Pythagoreans claimed that everything was integer. From a practical point of view, when precision is critical, they were right. Today all computers compute use ISO standard IEEE-754 floating-point numbers, as ratios and integer powers of integers, called *fractionals* $\mathbb{F}$. However, in theory the Pythagoreans were wrong. The error (difference) is a matter of precision, as determined by the $\mathbb{F}$'s.

Numerical Representations of $\mathbb{I}, \mathbb{R}, \mathbb{C}$: When doing numerical work, one must consider how to compute within the set of reals $\mathbb{R}$ (i.e., which contain irrationals $\mathbb{I}$). There can be no irrational number representation on a computer. The international standard of computation, IEEE-754 (2019) floating-point numbers,[15] is based on double-precision integer approximations of the magnitude and exponent for both rational and irrational numbers.

Numerical representation of rational numbers: The mantissa and the exponent of IEEE-754 numbers are based on two 64 bit integers, having sign and magnitude. The size of each integer depends on the precision of the number being represented. An IEEE-754 floating-point number is an approximation to a rational number because it has a binary (integer) mantissa, multiplied by 2 raised to the power of a binary (integer) exponent. For example, $\pi \approx \pm a 2^{\pm b}$ with $a, b \in \mathbb{Z}$. In summary, IEEE-754 floating-point rational numbers cannot represent either rational or irrational numbers, because that would require an infinite number of bits.

True floating-point numbers contain irrational numbers, which must be approximated by fractional numbers ($x \in \mathbb{F}$). For now we assume that $\boldsymbol{x} \in \mathbb{R}$ and we wish to approximate x as the ratio of two integers. This leads to the concept of fractional representation, which requires the definition of the *mantissa*, *base*, and *exponent*, where both the mantissa and the exponent are signed.

Numerical results must not depend on the base because numerical results needs to be independent of the representation. One could dramatically improve the resolution of the numerical representation by the use of the fundamental theorem of arithmetic. For example, one could factor the exponent into its primes and then represent the number as $2^a 3^b 5^c$ ($a, b, c \in \mathbb{Z}$). Such a representation would improve the resolution of the representation. But even so, the irrational numbers would be approximate. For example, base ten is natural using this representation, since $10^n = 2^n 5^n$.[16] Thus

$$\pi \cdot 10^5 \approx 314,159.27\ldots = 3 \cdot 2^5 5^5 + 1 \cdot 2^4 5^4 + 4 \cdot 2^3 5^3 + \cdots + 9 \cdot 2^0 5^0 + \cancel{2} \cdot \cancel{2^{-1}} 5^{-1} \cdots .$$

Since the number of bits used to represent the numbers must be finite on a computer, irrational numbers (e.g., π) cannot be precisely represented. We must resort to some sort of approximation. Using a rational representation, we can greatly improve the accuracy of any irrational number. An example is given in the next exercise using the irrational number π approximated using two prime numbers, 3 and 7. A much more accurate fractional representation is 355/113. The ratio of 355/113 over 22/7 is 1.26×10^{-3}.

It seems obvious that using a system other than base 10 (radix 10) can be confusing. It is not a notation we could easily adopt to for daily use. However since the computer keeps track of the decimal point, the radix remains invisible to the user.

Exercise #*6*

If we work in base 2 (binary arithmetic) and use the approximation $\pi \approx 22/7$, along with the Matlab/Octave command DEC2BIN(), show that the binary representation of $\hat{\pi}_2 \cdot 2^{17}$ is

$$\pi \cdot 2^{17} \approx 411,940_{10} = 64,924_{16} = 1,100,100,100,100,100,100_2.$$

[15] IEEE-754: `https://www.h-schmidt.net/FloatConverter/IEEE754.html`

[16] Base 10 is the accepted standard, simply because we have 10 fingers on two hands, each having 5 fingers.

The subscript 2 states that the approximation is base 2. Here the approximation is a sum over positive powers of 2, but we could also use negative powers of 2. The subscript 10 is base 10, while the subscript 16 is call hexidecimal (base 16).

Solution: First we note that this must be an approximation, since $\pi \in \mathbb{I}$, which cannot have an exact representation. A fractional ($\in \mathbb{F}$) approximation to π base on two prime (decimal) digits ($\hat{\pi}_2$), is:

$$\hat{\pi}_2 = 22/7 = 3 + 1/7 = [3; 7], \tag{2.5}$$

where `int64(fix(2^17*22/7))` = 411,940 and `dec2hex(int64(fix(2^17*22/7)))` = 64,924, and where 1 and 0 are multipliers of powers of 2, which are then added together:

$$411,940_{10} = 2^{18} + 2^{17} + 2^{14} + 2^{11} + 2^8 + 2^5 + 2^2 = 64,924_{16}.$$

To keep the analysis simple, exercise #6 uses only positive powers of 2, for example 2^{17} = $131{,}072_{10}$. The concept of the decimal point is replaced by an integer with the desired precision, called the radix. This scale factor may be thought of as moving the decimal point to the right (larger number) or left (smaller number). The mantissa fine-tunes the value about a scale factor (the exponent). In all cases the number is always an integer. Negative numbers are represented by an extra sign bit. Since the computer's hardware does the conversion, we don't need to master any radix calculations other than base 10 (the decimal system).

These are important exercises to sharpen your skills with a radix other than 10. Base 16 is called haxadecimal, and of course binary is in terms of $\{0,1\}$. Matlab/Octave has conversion programs to convert from one radix to another. These include `dec2hex` and `hex2dec`.

Exercise #*7*

Using Matlab/Octave and base 16 (i.e., hexadecimal) numbers, with $\hat{\pi}_2 = 22/7$, find (a) $\hat{\pi}_2 \cdot 10^5$ and (b) $\hat{\pi}_2 \cdot 2^{17}$.

1. $\hat{\pi}_2 \cdot 10^5$

   Solution: (a) Using the command `dec2hex(fix(22/7*1e5))` we get '4cbad'$_{16}$ since $22/7 \times 10^5 = 314,285.7\ldots$ and `hex2dec('4cbad')`= 314,285. (b) $2^{18} \cdot 11_{16}/7_{16}$

2. $\hat{\pi}_2 \cdot 2^{17}$ **Solution:** $2^{18} \cdot 11_{16}/7_{16}$

   .

The representation of numbers is not unique. For example, irrational complex numbers have approximate rational representations (i.e., $\pi \approx 22/7$). A better example is complex numbers $z \in \mathbb{C}$, which have many representations, as a pair of reals (i.e., $z = (x, y)$), or by Euler's formula, and matrices ($\theta \in \mathbb{R}$):

$$e^{\jmath\theta} = \cos\theta + \jmath\sin\theta \leftrightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

At a higher level, differentiable functions (analytic functions) may be represented by a single-valued Taylor series expansion, limited by its region of convergence (RoC).

Exercise #*8*

Write the first 11 primes, base 16.

Exercise #*9*

$x = 2^{17} \times 22/7$, using IEEE-754 double precision:[17]

Above the subscript represents the base (10, 2, 16, etc). Above the commas in the binary (0,1) string are to help visualize the quasi-periodic nature of the bit-stream. The numbers are stored in a 32-bit format, with 1 bit for the sign, 8 bits for the exponent, and 23 bits for the mantissa. An important alternate is format unsignedint64. Other useful formats are 'long engr, uint64, and unsigned long.'

[17] `https://www.h-schmidt.net/FloatConverter/IEEE754.html`

Perhaps a more instructive number is

$$\begin{aligned} x &= 4,793,490.0 && (2.6)\\ &= 0,100,1010,100,100,100,100,100,100,100,100_2 && (2.7)\\ &= 0x4a924,924_{16}, && (2.8)\end{aligned}$$

which has a repeating binary bit pattern of $((100))_3$, repeated 8 times. A more symmetrical example is

$$\begin{aligned} x &= 0x24924924_{16} && (2.9)\\ &= 00,100,100,100,100,100,100,100,100,100,100_2 && (2.10)\\ &= 6.344,131,191,146,900 \times 10^{-17}. && (2.11)\end{aligned}$$

Here the scale factor is 2^{24}.

It may be easier to find the period of rational numbers expressed a binary bit stream since a search for a repeating pattern consisting of only two symbols (0, 1) is more easily detected and searched.

Pythagoreans and Integers: The integer is the cornerstone of the Pythagorean doctrine—so much so that it caused a fracture within the Pythagoreans when it was discovered that not all numbers are rational. One famous proof of such irrational numbers comes from the spiral of Theodorus, as shown in Fig. 2.1, where the radius of each triangle has length $b_n = \sqrt{n}$ with $n \in \mathbb{N}$, and the long radius (the hypotenuse) is $c_n = \sqrt{1 + b_n^2} = \sqrt{1+n}$. This figure may be constructed using a compass and ruler by maintaining right triangles.

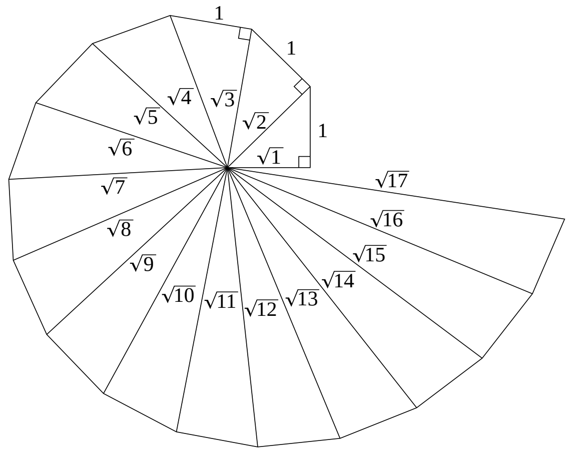

Figure 2.1: *The spiral of Theodorus, made from contiguous right triangles having lengths $a = 1$, $b_n = \sqrt{n}$, with $n \in \mathbb{N}$, and $c_n = \sqrt{n+1}$. In this way, each value of $c_n^2 = b_n^2 + a^2 \in \mathbb{N}$. This sequence of triangles generate the set $\{\sqrt{n}\} \in \mathbb{I}$, with $n \in \mathbb{N}$, and is easily generated using a compass and a ruler. Note the similarity with the Fibonacci spiral. (Adapted from* `https://en.wikipedia.org/wiki/spiral_of_Theodorus`.)

Public-key security: To make messages secure the message must be encoded in such a way that it is impossible to decoded. This is not easy. During WW-II methods were worked out, to allow one to send messages over an open network (radio link), that could be monitored by your enemy, but not decoded. These methods have a interesting history.[18, 19, 20]

Typically coding is done via a computer program which scrambles the message so it cannot be decoded without knowing a *secret key*. The science of this is known as *cryptography*. Early methods included Egyptian hieroglyphs (1900 B.C.) and a *Polybius Square* which was a 5x5 matrix to translate letters into 2-digit numbers. For example early codes simple scrambled the letters, such as swapping s and t, m with z. Such codes were easily cracked, and were quickly replaced by more robust ones.

Security tools depend on the use of number theory, which depends on the absolute precision of the integer. To encode something so that noone can break the code depends on the concept of a code having high entropy. Unfortunately the word entropy has several meanings, first in Claude Shannon's, which we label as entropyS, *theory of information*, and second, entropyH *thermodynamic theory of heat*.

[18] `https://en.wikipedia.org/wiki/Great_Cipher`
[19] `https://en.wikipedia.org/wiki/History_of_cryptography`
[20] `https://en.wikipedia.org/wiki/Classical_cipher`

While not ideal, the two different types of entropy S and H, are best understood from context. However this only works if you understand the concept.

I'm guessing that students in the 21 century are unfamiliar with the word *entropy*. The alternative meaning is *heat entropy*, a measure of the flow of heat (Allen, 1994, 1996), which has units of energy, measured in *Joules.*

One bit of Shannon entropy is the same as the toss of a coin, which is either heads or tails, either true (1) or faults (0). Probability measures seem to have no units since probability is dimensionless, thus the outcome of a coin toss is either *true* (T) or *false* (F). The associated math is called *Boolean algebra*.

The Shannon entropy for 10 bits is $2^{10} = 1024$ possible outcomes. An exhaustive search of 1024 outcomes is reasonable. Twenty bits is $2^{20} \approx 10^6$ possibilities, much harder to guess, but still possible. The realm of 100 bits is 1.27×10^{30} possibilities, seems impossible.

An important application of prime numbers is public-key encryption, which is essential for internet security applications (e.g., online banking). Decryption depends on factoring large integers formed from products of primes having thousands of bits.[21]

Security is based on the relative ease of multiplying large primes, coupled with the virtual impossibility of factoring them. One possible approach is to use the properties of roots of polynomials and Newton's method to generate the key (Allen, 2025).

A *trap-door function* is one where a computation is easy in one direction but its inverse is so difficult, it may be impossible. As an example think of a very long list of numbered (indexed) items. If I give you the index, you can look up the item quickly. But if I give you the item as a randomized set of characters, it would be hard (but not impossible) to find the index of the item. You might start by sorting on the list, which could be difficult. Consider the problem of sorting all the molecules in 1 liter of a mixture of several gases.

Public-key encryption is based on a trap-door function. If everyone were to switch from passwords to public-key encryption, the internet would be much more secure.[22]

Many people assume encryption is best done by a personal login and passwords. But passwords are fundamentally insecure because it is relatively easy for a computer to search all possible combinations of a computer's keyboard characters (including but not limited to: Shift, CapsLock, Ctrl, Alt, Fn, Esc). There is one strange key called the "Windows" key, perched between Ctrl and Alt. What does it do? It may be the most important key, that nobody uses. At one very confusing moment in keyboard history, the location of CapsLock and Ctrl were swapped. I spent at least a day trying to figure out how to swap them back, using the BIOS.

Exercise #*10*

Consider the following method for generating an irrational trap-door key. Define `PRIV` as any irrational number, e.g.: $\pi^\pi, \log_\pi, \sqrt{\pi_{101}}$. Then assume the public key `PUBL` is then a sequence of a large number (e.g. 10,00) digits, generated from `PRIV`, using the CFA (Exercise #8).

How would you propose to recover `PRIV` given `PUBL`?

Solution: In my view this is impossible because the space of irrational numbers is more than huge. Knowing the first 10 CFA integers tells you virtually nothing about `PRIV`.

It seems likely that NSA would not accepted such a scheme because it is "too good." I assume that NSA requires an encryption method that only the NSA can crack in a fixed amount of time, using the world's most powerful computer. The problem with this view is that the *bad guys* can use their own scheme rather than the RSA code.

Puzzles: Another application of integers is imaginative problems that use integers. An example is the classic Chinese *four stone problem:* Find the weights of four stones that can be used with a scale to weigh anything (e.g., salt, gold) between 0 and 40 grams (Assignment AE-2, Problem #5) The answer is not as interesting as the method, since the problem may be easily recast into a related one. This puzzle is best cast as a linear algebra problem with integer solutions. Again, once you know the trick, it is "easy." [23]

[21] The accepted method is the complexity of factoring numbers into the product of primes. Start with two (or more) primes, such as 5*3 and 5*11, and use the Euclidean (GCD) algorithm to find the common prime. Given the great difficulty in factoring numbers into their primes, this can serve as a "trap-door" function. Multiplying primes is cheap compared to factoring.

[22] `https://fas.org/irp/agency/dod/jason/cyber.pdf`

[23] Whenever someone tells you something is "easy," you should immediately appreciate that it is very hard, but once you learn a basic concept, the difficulty evaporates.

2.2 The role of physics in mathematics

Integers arose naturally in art, music, and science. Examples include the relationships between musical notes, the natural eigenmodes (tones) of strings and musical instruments. These relationships were so common that Pythagoras believed that to explain the physical world, one needed to understand integers. As discussed on §1 "all is number" was a seductive song.

It is best to view the relationships among acoustics, music, and mathematics as historical, since these topics inspired the development of mathematics. Today integers play a key role in quantum mechanics, again based on eigenmodes (§3.2). Eigenmodes follow from solutions of the Schrödinger equation, since the roots of the characteristic equation are purely imaginary numbers $\in \mathbb{N}$. When the real part of the eigenvalue is negative, the modes are naturally damped.

As discussed by Salmon (1946a,b, p. 201), Schrödinger's equation follows directly from the Webster horn equation. While Philip Morse (1948, p. 281) (a student of Arnold Sommerfeld) fails to make the direct link, he comes close to the same view when he shows that the real part of the horn resistance goes exactly to zero below a cutoff frequency (p. 256). He also discusses the eigenmodes trapped inside musical instruments, created by the horn flare. Notably, Morse cites Salmon (Morse, 1948, footnote 1, p. 271).

Engineers are so accustomed to working with real (vs. complex) numbers that they rarely acknowledge the distinction between real (i.e., irrational) and fractional numbers.[24] Integers arise in many contexts. One cannot master computer programming without understanding integer, hexadecimal, octal, and binary representations, since all numbers in a computer are represented in numerical computations in terms of rationals ($\mathbb{Q} = \mathbb{Z} \cup \mathbb{F}$).[25]

The primary reason integers are so important is their uniqueness. Every integer $n \in \mathbb{Z}$ is unique[26] and has the indexing property, which is essential for making lists that are ordered, so that one can quickly look things up. The alphabet also has this property (e.g., a book's index). However the indexing properties of prime number is much more sophisticated.

Because of the integer's absolute precision, the digital computer quickly overtook the analog computer once it was practical to make logic circuits that were fast. The first digital computer was thought to be the University of Pennsylvania's ENIAC in 1946. We now know that the code-breaking effort in Bletchley Park, England, under the guidance of Alan Turing, created the first digital computer (the Colossus), which was used to break the World War II German Enigma code. I recommend you watch *The Imitation Game*, released on December 24 2014. Due to the high secrecy of this war effort, the credit was first acknowledged 30 years later (1970), when the project was finally declassified.

There is no (zero) possibility of analog computing displacing digital computing, because of the importance of precision and speed. But even with binary representation, there is a nonzero probability of error. To deal with this, error-correcting codes were developed, further reducing the error by many orders of magnitude. Today error correction is a science. Billions of dollars have been invested to increase the density of error free bits, by a huge factor. Only a few years ago the terabyte drive was unheard of; today it is standard. In a few years petabyte drives will certainly become available. It is hard to comprehend how these will be used by individuals, yet they are essential for online (cloud) computing. The open question seems to be "Will AI computing move to the desktop?" The obvious answer seems to be yes. I believe it has already happened.

There seems to be an interesting conflict of interest in mathematics: 1) It needs to be precise. 2) Its utility is in the application to real-world problems. Can we have it both ways?

2.2.1 The three streams of mathematics

Modern mathematics is built on a hierarchical construct of fundamental theorems. The importance of such theorems cannot be overemphasized.

Gauss's and Stokes's laws play a major role in understanding and manipulating Maxwell's equations. Every engineering student needs to fully appreciate the significance of these key theorems. If necessary, memorize them. But memorization will not do over the long run, as each and every theorem must be fully understood. Fortunately most students already know several of these theorems, but perhaps not by name. In such cases, it is a matter of mastering the vocabulary.

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[24]Hopefully this is in a state of change today. The topic should be introduced at the high-school level.

[25]See Appendix 2.1.1 for a review of mathematical notation.

[26]Check out the history of the number $1729 = 1^3 + 12^3 = 9^3 + 10^3$ (Google for G.W. Hardy and Ramunujan).

The three streams of mathematics

1. **Number systems: Stream 1**
   - Arithmetic
   - Prime numbers
2. **Geometry: Stream 2**
   - Algebra
3. **Calculus: Stream 3** (Flanders, 1973)
   - Leibniz $\mathbb{R}^1$
   - Complex $\mathbb{C} \subset \mathbb{R}^2$
   - Vectors $\mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^\infty$
     - Gauss's law (divergence theorem)
     - Stokes's law (curl theorem, or Green's theorem)
     - Vector calculus (Helmholtz's decomposition theorem)

The fundamental theorems are naturally organized and may be thought of in terms of the three streams of Stillwell (2010). For Stream 1 we have the fundamental theorem of arithmetic and the prime number theorem. For Stream 2 there is the fundamental theorem of algebra, and for Stream 3 there are a host of theorems on calculus, ordered by their dimensionality. Some of these theorems seem trivial (e.g., the fundamental theorem of arithmetic). Others are more challenging, such as the fundamental theorem of vector calculus and Green's theorem.

Complexity should not be confused with importance. Each of these three theorems, as stated, is fundamental. Taken as a whole, they are a powerful way of summarizing mathematics.

2.2.2 Stream 1: Prime number theorems

In Figs. 2.2, 2.3 we test these observations about primes. Note how correlations are seen in the minimum of $1 - R_k$, which displays structure in the minimum values, while the upper-bound appears random.

From the figure it may be observed that the neighboring digits in a log-log plot of the $\sqrt{n \in \mathbb{N}}$ is quasi-random (Stillwell, 2010).[27] Here we are deal with simple electrical networks composed of inductors, resistors, and capacitors (Fig. 3.5, page 114), or mechanical networks consisting of masses, dashpots, and springs, or the pendulums used by Galileo in his studies of gravity (Figs. 1.3, page 11 and 5.7, page 202). These systems may be modeled as a Brune impedance which are always defined as the ratios of polynomials, expressed in terms of the Laplace frequency $s = \sigma + \omega\jmath$ (see §2.2.2, page 31 and §2.2.2, page 31).

Reciprocals of integers (e.g., primes) are always rational, by definition, because they always have a finite period, with a maximum period of $\pi_k - 1$. For example, $\pi_4 = 7$ has a period of $6 = \pi_4 - 1$. The period of 1/7 =((142857))$_6$, which by definition is $1/7 = 0.142857, 142857, 143857, \cdots$.

Given a prime, one may modify or bound its period. If the period is changed by a tiny amount, the corresponding variability in the prime may be easily computed. This does not seem to be well know. It may be demonstrated by the example $\pi_4 = 7$, which has the 6 digits period, 0.142857. To within IEEE-754 precision, 1/7=0.142857,142857,143. Thus this value of 1/7 is not periodic, since the third period is truncated to 143. Thus 1/0.142857142857143 cannot be equal to 7, since it does not have a period of 6. In fact 1/0.142857142857143= 6.999999999999994 $\neq$ 7 , with an error of 0.43×10^{-13}.

One way to see this is to expand any number using the CFA. For example irrational numbers and their CFA's are given by Octave/Matlab command $rat(e, 1e - 20)$:

$e \approx 3 + 1/(-4 + 1/(2 + 1/(5 + 1/(-2 + 1/(-7 + 1/(2 + 1/(9 + 1/(-2 + 1/(-11 + 1/(2 +$
$1/(13 + 1/(-2 +$
$1/(-12))))))))))))) + \epsilon_o$
where $\epsilon_o \approx 5.515, 095 \times 10^{-9} = \frac{25946}{9545}$, or in decimal notation, $e \approx 2.718, 281, 822, 943, 950, \cdots$.

The double–bracket notation for the CFA of e is [[3; -4, 2, 5, -2, -7, 2, 9, -2, -11, 2, 13, -2, -12]].

Also $\pi \approx 3.141, 592, 653, 589, 793, 238, 462, 643, 383, 278, 502, 884, 197, \cdots$.

For even more precision, one may use the Wolfram Language[28]

[27] 'Mathematics and Its History', Stillwell, (2010)

[28] https://reference.wolfram.com/language/guide/LanguageOverview.html

In summary, all *rational* numbers have the property that their reciprocals are periodic. Irrational numbers such as the square roots of primes, are not periodic. Reciprocals of primes always have a period $R_k = \pi_k - 1$. For example if $k = 12$, $\pi_{12} = 37$, thus $\pi_{12} - 1 = 36 = 6^2 = 2^2 \cdot 3^2$. To find the period, note that 10/37=0.270270270270270, which has a period of 3, which is a factor of 36. A second example, for $k = 13$, is $\pi_k = 41$. In this case $10/(\pi_{13} - 1) = 10/40 = 0.24390, 24390, 24390))_5$. This period is the largest prime factor of $41 - 1 = 40 = 5 * 2^3$.

> *Based on numerical examples, the general rule seems to be that every prime π_k has a period given by the largest factor of $\pi_k - 1$.*

There are two key fundamental theorems here

1. *The fundamental theorem of arithmetic*: The first and best know theorem is that every integer $n \in \mathbb{Z}$ may be uniquely factored into prime numbers, define as a number having no factors other than 1 and itself. For example, prime factors of 119 are 7 and 17.

   For example $1, 02, 559, 2953 = 7 \cdot 11 \cdot 317 \cdot 42017$ may be displayed using Octave/Matlab as a 64bit unsigned integer via the command UINT64(7*11*317*42017) and factored with factor1,025,592,953 on an Intel CPU. This may be increased by 30 and it still factors.

2. *The prime number theorem:* One would like to know how many primes there are. That is easy: $|\mathbb{P}| = \infty$ (the size of the set of primes is infinite). A better way of asking this question is "What is the average density of primes in the limit as $n \to \infty$"? This question was answered, for all practical purposes by Gauss, who computed the first three million primes. He discovered that the primes are equally likely on a log scale. This is nicely summarized by the couplet:

   > Chebyshev said, and I say it again: There is always a prime between *n* and *2n*.

   This alludes to the mathematician Pafnuty Chebyshev, who proved the *prime number theorem* in a novel way (Stillwell, 2010, p. 585). When the ratio of two frequencies (pitches) is 2, the relationship is called an *octave*. With a slight stretch of terminology, one could say there is at least one prime per Octave,[29] as shown as the lower bound of Fig. 2.2.

In modern western music, twice the frequency is defined as one Octave. The Octave is further divided into 12 smaller ratios called *semitones*, equal to $\sqrt[12]{2}$. Twelve semitones is one Octave. This scale is modeled after the physics of the ear (Allen, 1977). The density of the ratios of primes is a metric of the density of prime ratios.

Exercise #*11*

An interesting extension of Chebyshev's observation might be the density in primes per fraction of an Octave? An alternative is to study the ratio $\pi_{k+1}/\pi_{k-1} < 1$ as $k \to \infty$, or perhaps $\log_2 \pi_{n+1}/\pi_{n-1}$.

Solution: A graphical answer is Fig. 2.2, showing a log-log plot of $1 - R_k$. Note that the graph has a well defined lower bound, having a slope slightly steeper than the red dashed line. Jumps in k by multiples of 2 are labeled with **X**.

Figure 2.2 is a log-log plot of $R_k = \pi_{k+1}/\pi_{k-1}$ as a function of $\ln k$, namely a plot of $\log_{10}(\pi_{k+1}/\pi_{k+1})$ vs. $\log_{10} k$.[30] A linear fit to this log-log plot requires two points to determine the slope.

One might question the maximum number of primes per-Octave in the neighborhood of N, or ask for the fractions per Octave (factors of 2) for π_k as k increases. The maximum value of $R_k < 0.5$; thus Chebyshev's bound of 2 is conservative. As $k \to \infty$ the bound is exponentially compressed. The results of this calculation are shown in Fig. 2.2. When $k = 9580$, $1 - R_k \approx 2.0043 \times 10^4$. Note that by trial and error, by factoring each number, it is easy to find the prime near any number. Empirically it takes ≈5 tries to identify a new prime. Most integers have a small prime factor {3, 4, 5, 7, 11, 13}. Exceptions are rare, which is why very large primes can, in theory, be difficult to label.

One can use the Octave command $P = primes(1e4)$. Specific examples include $factor(9969) = [3, 3323]$, $factor(1745) = [5, 349]$, $P(2262) = 19997$, and $factor(19995) = [3, 5, 31, 43]$. These numerical values define a simple formula for $R(k)$ on this lower boundary. It can also be used to select the minimal prime ratios to allow us to find a formula for this minimum set of prime ratios.

[29] The number of primes grows as e^t as discussed at https://www.youtube.com/watch?v=qoJacpk_OXo

[30] The mean of $1-R_k$ is characterized by the red dashed line, regression line.

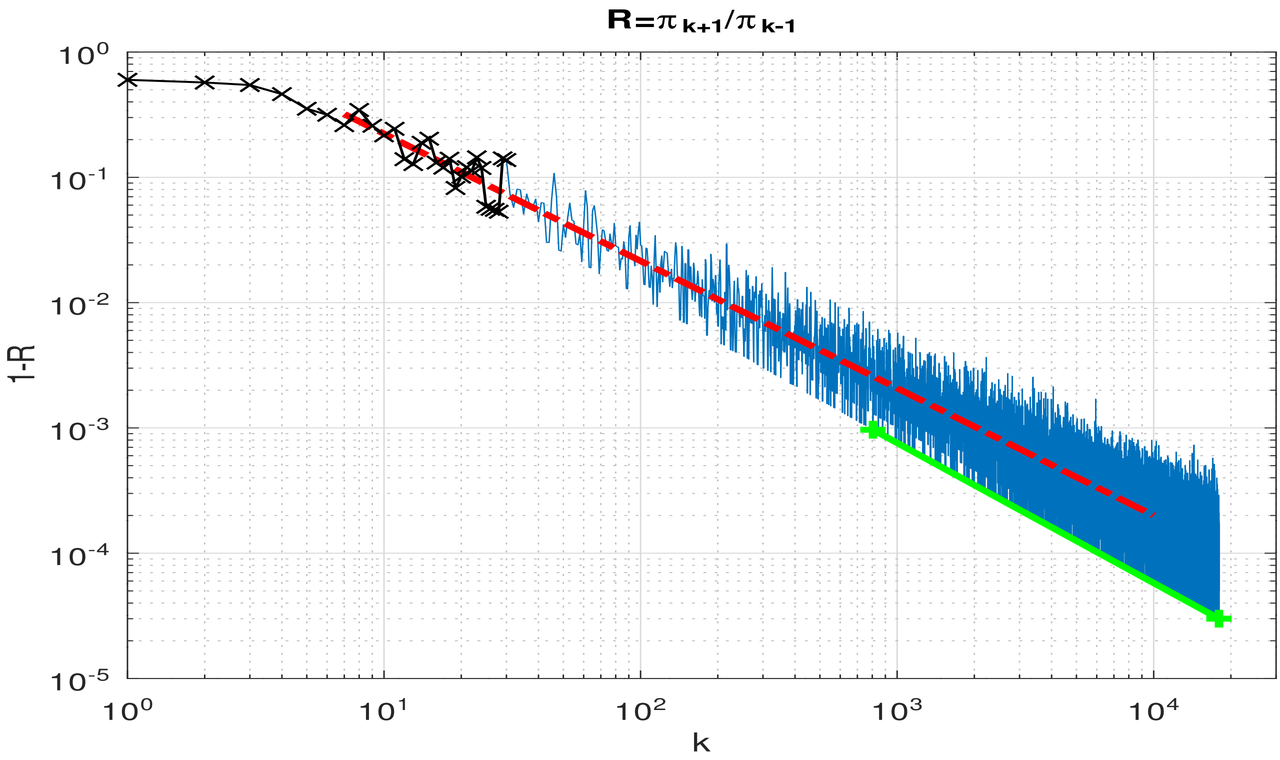


Figure 2.2: *This figure succinctly summarizes many theorems about prime numbers, and may add a new relationship. The figure is a log-log plot of $|1 - R_k|$. This is a demonstration that the number of primes grows slowly, but never goes to zero. This chart investigates the difference between primes $\ln(1 - R_k)$, which linearly decreases as a function $\ln k$. In the limit of large k, R_k goes to 1, thus $|1 - R_k| \to 0$. This is a direct application of the* prime number theorem. *The red-dashed line is the visual-mean of the distribution. It has a slope of 2/10 = 0.2 for $7 \le k \le 20$ and $\approx 2.0 \times 10^{-4}$ for $k = 2 \times 10^4$. Note the linear cyan regression line on the lower bound of the data, which has a slightly steeper slope than the red dashed line, on log-log coordinates. For $k \le 30$, the points are labeled with an* **x**. *The upper bound appears as random, unlike the lower cyan line, defining the slope. The first 10 odd counting numbers are ([3, 5, 7, 11, 13, 17, 19, 23, 29, 31]) are all primes. Their ratios $R_k = 7/3))_1 = 2+1/3, 11/5))_2 = 2+1/5, 13/7))_6 = 2-1/7, \cdots$ are all rational partial fraction expansions primes, having well defined reciprocal periods of $1/(\pi_k - 1)$. This explains the lower cyan regression line.*

2.2.3 Stream 2: Fundamental theorem of algebra

This theorem states that every polynomial in x of degree N,

$$P_N(x) = \sum_{k=0}^{N} a_k x^k, \tag{2.12}$$

has at least one root. When a root is factored out, the degree of the polynomial is reduced by 1. Applied recursively, a polynomial of degree N has N roots. Note there are $N+1$ coefficients (i.e., $[a_N, a_{N-1}, \dots, a_0]$). If we are interested only in the roots of $P_N(x)$, it is best to normalize such that $a_N = 1$, which defines a *monic polynomial*. Without more information about the roots, one is forced to assume they are irrational and complex $\in \mathbb{C}$.

2.2.4 Stream 3: Fundamental theorems of calculus

Next we deal with the theorems of Stream 3. We consider the several fundamental theorems of integration, starting with Leibniz's formula for integration on the real line ($\mathbb{R}$). Next we study complex integration (Cauchy's theorem), which is used to compute the Laplace transform $\mathcal{LT}$ and its inverse. Gauss's and Stokes's laws for $\mathbb{R}^2$ which require closed and open surface integration.

One cannot manipulate Maxwell's equations, fluid flow, or acoustics without understanding these concepts. Any problem that deals with the wave equation in more than one dimension requires this level of understanding. They are the basis of the derivation of the Kirchhoff voltage and current laws, first proposed by Newton for mechanics and acoustics.

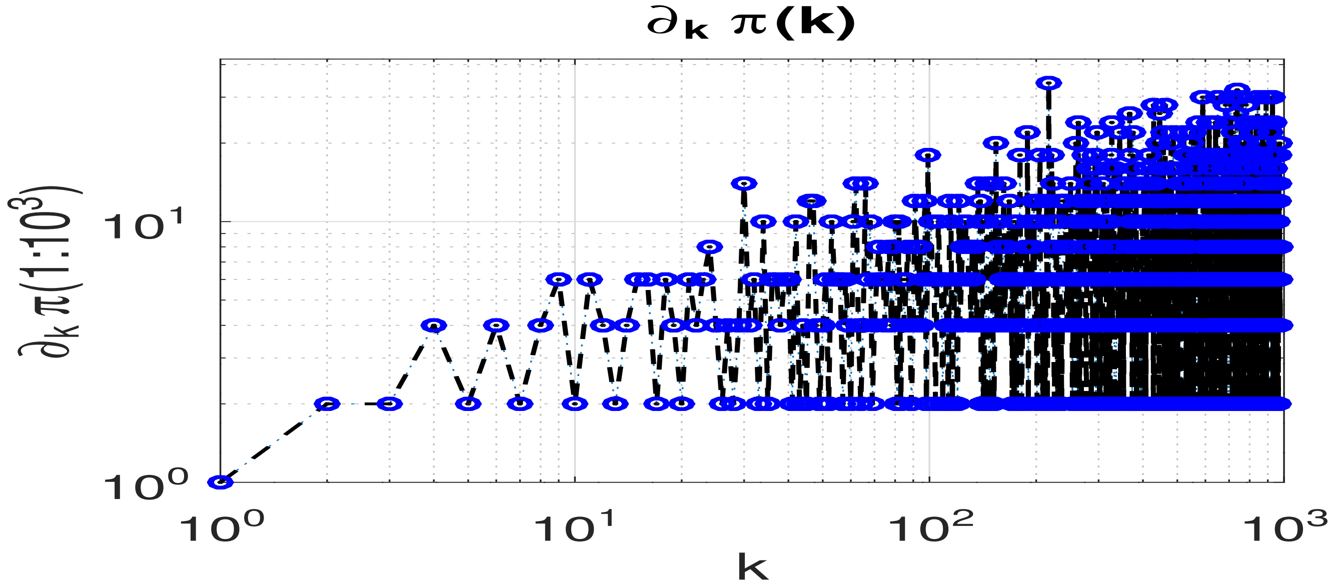


Figure 2.3: *This is a plot of the differences of primes ($\partial_k \pi_k$), as a function of k. The definition of $\partial_k \pi_k$ is $(\pi_{k+1} - \pi_k)$. The first 9 primes $\pi(k)$ are [2, 3, 5, 7, 11, 13, 17, 19, 23], and their first differences are ∂_k = [1, 2, 2, 4, 2, 4, 2, 4]. Excluding the first prime ($\pi_1 = 2$), all the primes must be odd since they cannot have 2 as a factor. Note that each row of the differences jump by factors of 2. Other than the first two jumps 2-1=1 and 3-2=1, they jump by 2, 4, 6, etc. For $k \geq 9$, $\partial_k \approx 6$. For $k \geq 30$, the maximum jumps approach 10. Above $k > 200$, $\partial_k \pi_k$ increase to above 10. Histograms as a function of k can be used to precisely quantify this one-sided distribution of primes. For $k \geq 2$ they seem approach a binomial distribution, since the mode is rising with k, and the frequency of the distribution of jumps by 2, 6, 8, rapidly increase. Precisely quantifying this trend should not be difficult.*

Next we define three basic vector operations: the *gradient* $\nabla()$, *divergence* $\nabla\cdot()$, and curl $\nabla\times()$, where

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{2.13}$$

is read as "del," or more precisely as "nabla." The hat over the three unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ means they are *unit vectors*, or notational $\|\hat{\mathbf{y}}\| = 1$.

Second-order operators include the *scalar Laplacian* $\nabla\cdot\nabla() = \nabla^2()$, and the divergence of the gradient (**DoG**), are be constructed from first-order operators. The most important of these is the *vector Laplacian* $\boldsymbol{\nabla}^2()$, and the gradient of the divergence (**gOd**), which is required when working with Maxwell's wave equations.

The first three operations are defined in terms of integral operations on a surface in one, two, or three dimensions, by taking the limit as that surface, or the volume contained within, goes to zero. These three differential operators are essential to comprehensive understand Maxwell's equations, the crown jewel of mathematical physics. As I'm sure you already know, mathematics plays a key role in physics, as does physics on math.

2.2.5 Other key mathematical theorems

In addition to the widely recognized fundamental theorems, there are a number of equally important theorems that have not yet been labeled as "fundamental."

The widely recognized *Cauchy integral theorem* is the primary example, since it is a stepping-stone to *Green's theorem* and the *fundamental theorem of complex calculus*, as discussed in Chapter 4. Once these fundamental theorems of integration have been mastered, you will be ready for the *complex frequency plane* ($s = \sigma + \omega\jmath \in \mathbb{C}$), which is required for the analysis of physical signals and systems.

Without the concepts of time and frequency we cannot develop an intuition for the Fourier and Laplace transforms. The Fourier transform covers signals, while the Laplace transform describes systems. Separating these two concepts based on their representations is an important starting place for understanding physics and the role of mathematics. However, these methods by themselves, do not provide the insight into physical systems that we need to be productive or creative. We need to master the tools of differential equations, and then partial differential equations, to fully appreciate the world that they describe. Electrical networks composed of inductors, capacitors, and resistors are equivalent (isomorphic) to mechanical systems, composed of masses, springs, and dash-pots. Newton's laws are analogous to those of Kirchhoff, which are the rules needed to analyze simple physical systems composed of linear (and nonlinear) sub-components. When lumped-element systems are taken to the limit in several dimensions, we obtain Maxwell's partial differential equations, the laws of continuum mechanics.

The ultimate goal of this text is to make you aware of, and productive in using these tools. This material can be best absorbed by treating it chronologically through history, so you can see how this body of knowledge came into existence, through the minds and hands of Galileo, Newton, Maxwell, and Einstein.

2.3 Applications of prime numbers

If someone asked you for a theory of counting numbers, I suspect you would laugh and start counting. It sounds like either a stupid question or a bad joke. Yet integers are a rich topic, so the question is not even slightly dumb. Birds and bees recognize small primes. Perhaps more surprising, Cicadas and other insects crawl out of hibernation on cycles of prime-number (e.g., 13- or 17-year cycles). If you have ever witnessed such an event (I have), you will never forget it. Somehow they know.

Complex-analytic functions Most important of all early functions of a complex variable is the Euler zeta function, first introduced by Euler based on his analysis of the *sieve of Eratosthenes*. Today this is known as the *Riemann zeta function* $\zeta(s)$. Its key property is that it is *complex–analytic*, with poles at the logs of the prime numbers. The properties of this function are both important and amazing. Many questions, and answers about primes, go back to the early Asian mathematicians (ca. 1500 BCE).[31] It seems possible to close the question of the roots of $\zeta(s)$ by computing the ROC for the known roots. If the are more roots in the right half plane, then Newton's method would find them.

A function is defined as *complex-analytic* if it obeys the Cauchy-Riemann conditions. One important property of each prime π_l may be studied by forming a few terms of the Taylor series

$$\frac{1}{1-\pi_l^{-1}} \approx 1 + \pi_l^{-k} + \pi_l^{-2k} + \pi_l^{-3k} \cdots . \tag{2.14}$$

For large primes this series converges rapidly. The recursive "sieve" method for finding primes was first devised by the Greek Eratosthenes (O'Neill, 2009). The three boxes below describe the sieve. The goal is to recursively recover the primes **2**, **3**, **5**, etc.

Summary: Starting from $\pi_1 = 2$, strike out all even numbers $2 \times (2, 3, 4, 5, 6, \ldots) = (4, 6, \ldots)$. Here we use the subscript to count the primes ($\pi_4 = 7$ and $\pi_5 = 11$). By definition, the multiples are products of the target prime (2 in our example) and every other integer ($n \geq 2$). In this way all the even numbers are removed in this first iteration. The next remaining integer (3 in our example) is identified as the second prime π_2. Then all the multiples of $\pi_2 = 3$ are removed. The next remaining number is $\pi_3 = 5$, so all multiples of $\pi_3 = 5$ are removed (i.e., $\cancel{10}$, $\cancel{15}$, $\cancel{25}$, . . .). This process is repeated until all the numbers of the list have been either canceled or identified as prime.

2.3.1 The importance of prime numbers

One of the key problems in the theory of numbers is factoring. A example is helpful. Suppose we know the primes 2, 3, 5, 7 and we wish to factor 41 and 411. One might assume that they are related given their similarity. First, both 41 and 411 are odd (neither are even). This is an important clue that they have no common 2. But what about all the other primes.

Two numbers with a common factor are called *coprime*. The simple test is to divide one into the other by long division. If there is a remainder, then the are not coprime. But this will be difficult if the common factor is a large prime. Before we continue you need to know that there is a solution to this question if you use OCTAVE.

It is most insightful to start with three large primes, to generate an example. If we compute the product of two large primes, 137 and 441 we get 60417. Prime 137 is the 33 prime and 441 we would have difficulty, since they are not coprime. The first is prime and $411 = 3^2 \cdot 7^2$ is not prime.

The best approach in this case is to factor the smaller number and then given its factors, see if any of them divide into the larger number. For this example it is important to know that 137 is prime, and 441=9*49, thus is said to be *composite*. The obvious solution is to divide the prime 137 into 441 and see if it results in a remainder. If the result has no remainder, then they have a common factor.

[31] https://www.youtube.com/watch?v=zlm1aajH6gY

1. Write out all the integers n from 2 to $N = 50$, and cross out multiples of $\pi_1 = 2$xn $n \cdot \pi_1 = 4, 6, 8, 10, 12, \ldots, 50$, namely all even n, e.g., `mod(n,`π_1`)`.

	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~

2. Let $k = 2$ and note that $\pi_2 = 3$. Cross out odd items: $n\pi_2$ $3 \cdot (3, 5, 7, \cancel{9}, \ldots, 45)$, that is, all `mod(n,`$\pi_2$`)`.

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	25	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	35	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~

3. Let $k = 3, \pi_3 = 5$. Cross out $n\pi_3$ $(25, 35)$ (remove `mod(n,5)`).

	2	**3**	~~4~~	**5**	~~6~~	**7**	~~8~~	~~9~~	~~10~~
11	~~12~~	**13**	~~14~~	~~15~~	~~16~~	**17**	~~18~~	**19**	~~20~~
~~21~~	~~22~~	**23**	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	**29**	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	**37**	~~38~~	~~39~~	~~40~~
41	~~42~~	**43**	~~44~~	~~45~~	~~46~~	**47**	~~48~~	~~49~~	~~50~~

4. Finally cross out $n\pi_4$: (49). There are 15 primes less than $N = 50$: $\pi_k = 2, 3, 5, \cdots, 47$ (in red). By definition, every prime ≥ 3 is odd.

Figure 2.4: *This figure consists of thee boxes which define the* SIEVE OF ERATOSTHENES *for $N = 50$. The sieve is the only known method to identify primes $\pi_k = \zeta_k \in \mathbb{P}$. See the introduction in the Preface, and then §1.1 on primes, for a full discussion of the theory. In the first box we write down the positive integers as a 5x50 matrix, after which we cross off all the even numbers, which are multiples of the first prime $\pi_1 = 2$.*

Two famous examples of irrational numbers are the expansion of π and $\sqrt{\pi}$. With octave we find the following: $s = rat(\pi) \Rightarrow s = 3 + 1/(7 + 1/16)$, which of course is an approximation.

The solution is to form the *partial fraction expansion* using Octaves $rat(a, b)$ function. The following example is helpful: $rat(pi, 1e - 7)$
ans = 3 + 1/(7 + 1/(16 + 1/(-294))) This is the same as the first example, but with a very small negative correction (-1/294). We may conclude that pi is not rational (it is irrational). This construction works because we asked for the error ($1e - 7 = 10^{-7}$). The unanswered question here is, how does octave compute the error if it cannot determine the exact answer? The answer is that Octave has internally saved the best it can do for π, which is an important constant. One may ask Octave what this limit is with the command $eps = 2.2204 \times 10^{-16}$. The amazing thing is it can compute $eps^{20} = 8.4880e - 314$, which is a very small number. It seems likely this is a stored number, like π. This flexibility is useful and important.

A prime may be used to find multiples of that prime by dividing it into a number. When the numbers are huge can be difficult or even impossible if the numbers are larger than eps, or smaller than $1/eps$.

Note: Octave is a free open-source version of *Matlab*. By free I mean it is free to download at zero cost. That is not the case for $Matlab$, which is expensive.

Finally, a very important command is *whose*. which returns all the internal variables and their type. The help command is of special use when your in learning mode, or if you have forgotten the syntax of a command. There is even a *help help*. Who ever wrote this tool deserves to go to heaven. John W. Eaton is the primary author and creator of GNU Octave. Bless you John.

Note that eps, by definition, is not prime. Manual factoring a number by repeated trials is labor intensive, thus it is not a reasonable option. The primes are identified by this recur*sieve* elimination (hence the term *sieve*). It follows that prime numbers are the key that unlocks the theory of numbers (*number theory*), defining the *fundamental theorem of arithmetic* (FTA).

As the word *sieve* implies, this process takes a heavy toll on the integers, rapidly pruning the non-primes. In four iterations of the sieve algorithm, all the primes less than $N = 50$ are identified in red. The final set of primes is displayed in step 4 of Fig. 2.4.

It is likely that the first insight into the counting numbers started with the sieve shown in Fig. 2.4. A sieve answers the question "How can one identify the prime numbers?" The answer may be found by looking for irregular patterns in the counting numbers.

Once a prime greater than $N/2$ has been identified (25 in the example), the recursion stops, since twice that prime is greater than N, the maximum number under consideration. Thus once $\sqrt{49}$ has been reached, all the primes have been identified.

There are various schemes for making the sieve more efficient. For example, the recursion $n\pi_k = (n-1)\pi_k + \pi_k$ might speed up the process, by replacing the multiplication with additions. However this is not worth the effort. The overhead will eat up any gain.

2.3.2 Two fundamental theorems of primes

Early theories of numbers revealed two fundamental theorems (§2.2.3 and §2.2.4). The first of these is the fundamental theorem of arithmetic, which says that every integer $n \in \mathbb{N}$ greater than 1 may be uniquely factored into a product of primes:

$$n = \prod_{k=1}^{K} \pi_k^{\beta_k}, \tag{2.15}$$

where $k = 1, \ldots, K$ indexes the integer's K prime factors $\pi_k \in \mathbb{P}$. To make the notation compact we define the *multiplicity* β_k of each prime factor π_k. Our demonstration of this is empirical, using the Matlab/Octave `factor(N)` routine, which factors N. What seems amazing is the unique nature of this theorem. Each counting number is uniquely represented as a product of primes. No two integers can share the same factorization. Save the numbers in factored form. To remove a factor, set $\beta_k = 0$. It seems best to save the numbers in factored form, so you never need to factor them.

Factoring is much more expensive than division. This is not due to the higher cost of division over multiplication, which is less than a factor of 2.[32] Dividing the product of two primes, given one, is trivial, slightly more expensive that multiplying. Factoring the product of two primes is nearly impossible, as one needs to know what to divide by. Factoring means dividing by some integer and obtaining another integer with remainder zero.

This brings us to the prime number theorem (PNT). The security problem is the reason these two theorems are so important: (1) Every integer has a unique representation as a product of primes, and (2) the density of primes is large. Thus security reduces to the "needle in the haystack problem" due to the cost of a deep search. One could factor a product of primes $N = \pi_k \pi_l$ by doing M divisions, where M is the number of primes less than $\sqrt{N}$. This assumes the primes less than $\sqrt{N}$ are known. However, most integers are not a simple product of two primes. An exception is the product of roots or logs of primes.

An important tool in security systems is the *Shannon entropy* S. But the utility of using prime factorization has to do with their density. If we were simply looking up a few numbers from a short list of primes, it would be easy to factor them. But given that their density is logarithmic (>1 per-Octave, as shown in Fig. 2.2), factoring comes at a very high computational cost compared to a table lookup. Obviously the people who make computers know everything I have said. This book is about math and physics, not computers.

2.3.3 The Euclidean algorithm (Greatest common divisor)

The Euclidean algorithm is a systematic method known to all educated people, on how to do long division. The largest common integer factor k between two integers n and m (divisor and dividend) is denoted $k = \gcd(n, m)$, where $n, m, k \in \mathbb{N}$ (Graham et al., 1994). For example, $15 = \text{GCD}(30, 105)$ since, when factored, $(30, 105) = (2 \cdot 3 \cdot 5,\ 7 \cdot 3 \cdot 5) = 3 \cdot 5 \cdot (2, 7) = 15 \cdot (2, 7)$. Thus the GCD is 15. Two integers are said to be *coprime* if their GCD is 1 (i.e., they have no common prime factor). The Euclidean algorithm was first known to the Chinese (i.e., not discovered by Euclid) (Stillwell, 2010, p. 41).

The Euclidean algorithm is best explained by a trivial example: Consider the two numbers 6 and 9. At each step the smaller number (6) is subtracted from the larger (9) and the smaller number and the difference (the remainder) are saved. This process continues until the two resulting numbers are equal, which is the GCD. For our example, $9 - 6 = 3$, leaving the smaller number 6 and the difference 3. Repeating this, we get

[32] `https://streamcomputing.eu/blog/2012-07-16/how-expensive-is-an-operation-on-a-cpu/`

Examples of the GCD $l = \mathrm{gcd}(m, n)$

- Examples ($m, n, l \in \mathbb{Z}$):
  - $5 = \mathrm{gcd}(13 \cdot 5, 11 \cdot 5)$. The GCD is the common factor 5.
  - $(13 \cdot 10, 11 \cdot 10) = 10\,\mathrm{gcd}(130, 110) = 10 = 2 \cdot 5$ is not prime
  - $\mathrm{gcd}(1234, 1024) = 2$, since $1234 = 2 \cdot 617$ and $1024 = 2^{10}$
  - $\mathrm{gcd}(\pi_k\pi_m, \pi_k\pi_n) = \pi_k$
  - $l = \mathrm{gcd}(m, n)$ is the part that cancels in the fraction $m/n \in F$
  - $m/\mathrm{gcd}(m, n) \in \mathbb{Z}$
- *Coprimes* ($m \perp n$) are numbers that have no distinct common factors; that is, $\mathrm{gcd}(m, n) = 1$
  - The GCD of two primes is always 1: $\mathrm{gcd}(13, 11) = 1$, $\mathrm{gcd}(\pi_m, \pi_n) = 1$ ($m \neq n$)
  - $m = 7 \cdot 13, n = 5 \cdot 19 \Rightarrow (7 \cdot 13) \perp (5 \cdot 19)$
  - If $m \perp n$, then $\mathrm{gcd}(m, n) = 1$.
  - If $\mathrm{gcd}(m, n) = 1$, then $m \perp n$.

$6 - 3 = 3$, leaving the smaller number 3 and the difference 3. Since these two numbers are the same, we are done; thus $3 = \mathrm{gcd}(9, 6)$. We can verify this result by factoring [e.g., $(9, 6) = 3(3, 2)$]. The value may also be numerically verified using the Matlab/Octave GCD command `gcd(6,9)`, which returns 3. Thus the GCD reduces to the definition of long division.

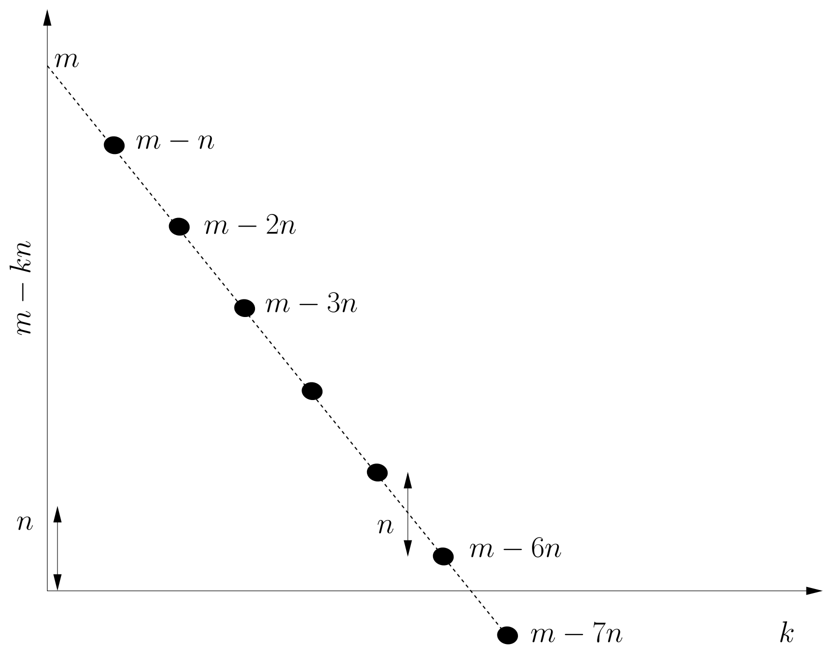


Figure 2.5: *The Euclidean algorithm* `mod(m,n)`*, better known as the* greatest common denominator *(GCD), recursively finds a value for k by subtracting n from m until the remainder is $m - kn \leq 0$. For the case depicted here, the value of k renders the* remainder $m - 6n < n$. *The* turning-point *satisfies the linear relationship $m - \alpha n = 0$ with $\alpha \in \mathbb{R}$. A closely related method, know as* Newton's method, *directly solves the much more difficult problem of finding roots of polynomials.*

Direct matrix method: The GCD may be written as a matrix recursion given the starting vector $(m_0, n_0)^T$. The recursion is then

$$\begin{bmatrix} m_{k+1} \\ n_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_k \\ n_k \end{bmatrix}. \tag{2.16}$$

This recursion continues until $m_{k+1} \leq n_{k+1}$, at which point m and n are be swapped. The process is repeated until $m_k = n_k$, which equals the GCD. We call this the *direct method* (see Fig. 2.5). The direct method is inefficient because it recursively subtracts n_k many times until the resulting m_k is less than n_k. It also must test for $m \leq n$ after each subtraction and then swap them if $m_k < n_k$. If they are equal, we are done.

The GCD's *turning-point* may be defined using the linear interpolation $m - \alpha n = 0, \alpha \in \mathbb{R}$, where the solid line cross the abscissa in Fig. 2.5. If, for example, $l = 6 + 43/97 \approx 6.443298\ldots$, then $6 = \lfloor m/n \rfloor < n$

and $\alpha \in \mathbb{F} \in \mathbb{R}$. Thus nonlinear arithmetic fundamental to the GCD. The `floor()` functions finds the turning-point, where we swap the two numbers, since by definition, $m > n$. In this example, $6 = \lfloor l \rfloor$.

Exercise #*12*

Show that

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}.$$

Solution: To prove this let $n = 2$ and then 3. Each recursive multiplication adds 1 to the upper right corner.

Why is the GCD important? The utility of the GCD algorithm arises directly from the fundamental difficulty in factoring large integers. Computing the GCD using the Euclidean algorithm costs less than factoring when finding the coprime factors, which is extremely expensive. The utility surfaces when the two numbers are composed of very large primes.

When two integers have no common factors, they are said to be *coprime* and their GCD is 1. The ratio of two integers that are coprime is automatically in reduced form (they have no common factors). For example, $4/2 \in \mathbb{Q}$ is not reduced, since $2 = \gcd(4, 2)$ (with a zero remainder). Canceling out the common factor 2 gives the reduced form $2/1 \in \mathbb{F}$. Thus if we wish to form the ratio of two integers, we first compute the GCD, then remove it from the two numbers to form the ratio. This ensures that the rational number is in its reduced form ($\in \mathbb{F}$ rather than $\in \mathbb{Q}$). If the GCD were 10^3 digits, it is obvious that any common factor would need to be be removed, thus greatly simplifying further computation. This can make a huge difference when using IEEE-754.

The floor function and the GCD are related in an important way, as discussed next. Later we shall discuss Netwon's method for finding roots of analytic functions, such a polynomials. The two methods are closely related, since they both use Newton's method.

Indirect matrix method: As further discussed in Appendix F, a much more efficient method uses the `floor()` function, which is called *division with rounding*, or simply the *indirect method.* Specifically the GCD may be written in a single step as

$$\begin{bmatrix} m \\ n \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & -\left\lfloor \frac{m}{n} \right\rfloor \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix}_k. \tag{2.17}$$

While it is not obvious, the above matrix is Eq. 2.16 to the power $\lfloor m/n \rfloor$, followed by swapping the inputs (the smaller number must always be on the bottom). The swap is implemented by the floor function.

The GCD and multiplication: Multiplication is simply recursive addition, and finding the GCD takes advantage of this fact. For example, $3 \times 2 = 3 + 3 = 2 + 2 + 2$. Since division is the inverse of multiplication, it must be recursive subtraction.

The GCD and long division: When we learn how to divide a smaller number into a larger one, we must learn how to use the GCD. For example, suppose we wish to compute $110 \div 6$ (110/6). We start by finding out how many times 6 goes into 11. Since $6 \times 2 = 12$, which is larger than 11, the answer is 1. This is of course the floor function (e.g., $\lfloor 11/6 \rfloor = 1$). We then subtract 6 from 11 to find the remainder 5.

Example: Start with the two integers $[873, 582]$. Factoring these gives $[\pi_{25} \cdot 3^2, \pi_{25} \cdot 3 \cdot 2]$. Given the factors, we see that the GCD is the 25 prime $\pi_{25} = 97$. When we take the ratio of the two numbers, the common factor $(97 \cdot 3)$ cancels:

$$\frac{873}{582} = \frac{\cancel{\pi_{25}} \cdot \cancel{3} \cdot 3}{\cancel{\pi_{25}} \cdot \cancel{3} \cdot 2} = \frac{3}{2} = 1 + \frac{1}{2} = 1.5.$$

Dividing 582 into 873, we numerically obtain the answer $1.5 \in \mathbb{F}$. Solving this ratio in your head may not be easy. Octave gives $rat(873/582) = 2 - 1/2 = 1 + 1/2 = 3/2$.

Exercise #*13*

What is meant by "to reach the turning-point" when using the Euclidean algorithm?

Solution: When $m/n - \lfloor m/n \rfloor < n$, we are close to turning-point, and there is a small positive remainder, less than n. When the remainder is zero (i.e., $m/n - \lfloor m/n \rfloor = 0$), we are at the GCD.

Exercise #*14*

Show that in Matlab/Octave `rad(873/582)` $= 2 - 1/2$), while `rats(873/582)=3/2`, which is the correct answer. Hint: Factor the two numbers and cancel out the GCD.

Solution: Since

$$\texttt{factor(873)} = 3 \cdot 3 \cdot 97 \text{ and } \texttt{factor(582)} = 2 \cdot 3 \cdot 97,$$

the GCD is $3 \cdot 97$. Thus `3/2 = 1 + 1/2` is the correct answer. For some reason Matlab/Octave gives `rat(3/2)= 2+1/-2` , which is correct. Since the `rat(·)` function produces negative numbers, rounding up must have been employed.

For example (873/581)-3/2 over 1.5 has a relative error of 0.172%. I recommend you try this exercise. It gives insight into how Octave implements the computation.

Exercise #*15*

Divide 10 into 99. The floor function (`floor(99/10)`) must be used, followed by the remainder function (`rem(99,10)`). In this case rounding would produce large error.

Solution: When we divide a smaller number into a larger one, we must first use the floor, followed by the remainder. For example, 99/10 = 9 + 9/10 has a floor of 9 and a remainder of 9/10=0.9.

Graphical description of the GCD: The Euclidean algorithm is best viewed graphically. In Fig. 2.5 we show what is happening as one approaches the turning-point, at which point the two numbers must be swapped to keep the difference positive, which is addressed by the upper row of Eq. 2.17.

Here is a simple Matlab/Octave code to find `l=gcd(m,n)` based on the Stillwell (2010) definition:

```
  %~/M/gcd0.m
function k = gcd(m,n)
 while m ~=0
  A=m; B=n;
  m=max(A,B); n=min(A,B); %m>n
  m=m-n;
 endwhile %m=n
  k=A;
```

This program loops until $m = 0$.

Coprimes: When the GCD of two integers is 1, the only common factor is 1. This is of key importance when trying to find common factors between the two integers. When `1=gcd(m,n)`, the two integers are said to be *coprime* or *relatively prime*, which is may be written as $m \perp n$. By definition, the largest common factor of coprimes is 1. But since 1 is not prime, the two integers have no common primes. It can be shown (Stillwell, 2010, pp. 41–4) that when $a \perp b$, there exist $m, n \in \mathbb{Z}$ such that

$$am + bn = \gcd(a, b) = 1.$$

Dividing by mn gives

$$\frac{a}{n} + \frac{b}{m} = \frac{1}{mn}.$$

This linear equation may be related to the addition of two fractions that have coprime numerators ($a \perp b$). For example,

$$\frac{a}{m} + \frac{b}{n} = \frac{an + bm}{mn}.$$

Exercise #*16*
Show that

$$\begin{bmatrix} 0 & 1 \\ 1 & -\lfloor \frac{m}{n} \rfloor \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{\lfloor \frac{m}{n} \rfloor}$$

.

Solution: This exercise uses the results of the earlier Exercise # 10, times the row-swap matrix.

2.3.4 Continued fraction expansion algorithm (CFA)

As shown in Fig. 2.6, the *continued fraction expansion algorithm* (CFA) starts from a single real decimal number $x_o \in \mathbb{R}_o$ and recursively approximates it as a fraction $x \in \mathbb{F}$ (Graham et al., 1994). Thus the CFA may be used for forming rational approximations to any real (rational or irrational) number, to any desired accuracy. The famous example is the highly accurate approximation to the irrational number $\pi \approx 22/7$, well known to Chinese mathematicians.

The CFA is not to be confused with the *Euclidean algorithm* (i.e., GCD), which operates on a pair of integers $m, n \in \mathbb{N}$ and returns the *greatest common divisor* $k \in \mathbb{N}$, such that $m/k, n/k \in \mathbb{F}$ are coprime, thus reducing the ratio to its irreducible form (i.e., $m/k \perp n/k$). It is important because this result is done without factoring either m or n.

Despite this seemingly irreconcilable difference between the GCD and CFA, the two are closely related—so close that Gauss called the GCD the CFA (Stillwell, 2010, p. 48).

It is not clear how Gauss could be so "confused." One is forced to assume that Gauss had some deeper insight into this relationship. If so, it would be valuable to understand that insight[33].

Since Eq. 2.17 may be inverted, the process may be reversed, which is closely related to the CFA as discussed in Fig. 2.6. There is no way to read Gauss's mind to explore this possible insight. But it is interesting to consider the possibilities on any possible insight he might have had. Or is our observation true that these two algorithms are inverses of each other.

Definition of the CFA

1. Start with $n = 0$ and the positive input target $x_0 \in \mathbb{R}^+$. $\qquad n = 0, m_0 = 0, x_0 = \pi$
2. Rounding: Let $m_n = \lfloor x_n \rceil \in \mathbb{N}$. $\qquad m_0 = \lfloor \pi \rceil = 3$
3. The input vector is then $[m_n, x_n]^T$. $\qquad [3, \pi]^T$
4. Remainder: $r_n = x_n - m_n \qquad (-0.5 \le r_n \le 0.5)$ $\qquad r_0 = \pi - 3 \approx 0.1416$
5. Reciprocate:

$$x_{n+1} = \begin{cases} 1/r_n, \ n \leftarrow n+1; \text{ go to step 2} & r_n \ne 0 \\ 0, \quad \text{terminate} & r_n = 0 \end{cases} \qquad \begin{array}{l} x_2 = 1/0.14159 = 7.06\ldots \\ \textit{Output: } [m_n, x_{n+1}]^T = [3, 7.06]^T \end{array}$$

Figure 2.6: *Definition of the CFA of any positive number, $x_0 \in \mathbb{R}^+$,. Numerical values for $n = 0, x_0 = \pi, m_0 = 0$ are on the right. For $n = 1$ the input vector is $[m_1, x_2]^T = [3, 7.0626]^T$. If at any step the remainder is zero, the algorithm terminates (step 5). Convergence is guaranteed. The recursion may continue to any desired accuracy, and terminates if $r_n = 0$.*

Notation: Writing out all the fractions can become tedious. For example, expanding $e = 2.7183\ldots$ using the Matlab/Octave command `rat(exp(1))` gives the approximation

$$\begin{aligned} \exp(1) &= 3 + 1/(-4 + 1/(2 + 1/(5 + 1/(-2 + 1/(-7))))) - o\left(1.75 \times 10^{-6}\right) \\ &= [3; -4, 2, 5, -2, -7] - o(1.75 \times 10^{-6}). \end{aligned}$$

[33] In some sense, to be explored below, the GCD and CFA are inverse of each other.

Here we use a compact bracket notation, $\hat{e}_6 \approx [3; -4, 2, 5, -2, -7]$, where $o()$ indicates the order of the error of the CFA expansion.

Since entries are negative, we may deduce that Matlab/Octave uses rounding arithmetic (this does not seem to be documented). Note that the leading integer part m_0 may be labeled by a semicolon.[34] If the steps are carried further, the values of $m_n \in \mathbb{Z}$ give increasingly more accurate rational approximations. The five rounding schemes are discussed in Appendix 2.3.4.

Exercise #*17*

Let $x_0 \equiv \pi \approx 3.14159\ldots$. As shown in Fig. 2.7, $a_0 = 3$, $r_0 = 0.14159$, $x_1 = 7.065 \approx 1/r_0$, and $a_1 = 7$. If we were to stop here, we would have

$$\widehat{\pi}_2 = 3 + \frac{1}{7 + \cancel{0.0625}\ldots} = 3 + \frac{1}{7} = \frac{22}{7}. \tag{2.18}$$

This approximation of $\widehat{\pi}_2 = 22/7$ has a relative error of 0.04%

$$\frac{22/7 - \pi}{\pi} \approx 4 \times 10^{-4}.$$

Exercise #*18*

For a second level of approximation one may continue by reciprocating the remainder $1/0.0625 \approx 15.9966$, which rounds to 16 with a small remainder of $\approx -1/300$. Namely an excellent third order approximation to $\pi = 3.141592\cdots$ is

$$\widehat{\pi}_3 \approx 3 + 1/(7 + 1/16) = 3 + 16/(7 \cdot 16 + 1) = 3 + 16/113 = 355/113,$$

which has a relative error of 8.5×10^{-8}. If floor rounding is used ($\lfloor 15.9966 \rfloor = 15$) the resulting in a very poor rational approximation for the same number of terms. Thus there can be a dramatic difference depending on the rounding scheme, which, for clarity, should be specified, not inferred.

Rational approximation examples

$$\begin{aligned}
\hat{\pi}_2 &= \frac{22}{7} = [3; 7] && \approx \hat{\pi}_2 + o(1.3 \times 10^{-3}) \\
\hat{\pi}_3 &= \frac{355}{113} = [3; 7, 16] && \approx \hat{\pi}_3 - o(2.7 \times 10^{-7}) \\
\hat{\pi}_4 &= \frac{104,348}{33,215} = [3; 7, 16, -249] && \approx \hat{\pi}_4 + o(3.3 \times 10^{-10})
\end{aligned}$$

Figure 2.7: *The expansion of π to various orders of approximation by use of the CFA, along with the error. For example, $\hat{\pi}_2$ =22/7 has an absolute error ($|22/7 - \pi|$) of about 0.13%. It is a trivial observation that a change to any of the numbers in these approximations would change the valued. They are all approximations to π, as shown on the right of each definition. The addition of such a large number (249) in $\hat{\pi}_4$ results in an impressive reduction by three orders of magnitude. It was once rumored that the Indiana state legislature attempted to redefine π to be equal to 22/7, until it was ultimately pointed out that this was not possible.*

Exercise #*19*

Find the CFA using the floor function, to the 12th order.

Solution: When expressed in bracket notation, $\widehat{\pi}_{12} = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1]$. It is interesting that both Octave/Matlab give the same result.

Exercise #*20*

Matlab/Octave's `rat(π,1e-16)` gives:

[34] Unfortunately Matlab/Octave do not support the bracket notation.

`3 + 1/(7 + 1/(16 + 1/(-294 + 1/(3 + 1/(-4 + 1/(5 + 1/(-15 + 1/(-3)))))))).`

In bracket notation,

$$\hat{\pi}_9 = [3; 7, 16, -294, 3, -4, 5, -15, -3].$$

The error must be smaller with rounding than with truncation error, which is 266.764×10^{-9} ($|\pi - \hat{\pi}_9| \leq 4.41 \times 10^{-16}$), which is slightly larger than IEEE-754.

Exercise #*21*

Based on the several examples given above, which rounding scheme is the most accurate? Explain why.

Solution: Rounding results in a smaller remainder at each iteration and thus results in a smaller net error and faster convergence. This is because the error due to rounding has a smaller mean that that for truncation. When using floor truncation, the CFA always gives positive coefficients, which could have useful applications.

When the CFA is applied and the expansion terminates ($r_n = 0$), the target is rational. When the expansion does not terminate (which is not always easy to determine, as the remainder may be ill-conditioned due to small numerical rounding errors), the number is irrational. Thus the CFA has important theoretical applications with irrational numbers. You may explore this using Matlab's `rats(pi)` command.

In addition to these five basic rounding schemes, there are two other important $\mathbb{R} \to \mathbb{N}$ functions (i.e., mappings) that will be needed later: `mod(x,y)` and rem(x,y) with $x, y \in \mathbb{R}$. The base-10 numbers may be generated from the counting numbers using `y=mod(x,10)`.

Exercise #*22*

1. Show how to generate a base-10 real number $y \in \mathbb{R}$ from the counting numbers $\mathbb{N}$ using the `m=mod(n,10)`$+ k10$ with $n, k \in \mathbb{N}$.

   Solution: Every time n reaches a multiple of 10, m is reset to 0 and the next digit to the left is increased by 1 by adding 1 to k, generating the digit pair km. Thus the `mod()` function forms the underlying theory behind decimal notation.

2. How would you generate binary numbers (base 2) using the `mod(x,b)` function? **Solution:** Use the same method as in part 1, but with $b = 2$.

3. How would you generate hexadecimal numbers (base 16) using the `mod(x,b)` function? **Solution:** Use the same method as in part 1, but with $b = 16$.

4. Write the first 19 numbers in hex notation, starting from zero. **Solution:** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, 10, 11, 12. Recall that $10_{16} = 16_{10}$, thus $12_{16} = 18_{10}$, resulting in a total of 19 numbers if we include 0.

5. What is FF_{16} in decimal notation? **Solution:** `hex2dec('ff')` $= 255_{10}$

Symmetry: A continued fraction expansion algorithm (CFA) can have a high degree of recursive symmetry . For example, consider the CFA §2.3.4.

$$R \equiv \frac{1+\sqrt{5}}{2} = 1 + \frac{1}{1 + \frac{1}{1+\dots}} = 1.618033988749895\dots. \tag{2.19}$$

Here the remainder of the CFA is always 1, thus the sequence fails to terminate, proving that $\sqrt{5} \in \mathbb{I}$.

A similar example is true for any irrational number, such as $\sqrt{2}$, result in in $R \equiv$ `rat(1+sqrt(2))`, which gives $R = [2; 2, 2, 2, \dots]$. The periodic CFA expansion verifies that the number must be in $\mathbb{I}$. I am not aware of a method that proves that number never repeats (that it is irrational). Any such proof will become increasingly difficult as the period approaches ∞.

Note that it is impossible to bound the period of R unless it is directly observed, as in the case of 1/7 with a period of $7 - 1 = 6$. The only way I know to prove that a number is irrational, is to show that its reciprocal is not periodic. For long periods, this is not practical (it can be impossible).

When we expand a target irrational number ($x_0 \in \mathbb{I}$) and the CFA is truncated, the resulting rational fraction approximates the irrational target. From the above example, if we truncate at three coefficients ($[1; 1, 1]$), we obtain

$$1 + \frac{1}{1 + \frac{1}{1+0}} = 1 + 1/2 = 3/2 = 1.5 \neq \frac{1+\sqrt{5}}{2} + 0.118\ldots,$$

where 0.118 is the approximate error.

Truncation after six steps gives

$$[1; 1, 1, 1, 1, 1, 1] = 13/8 \approx 1.6250 \neq \frac{1+\sqrt{5}}{2} + 0.0070\ldots.$$

Because all the residues are 1, this example converges very slowly. The expansion of $\pi \in I$ has much faster convergence.

The reciprocals of all primes are rational. I have failed to find this stated as a theorem. If someone knew of an exception, it would be published.

In summary: Every rational number $m/n \in \mathbb{F}$, with $m > n > 1$, may be uniquely expanded as a continued fraction, with coefficients a_k determined using the CFA. When the target number is irrational ($x_0 \in \mathbb{Q}$), the CFA does not terminate; thus each step produces a more accurate rational approximation, converging only in the limit as $n \to \infty$.

Thus the CFA expansion is an algorithm that can, in theory, determine when the target is rational, but with an important caveat: One must determine whether the expansion terminates. This may not be obvious. The fraction $1/3 = 0.33333\ldots$ is an example of such a target, where the CFA terminates yet the fraction repeats.

It must be that

$$1/3 = 3 \times 10^{-1} + 3 \times 10^{-2} + 3 \times 10^{-3} + \ldots.$$

Here $3 \cdot 3 = 9$. As a second example,[35]

$$1/7 = 0.142857, 142857, 142857, 142857 \cdots = 14,2857 \times 10^{-6} + 142,857 \times 10^{-12} + \ldots.$$

There are several notations in the Number Theory literature for repeating decimals, such as $1/7 = 0.1\overline{142857}$ and $1/7 = 0.1((142857))_6$. Note that $142,857 = 999,999/7$. Related identities include $1/11 = 0.0909090909\ldots$. However Octave gives $11 \times 0.090909 = 0.999999999900000$, not 1.

When the sequence of digits repeats, the sequence is predictable, thus is rational, because it has a finite period. But it is impossible to be sure that it repeats, because the period can be unbounded.

In summary: We conclude that the sum of is repeating decimal cannot not be expressed in closed form other than as a fraction. By definition 1/7 is rational, however when we to express it as a number, the series converges, but only in the limit as $n \to \infty$.

There are parallels between factoring numbers as products of primes and the CFA. However this may not be documented in the number theory literature. For example factoring the product of several very large primes, can be tedious, and may even be impossible. This may be why it is useful in encryption algorithms.

Exercise #*23*

Discuss the relationship between the CFA and the transmission line modeling method.

Solution: The solution is detailed in Appendix 2.1.1.

Greatest common divisors

Consider using the *Euclidean algorithm* to find the *greatest common divisor* (i.e., GCD; the largest common prime factor) of two numbers (Allen 2020, p. 42). This algorithm may be performed using one of two methods:

[35] Using a signal processing approach, we could take the Fourier transform of the target number, represented as a sequence. This could help to identify an underlying periodic component. For example, the number $1/7 \leftrightarrow [[1, 4, 2, 8, 5, 7]]_6$ has a 50 [dB] notch at 0.8π [rad], due to its six-digit periodicity, carried to 15 digits (Matlab/Octave maximum precision), Hamming-windowed, and zero padded to 1024 samples.

Method	Division	Subtraction
On each iteration...	$a_{i+1} = b_i$ $b_{i+1} = a_i - b_i \cdot \text{floor}(a_i/b_i)$	$a_{i+1} = \max(a_i, b_i) - \min(a_i, b_i)$ $b_{i+1} = \min(a_i, b_i)$
Terminates when	$b = 0$ (GCD $= a$)	$b = 0$ (GCD $= a$)

The division method (Matlab's floor function) (Eq. 2.1, §2.1.2, Ch. 2) is preferred because the subtraction method may require a huge number of itterations steps.

Problem # *1: Understanding the* Euclidean algorithm *(GCD)*

– *1.1: Find the prime factors of* $a = 85$ *and* $b = 15$.

Solution: From Octave's `factor()` we find $85 = 17 \cdot 5$, $15 = 3 \cdot 5$.

– *1.2: What is the greatest common prime factor of* $a = 85$ *and* $b = 15$*?*

Solution: The largest common factor `gcd(85,15)` is 5.

– *1.3: By hand, perform the Euclidean algorithm for* $a = 85$ *and* $b = 15$.

Solution: Division method:

$$a_1 = 15 \qquad b_1 = 85 - 15\left\lfloor \cancelto{75}{\frac{85}{15}} \right\rfloor = 10$$

$$a_2 = 10 \qquad b_2 = 15 - 10\left\lfloor \cancelto{2}{\frac{15}{10}} \right\rfloor = 5$$

$$a_3 = 5 \qquad b_3 = 10 - 5\left\lfloor \cancelto{2}{\frac{10}{5}} \right\rfloor = 0$$

$\therefore$ gcd$= 5$

Subtraction method:

$$\begin{aligned}
a_1 &= 85 - 15 = 70 & b_1 &= 15\\
a_2 &= 70 - 15 = 55 & b_2 &= 15\\
a_3 &= 55 - 15 = 40 & b_3 &= 15\\
a_4 &= 40 - 15 = 25 & b_4 &= 15\\
a_5 &= 25 - 15 = 10 & b_5 &= 15\\
&\text{swap} & &\\
a_6 &= 15 - 10 = 5 & b_6 &= 10\\
a_7 &= 10 - 5 = 5 & b_7 &= 5\\
&\text{terminate} & &
\end{aligned}$$

$\therefore$ gcd$= 5$

– *1.4: By hand, perform the Euclidean algorithm for* $a = 75$ *and* $b = 25$. *Is the result a prime number?*

Solution: Division method:

$$a_1 = 25 \qquad b_1 = 75 - 25\left\lfloor \cancelto{3}{\frac{75}{25}} \right\rfloor = 0$$

Subtraction method:

$$\begin{aligned}
a_1 &= 75 - 25 = 50 & b_1 &= 25\\
a_2 &= 50 - 25 = 25 & b_2 &= 25
\end{aligned}$$

$\therefore$ gcd$= 25$
The result is $25 = 5^2$, the *square* of a prime number.

– 1.5: Consider the first step of the GCD division algorithm when $a < b$ (e.g., $a = 25$ and $b = 75$). What happens to a and b in the first step? Does it matter if you begin the algorithm with $a < b$ rather than $b < a$?
Solution: If $a < b$, the first step of the division algorithm swaps the terms ($a \to b$ and $b \to a$).

– 1.6: Describe in your own words how the GCD algorithm works. Try the algorithm using numbers that have already been divided into factors (e.g., $a = 5 \cdot 3$ and $b = 7 \cdot 3$).
Solution: Division method:

$$a_1 = 5 \cdot 3 \qquad b_1 = 7 \cdot 3 - 5 \cdot 3 \lfloor \cancelto{1}{\frac{7 \cdot 3}{5 \cdot 3}} \rfloor = 2 \cdot 3$$

$$a_2 = 2 \cdot 3 \qquad b_2 = 5 \cdot 3 - 2 \cdot 3 \lfloor \cancelto{2}{\frac{5 \cdot 3}{2 \cdot 3}} \rfloor = 1 \cdot 3$$

$$a_3 = 1 \cdot 3 \qquad b_3 = 2 \cdot 3 - 1 \cdot 3 \lfloor \cancelto{2}{\frac{2 \cdot 3}{1 \cdot 3}} \rfloor = 0$$

Subtraction method:

$$\begin{aligned}
a_1 &= 7 \cdot 3 - 5 \cdot 3 = 2 \cdot 3 & b_1 &= 5 \cdot 3 \\
a_2 &= 5 \cdot 3 - 2 \cdot 3 = 3 \cdot 3 & b_2 &= 2 \cdot 3 \\
a_3 &= 3 \cdot 3 - 2 \cdot 3 = 1 \cdot 3 & b_3 &= 2 \cdot 3 \\
a_4 &= 2 \cdot 3 - 1 \cdot 3 = 1 \cdot 3 & b_4 &= 1 \cdot 3
\end{aligned}$$

The algorithm iteratively converges on the GCD by subtracting out multiples of the GCD until only the GCD is left.

– 1.7: Find the GCD of $2 \cdot \pi_{25}$ and $3 \cdot \pi_{25}$.
Solution: π_{25}
.

Problem # 2: *Coprimes*

– 2.1: Define the term coprime.
Solution: when two integers have no common factors the are said to be *coprime*

– 2.2: How can the Euclidean algorithm be used to identify coprimes?
Solution: If `gcd(a,b)=1` they only have 1 as a common factor, thus they are coprime.

– 2.3: Give an important application of the Euclidean algorithm.
Solution: The obvious application is in network security using two-key encryption methods. Given two integers $n, d \in \mathbb{Z}$, if we wish to reduce the fraction n/d, we must cancel the common factors. Example: If $n = 9, d = 6$ then $9/6 = (3 \cdot 3)/(2 \cdot 3) = 3/2$, where the GCD, 3, may be identified using the Euclidean algorithm. While this fraction may be easily simplified via inspection, the GCD algorithm could be very helpful for larger numbers n, d.

– 2.4: Write a Matlab function, `function x = my_gcd(a,b)`, that uses the Euclidean algorithm to find the GCD of any two inputs `a` and `b`. Test your function on the (a, b) combinations from the previous problem. Include a printout (or hand-write) your algorithm to turn in.
Hints and advice:

- Don't give your variables the same names as Matlab functions! Since `gcd` is an existing Matlab/Octave function, if you use it as a variable or function name, you won't be able to use `gcd` to check your `gcd()` function. Try `clear all` to recover from this problem.

- Try using a "while" loop for this exercise (see Matlab documentation for help).

- You may need to use some temporary variables for `a` and `b` in order to perform the algorithm.

Solution: Division method:

```
function x = my_gcd(a,b)
while b>0
atmp= a; btmp = b;
a = btmp; b = atmp-btmp*floor(atmp/btmp);
end
```

Subtraction method:

```
function x = my_gcd(a,b)
while a~=b
atmp= a; btmp = b;
a = max(atmp,btmp) - min(atmp,btmp); b = min(atmp,btmp);
end
```

Algebraic generalization of the GCD (Euclidean) algorithm

Problem # *3: In this problem we are looking for integer solutions* $(m, n) \in \mathbb{Z}$ *to the equations* $ma + nb = gcd(a, b)$ *and* $ma + nb = 0$ *given positive integers* $(a, b) \in \mathbb{Z}^+$*.* This requires that either m or n is negative. The solution may be found using the Euclidean algorithm.

Example: `gcd(2,3)= 1`: For $(a, b) = (2, 3)$, the result is

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & 1\\3 & -2\end{bmatrix}}_{m \quad n}\begin{bmatrix}2\\3\end{bmatrix}.$$

From the above equation we find the solution (m, n) to the integer equation

$$2m + 3n = \texttt{gcd}(2, 3) = 1;$$

namely, $(m, n) = (-1, 1)$ (i.e., $-2 + 3 = 1$). There is also a second solution $(3, -2)$ (i.e., $3 \cdot 2 - 2 \cdot 3 = 0$). Thus these two solutions are a pair, and the solution exists only if (a, b) are coprime ($a \perp b$).

– 3.1: By inspection, find at least one integer pair (m, n) *that satisfies* $12m + 15n = 3$*.*
Solution: By inspection, $(m, n) = (-1, 1)$ is one solution.

– 3.2: Using matrix methods for the Euclidean algorithm, find integer pairs (m, n) *that satisfy* $12m + 15n = 3$ *and also for* $12m + 15n = 0$*. Show your work!!!*
Solution: If we look for solutions in terms of (m, n) the both cases have solutions. For the first case where RHS = 3, $m = -1$ and $n = 1$, and $-12 + 15 = 3$. For the case of the RHS = 0, $m = 5$ and $n = -4$, thus $12 * 5 - 15 * 4 = 0$.

– 3.3: Does the equation $12m + 15n = 1$ *have integer solutions for* n *and* m*? Why or why not?*
Solution: No, because $\gcd(12, 15) = gcd(3 * 4, 3 * 5) = 3$, not 1. Thus there are no Diophantine solutions to this equation.

Problem # *4: Matrix approach:*
It can be difficult to keep track of the a's and b's when the algorithm has many steps. We need an alternative way to run the Euclidean algorithm using matrix algebra. Matrix methods provide a more transparent approach to the operations on (a, b). Thus the Euclidean algorithm can be classified in terms of standard matrix operations. Write out the indirect matrix approach discussed at the end of §2.3.3 (Eq. 2.17).

Solution: Division method:
Define

$$\begin{bmatrix} a \\ b \end{bmatrix}_0 = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}, \qquad \begin{bmatrix} a \\ b \end{bmatrix}_{i+1} = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix}_i \begin{bmatrix} a \\ b \end{bmatrix}_i$$

Prime numbers

Problem # *5: Every integer may be written as a product of primes.*

– 5.1: Write the numbers $1,000,000$, $1,000,004$, *and* $999,999$ *in the form* $N = \prod_k \pi_k^{\beta_k}$.

Hint: Use Matlab/Octave to find the prime factors.

Solution: $1,000,000 = 2^6 \cdot 5^6$
$1,000,004 = 2^2 \cdot 53^2 \cdot 89$
$999,999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$

– 5.2: Give a generalized formula for the natural logarithm of a number $\ln(N)$ *in terms of its primes* π_k *and their multiplicities* β_k. *Express your answer as a* sum *of terms.*

Solution: $\ln N = \sum_k \beta_k \ln(\pi_k)$

Problem # *6: Using the computer*

– 6.1: Explain why the following brief Matlab/Octave program returns the prime numbers π_k *between 1 and 100.*

```
n=2:100;
k = isprime(n);
n(k)
```

Solution: The first line $n = 2 : 100$ defines the row vector $n = [2, 3, 4, \cdots, 100]$. The second line creates a row vector the same length as n, with entries of 1 if the element is prime and zero if the element is not prime. The third line $n(k)$ prints out $n()$ if $k = 1$, namely it is a list of all the primes from 2 to 100. Run this program without the ';' at the end of each line, and to see what it is doing.

– 6.2: How many primes are there between 2 *and* $N = 100$?

Solution: `length(n(k))` returns 25. Thus there are 25 primes less than 100 ($N/4$, on average).

Problem # *7: Prime numbers may be identified using a sieve.*

– 7.1: To find the period of any prime π_k, *print out the reciprocal of the prime.*

Solution: If the period is larger than IEEE-754, then we need to solve for the period by a recurrsive approximation method. To do this subtract the first approximation for the reciprocal. Then recompute the reciprocal with the estimate of the remainder removed. At each step, the error will be reduced. After several itterations, the reciprocal will start to repeat. At this point the period has been identified.

– 7.2: By hand, complete the sieve of Eratosthenes for $n = 1, \ldots, 49$. Start by writing out a table of the integers 1-50, as 5 rows of 10 numbers. Starting with the first prime, $p_k = 2$, $k = 1$, circle it *and* cross out *all multiples (e.g., $2\pi_k = r, 3\pi_k = 6, \cdots, 24 * \pi_2 = 48.$). Then repeat for the second, third, and higher primes π_2. When done, only the circled primes should remain. Be sure you look up the definition of a prime.*

Solution: Note: The number 1 should not be circled since it is *not* a prime.

– 7.3: What is the largest number you need to consider before only primes remain? Look up the definition of the Matlab/Octave floor *function (e.g, $\lfloor \pi \rfloor = 3$).*

Solution: $\lfloor \sqrt{50} \rfloor = \lfloor 7.0711 \rfloor = 7.$

– 7.4: Generalize: For $n = 1, \ldots, N$, what is the largest number you need to consider before only the primes remain?

Solution: floor($\sqrt{N}$)

– 7.5: Write each of these numbers as a product of primes: 22, 30, 34, 43, 44, 48, 49.

Solution: *22*= $2 \cdot 11 = \pi_1 \pi_5$
30= $2 \cdot 3 \cdot 5 = \pi_1 \pi_2 \pi_3$
34= $2 \cdot 17 = \pi_1 \pi_7$
43= π_{14}
44= $4 \cdot 11 = \pi_1^2 \pi_5$
48= $4 \cdot 12 = 4^2 \cdot 3 = \pi_1^4 \pi_2$
49= $7^2 = \pi_4^2$

– 7.6: Find the largest prime $\pi_k \leq 100$. Do not use Matlab/Octave other than to check your answer. Hint: Write the numbers starting with 100 and count backward: 100, 99, 98, 97, ... Cross off the even numbers, leaving $99, 97, 95, \ldots$. Pull out a factor (only one is necessary to show that it is not prime).

Solution: 99=11*9, $\pi_{25} = 97$.

– 7.7: Find the largest prime $\pi_k \leq 1000$. Do not use Matlab/Octave other than to check your answer.

Solution: Write out the numbers starting with 1000 and counting backwards: 1000, 999, 998, 997, $\cdots$. Cross off the even numbers, leaving $999, 997, 995, \cdots$. Pull out a factor (only one is necessary to show that it is not prime). $9 \cdot 111, 997 = \pi_{168}, 5 \cdot 199 = \pi_3 \cdot \pi_{46}$.

– 7.8: Explain why $\pi_k^{-s} = e^{-s \ln \pi_k}$.

Solution: This follows from the identify $z^a = e^{a \ln z}$ with $a, z \in \mathbb{C}$.

Problem # *8:CFA of ratios of large primes*

– 8.1: Expand 23/7 as a continued fraction. Express your answer in bracket notation (e.g., $\pi = [3., 7, 16, \cdots]$). Show your work. **Solution:** $23/7 = (21+2)/7 = 3 + 2/7 = 3 + 1/(6+1)/2 = 3 + 1/(6 + 1/2)$. In bracket notation $23/7 = [3., 6, 2]$. Matlab gives `rat(23/7)` $= 3 + 1/(4 + 1/(-2))$, or [1., 4, -2] because rounding $7/2$ can be taken as either 3+1/2 or 4-1/2.

– 8.2:Starting from the primes below 10^6, form the CFA of π_j/π_k with $j = 78498$ and $k < j$.

Solution: First generate 10^6 primes with the matlab command `π=primes(11+1e6)`.

The length of π is $j = 78499$, $\pi(j) = 1,000,003$, $\pi(j-1) = 999,983$ and $\pi(j-2) = 999,979$.

Let the target fraction be

$$T = \frac{\pi(end-1)}{\pi(end-2)} = \frac{999983}{999979} = 1.000004000084002.$$

Finding the CFA of T gives

$$rat(T) = 1 + 1/249995 = [1; 249995].$$

Factoring this integer gives `factor(249995)=5*49999`.

– 8.3: Look at other ratios of prime numbers and look for a pattern in the CFA of the ratios of large primes. What is the most obvious conclusion? **Solution:** *The CFA terminates in only one term, as in the above example.*

– 8.4: Try the Matlab/Octave functions `rats(23/7)`, `rats(3.2857)`, and `rats(3.2856`
What an you conclude?

Solution: This function is similar to the CFA but uses rounding rather than truncation arithmatic. `rats(3.2857)=32`
but `rats(23/7)=23/7` because it rounds to 23/7, whereas `rats(3.2856)`= 4107/1250 because it does not.

– 8.5: Can $\sqrt{2}$ be represented as a finite continued fraction? Why or why not?

Solution: No, because it is irrational.

– 8.6: What is the CFA for $\sqrt{2} - 1$?

$$\text{Hint:} \qquad \sqrt{2} + 1 = \frac{1}{\sqrt{2}-1} = [2; 2, 2, 2, \cdots].$$

Solution: $1 + \sqrt{2} = 2 + 1/(2 + 1/(2 + \cdots))$ or $[2., 2, 2, 2, \cdots]$, thus

$$\sqrt{2} - 1 = [2., 2, 2, 2, \cdots] - 2 = 0 + 1/(2 + 1/(2 + 1/(2 + \cdots))).$$

Repeat this for $2 + \sqrt{5}$, $2 + \sqrt{3}$

– 8.7: Show that

$$\frac{1}{1-\sqrt{a}} = a^{\frac{11}{2}} + a^{\frac{9}{2}} + a^{\frac{7}{2}} + a^{\frac{5}{2}} + a^{\frac{3}{2}} + \sqrt{a} + a^5 + a^4 + a^3 + a^2 + a + 1 = 1 - a^6$$

```
syms a,b
b= taylor(1/( 1-sqrt(a) ))
simplify((1-sqrt(a))*b) = 1-a^6
```

Use symbolic analysis to show this, then explain. **Solution:** This seems like a very unlikely relationship. Unexpectedly the coefficents of this expansion are all 1, leading to a is a sixth degree polynomial. It is obvioiusly related to the six complex roots of unity. Thus we may find the companion matrix, followed by an eigen solution. This seems to be a Taylor expansion of six roots of unity, expressed in terms of removable singularities. See Cotes Theorem (1716) (Stillwell, 2010, p. 289).

2.3.5 Pythagorean triplets (Euclid's formula)

Euclid's formula is a method for finding three integer lengths $[a, b, c] \in \mathbb{N}$ that satisfy Eq. 1.1. It is important to ask "Which set are the lengths $[a, b, c]$ drawn from?" There is a huge difference, both practical and theoretical, when they come from pairs of real numbers $\mathbb{R}$ or from the counting numbers $\mathbb{N}$. We must be careful if we need a negative length.

Given $p, q \in \mathbb{N}$ with $p > q$, the three lengths $[a, b, c] \in \mathbb{N}$ of Eq. 1.1 (p. 3) are given by

$$a = p^2 - q^2, \qquad b = 2pq, \qquad c = p^2 + q^2. \tag{2.20}$$

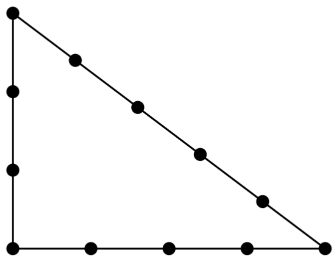


Figure 2.8: *Beads on a string form perfect right triangles when the number of unit lengths between beads on each side satisfies Eq. 1.1. For example, when $p = 2, q = 1$, the sides are $[3, 4, 5]$.*

This result may be directly verified, since

$$[p^2 + q^2]^2 = [p^2 - q^2]^2 + [2pq]^2$$

or

$$p^4 + q^4 + \cancel{2p^2q^2} = p^4 + q^4 - \cancel{2p^2q^2} + \cancel{4p^2q^2}.$$

Thus, Eqs. 1.1 are easily proved. Deriving Euclid's formula (see AE-2, problem #15). is obviously much more difficult, and is similar to the proof of Pell's equation.

Table 2.1: *Table of Pythagorean triplets computed from Euclid's formula, Eq. 1.1, for various $[p, q]$. The last three columns are the first, fourth, and penultimate values of Plimpton-322, along with their corresponding $[p, q]$. In all cases $c^2 = a^2 + b^2$ and $p = q + l$, where $l = \sqrt{c - b} \in \mathbb{N}$.*

q	1	1	1	2	2	2	3	3	3	5	54	27
l	1	2	3	1	2	3	1	2	3	7	71	23
p	2	3	4	3	4	5	4	5	6	12	125	50
a	3	8	15	5	12	21	7	16	27	119	12709	1771
b	4	6	8	12	16	20	24	30	36	120	13500	2700
c	5	10	17	13	20	29	25	34	45	169	18541	3229

A well-known example is the right triangle depicted in Fig. 2.8, defined by the integer lengths $[3, 4, 5]$ that have angles $[0.54, 0.65, \pi/2]$ [rad], which satisfies Eq. 1.1. As quantified by Eq. 1.1, there are an infinite number of Pythagorean triplets (PTs). Furthermore, the seemingly simple triangle that has angles of $[30, 60, 90] \in N$ [deg] (i.e., $[\pi/6, \pi/3, \pi/2] \in \mathbb{I}$ [rad]) has one irrational ($\mathbb{I}$) length ($[1, \sqrt{3}, 2]$).

The set from which the lengths $[a, b, c]$ are drawn was not missed by the early Asians and was documented by the Greeks. Any equation whose solution is based on integers is called a *Diophantine equation*, named for the Greek mathematician Diophantus of Alexandria (ca. 250 CE) (Fig. 1.1)

A clay tablet from the 19th century BCE with the numbers engraved on it, as shown in Fig. 1.1, was discovered in Mesopotamia, and cataloged in 1922 by George A. Plimpton.[36] These numbers are a and c pairs from PTs $[a, b, c]$. Given this discovery, it is clear that the Pythagoreans were following those who came long before them. Recently a second tablet, dating between 350 and 50 BCE, has been reported, that indicates calculations of the apparent motion of Jupiter based on a trapezoidal graph of the rate.[37] It is interesting that PTs play a role on atomic physics, as discussed in Appendix (pp. 206–210.)

[36] `https://www.nytimes.com/2010/11/27/arts/design/27tablets.html`

[37] `https://www.nytimes.com/2016/01/29/science/babylonians-clay-tablets-geometry-astronomy-jupiter.html`

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
481	769
4961	8161
45	75
1679	2929
161	289
1771	3229
56	106

Figure 2.9: *Plimpton-322, a clay tablet from 1800 BCE that displays a and c values of the Pythagorean triplets $[a, b, c]$, with the property $b = \sqrt{c^2 - a^2} \in \mathbb{N}$. Several of the c values are primes, but not the a values. The clay is item 322 (item 3 from 1922) from the collection of George A. Plimpton.*

Pell's equation (see p. 52)

$$x_n^2 - Ny_n^2 = (x_n - \sqrt{N}y_n)(x_n + \sqrt{N}y_n) = 1, \tag{2.21}$$

with non square $N \in \mathbb{N}$ and $x_n, y_n \in \mathbb{N}$ as unknowns, has a venerable history in both physics and mathematics. Given its factored form, it is obvious that every solution x_n, y_n has the asymptotic property

$$\left.\frac{x_n}{y_n}\right|_{n\to\infty} \to \pm\sqrt{\pm N}. \tag{2.22}$$

It is believed that Pell's equation is directly related to the Pythagorean theorem, since both are simple binomials that have integer coefficients (Stillwell, 2010, p. 48), with Pell's equation being the hyperbolic version of Eq. 1.1. For example, with $N = 2$, a solution is $x = 17, y = 12$ (i.e., $17^2 - 2 \cdot 12^2 = 1$).

A 2×2 matrix recursion algorithm, likely due to the Chinese and used by the Pythagoreans to investigate $\sqrt{N}$, is

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n, \tag{2.23}$$

where we indicate the index outside the vectors.

Starting with the trivial solution $[x_o, y_o]^T = [1, 0]^T$ (i.e., $x_o^2 - Ny_o^2 = 1$), additional solutions of Pell's equations are determined, having the property $\boldsymbol{x}_n/y_n \to \sqrt{N} \in \mathbb{F}$, motivated by Euclid's formula for Pythagorean triplets (Stillwell, 2010, p. 44).

Note that Eq. 2.23 is a 2×2 linear matrix composition method, since the output of one matrix multiplication is the input to the next.

Asian solutions: The first solution of Pell's equation was published in about 628 CE by Brahmagupta, who first discovered the equation (Stillwell, 2010, p. 46). Brahmagupta's novel solution also used the composition method, but in a different way from Eq. 2.23. Then in 1150 CE, Bhaskara II independently obtained solutions using Eq. 2.23 (Stillwell, 2010, p.69). This is the composition method we shall explore here, as summarized in Appendix B, Table A.1.

The best way to see how this recursion results in solutions to Pell's equation is by example. Initializing the recursion with the trivial solution $[x_0, y_0]^T = [1, 0]^T$ gives

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_0 \qquad 1^2 - 2 \cdot 1^2 = -1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_1 \qquad 3^2 - 2 \cdot 2^2 = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_3 = \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 \qquad (7)^2 - 2 \cdot (5)^2 = -1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_4 = \begin{bmatrix} 17 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix}_3 \qquad 17^2 - 2 \cdot 12^2 = 1$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_5 = \begin{bmatrix} 41 \\ 29 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix}_4 \qquad (41)^2 - 2 \cdot (29)^2 = -1$$

Solution to Pell's equation: By multiplying the matrix by $1\jmath$, all the solutions ($\boldsymbol{x}_k \in \mathbb{C}$) to Pell's equation are determined. The $1\jmath$ factor alters the sign so every iteration yields a solution. For $N = 2$, $n = 0$ (the initial solution), $[x_0, y_0]$ is $[1, 0]_0$, $[x_1, y_1] = \jmath[1, 1]_1$, and $[x_2, y_2] = -[3, 2]_2$. These are easily checked using this recursion.

The solution for $N = 3$ is given in Appendix A.2.1. Table A.1 shows that every output of this modified matrix recursion gives solutions to Pell's equation: $[1, 0], [1, 1], [4, 2], [10, 6], \ldots, [76, 44], \ldots$.

At each iteration, the ratio x_n/y_n approaches $\sqrt{2}$, converging like the CFA to $\sqrt{N}$. The value of $41/29 \approx \sqrt{2}$, with a relative error of <0.03%.

2.3.6 Fibonacci sequence

Another classic problem, also formulated by the Chinese, is the Fibonacci sequence, generated by the relationship

$$f_{n+1} = f_n + f_{n-1}. \tag{2.24}$$

Here the next number $f_{n+1} \in \mathbb{N}$ is the sum of the previous two. If we start from $[0, 1]$, this linear recursion equation leads to the Fibonacci sequence $f_n = [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots]$. Alternatively, if we define $y_{n+1} = x_n$, then Eq. 2.24 may be compactly represented by a 2×2 companion matrix recursion.

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}_n, \tag{2.25}$$

which has eigenvalues $(1 \pm \sqrt{5})/2$.

The correspondence of Eqs. 2.24 and 2.25 is easily verified. Starting with $[x, y]_0^T = [0, 1]^T$, we obtain for the first few steps:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_0, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_1, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_2, \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3, \quad \ldots.$$

From the above, $x_n = [0, 1, 1, 2, 3, 5, \ldots]$ is the Fibonacci sequence, since the next x_n is the sum of the previous two, and the next y_n is x_n.

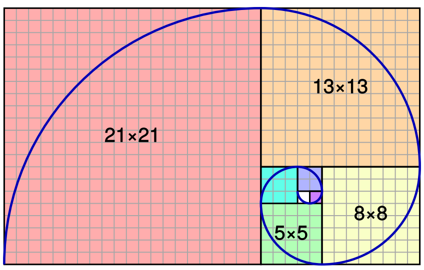


Figure 2.10: *A construction called the Fibonacci spiral. Note how it is constructed of squares that have areas given by the squares of the Fibonacci numbers. In this way, the spiral is smooth and the radius increases as the Fibonacci numbers (e.g., $8 = 3 + 5$, $13 = 5 + 8$, etc.). (Adapted from* `https://en.wikipedia.org/wiki/Golden_spiral`*)*

Exercise #*24*

Use the Octave/Matlab command `compan(c)` to find the companion matrix of the polynomial coefficients defined by Eq. 2.24.

Solution: Using Matlab/Octave: `f=[1, -1, -1]; C=compan(f);`

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.26}$$

Exercise #*25*

Find the eigenvalues of matrix C.

Solution: The characteristic equation is

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

or $\lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - 1/4 - 1 = 0$, which has roots $\lambda_\pm = (1 \pm \sqrt{5})/2 \approx \{1.618, -0.618\}$.

The mean-Fibonacci sequence: Suppose that the Fibonacci sequence recursion is replaced by the mean of the last two values–namely, let

$$f_{n+1} = \frac{f_n + f_{n-1}}{2}. \tag{2.27}$$

This seems like a small change. But how does the solution differ? To answer this question it is helpful to look at the corresponding 2×2 matrix.

Exercise #*26*

Find the 2×2 matrix corresponding to Eq. 2.27. The 2×2 matrix may be found using the *companion matrix* method.

Solution: Using Matlab/Octave code, we have `f=[1, -1/2, -1/2]; C=compan(f);` which returns

$$C = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}. \tag{2.28}$$

Exercise #*27*

Find the steady-state solution for the mean-Fibonacci, starting from $[1, 0]_0$. State the nature of both solutions.

Solution: By inspection one steady-state solution is $[1, 1]^T_\infty$ or $f_n = 1$. To find the full solution, we need to find the two eigenvalues, defined by

$$\det \begin{bmatrix} 1/2-\lambda & 1/2 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda/2 - 1/2 = (\lambda - 1/4)^2 - (1/4)^2 - 1/2 = 0.$$

Thus $\lambda_\pm = (1 \pm 3)/4 = [1, -0.5]$. The first solution converges to 1 while the second solution is $(-1/2)^n$, which changes sign at each time step and quickly converges to zero. The full solution is given by $\boldsymbol{e}\Lambda^n\boldsymbol{e}^{-1}[1, 0]^T$ (see Appendix A.

Relationships to digital signal processing: Today we recognize Eq. 2.24 as a discrete difference equation, which is a pre-limit (pre–Stream 3) recursive form of a differential equation. The 2×2 matrix form of Eq. 2.24 is an early precursor to 17th- and 18th-century developments in linear algebra. Thus the Greeks' recursive solution for the $\sqrt{2}$ and Bhaskara's solution of Pell's equation are early precursors to discrete-time signal processing as well as to calculus.

There are strong similarities between Pell's equation and the Pythagorean theorem. As we shall see, Pell's equation is related to the geometry of a hyperbola, just as the Pythagorean equation is related to the geometry of a circle. We shall show, as one might assume, that there is a counterpart to Euclid's formula for the case of Pell's equations, since these are all conic sections with closely related conic geometry. As we have seen, the solutions involve $\sqrt{-1}$. The derivation is a trivial extension of that for Euclid's formula for Pythagorean triplets. The early solution of Brahmagupta was not related to this simple formula.

2.3.7 Diagonalization of a matrix (eigenvalue/eigenvector decomposition)

As derived in Appendix A, the most efficient way to compute A^n is to diagonalize the matrix A by finding its eigenvalues and eigenvectors.

The eigenvalues λ_k and eigenvectors $\vec{e}_k$ of a square matrix A are related by

$$A\vec{e}_k = \lambda_k \vec{e}_k, \tag{2.29}$$

such that multiplying an eigenvector $\vec{e}_k$ of A by the matrix A is the same as multiplying by a scalar, $\lambda_k \in \mathbb{C}$ (the corresponding eigenvalue). The complete eigenvalue problem may be written as

$$AE = E\Lambda. \tag{2.30}$$

If A is a 2×2 matrix,[38] the matrices E and Λ (of eigenvectors and eigenvalues, respectively) are

$$E = [\vec{e}_1, \vec{e}_2] \qquad \text{and} \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{2.31}$$

Thus the matrix equation $AE = [A\vec{e}_1 \;\; A\vec{e}_2] = [\lambda_1\vec{e}_1 \;\; \lambda_2\vec{e}_2] = E\Lambda$ contains Eq. 2.31 for each eigenvalue-eigenvector pair λ_1, λ_2.

The diagonalization of the matrix A refers to the fact that the matrix of eigenvalues, Λ, has nonzero elements only on the diagonal. The key result is found by post-multiplication of the eigenvalue matrix by E^{-1}, giving

$$AEE^{-1} = A = E\Lambda E^{-1}. \tag{2.32}$$

If we now take powers of A, the nth power of A is

$$\begin{aligned} A^n &= (E\Lambda E^{-1})^n \\ &= E\Lambda E^{-1} E\Lambda E^{-1} \ldots E\Lambda E^{-1} \\ &= E\Lambda^n E^{-1}. \end{aligned} \tag{2.33}$$

This is a very powerful result because the nth power of a diagonal matrix is:

$$\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}.$$

Thus, from Eq. 2.33 we can calculate A^n using only two matrix multiplications:

$$A^n = E\Lambda^n E^{-1}.$$

2.3.8 Finding the eigenvalues:

The eigenvalues λ_k are determined from Eq. 2.31, by factoring out $\vec{e}_k$:

$$\begin{aligned} A\vec{e}_k &= \lambda_k \vec{e} \\ (A - \lambda_k I)\vec{e}_k &= \vec{0}. \end{aligned}$$

Matrix $I = [1, 0; 0, 1]^T$ is the identity matrix, having the dimensions of A, with elements δ_{ij} (i.e., diagonal elements $\delta_{11,22} = 1$ and off-diagonal elements $\delta_{12,21} = 0$).

[38] These concepts naturally generalize to higher dimensions.

The vector $\vec{e}_k$ is not zero, yet when operated on by $A - \lambda_k I$, the result must be zero. The only way this can happen is if the operator is degenerate (has no solution) — that is,

$$\det(A - \lambda I) = \det \begin{bmatrix} (a_{11} - \lambda) & a_{12} \\ a_{21} & (a_{22} - \lambda) \end{bmatrix} = 0. \tag{2.34}$$

This means that the two equations have the same roots (the equation is degenerate).

This determinant equation results in a second-degree polynomial in λ:

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0,$$

the roots of which are the eigenvalues of the matrix A.

The details: An eigenvector $\vec{e}_k$ can be found for each eigenvalue λ_k from Eq. 2.31 (p. 55),

$$(A - \lambda_k I)\vec{e}_k = \vec{0}.$$

The left side of the above equation becomes a column vector, where each element is an equation in the elements of $\vec{e}_k$, set equal to 0 on the right side. These equations are always degenerate, since the determinant is zero. Thus the two equations have the same slope (see Appendix A.3 p. 264).

Solving for the eigenvectors is often confusing because they have arbitrary magnitudes, $||\vec{e}_k|| = \sqrt{\vec{e}_k \cdot \vec{e}_k} = \sqrt{e_{k,1}^2 + e_{k,2}^2} = d$. From Eq. 2.31, we can determine only the relative magnitudes and signs of the elements of $\vec{e}_k$, so we have to choose a magnitude d. It is common practice to normalize each eigenvector to have unit magnitude ($d = 1$).

2.4 Summary

1. Number systems are the underbelly of mathematics. Their importance was first appreciated by the early Chinese culture.

2. It took thousands of years for this ideas to peculate to the west. The actual history is unknown. Our historical timelines in the first few figures attempt to tell this story. In the next chapter we continue this history. It is clear that the destruction of the library of Alexandria was due to recognition of the power of this specific knowledge. Maps were a key factor in this cerebral knowledge.

3. Counting numbers and integers were obvious, but the finer points, such as prime numbers, rational and irrational numbers, and ultimately complex numbers were not as obvious. Newton called them *imaginary* because he assumed they were not useful. Of course once algebra became part of the story, and roots of polynomials were explored, the complex number could no longer be ignore. It was simply too late to give the concept a more reasonable name. Complex numbers were in no way *imaginary*, but they could not be called *real*, so they were stuck with the strange name. Today a complex number has a real and imaginary part, both of which are well defined numbers numbers $[\lambda_1, \lambda_2]^T \in \mathbb{C}$,

Pythagorean triplets

Problem # *9: Euclid's formula for the Pythagorean triplets a, b, c is $a = p^2 - q^2$, $b = 2pq$, and $c = p^2 + q^2$.*

– 9.1: What condition(s) must hold for p and q such that a, b, and c are always positive and nonzero?

Solution: $p > q > 0$ (strictly greater than)■

– 9.2: Solve for p and q in terms of a, b, and c.

Solution:

Method 1: Given a, c, one may find p, q via matrix operations by solving the *nonlinear system of equations* for p, q.

First solve linear system of equations for p^2, q^2:

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p^2 \\ q^2 \end{bmatrix}$$

Inverting this 2x2 matrix gives (the determinant $\Delta = 2$)

$$\begin{bmatrix} p^2 \\ q^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.$$

Thus $p = \pm\sqrt{(a+c)/2}$, $q = \pm\sqrt{(c-a)/2}$.

Method 2: The algebraic approach is:

$$a + c = (p^2 - q^2) + (p^2 + q^2) = 2p^2$$

$$-a + c = -(p^2 - q^2) + (p^2 + q^2) = 2q^2,$$

Thus $p = \sqrt{(a+c)/2}$, $q = \sqrt{(c-a)/2}$, where $p, q \in \mathbb{N}$. Method 1 seems more "transparent" than Method 2. ■

Problem # *10: The ancient Babylonians (ca. 2000 BCE) cryptically recorded* (a, c) *pairs of numbers on a clay tablet, archeologically denoted* Plimpton-322*).*

– 10.1: Find JBA p *and* q *for the first five pairs of* a *and* c *shown here from Plimpton-322.*

Table 1: First five (a,c) pairs of Plimpton-322.

Solution:

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97

■ Find a formula for a in terms of p and q. **Solution:**

$$\begin{array}{ll}
(a,c) = (119, 169) & (p,q) = \pm(12, 5) \\
(a,c) = (3367, 4825) & (p,q) = \pm(64, 27) \\
(a,c) = (4601, 6649) & (p,q) = \pm(75, 32) \\
(a,c) = (12709, 18541) & (p,q) = \pm(125, 54) \\
(a,c) = (65, 97) & (p,q) = \pm(9, 4)
\end{array}$$

■

– 10.2: Based on Euclid's formula, show that $c > (a, b)$*.*

Solution: $c - a = (p^2 + q^2) - (p^2 - q^2) = 2q^2$
Because $2q^2$ is always positive, $c > a$
$c - b = (p^2 + q^2) - 2pq = (p - q)^2 > 0$
Note that by the definition of $p, q \in \mathbb{N}$, $p > q$. ■

– 10.3: What happens when $c = a$*?*

Solution: Then its not a triangle since $b = 0$. The triangle is degenerate.■

– 10.4: Is $b + c$ a perfect square? Discuss.

Solution: $b + c = p^2 + 2pq + q^2 = (p + q)^2$. Since p and q are integers, $b + c$ will always be a perfect square ($\sqrt{b + c}$ will always be an integer). ■

Pell's equation:

Problem # *11: Pell's equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{I}$. We seek integer solutions of*

$$x^2 - Ny^2 = 1.$$

As shown in §A.2, the solutions x_n, y_n for the case of $N = 2$ are given by the linear 2×2 matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1\jmath = \sqrt{-1} = e^{j\pi/2}$. It follows that the general solution to Pell's equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{\jmath\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

To calculate solutions to Pell's equation using the matrix equation above, we must calculate

$$A^n = e^{\jmath\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n = e^{\jmath\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

which becomes tedious for $n > 2$.

– 11.1: Find the companion matrix and thus the matrix A that has the same eigenvalues as Pell's equation. Hint: Use Matlab's function `[E,Lambda] = eig(A)` to check your results!

Solution: The companion matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

■

– 11.2: Solutions to Pell's equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell's equation is relevant to $\sqrt{2}$.

Solution: As discussed in §2.5.2, as the iteration n increases, the ratio of the x_n/y_n approaches $\sqrt{2}$. ■

– 11.3: Find the first three values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell's equation for $N = 2$. By hand, find the eigenvalues $\lambda_\pm$ of the 2×2 Pell's equation matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Solution: The eigenvalues are given by the roots of the equation $(1 - \lambda_\pm)^2 = 2$. Thus $\lambda_\pm = 1 \pm \sqrt{2} = \{2.1412, -.4142\}$■

– 11.4: By hand, show that the matrix of eigenvectors, E, is

$$E = [\vec{e}_+ \;\; \vec{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}.$$

Solution: The eigenvectors $\vec{e}_\pm$ may be found by solving

$$A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda_\pm \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \rightarrow (A - \lambda_\pm I) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$$

For λ_+, this gives

$$0 = \begin{bmatrix} 1-(1+\sqrt{2}) & 2 \\ 1 & 1-(1+\sqrt{2}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

which gives the relation between the elements of $\vec{e}_+$, e_1, e_2, as $e_1 = \sqrt{2}e_2$.

The eigenvectors are defined to be unit length and orthogonal, namely

1. $||\vec{e}_k||^2 = \vec{e}_k \cdot \vec{e}_k = 1$
2. $\vec{e}_+ \cdot \vec{e}_- = 0$.

Once we normalize $\vec{e}_+$ to have unit length, we obtain the first eigenvector

$$\vec{e}_+ = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

Repeating this for λ_- gives

$$\vec{e}_- = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

Thus, the matrix of eigenvalues is

$$E = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$

■

– 11.5: Using the eigenvalues and eigenvectors you found for A, verify that

$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

Solution: Using the formula for a matrix inverse, we find

$$E^{-1} = \frac{1}{\det(E)} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} = \frac{3}{-2\sqrt{2}} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}$$

Thus

$$\begin{aligned} E^{-1}AE &= \frac{-\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix} \\ &= \frac{-1}{2\sqrt{2}} \begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} (-\sqrt{2}+2) & (\sqrt{2}+2) \\ (-\sqrt{2}+1) & (\sqrt{2}+1) \end{bmatrix} \\ &= \begin{bmatrix} 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{bmatrix} = \Lambda \end{aligned}$$

■

– 11.6: Once you have diagonalized A, use your results for E and Λ to solve for the $n = 10$ solution $(x_{10}, y_{10})^T$ to Pell's equation with $N = 2$.

Solution: $x_{10} = -3363$ and $y_{10} = -2378$. Note this formulation gives the negative solution, but since the values for $n = 10$ are real, when they are squared in Pell's equation, it makes no difference whether they are negative or positive.■

The Fibonacci sequence

The Fibonacci sequence is famous in mathematics and has been observed to play a role in the mathematics of genetics. Let x_n represent the Fibonacci sequence,

$$x_{n+1} = x_n + x_{n-1}, \tag{2.35}$$

where the current input sample x_n is equal to the sum of the previous two inputs. This is a "discrete time" recurrence relationship. To solve for x_n, we require some initial conditions. In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ for $n = 0, 1, 2, 3, \ldots$.

Equation 2.35 is equivalent to the 2×2 matrix equations

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.36}$$

Problem # *12: Here we seek the general formula for x_n. Like Pell's equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast x_n as a 2×2 matrix relationship and then proceed, as we did for the Pell case.*

– 12.1: Show that the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$ may be generated by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.37}$$

Solution: Using the Matrix Eigen-equation, powers of the eigen equation $\mathbf{A}^n = \mathbf{E}\Lambda^n\mathbf{E}^{-1}$. The final solution is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \boldsymbol{e} \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n \boldsymbol{e}^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{2.38}$$

■

– 12.2: What is the relationship between y_n and x_n?

Solution: This equation says that $x_n = x_{n-1} + y_{n-1}$ and $y_n = x_{n-1}$. The latter equation may be rewritten as $y_{n-1} = x_{n-2}$. Thus

$$x_n = x_{n-1} + x_{n-2}$$

as requested. ■

– 12.3: Write a Matlab/Octave program to compute x_n using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is x_{40}? Note: Consider using the eigenanalysis of A, described by Eq. 2.33 of the text.

Solution: You can try something like:

```
function xn = fib(n)
A = [1 1; 1 0]; [E,D] = eig(A); xy = E*D^n*inv(E)*[1; 0];
xn = xy(1);
```

For this initial condition, $x_{40} = 165,580,141 = \frac{1}{\sqrt{5}}\left(\frac{(1+\sqrt{5}}{2}\right)^{41}$. ■

– 12.4: Using the eigenanalysis of the matrix A (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \tag{2.39}$$

– 12.5: What are the eigenvalues $\lambda_\pm$ of the matrix A?

Solution: The eigenvalues of the Fibonacci matrix are given by

$$det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0,$$

thus $\lambda_\pm = \frac{1\pm\sqrt{5}}{2} = [1.618, -0.618]$. ■

– 12.6: How is the formula for x_n related to these eigenvalues? Hint: Find the eigenvectors.

Solution: The eigenvectors (determined from the equation $(A - \lambda_\pm I)\vec{e}_\pm = \vec{0}$, and normalized to 1) are given by

$$\vec{e}_+ = \begin{bmatrix} \frac{\lambda_+}{\sqrt{\lambda_+^2+1}} \\ \frac{1}{\sqrt{\lambda_+^2+1}} \end{bmatrix} \qquad \vec{e}_- = \begin{bmatrix} \frac{\lambda_-}{\sqrt{\lambda_-^2+1}} \\ \frac{1}{\sqrt{\lambda_-^2+1}} \end{bmatrix} \qquad E = [\vec{e}_+ \quad \vec{e}_-]$$

From the eigenanalysis, we find that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = E \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} \frac{1}{(e_{11}e_{22} - e_{12}e_{21})} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving for x_n we find that

$$\begin{aligned} x_n &= \frac{1}{(e_{11}e_{22} - e_{12}e_{21})} \left(\lambda_+^n e_{11} e22 - \lambda_-^n e_{12} e_{21} \right) \\ &= \frac{1}{\frac{\sqrt{5}}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}} \left[\lambda_+^n \left(\frac{\lambda_+^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}} \right) - \lambda_-^n \left(\frac{\lambda_-^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}} \right) \right] \\ &= \frac{1}{\sqrt{5}} \left[\lambda_+^{n+1} - \lambda_-^{n+1} \right] \end{aligned}$$

■

– 12.7: What happens to each of the two terms

$$\left[\frac{1 \pm \sqrt{5}}{2} \right]^{n+1}$$

as $n \to \infty$?

Solution: $[\frac{1+\sqrt{5}}{2}]^{n+1} \to 0$ and $[\frac{1+\sqrt{5}}{2}]^{n+1} \to \infty$ ■

– 12.8: What happens to the ratio x_{n+1}/x_n as $n \to \infty$ for

$$x_n \approx \left[\frac{(1+\sqrt{5})}{2} \right]^{n+1} ? \tag{2.40}$$

Solution: $x_{n+1}/x_n \to (1+\sqrt{5})/2$, because $\left((1-\sqrt{5})/2\right)^n \to 0$ ■

Problem # *13: Replace the Fibonacci sequence with the average of the previous two values in the sequence.*

$$x_n = \frac{x_{n-1} + x_{n-2}}{2},$$

– 13.1: What matrix A is used to calculate this sequence?

Solution:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

■

– 13.2: Modify your computer program to calculate the new sequence x_n. What happens as $n \to \infty$?

Solution: As $n \to \infty$, $x_n \to 2/3$ ■

– 13.3: What are the eigenvalues of the modified A? How do they relate to the behavior of x_n as $n \to \infty$? Hint: You can expect the closed-form expression for x_n to be similar to Eq. 2.39.

Solution: The eigenvalues are $\lambda_+ = 1$ and $\lambda_- = -0.5$. From Eq. 2.33, the expression for A^n is

$$A^n = (E\Lambda E^{-1})^n = E\Lambda^n E^{-1} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n = \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix}.$$

The solution is the sum of two sequences, one a constant and the other an oscillation that quickly fades. As $n \to \infty$, $\lambda_+^n = 1^n \to 1$ and $\lambda_-^n = (-1/2)^n \to 0$. The solution becomes

$$x_n = \frac{2}{3}\left[\lambda_+^n - \lambda_-^n\right] = \frac{2}{3}\left[1^n - (-1)^n\right] \to \frac{2}{3}.$$

■

Problem # *14: Consider the expression*

$$\sum_{1}^{N} f_n^2 = f_N f_{N+1}.$$

– 14.1: Find a formula for f_n that satisfies this relationship. Hint: It holds for the Fibonacci recursion formula. Calculate the eigen values, and explain what is going on (Allen, 2025).

Solution: Write this out for N and $N-1$:

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_{N-1}^2 + f_N^2 &= f_N f_{N+1} \\ f_1^2 + f_2^2 + \cdots + f_{N-1}^2 \qquad &= f_{N-1} f_N \end{aligned}$$

Subtracting gives

$$\begin{aligned} &f_N^{\cancel{2}} = \cancel{f_N} f_{N+1} - f_{N-1}\cancel{f_N} = \cancel{f_N}(f_{N+1} - f_{N-1}) \\ &f_N = f_{N+1} - f_{N-1} \end{aligned}$$

Thus the relation only holds for the Fibonacci recursion formula. ■

Problem # *15: The CFA may be writen as a matrix recursion. For this we adopt a special notation, unlike other matrix notations,*[39] *with $k \in \mathbb{N}$:*

$$\begin{bmatrix} n \\ x \end{bmatrix}_{k+1} = \begin{bmatrix} 0 & \lfloor x_k \rfloor \\ 0 & \frac{1}{x_k - \lfloor x_k \rfloor} \end{bmatrix} \begin{bmatrix} n \\ x \end{bmatrix}_k.$$

This equation says that $n_{k+1} = \lfloor x_k \rfloor$ and $x_{k+1} = 1/(x_k - \lfloor x_k \rfloor)$. It does *not* mean that $n_{k+1} = \lfloor x_k \rfloor x_k$, as would be implied by standard matrix notation. The lower equation says that $r_k = x_k - \lfloor x_k \rfloor$ is the *remainder*—namely, $x_k = \lfloor x - k \rfloor + r_k$ (Octave/Matlab's `rem(x, floor(x))` function), also known as `mod(x,y)`.

– 15.1: Start with $n_0 = 0 \in \mathbb{N}$, $x_0 \in \mathbb{I}$, $n_1 = \lfloor x_0 \rfloor \in \mathbb{N}$, $r_1 = x - \lfloor x \rfloor \in \mathbb{I}$, and $x_1 = 1/r_1 \in \mathbb{I}$, $r_n \neq 0$. For $k = 1$ this generates on the left the next CFA parameter $n_2 = \lfloor x_1 \rfloor$ and $x_2 = 1/r_2 = 1/(x_0 - \lfloor x_0 \rfloor)$ from n_0 and x_0. Find $[n, x]_{k+1}^T$ for $k = 2, 3, 4, 5$.

Solution: If $x_0 = \pi$, then $n_1 = \lfloor \pi \rfloor = 3$, $r_1 = \pi - n_1 = 0.14159\cdots$, and $x_1 = 1/r_1 \approx 7.06$:

$$\begin{bmatrix} 3 \\ 7.06251 \end{bmatrix}_1 = \begin{bmatrix} 0 & \lfloor \pi \rfloor \\ 0 & \frac{1}{\pi - \lfloor \pi \rfloor} \end{bmatrix} \begin{bmatrix} 0 \\ \pi \end{bmatrix}_0$$

and for $n = 2$

$$\begin{bmatrix} 7 \\ 15.99659 \end{bmatrix}_2 = \begin{bmatrix} 7 \\ \frac{1}{0.06251} \end{bmatrix}_2 = \begin{bmatrix} 0 & 7 \\ 0 & \frac{1}{7.0625-7} \end{bmatrix} \begin{bmatrix} 3 \\ 7.06251 \end{bmatrix}_1$$

For $n = 3$, $\pi_3 = [n_1; n_2, n_3] = [3; 7, 15]$. Continuing $n_4 = \lfloor 1.003418 \rfloor = 1$ and $n_5 = 292$. ■

[39] This notation is highly nonstandard due to the nonlinear operations. The matrix elements are *derived* from the vector rather than multiplying them. These calculation may be done with the help of Matlab/Octave.

Chapter 3

Stream 2: Algebraic Equations

3.1 Algebra and geometry as physics

Stream 2 is geometry, which led to the merging of Euclid's geometrical methods and the development of algebra by al-Khwarizmi in 830 CE (Fig. 1.1). This migration of ideas led Descartes and Fermat to develop analytic geometry (Fig. 1.2.

The mathematics up to the time of the Greeks, documented and formalized by Euclid, served students of mathematics for more than two thousand years. Algebra and geometry were, at first, independent lines of thought. When merged, the focus returned to the Pythagorean theorem. Algebra generalized the analytic conic section into the complex plane, greatly extending the geometrical approach as taught in Euclid's *Elements*. With the introduction of algebra, numbers, rather than lines, could be used to reproduce geometrical lengths in the complex plane. Thus the appreciation for geometry grew after the addition of rigorous analysis using numbers.

History of Mathematics after the 15th Century

16th Bombelli 1526–1572; Galileo 1564–1642; Kepler 1571–1630; Mersenne 1588–1648;

17th Huygens 1629–1695; Newton 1642–1727[a], *Principia* 1687; Bernoulli, Jakob 1655–1705; Bernoulli, Johann 1667–1748; Fermat, Pierre de 1607–1665; Pascal, Blaise 1623–1662; Descartes, René 1596–1648

18th Bernoulli, Daniel 1700–1782; Euler, Leonhard 1707–1783; d'Alembert, Jean le Rond 1717–1783; Lagrange, Joseph-Louis 1736–1833; Laplace 1749–1827; Fourier 1768–1830; Gauss 1777–1855; Cauchy 1789–1857

19th Helmholtz 1821–1894; Kelvin 1824–1907; Kirchhoff 1824–1887; Riemann 1826–1866; Maxwell 1831–1879; Rayleigh 1842–1919; Heaviside 1850–1925; Poincare 1854–1912; Hilbert 1862–1942; Einstein 1879–1955; Fletcher 1884–1981; Sommerfeld 1886–1951; Brillouin 1889–1969; Nyquist 1889–1976

20th Bode 1905–1982

[a]Born Dec 25, 1642, Julian calendar

Example: Given the polynomial $P_2 = 1 - s^2$ we may use Newton's method to find the roots. Since $P_2'(s) = -2s$, Newton's iteration becomes

$$s_{n+1} = s_n + \frac{1 - s_n^2}{2s_n}.$$

From the Gauss-Lucas theorem for the case of $N = 2$, the root of $P_2'(s)$ is always the average of the roots of $P_2(s)$. To start the iteration ($n = 0$) we need an initial random guess, s_0. The only place we may not start is at a root of P_N'.

For our case where $P_2(s) = 1 - s^2$,

$$s_1 = s_0 + \frac{1 - s_0^2}{2s_0} = s_0 + \frac{1}{2}\left(-s_0 + 1/s_0\right).$$

Exercise #*1*

Let $P_2(s) = 1 - s^2$. Choose the starting point as $s_0 = 1/2$. Draw a graph describing the first step of the iteration.

Solution: We start with an $s \in \mathbb{C}$ coordinate system and put points at $s = (1/2, 0)$ and the vertex of $P_2(s)$; that is, $(0, 1)$ $(P_2(0) = 1)$. Then we draw $1 - s^2$, along with a line from s to s.

Exercise #*2*

Starting from Exercise #2, what does the iteration converge to? What are the roots of P_2?

Solution: First we must find $P_2'(s) = -2s$. Thus the equation we will iterate is

$$s_{n+1} = s_n + \frac{1 - s_n^2}{2s_n} = \frac{s_n^2 + 1}{2s_n} = (s_n + 1/s_n)/2.$$

By hand,

$$\begin{aligned}
s_0 &= 1/2 \\
s_1 &= \frac{(1/2)^2 + 1}{2(1/2)} = \frac{1}{4} + 1 = 5/4 = 1.25 \\
s_2 &= \frac{(5/4)^2 + 1}{2(5/4)} = \frac{(25/16) + 1}{10/4} = \frac{1}{2}\left(\frac{5}{4} + \frac{4}{5}\right) = \frac{41}{40} = 1.025.
\end{aligned}$$

These estimates rapidly approach the positive real root $s = 1$. Note that if one starts at the root of $P'(s) = 0$ (i.e., $s_0 = 0$), the first step is indeterminate.

Exercise #*3*

Write an Octave/Matlab script to check your answer for part Exercise #3.

Solution:

```
s=1/2;
for n = 1:3
s = s+(1-s*s)/(2*s);
end
```

Exercise #*4*

For $n = 4$, what is the absolute difference between the root and the estimate, $|s_r - s_4|$?

Solution: 4.6E-8 (very small!)

Exercise #*5*

What happens if $s_0 = -1/2$?

Solution: The solution converges to the negative root, $s = -1$.

Exercise #*6*

Does Newton's method work for $P_2(s) = 1 + s^2$ (Kelley, 2003)? Hint: What are the roots in this case?

Solution: In this case $P_2'(s) = +2s$; thus the iteration gives

$$s_{n+1} = s_n - \frac{1 + s_n^2}{2s_n}.$$

The roots are purely imaginary, $s_\pm = \pm 1\jmath$. Newton's method works fine as long as you use complex arithmetic.

Exercise #7

What if you let $s_0 = 1 + \jmath$ for the case of $P_2(s) = 1 + s^2$?

Solution:

```
s=1+j;
for n = 1:4
s = s-(1+s*s)/(2*s);
end
```

After 4 steps $s_4 = -0.0000046418 + 1.0000021605i$. After 6 steps $s_6 = 8.46e - 23 + i$. On the 7th step the result is exact.

If you use only real arithmetic, obviously Newton's method fails, because there is no way for the answer to become complex. If, like Newton, you didn't believe in complex numbers, your method would fail to converge to the complex roots (i.e., Real in = Real out). This is because Octave/Matlab assumes $x \in \mathbb{R}$ if it is initialized as $\mathbb{R}$. By starting with a complex initial value, we fix the Real in = Real out problem.

Basic properties of polynomials

In some sense polynomials such as $P_N(z)$ are the simplest constructions used in algebra, and a summary of their most basic properties is helpful.

1. The degree of the polynomial is N.
2. In mathematical physics and engineering it is common to have real coefficients a_n, but complex coefficients are possible.
3. The coefficients of every polynomial are determined by its Taylor series—namely, Eq. 3.2.3,
4. If the coefficients are real and positive, then the $P_N(x)$ is positive and real if $x \geq 0$
5. The fundamental theorem of algebra states that $P_N(z)$ has exactly N roots.
6. The number of coefficients of the monomial $M_N(x)$ is equal to N, thus the number of roots.
7. The roots of polynomials with positive and real coefficients typically have complex roots—that is, if $P_N(z_k) = 0$, then $z_k \in \mathbb{C}$.
8. The region of convergence (RoC) of every polynomial about the expansion point is infinite.
9. The roots of the derivative of a polynomial lie within the convex hull defined by the roots of $P_N(z)$, as described by the Gauss-Lucas theorem (see discussion below Eq. 3.1).
10. The eigenvalues of the companion matrix are identical to the roots as the monic $M_N(x)$.
11. Poles in each of the four quadrants define the properties of its impulse response.
12. Poles in the left half plane are causal ($h(t) = 0$ for $t < 0$).
13. Poles in the right half plane are anti-causal ($h(t) = 0$ for $t > 0$).

These last two constraints require that for any function of complex frequency $H(s)$, the impulse response $h(t)$ is uniquely defined. As a result, the order of the poles specified in $P_N(s)$ is irrelevant to the properties of the impulse response

$$h_N(t) = \sum_k h_k(t) \leftrightarrow H_N(s) = \sum_k H_1(s - s_k),$$

where $h_k(t) \leftrightarrow H_1(s - s_k)$ due to linear superposition.

Exercise #8

Find the logarithmic derivative of $f(x)g(x)$.

Solution: From the definition of the logarithmic derivative and the chain rule for the differentiation of a product, we have

$$\begin{aligned}\frac{d}{dx}\ln f(x)g(x) &= \frac{d}{dx}\ln f + \frac{d}{dx}\ln g\\ &= \frac{1}{f}\frac{d}{dx}f + \frac{1}{g}\frac{d}{dx}g.\end{aligned}$$

Example: If we assume that function $P_3(s) = (s-a)^2/(s-b)^\pi$, then

$$\ln P_3(s) = 2\ln(s-a) - \pi\ln(s-b)$$

and

$$\frac{d}{ds}\ln P_3(s) = \frac{2}{s-a} - \frac{\pi}{s-b}.$$

From this simple example we find the residues to be 2 and $-\pi$.

Reduction by the logarithmic derivative to simple poles: As shown for $P_3(s)$ of the previous example, a function that has poles of arbitrary degree (i.e., π in the example) may be reduced to the sum of two functions having simple poles by taking the logarithmic derivative, since

$$L_N(s) = \frac{N(s)}{D(s)} = \frac{d}{ds}\ln P_N(s) = \frac{P_N'(s)}{P_N(s)}. \tag{3.1}$$

Here the polynomial is the denominator $D(s) = P_N(s)$, while the numerator $N(s) = P_N'(s)$ is the derivative of $N(s)$. This can greatly reduce the algebra when calculating residues, since the logarithmic derivative reduces higher-order poles, even those of irrational degree, to simple poles (those of degree 1).

The logarithmic derivative $L_N(s)$ has the following special properties:

1. $L_N(s)$ has simple poles s_p and zeros s_z.
2. The poles of $L_N(s)$ are the zeros of $P_N(s)$.
3. The zeros of $L_N(s)$ (i.e., $P_N'(s_z) = 0$) are the zeros of $P_N'(s)$.
4. $L_N(s)$ is analytic everywhere other than at its poles.
5. Since the zeros of $P_N(s)$ are simple (no second-order poles), the zeros of $L_N(s)$ always lie close to the line connecting the two poles. One may easily demonstrate the truth of the statement numerically, and it has been quantified by the Gauss-Lucas theorem, which specifies the relationship between the roots of a polynomial and those of its derivative. Specifically, the roots of P_{N-1}' lie inside the convex hull of the roots of P_N.
6. The eigenvalues s_k of the companion matrix are equal to the roots of the monomial $M_N(s)$ To understand the meaning of the term *convex hull*, consider the following construction: If stakes are placed at each of the N roots of $P_N(x)$, and a string is then wrapped around the stakes, with all the stakes inside the string, the convex hull is then the closed set inside the string. One can begin to imagine how the $N-1$ roots of the derivative must evolve with each set inside the convex hull of the previous set. This concept may be recused to smaller values of N.
7. Newton's method may be expressed in terms of the reciprocal of the logarithmic derivative, since

   $$s_{k+1} = s_k + \eta L_N^{-1}(s_k),$$

   where the *step size* η is used to control the rate of convergence of the algorithm. If the step size is too large, the root-finding path may jump to a different domain of convergence and thus a different root of $P_N(s)$ (Allen (2025)).

8. Not surprisingly, given all the special proprieties, $L_N(s)$ plays an key role in mathematical physics.

Euler's product formula: Counting may be written as a linear recursion simply by adding 1 to the previous value, starting from 0. The even numbers may be generated by adding 2, starting from 0. Multiples of 3 may be similarly generated by adding 3 to the previous value, starting from 0. Such recursions are fundamentally related to prime numbers $\pi_k \in \mathbb{P}$, as first investigated by Euler. This logic is the basis of the sieve. The basic idea is both simple and important, taking almost everyone by surprise, likely even Euler. It is related to the old idea that the integers may be generated by the geometric series when viewed as a recursion.

Example: Let's look at counting modulo prime numbers. For example, if $k \in \mathbb{N}$, then

$$k \cdot \mathrm{mod}(k,2), \quad k \cdot \mathrm{mod}(k,3), \quad k \cdot \mathrm{mod}(k,5)$$

are all multiples of the primes $\pi_1 = 2$, $\pi_2 = 3$, and $\pi_3 = 5$.

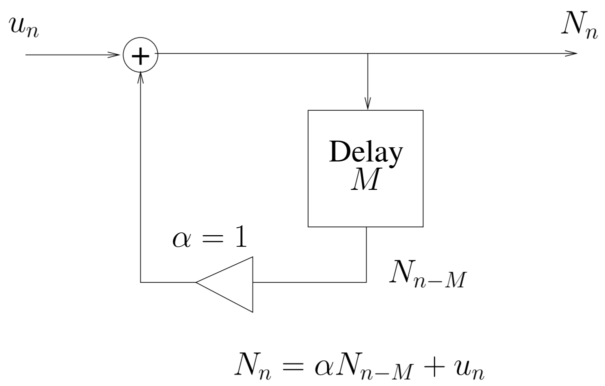


Figure 3.1: *This feedback network describes the linear discrete-time difference equation with delay M [s] given by Eq. 3.2. If $M = 1$ this circuit acts as an integrator. When the input is a step function, the output will be $N_n = nu_n = [0, 1, 2, 3, \ldots]$. Such discrete–time circuits are called digital filters.*

To see this, we define the *step function* $u_n = 0$ for $n < 0$ and $u_n = 1$ for $n \geq 0$ and the *counting number function* $N_n = 0$ for $n < 0$. The counting numbers may be recursively generated from the recursion

$$N_{n+1} = N_{n-M} + u_n, \tag{3.2}$$

which for $M = 1$ gives $N_n = n$. For $M = 2$, $N_n = 0, 2, 4, \ldots$ gives the even numbers.

As was first published by Euler in 1737, one may recursively factor out the leading prime term, resulting in Euler's product formula. Based on the argument given in the discussion of the sieve, one may automate the process. The lowest number on this list is the next prime. One may then recursively generate all the multiples of this new prime and remove them from the list. Any numbers that remain are candidates for the remaining primes.

The observation that this procedure may be automated with a recursive filter, such as that shown in Fig. 3.1, implies that it may be transformed into the frequency domain and described in terms of its poles. For example, the poles of the filter shown in Fig. 3.1 may be determined by taking the z-transform of the recursion equation and solving for the roots of the resulting polynomial.

Exercise #9

Show that $N_n = n$ follows from the above recursion.

Solution: If $n = -1$, we have $N_n = 0$ and $u_n = 0$. For $n = 0$ the recursion gives $N_1 = N_0 + u_0$; thus $N_1 = 0+1$. When $n = 1$, we have $N_2 = N_1 + 1 = 1 + 1 = 2$. For $n = 2$, the recursion gives $N_3 = N_2 + 1 = 3$. Continuing the recursion, we find that $N_n = n$. Today we denote a recursions of this form as a *digital-filter*. The state diagram for N_n is shown in Fig. 3.1.

To start the recursion, we define $u_n = 0$ for $n < 0$. Thus $u_0 = u_{-1} + 1$. But since $u_{-1} = 0$, $u_0 = 1$. The counting numbers follow from this recursion. A more understandable notation is convolution of the step function with itself—namely,

$$nu_n = u_n \star u_n = \sum_{m=0}^{\infty} u_m u_{m-n} \leftrightarrow \frac{1}{(1-z)^2},$$

which says that the counting numbers $\hat{\mathbf{n}} \in \mathbb{N}$ are easily generated by convolution, which corresponds to a second-order pole at $z = 1$ in the z-transform frequency domain (see Sec. 3.3.1).

Exercise #*10*

Write an Octave/Matlab program that generates the odd numbers $N_n = \{1, 0, 3, 0, 5, 0, 7, 0, 9, \ldots\}$ by removing the even numbers.

Solution:

```
M=50; N=(0:M-1);
u=ones(1,M); u(1)=0;
Dem=[1 1]; Num=[1];
n=filter(Num,Dem,u);
y2=n.*N; F1=N-y2
```

which gives: $F1 = [0, 1, 0, 3, 0, 5, 0, 7, 0, 9, 0, \ldots]$.

An alternative is to use the `mod(n,N)` function:

```
M=20; L=2; n=0:M; k=mod(n,L); m=(k==0).*n;
```

which generates the even numbers.

Exercise #*11*

Write a program to down-sample N_n by 2:1.

Solution:

```
N=[1 0 3 0 5 0 7 0 9 0 11 0 13 0 15]
M=N(1:2:end);
```

which gives: $M = [1, 3, 5, 7, 9, 11, 13, 15, \ldots]$

For the next step toward a full sieve (Fig. 2.4), we would need to generate all the multiples of 3 (the second prime) and remove these these from the list.

3.1.1 Matrix formulation of a polynomial

There is a simple relationship between every constant coefficient differential equation, its characteristic polynomial, and the equivalent matrix form of that differential equation, known as the *companion matrix*. The roots of the monic polynomial are the eigenvalues of the companion matrix C_N (Horn and Johnson, 1988, p. 147).

The companion matrix: The $N \times N$ *companion matrix* is defined as

$$C_N = \begin{bmatrix} 0 & & & & & & -c_0 \\ 1 & 0 & & & \mathbf{0} & & -c_1 \\ 0 & 1 & 0 & & & & -c_2 \\ \vdots & 0 & 1 & 0 & \cdots & & \vdots \\ & & \cdots & \ddots & 0 & & \vdots \\ & \mathbf{0} & & & 1 & 0 & -c_{N-2} \\ & & & & 0 & 1 & -c_{N-1} \end{bmatrix}_{N\times N}. \tag{3.3}$$

The constants c_{N-n} are from the monic polynomial of degree N,

$$\begin{aligned} P_N(s) &= s^N + c_{N-1}s^{N-1} + \cdots + c_2 s^2 + c_1 s + c_0 \\ &= s^N + \sum_{n=0}^{N-1} c_n s^n, \end{aligned}$$

which has as its coefficient vector

$$\mathbf{c}_N = [1, c_{N-1}, c_{N-2}, \ldots, c_1, c_0]^T.$$

Any transformation of a matrix that leaves the eigenvalues invariant (e.g., C_N^T) results in an equivalent definition of C_N. Note that the Octave/Matlab companion matrix function `C=compan(A)` returns the coefficient vector along the top row rather than the right-most column.

Example: Matlab/Octave returns the companion matrix in a different format with the coefficients along the top row rather than on the final column. For example if $P_3(s) = [1, c_2, c_1, c_0]$ then Octave/Matlab returns

$$P_3(s) = \begin{bmatrix} -c_2 & -c_1 & -c_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Exercise #*12*

Show that the eigenvalues of the 3×3 companion matrix are the same as the roots of $P_3(s)$.

Solution: Expanding the determinant of $C_3 - sI_3$ along the rightmost column, we get

$$P_3(s) = -\begin{vmatrix} -s & 0 & -c_0 \\ 1 & -s & -c_1 \\ 0 & 1 & -(c_2+s) \end{vmatrix} = c_0 + c_1 s + (c_2+s)s^2 = s^3 + c_2 s^2 + c_1 s + c_0.$$

Setting this to zero gives the requested result.

Exercise #*13*

Find the companion matrix for the *Fibonacci sequence* defined by the difference equation

$$f_{n+1} = f_n + f_{n-1},$$

initialized with $f_n = 0$ for $n < 0$ and $f_0 = 1$.

Solution: Taking the z-transform gives the polynomial $(z^1 - z^0 - z^{-1})F(z) = 0$. The coefficient vector $\mathbf{c} = [1, -1, -1]^T$. The corresponding Fibonacci companion matrix is

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The Matlab/Octave companion matrix routine `compan(C)` uses an alternative definition, but has the same eigenvalues.

Example: A polynomial is represented in Matlab/Octave in terms of its coefficient vector. When the polynomial vector for the poles of a differential equation is normalized such that $c_N = 1$, thus

$$\boldsymbol{c}_N = [1, c_{N-1}, c_{N-2}, \ldots, c_0]^T.$$

This normalization guarantees the requirement that the leading term is not zero, and that the number of roots (N) is equal to the degree of the monic polynomial.

3.1.2 Working with polynomials in Matlab/Octave

In Matlab/Octave there are eight key related functions. It is important to try these examples, to assure that you fully understand them.

1. `R=roots(A)`: Vector $\mathrm{A} = [a_N, a_{N-1}, \ldots, a_0] \in \mathbb{C}$ are the complex coefficients of polynomial $P_N(z) = \sum_{n=0}^N a_n z^n \in \mathbb{C}$, where $N \in \mathbb{N}$ is the degree of the polynomial. If we manually set $a_N = 1$ by dividing A/a_N, we guaranteeing it cannot be zero.

   Example: `roots([1, -1])=1`

2. `y=polyval(A,x)`: This evaluates the polynomial defined by vector $\mathtt{A} \in \mathbb{C}^N$ evaluated at $\mathtt{x} \in \mathbb{C}$, returning vector $\mathtt{y(x)} \in \mathbb{C}$. If `R` is the vector of roots $[z_1, z_2, \ldots, z_n] \in \mathbb{C}$ then `polyval(A,R)=0`.

   Example: `polyval([1 -1],1)=0,polyval([1, 1],3)=4`

3. `P=poly()`: This is the inverse of `root()`, returning a vector of polynomial coefficients $\mathtt{P} \in \mathbb{C}^N$ of the corresponding characteristic polynomial, starting from either a vector of roots R or a matrix A, for example, defined with the roots on the diagonal. The characteristic polynomial is defined as the determinant of $|A - \lambda I| = 0$ that has roots R.

   Example: `poly([1])=[1, -1],poly([1,2])=[1,-3,2]`

   Due to IEEE-754 scaling issues, this can give strange results that are numerically correct, but within the limits of IEEE-754 accuracy.

4. `R=polyder(C)`: This routine takes the N coefficients of polynomial C and returns the $N-1$ coefficients of the derivative $P'(x)$ of the polynomial. This is useful when working with Newton's method, where each step is proportional to $P_N(x)/P'_{N-1}(x)$.

   Example: `polyder([1,1,1,1,1,1])= [5 4 3 2 1]`

5. `[K,R]=residue(N,D)`: Given the ratio of two polynomials `N,D`, `residue(N,D)` returns vectors `K,R` such that
$$\frac{N(s)}{D(s)} = \sum_k \frac{K_k}{s - s_k}, \tag{3.4}$$
   where $s_k \in \mathbb{C}$ are the roots of the denominator `D` polynomial and $\mathtt{K} \in \mathbb{C}$ is a vector of residues, which characterize the roots of the numerator polynomial $N(s)$. This is one of the most valuable time-saving routines I know Allen (2025). The Octaves command `help residue` is informative.

   Example: `residue(2, [1 0 -1])= [1 -1]`

6. `C=conv(A,B)`: Vector $\mathtt{C} \in \mathbb{C}^{N+M-1}$ contains the polynomial coefficients of the convolution of the two vectors of coefficients of polynomials $\mathtt{A,B} \in \mathbb{C}^N$ and $\mathtt{B} \in \mathbb{C}^M$.

   Example: `[1, 2, 1]=conv([1, 1], [1, 1])`

7. `[C,R]=deconv(N,D)`: Vectors $\mathtt{C,\ N,\ D} \in \mathbb{C}$. This operation uses long division of polynomials to find $C(s) = N(s)/D(s)$ with remainder $R(s)$, where `N = conv(D,C)+R`, which is
$$C = \frac{N}{D} \text{ with remainder } R. \tag{3.5}$$

   Example: By defining the coefficients of two polynomials as $A = [1, a_1, a_2, a_3]$ and $B = [1, b_1, b + 2]$, we can find the coefficients of the product from `C=conv(A,B)` and recover B from C with `B=deconv(C,A)`.

8. `A=compan(D)`: Vector $D = [1, d_{N-1}, d_{N-2}, \ldots, d_0]^T \in \mathbb{C}$ contains the coefficients of the monic polynomial
$$D(s) = s^N + \sum_{k=1}^{N} d_{N-k} s^k,$$
   and A is the companion matrix of vector D (Eq. 3.3), then by definition, the eigenvalues of A are the roots of the monic polynomial $D(s)$.

Example: `compan([1 -1 -1])= [1 1; 1 0]`

Exercise #*14*

Practice the use of Matlab's/Octave's related functions that manipulate roots, polynomials, and residues: `root()`, `conv()`, `deconv()`, `poly()`, `polyval()`, `polyder()`, `residue()`, `compan()`.

Solution: We may try Newton's method for various polynomials. Try `N=poly(R)` to provide the coefficients of a polynomial given the roots R. Then use `root()` to factor the resulting polynomial. Finally, use Newton's method and show that the iteration converges to the nearest root.[1]

3.2 Eigen-analysis

At this point we turn a corner in the discussion toward the important topic of eigen-analysis, which starts with the computation of the eigenvalues of a matrix, and their eigenvectors. Eigenvectors are mathematical generalizations of resonances, or modes, naturally found in physical systems.

When you pluck the string of a violin or guitar, or hammer a bell or tuning fork, there are natural resonances that occur. These are the eigenmodes of the instrument. The frequency of each mode is related to the eigenvalue, which in physical terms is the frequency of the mode. But this idea goes way beyond simple acoustical instruments. Wave-guides and atoms are resonant systems. The resonances of the hydrogen atom are called the Lyman series, a special case of the Rydberg series and Rydberg atom (Bohr, 1954; Gallagher, 2005).

Thus this stream runs deep in both physics and eventually mathematics. In some real sense, eigen-analysis was what the Pythagoreans were seeking to understand. This relationship is rarely spoken about in the literature, but once you see it, it can never be forgotten, as it colors your entire view of all aspects of modern physics.

3.2.1 Eigenvalues of a matrix

The method for finding eigenvalues is best described with an example.[2] Starting from the matrix Eq. 2.28, the eigenvalues are defined by the eigen-matrix equation

$$\frac{1}{2}\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} = \lambda\begin{bmatrix}e_1\\ e_2\end{bmatrix}. \tag{3.6}$$

The unknowns here are the eigenvalue λ and the eigenvector $\boldsymbol{e} = [e_1, e_2]^T$. First we find λ by subtracting the right from the left:

$$\frac{1}{2}\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} - \lambda\begin{bmatrix}e_1\\ e_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1-2\lambda & 1\\ 2 & -2\lambda\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} = 0.$$

The only way this equation for $\boldsymbol{e}$ can have a non-trivial ($e_1 = e_2 = 0$) solution is if the matrix is singular. If it is singular, the determinant of the matrix is zero.

Example: The determinant in the above equation is the product of the diagonal elements minus the product of the off-diagonal elements, which results in the quadratic equation

$$-2\lambda(1-2\lambda) - 2 = 4\lambda^2 - 2\lambda - 2 = 0.$$

Completing the square gives

$$(\lambda - 1/4)^2 - (1/4)^2 - 1/2 = 0;$$

thus the roots (i.e., eigenvalues) are $\lambda_\pm = \frac{1\pm 3}{4} = \{1, -1/2\}$.

Exercise #*15*

Expand Eq. 3.2.1 and recover the quadratic equation.

Solution:

$$(\lambda - 1/4)^2 - (1/4)^2 - 1/2 = \lambda^2 - \lambda/2 + \cancel{(1/4)^2 - (1/4)^2} - 1/2 = 0.$$

[1] A Matlab/Octave program that does this may be downloaded from (right-click your mouse on the file) `https://jontalle.web.engr.illinois.edu/uploads/493/M/NewtonJPD.m`.

[2] Appendix A is an introduction to the topic of eigen-analysis for 2×2 matrices.

Thus completing the square is the same as the original equation.

Exercise #*16*

Find the eigenvalues of the matrix of Eq. 2.23.

Solution: This is a minor variation on the previous example. Briefly, we have

$$\det \begin{bmatrix} 1-\lambda & N \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - N = 0.$$

Thus $\lambda_\pm = 1 \pm \sqrt{N}$.

Exercise #*17*

Starting with Eq. 3.6 and initial conditions $[x_1, y_1]^T = [1, 0]^T$, compute the first five values of $[x_n, y_n]^T$.

Solution: Here is a Matlab/Octave code for computing $[x_n, y_n]^T$:

```
x=[1;0];
A=[1 1;2 0]/2;
for k=1:10; x(k+1)=A*x(:,k); end
```

which gives the rational ($x_n \in \mathbb{Q}$) sequence: $1, 1/2, 3/4, 5/8,\ 11/2^4, 21/2^5, 43/2^6,\ 85/2^7,\ 171/2^8,\ 341/2^9,\ 683/2^{10}, \ldots$.

Exercise #*18*

Show that the solution to the mean-Fibonacci sequence (Eq. 2.27) is bounded, unlike that of the Fibonacci sequence. Explain what is going on.

Solution: Because the next value is the mean of the last two, the sequence is bounded. To see this one needs to compute the eigenvalues of the matrix in Eq. 2.28.

The key to the analysis of such equations is called eigen-analysis, or the modal-analysis method (see Appendix A). The eigenvalues (eigen-frequencies) are also known as resonant frequencies in engineering and eigenmodes in physics. Eigenmodes describe the naturally occurring “ringing” found in physical wave-dominated boundary value problems and in resonant circuits. Each mode’s eigenvalue quantifies the mode’s natural complex frequency $s_k = \sigma_k + \omega_k \jmath$.

Complex eigenvalues result in damped modes having frequencies $s_k \in \mathbb{C}$, which decay in time as $\tau_k = 1/\sigma_k$ due to energy losses, as determined by σ_k.

Two modes that have exactly the same frequency are said to be degenerate. This is a very special condition representing a very high degree of symmetry. When two modes are slightly different in frequency, one hears a beating of the modes at the difference frequency (they are not degenerate). If they have different damping, the beats will die away as determined by the smaller time constant (e.g., $\min \sigma_1, \sigma_2$).

Common examples include tuning forks, pendulums, bells, and the strings of musical instruments (such as guitar and fiddles), all of which (except for tuning forks) have hundreds of modes (Fletcher and Rossing, 2008; Morse, 1948). For those interested in musical acoustics, these books are excellent.

3.2.2 Cauchy’s theorem and eigenmodes

Cauchy’s residue theorem is used to find the time-domain response of each frequency-domain complex eigenmode. Thus eigen-analysis and eigenmodes of physics are the same thing but are described using different notional methods.[3] The eigen-analysis method is summarized in Appendix A.3.

[3]During the discovery or creation of quantum mechanics, two alternatives were developed: Schrödinger’s differential equation method and Heisenberg’s matrix method. Eventually it was claimed that the two methods were equivalent. After reading their published exchange, it seems there was no final agreement at all. I suspect the more astute Heisenberg was correct, the two methods are quite different.

Taking a simple example of a 2×2 matrix $\mathcal{T} \in \mathbb{C}$, we start from the definition of the two eigen-equations

$$\mathcal{T}\boldsymbol{e}_{\pm} = \lambda_{\pm}\boldsymbol{e}_{\pm} \tag{3.7}$$

corresponding to two eigenvalues $\lambda_{\pm} \in \mathbb{C}$ and two 2×1 eigenvectors $\boldsymbol{e}_{\pm} \in \mathbb{C}$.

Example: Assume that $\mathcal{T}$ is the Fibonacci matrix in Eq. 2.25. (see p. 53).

The eigenvalues $\lambda_{\pm}$ may be merged into a 2×2 diagonal eigenvalue matrix

$$\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix},$$

while the two eigenvectors $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ are merged into a 2×2 eigenvector matrix

$$\boldsymbol{e} = [\boldsymbol{e}_+, \boldsymbol{e}_-] = \begin{bmatrix} e_1^+ & e_1^- \\ e_2^+ & e_2^- \end{bmatrix}, \tag{3.8}$$

corresponding to the two eigenvalues. Using matrix notation, we can write this compactly as

$$\mathcal{T}\boldsymbol{e} = \boldsymbol{e}\Lambda. \tag{3.9}$$

Note that while $\lambda_{\pm}$ and $\boldsymbol{e}_{\pm}$ commute, $\boldsymbol{e}\Lambda \neq \Lambda\boldsymbol{e}$.

From Eq. 3.9 we may obtain two very important relations:

1. the diagonalization of $\mathcal{T}$,

$$\Lambda = \boldsymbol{e}^{-1}\mathcal{T}\boldsymbol{e}, \tag{3.10}$$

   and

2. the eigen-expansion of $\mathcal{T}$,

$$\mathcal{T} = \boldsymbol{e}\Lambda\boldsymbol{e}^{-1}, \tag{3.11}$$

   which is used for computing powers of $\mathcal{T}$ (i.e., $\mathcal{T}^{100} = \boldsymbol{e}^{-1}\Lambda^{100}\boldsymbol{e}$).

Example: If we take

$$\mathcal{T} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

then the eigenvalues are given by $(1 - \lambda_{\pm})(1 + \lambda_{\pm}) = -1$; thus $\lambda_{\pm} = \pm\sqrt{2}$. This method of eigen-analysis is discussed in Appendix A.2.

Exercise #*19*

Show that the geometric series formula holds for 2×2 matrices. Starting with the 2×2 identity matrix I_2 and $a \in \mathbb{C}$, with $|a| < 1$, show that

$$I_2(I_2 - aI_2)^{-1} = I_2 + aI_2 + a^2I_2^2 + a^3I_2^3 + \cdots.$$

Solution: Multiply both sides by $I_2 - aI_2^k$ results in an identity

$$\begin{aligned} I_2 &= I_2 + aI_2 + a^2I_2^2 + a^3I_2^3 + \cdots - aI_2(aI_2 + a^2I_2^2 + a^3I_2^3 + \cdots) \\ &= [1 + (a + a^2 + a^3 + \cdots) - (a + a^2 + a^3 + a^4 + \cdots)]I_2 \\ &= I_2. \end{aligned}$$

This equality requires that the two series converge, but only if $|a| < 1$.

When the matrix $\mathcal{T}$ is not a square matrix, Eq. 3.11 may be generalized as

$$\mathcal{T}_{m,n} = \mathbf{U}_{m,m}\Lambda_{m,n}\mathbf{V}^{\dagger}_{n,n}.$$

This useful generalization of eigen-analysis is called singular value decomposition (SVD). To see this use the Matlab/Octave command `[U,L,V]=svd(A)` where A is a rectangular (non-square) matrix.

Exercise #*20*
Verify that $\Lambda = \boldsymbol{e}^{-1}\boldsymbol{a}\boldsymbol{e}$.

Solution: We shall work with the unnormalized eigen-matrix $c\boldsymbol{e}$, where $c = \sqrt{\sqrt{2}^2+1} = \sqrt{3}$. To compute the inverse of $c\boldsymbol{e}$: 1) swap the diagonal values, 2) change the sign of the off diagonals, and 3) divide by the determinant Δ:

$$(c\boldsymbol{e})^{-1} = \frac{1}{2c\sqrt{2}}\begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix} = \frac{1}{2c}\begin{bmatrix} 0.707 & 1 \\ -0.707 & 1 \end{bmatrix}.$$

We wish to show that $\Lambda = \boldsymbol{e}^{-1}\boldsymbol{a}\boldsymbol{e}$

$$\frac{1}{2\cancel{c}^{1}}\begin{bmatrix} 0.707 & 1 \\ -0.707 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\cancel{c}^{1}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$$

which is best verified with Matlab/Octave.

Exercise #*21*
Verify that $\boldsymbol{a} = \boldsymbol{e}\Lambda\boldsymbol{e}^{-1}$.

Solution: We wish to show that

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \frac{1}{\cancel{\sqrt{3}}}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix} \cdot \frac{\cancel{\sqrt{3}}}{2\sqrt{2}}\begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix}.$$

All the above solutions have been verified with Octave.

Eigen-matrix diagonalization is helpful in generating solutions for finding the solutions of Pell's and Fibonacci's equations using transmission matrices.

Example: If the matrix corresponds to a transmission line, the eigenvalues have units of seconds [s]

$$\begin{bmatrix} V^+ \\ V^- \end{bmatrix}_n = \begin{bmatrix} e^{-sT_o} & 0 \\ 0 & e^{sT_o} \end{bmatrix}\begin{bmatrix} V^+ \\ V^- \end{bmatrix}_{n+1}. \tag{3.12}$$

In the time domain the forward traveling wave $v^+_{n+1}(t-(n+1)T_o) = v^+_n(t-nT_o)$ is delayed by T_o. Two applications of the matrix delays the signal by $2T_o$.

Summary: The GCD (Euclidean algorithm), Pell's equation, and the Fibonacci sequence may all be written as compositions of 2×2 matrices. Pell's equation and the Fibonacci sequence are special cases of the 2×2 matrix composition

$$\begin{bmatrix} x \\ y \end{bmatrix}_{n+1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}_n.$$

This is an important and common thread of these early mathematical findings. This 2×2 linearized matrix recursion plays a special role in physics, mathematics, and engineering because one-dimensional system equations are solved using the 2×2 eigen-analysis method. More than several thousand years of mathematical trial and error set the stage for this breakthrough. But it took even longer to be fully appreciated.

The key idea of the 2×2 matrix solution, widely used in modern engineering, can be traced back to Brahmagupta's solution of Pell's equation for arbitrary N. Brahmagupta's recursion, identical to that of the Pythagoreans' $N=2$ case (see Eq. 2.23, eventually led to the concept of linear algebra, defined by the simultaneous solutions of many linear equations. The recursion by the Pythagoreans (6th century BCE) predated the creation of algebra by al-Khwarizmi (ninth century CE), as seen in Fig. 1.1.

3.2.3 Taylor series

An analytic function is one that meets these criteria:

1. It may be expanded in a Taylor series:

$$P(x) = \sum_{n=0}^{\infty} c_n (x - x_o)^n, \quad \text{where} \quad x \in \mathbb{R}. \tag{3.13}$$

2. It converges for $|x - x_o| < 1$, called the region of convergence (RoC), with coefficients c_n.

3. The Taylor series coefficients c_n are defined by taking derivatives of $P(x)$ and evaluating them at the expansion point x_o—namely,

$$c_n = \frac{1}{n!} \frac{d^n}{dx^n} P(x) \bigg|_{x=x_o}. \tag{3.14}$$

4. Although $P(x)$ may be multi-valued, the Taylor series is always single-valued.

Exercise #*22*

Verify that c_0 and c_1 of Eq. 3.13 follow from Eq. 3.14.

Solution: To obtain c_0, for $n = 0$, there is no derivative (d^0/dx^0 indicates no derivative is taken), so we must simply evaluate $P(x - x_o) = c_0 + c_1(x - x_o) + \cdots$ at $x = x_o$, leaving c_0. To find c_1, we take one derivative, which results in $P'(x) = c_1 + 2c_2(x - x_o)) + \cdots$. Evaluating this at $x = x_o$ leaves c_1. Each time we take a derivative we reduce the degree of the series by 1, leaving the next constant term.

Exercise #*23*

Suppose we truncate the Taylor series expansion to N terms. What is the name of such functions?

Solution: When an infinite series is truncated, the resulting function is an Nth-degree polynomial:

$$P_N(x) = \sum_{n=0}^{N} = c_0 + c_1(x - x_o) + c_2(x - x_o)^2 + \cdots + c_N(x - x_o)^N.$$

We can find c_0 by evaluating $P_N(x)$ at the expansion point x_o, since from the above formula $P_N(x_o) = c_0$. From the Taylor formula, $c_1 = P'_N(x)|_{x_o}$.

Exercise #*24*

How many roots do $P_N(x)$ and $P'_N(x)$ have?

Solution: According to the fundamental theorem of algebra, $P_N(x)$ has N roots and $P'_N(x)$ has $N - 1$ roots. The Gauss-Lucas theorem states that the $N - 1$ roots of $P'_N(x)$ lie inside the convex hull of the N roots of $P_N(x)$.

Exercise #*25*

Would it be possible to find an inverse Gauss-Lucas theorem, that states where the roots of the integral of a polynomial might be?

Solution: To the best of my knowledge this problem has not been addressed. However it seems a question worthy of significant thought.

With each integral there is a new degree of freedom that must be accommodated. Thus this problem is difficult. But since there is only one extra degree of freedom, it does not seem intractable. To solve this problem a constraint is needed.

Properties: The Taylor formula is a prescription for how to uniquely define the coefficients c_n. Without the Taylor series formula, we would have no way of determining c_n. The proof of the Taylor formula is transparent; The coefficients may be determined by simply taking successive derivatives of Eq. 3.13 and then evaluating the result at the expansion point. If $P(x)$ is analytic, then this procedure will always be successful. If $P(x)$ fails to have a derivative of any order, then the function is not analytic and Eq. 3.13 is not valid.

The Taylor series representation of $P(x)$ has special applications for solving differential equations for these reasons:

1. It is single-valued.
2. The series is valid inside the RoC (an open set).
3. All its derivatives and integrals are uniquely defined.
4. It may be continued into the complex plane by extending $x \in \mathbb{C}$. This extension is necessary because the eigenvalues are typically in $\mathbb{C}$. In fact the only reasonable eigenvalues must be complex, having negative real parts. If the real part of λ_k is zero, the solution is loss-less, thus never dies away, which is non-physical in the macroscopic world.[4] If it is positive, the solution is unstable (blows up). Typically this involves expanding the series about a different expansion point.

Analytic continuation: A limitation of the Taylor series expansion is that it is not valid outside of its RoC. One method for working with this limitation is to move the expansion point. This is called *analytic continuation*. However, analytic continuation is a nontrivial operation because: (1) It requires manipulating an infinite number of derivatives of $P(x)$, (2) at the new expansion point x_o, where (3) $P(x - x_o)$ may not have derivatives, due to possible singularities. (4) Thus one needs to know where the singularities of $P(s)$ are in the complex s plane. Due to these many problems analytic continuation is rarely used, other than as a theoretical concept. In Appendix 3.2.3 (p. 76) we consider a novel definition of analytic continuation.

Every Taylor series is a single valued representation because powers of the variable are single valued. Single valuedness is key feature to the series representation. However functions have regions where the series is not valid. This is best seen with a simple example using the geometric series

$$f(s) = \frac{1}{1 - as} = \sum_{n=0}^{\infty} (as)^n, \quad |as| < 1$$

with $a, s \in \mathbb{C}$. Note that $f(s)$ has a pole at $s_o = 1/a$ and residue $1/a$.

However $f(s)$ is perfectly well defined for $|as| > 1$, for which it has a different series expansion. If we let $s = 1/z$ we find

$$f(z) = \frac{1}{1 - a/z} = \frac{-z/a}{1 - z/a} = -\frac{z}{a} \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n, \quad |z/a| < 1.$$

Expressed in terms of $z = 1/s$ we have that $|sa| > 1$.

Thus the first expansion is good inside the circle $|s| < 1/a$ while the second is valid outside the circle. While each series is single valued within its RoC, $f(s)$ is valid everywhere, except at the pole $s = 1/a$, where it is singular.

Example: The similar case is the geometric series $P(x) = 1/(\jmath - x)$ about the expansion point $x = 1$. The function $P(x)$ is defined everywhere, except at the singular point $x = \jmath$, whereas the geometric series is valid for $|x| < 1$. However $P(x)$ is valid for $|x| > 1$. For example $P(10) = 1/(\jmath - 10) = (\jmath + 10)/(\jmath + 10)(\jmath - 10) = -(\jmath + 10)/101$.

Role of the Taylor series: The Taylor series plays a key role in the mathematics of differential equations and their solution, as the coefficients of the series uniquely determine the analytic series representation via its derivatives. The implications and limitations of the power series representation are very specific: If the series fails to converge (i.e., outside the RoC), it is meaningless.

Every differential equation has as many independent solutions as it has eigenvalues. To obtain these solutions we must use the Taylor series, with its single-value property, to uniquely represents each solution. The

[4]Quantum eigen-states are lossless.

general solution is then the weighted sum over the independent solutions. This theory trivially follows from the Cauchy residue theorem CT-3 (Eq. 3.2.3).

Starting from a differential equation, it may be transformed to a matrix equation using the companion matrix (§3.1.1,) having K eigenvalues $\lambda_1, \cdots, \lambda_k, \cdots, \lambda_K$, with a general solution

$$f(t; C_k) = \sum_{k=1}^{K} C_k e^{s_k t}.$$

The constants $C_k \in \mathbb{C}$ are determined using the initial conditions.

A very important fact about the RoC: It is relevant to only the series, not the function being expanded. Typically the function has a pole at the radius of the RoC, beyond which the series fails to converge. However, the function being expanded is valid everywhere (other than at its poles). This point has been inadequately explained in many text books. In addition, the RoC is the region of divergence (RoD), which is the RoC's complement.

The Taylor series does not need to be infinite to converge to the function it represents, since it obviously works for any polynomial $P_N(x)$ of degree N. But in the finite case ($N < \infty$), the RoC is infinite and the series is the function $P_N(x)$ exactly, everywhere. Of course, $P_N(x)$ is a polynomial of degree N. When $N \to \infty$, the Taylor series is valid only within the RoC, and it is (typically) the representation of the reciprocal of a polynomial.

These properties are both the curse and the blessing of the analytic function. On the positive side, analytic functions are the ideal starting point for solving differential equations, which is exactly how they were used by Newton and many others. Analytic functions are "smooth," since they are infinitely differentiable, with coefficients given by Eq. 3.14. They are single-valued, so there can be no ambiguity in their interpretation. On the negative side, they only represent the function within the RoC, which depends on the expansion point.

Two well-known analytic functions are the geometric series ($|x| < 1$)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \tag{3.15}$$

and the exponential series ($|x| < \infty$)

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n. \tag{3.16}$$

Exercise #*26*

Provide the Taylor series expression for the following functions:

$$F_1(x) = \int^x \frac{1}{1-x}dx \tag{3.17}$$

Solution: $F_1(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$

$$F_2(x) = \frac{d}{dx}\frac{1}{1-x} \tag{3.18}$$

Solution: $F_2(x) = 1 + 2x + 3x^2 + \cdots$

$$F_3(x) = \ln \frac{1}{1-x} \tag{3.19}$$

Solution: $F_3(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \cdots$

$$F_4(x) = \frac{d}{dx} \ln \frac{1}{1-x} \tag{3.20}$$

Solution: $F_4(x) = 1 + x + x^2 + x^3 + \cdots$

Exercise #*27*

Using symbolic manipulation (Matlab, Octave, Mathematica), expand the function $F(s)$ in a Taylor series and find the recurrence relationships among the Taylor coefficients c_n, c_{n-1}, c_{n-2}. Assume $a \in \mathbb{C}$ and $T \in \mathbb{R}$.

$$F(s) = e^{as}$$

Solution: A Google search on *octave syms taylor* is useful. The Matlab/Octave code to expand this in a Taylor series is

```
syms s
taylor(exp(s),s,0,'order',10)
```

Exercise #*28*

Find the coefficients of the following functions by the method of Eq. 3.14 and give the RoC.

1. $w(x) = \frac{1}{1-x\jmath}$.

   Solution: From a straightforward expansion we know the coefficients are

   $$\frac{1}{1-x\jmath} = 1 + x\jmath + (x\jmath)^2 + (x\jmath)^3 + \cdots = 1 + x\jmath - x^2 + -\jmath x^3 + \cdots.$$

   Working this out using Eq. 3.14 is more work:
   $c_0 = \frac{1}{0!}w\big|_0 = 1$; $c_1 = \frac{1}{1!}\frac{dw}{dx}\Big|_0 = -\frac{-\jmath}{(1-x\jmath)^2}\Big|_{x=0} = \jmath$; $c_2 = \frac{1}{2!}\frac{d^2w}{dx^2}\Big|_0 = \frac{1}{2!}\frac{-2}{(1-x\jmath)^3}\Big|_0 = -1$;
   $c_3 = \frac{1}{3!}\frac{d^3w}{dx^3}\Big|_0 = \frac{-\jmath}{(1-x\jmath)^4}\Big|_0 = -j$.
   However, if we take derivatives of the series expansion, it is much easier and we can even figure out the term for c_n:
   $c_0 = 1$; $c_1 = \frac{d}{dx}\sum(\jmath x)^n\Big|_0 = j$; $c_2 = \frac{1}{2!}\frac{d^2}{dx^2}\sum(jx)^n\Big|_0 = 2(\jmath)^2$;
   $c_3 = \frac{1}{3!}\frac{d^3}{dx^3}\sum(jx)^n\Big|_0 = (\jmath)^3 = -\jmath$;
   $\cdots$,
   $c_n = \frac{1}{n!}\jmath^n n! = \jmath^n$.
   The RoC is $|xj| = |x| < 1$.

2. $w(x) = e^{x\jmath}$.

   Solution: $c_n = \frac{1}{n!}\jmath^n$. The RoC is $|x| < \infty$. Functions with an RoC of ∞ are called *entire*. Thus $c_n = \jmath c_{n-1}/n$.

Exercise #*29*

Show that $Z(s) = 1/\sqrt{s}$ is positive-real but not a Brune impedance.

Solution: Since it may not be written as the ratio of two polynomials, it is not in the Brune impedance class. If we write $Z(s) = |Z(s)|e^{\phi\jmath}$ in polar coordinates, since $-\pi/4 \le \phi \le \pi/4$ when $|\angle s| < \pi/2$, $Z(s)$ satisfies the Brune condition and thus is positive-real. A proof might be to state that $w(s) = s^2$ is complex-analytic.

Determining the region of convergence (RoC): Determining the RoC for a given analytic function is quite important and may not always be obvious. In general the RoC is a circle whose radius extends from the expansion point out to the nearest pole. Thus when the expansion point is moved, the RoC changes, since the location of the pole is fixed.

Example: For the geometric series (Eq. 3.15), the expansion point is $x_o = 0$ and the RoC is $|x| < 1$, since $1/(1-x)$ has a pole at $x = 1$. We may move the expansion point by a linear transformation—for example, by replacing x with $z+3$. Then the series becomes $1/((z+3)-1) = 1/(z+2)$, so the RoC becomes 3 because in the z plane the pole has moved to -2.

Example: A second important example is the function $1/(x^2+1) \in \mathbb{C}$, which has the same RoC as the geometric series, since it may be expressed in terms of its residue expansion (also called its partial fraction expansion), but perhaps best known in the literature as *vector fitting*

$$\frac{1}{x^2+1} = \frac{1}{(x+1\jmath)(x-1\jmath)} = \frac{1}{2\jmath}\left(\frac{1}{x-1\jmath} - \frac{1}{x+1\jmath}\right).$$

Each term has an RoC of $|x| < |1\jmath| = 1$. The amplitude of each pole is called the *residue*, defined in Eq. 3.2.3. The residue for the pole at $1\jmath$ is $-1/2\jmath$.

The roots must be found by first factoring the polynomial (e.g., Newton's method (Allen, 2025)). Once the roots are known, the residues (amplitudes) are found using using simple algebra. A generazation of this method is known as *vector fitting*.

In summary, the function $1/(x^2+1)$ is the sum of two geometric series, with poles at $\pm 1\jmath$, which is not initially obvious because the roots are complex and conjugate. Only when the function is factored does it become clear what is going on.

Exercise #*30*
What day and year was Newton born

Exercise #*31*
Verify that the above expression is correct, to show that the residues are $\pm 1/2\jmath$.

Solution: We cross-multiply and cancel, leaving 1, as required. The RoC is the coefficient on the pole. Thus the residue of the pole at $x\jmath$ is $\jmath/2$.

Exercise #*32*
Find the residues α, β by solving the 2x2 matrix equation for $[\alpha, \beta]$.

$$\begin{bmatrix} \frac{1}{\pi_i} & 0 \\ 0 & \frac{1}{\pi_j} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{\pi_i \pi_j} \\ \frac{1}{\pi_i \pi_j}. \end{bmatrix}$$

Solution:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \pi_i \pi_j \begin{bmatrix} \frac{1}{\pi_j} & 0 \\ 0 & \frac{1}{\pi_i} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \pi_i \\ \pi_j \end{bmatrix}.$$

Exercise #*33*
Find the residue of $\frac{d}{dz} z^\pi$.

Solution: Taking the derivative gives $\pi z^{\pi-1}$, which has a pole at $z=0$. Applying the formula for the residue (Eq. 3.2.3, we find

$$c^{-1} = \pi \lim_{z \to 0} z z^{\pi-1} = \pi \lim_{z \to 0} z^\pi = 0.$$

Thus the residue is zero.

3.2.4 Complex–Analytic functions $\in ; C$

Any function that has a complex Taylor series ($\in \mathbb{C}$) expansion is called a *complex–analytic function* (or simply an *analytic function*). Within the RoC, the series expansion defines a single-valued function. Polynomials $1/(1-x)$ and e^x are examples of analytic functions that are real functions of their real argument x.

Every analytic function has a corresponding differential equation, which is determined by the coefficients a_k of the analytic power series. An example is the exponential, which has the property that it is the eigen-function of the derivative operation

$$\frac{d}{dx}e^{ax} = ae^{ax},$$

which may be verified using Eq. 3.16. This relationship is a common definition of the exponential function, which is special because it is the eigen-function of the derivative.

The complex–analytic power series (i.e., complex–analytic functions) may also be integrated term by term, since

$$\int^x f(x)dx = \sum \frac{a_k}{k+1}x^{k+1}. \tag{3.21}$$

Newton took full advantage of this property of the analytic function and used the analytic series (Taylor series) to solve analytic problems, especially for working out integrals. This enabled him to solve differential equations. To fully understand the theory of differential equations, one must master single-valued analytic functions and their analytic power series.

Single- vs. multi-valued functions: Polynomials and their ∞-degree extensions (analytic functions) are single-valued: For each x there is a single value for $P_N(x)$. The roles of the domain and co-domain may be swapped to obtain an inverse function with properties that can be very different from those of the function. For example, $y(x) = x^2 + 1$ has the inverse $x = \pm\sqrt{y-1}$, which is double-valued and complex when $y < 1$. Periodic functions such as $y(x) = \sin(x)$ are even more "exotic," since $x(y) = \arcsin(x) = \sin^{-1}(x)$ has an infinite number of $x(y)$ values for each y. This problem was first addressed in Bernhard Riemann's 1851 PhD thesis, written while he was working with Gauss.

Exercise #*34*

Let $y(x) = \sin(x)$. Then $dy/dx = \cos(x)$. Show that $dx/dy = \pm 1/\sqrt{1-y^2}$.

Solution: Since $\sin^2 x + \cos^2 x = 1$, it follows that $y^2(x) + (dy/dx)^2 = 1$. Thus $dy/dx = \pm\sqrt{1-y^2}$. Taking the reciprocal gives the result.

To fully understand this, Google "implicit function theorem" (D'Angelo, 2017, p. 104).

Exercise #*35*

Evaluate the integral

$$I(y) = \int^y \frac{dy}{\sqrt{1-y^2}}.$$

Solution: From the previous Exercise we know that

$$x(y) = \int^x dx = \int^y \frac{dy}{\sqrt{1-y^2}}.$$

But since $y(x) = \sin(x)$, it follows that $x(y) = \sin^{-1} y = \arcsin(y)$.

Exercise #*36*

Find the Taylor series coefficients of $y = \sin(x)$ and $x = \sin^{-1}(y)$. Note that $\log e^s = s$ and

$$\sin(\sin^{-1}(s)) = \sin^{-1}(\sin(s)) = 1.$$

Hint: Use symbolic Octave. Note $\sin^{-1}(y) = \arcsin(y)$.

Solution: `syms s;taylor(sin(s),'order',10);`

$$\sin(s) = s - s^3/3! + s^5/5! - s^7/7! + \cdots$$

and `syms s;taylor(asin(s),'order',15);`

$$\arcsin(s) = s + \frac{1}{6}s^2 + \frac{3}{40}s^5 + \frac{5}{112}s^7 + \frac{35}{1152}s^9 + \frac{63}{2816}s^{11} + \frac{231}{13312}s^{13} + \cdots$$
$$= s + \frac{1}{3\cdot 2^1}s^3 + \frac{3}{5\cdot 2^3}s^5 + \frac{5}{7\cdot 2^4}s^7 + \frac{7\cdot 5}{9\cdot 2^7}s^9 + \frac{7\cdot 3^2}{11\cdot 2^8}s^{11} + \frac{3\cdot 7\cdot 11}{13\cdot 2^{10}}s^{13} + \cdots$$

Note that every complex–analytic function may be expanded in a Taylor series, within its RoC. It follows that the inverse is also complex–analytic, as demonstrated in this case using symbolic algebra.

Exercise #*37*
What is the necessary condition such that if $dy/dx = F(x)$, then $dx/dy = 1/F(x)$?

Solution: This will be true when $df(x)/dx = F(x)$ is complex–analytic because the Fundamental Theorem of Complex Calculus (FTCC) defines the anti-derivative. In this case $dy/dx = (dx/dy)^{-1}$ (except at singular points, where it is not analytic).

3.2.5 Polynomials as complex–analytic functions

If $x \in \mathbb{R}$ is replaced by the Laplace frequency $s = \sigma + \omega\jmath \in \mathbb{C}$, then

$$F(s) = \sum_{n=0}^{\infty} c_n(s - s_o)^n \in \mathbb{C}, \tag{3.22}$$

namely $F(s)$ is *complex–analytic*.

The germinal example is the exponential

$$e^{st} = e^{(\sigma+\omega\jmath)t} = e^{\sigma t}e^{\jmath\omega t} = e^{\sigma t}\left[\cos(\omega t) + \jmath\sin(\omega t)\right]. \tag{3.23}$$

Taking the real and imaginary parts give

$$\Re\{e^{st}\} = e^{\sigma t}\frac{e^{\omega\jmath t} + e^{-\omega\jmath t}}{2} = e^{\sigma t}\cos(\omega t)$$

and $\Im\{e^{st}\} = e^{\sigma t}\sin(\omega t)$. Once the argument is allowed to be complex, it becomes obvious that the exponential and circular functions are fundamentally related. This exposes the family of entire circular functions [i.e., $e^s, \sin(s), \cos(s), \tan(s), \cosh(s), \sinh(s)$] and their inverses [$\ln(s), \arcsin(s), \arccos(s), \arctan(s), \cosh^{-1}(s), \sinh^{-1}(s)$], first fully elucidated by Euler in about 1750 (Stillwell, 2010, p. 315).

Note that because $\sin(\omega t)$ is periodic, its inverse must be multi-valued. What was needed is some systematic way to account for this multi-valued property. This extension to multi-valued functions is called a *branch cut*, invented by Riemann in his 1851 PhD thesis, supervised by Gauss in the final years of Gauss's long life.

The Taylor series of a complex–analytic function: However, there is a fundamental problem: We cannot formally define the Taylor series for the coefficients c_k until we have defined the derivative with respect to the complex variable $dF(s)/ds$, with $s \in \mathbb{C}$. Thus simply substituting s for x in an analytic function leaves a major hole in one's understanding of the complex–analytic function.

It was Cauchy in 1814 (Fig. 1.5), who uncovered the much deeper relationships within complex–analytic functions by defining differentiation and integration in the complex plane, leading to several fundamental theorems of complex calculus, including the fundamental theorem of complex calculus and Cauchy's formula.

There seems to be some disagreement as to the status of multi-valued functions: Are they functions, or is a function strictly single-valued? If so, then we are missing out on a host of interesting possibilities, including all the inverses of nearly every complex–analytic function. For example, the inverse of a complex–analytic function is a complex–analytic function (e.g., e^s and $\log(s)$).

Impact of complex–analytic mathematics on physics: It seems likely, if not obvious, that the success of Newton was his ability to describe physics using mathematics. He was inventing new mathematics at the same time he was explaining new physics. The same might be said for Galileo. It seems likely that Newton was extending the successful techniques and results of Galileo's work on gravity (Galileo, 1638). Galileo died on January 8, 1642, and Newton was born January 4, 1643, just short of one year later. Obviously Newton was well aware of Galileo's great success and naturally would have been influenced by him.

The application of complex–analytic functions to physics was dramatic, as may be seen in the six volumes on physics written by Arnold Sommerfeld (1868–1951), and from the productivity of his many (36) students (e.g., Debye, Lenz, Ewald, Pauli, Guillemin, Bethe, Heisenberg, Morse, and Seebach, to name a few), notable coworkers (Leon Brillouin), and others (John Bardeen) upon whom Sommerfeld had a strong influence. Sommerfeld is famous for training many students who were awarded the Nobel Prize in Physics, yet he never won a Nobel Prize (the prize is not awarded in mathematics). Sommerfeld brought mathematical physics (the merging of physical and experimental principles via mathematics) to a new level with the use of complex integration of analytic functions to solve otherwise difficult problems, thus following the lead of Newton, who used real integration of Taylor series to solve differential equations (Brillouin, 1960, Ch. 3 by Sommerfeld).

Exercise #*38*

What day and year was Issac Newton born? Hint: This is sort of a trick question.

Solution: He was born on Christmas day, Dec 25, 1642. However this is the date according to the Julian calendar, active during his life time. However the Church changed the calendar to the Gregorian (New) calendar, having its dates ten days ahead of Julian calendar dates;

The question is, why would they do such an outrageous deed? In my view the only obvious answer that the church was scared to death of labeling Newton as the new-born Christ. I suspect you will not find my laughable personal conspiracy theory anywhere on the internet.

3.2.6 Brune impedance

A special family of functions is formed from ratios of two polynomials $Z(s) = N(s)/D(s)$ commonly used to define an impedance $Z(s)$, called a *Brune impedance*. Impedance functions are a special class of complex–analytic functions because they must have a non-negative real part

$$\Re Z(s) = \Re \frac{N(s)}{D(s)} \geq 0$$

so as to obey conservation of energy. A physical Brune impedance cannot have a negative resistance (the real part); otherwise, it would act like a power source, violating conservation of energy. Most impedances used in engineering applications are in the class of Brune impedances, defined by the ratio of two polynomials of degrees M and N:

$$Z_{\text{Brune}}(s) = \frac{P_M(s)}{P_N(s)} = \frac{s^M + a_1 S^{M-1} + \cdots + a_0}{s^N + b_1 S^{N-1} + \cdots + b_0}, \tag{3.24}$$

where $M = N \pm 1$ (i.e., $N = M \pm 1$). This fraction of polynomials is sometimes known as a *Padé approximation*, with poles and zeros, defined as the complex roots of the two polynomials. The key property of the Brune impedance is that the real part of the impedance is non-negative (positive or zero) in the right s half-plane:

$$\Re Z(s) = \Re\left[R(\sigma,\omega) + jX(\sigma,\omega)\right] = R(\sigma,\omega) \geq 0 \qquad \text{for} \quad \Re s = \sigma \geq 0. \tag{3.25}$$

Since $s = \sigma + \omega\jmath$, the complex frequency (s) right half-plane (RHP) corresponds to $\Re s = \sigma \geq 0$. This condition defines the class of positive-real functions, also known as the *Brune condition*, which is frequently written in the abbreviated form

$$\Re Z(\Re s \geq 0) \geq 0. \tag{3.26}$$

As a result of this positive-real (PR) constraint, the subset of Brune impedances (those given by Eq. 3.24 and satisfying Eq. 3.25) must be complex–analytic in the entire right s half-plane. This is a powerful constraint that places strict limitations on the locations of both the poles and the zeros of every positive-real Brune impedance.

A little history: The key idea that every impedance $Z(s)$ must be complex–analytic and its real part be non-negative ($\Re Z(s) \geq 0$) for $\sigma = \Re s > 0$, as first proposed by Otto Brune in his PhD thesis at MIT. His supervised was Ernst A. Guillemin, an MIT electrical engineering professor who played an important role in the development of circuit theory, who was a student of the most important circuit theory scientists of all, Arnold Sommerfeld.[5] [6]

MIT professors caught more wind than UIUC due to the fame of Norbert Wiener and Vannevar Bush. Brune's primary, but non-MIT advisor, was the brilliant German professor W. Cauer, who was trained in 19th-century German mathematics, likely by Sommerfeld (Brune, 1931a). German science was way ahead of that of the US before WWII. When Germany and Japan lost the two wars, science centers flipped. For example, Albert Einstein came to Princeton and Japan refused anything even smelling like nuclear.[7] [8]

We on the other-hand tested both A and H bombs. Watch the amazing movie *Oppenheimer*, especially the last 3 minutes. Do not watch the end first! Amen.

3.3 Polynomial analysis

Following the exploration of algebraic relationships by Fermat and Descartes, the first theorem was being formulated by d'Alembert. The idea behind this theorem is that every polynomial of degree N (Eq. **??**) has at least one root. Every polynomial may be written as the product of a monomial root and a second polynomial of degree of $N-1$. By the recursive application of this concept, it is clear that every polynomial of degree N has N roots. Today this result is known as the *fundamental theorem of algebra:*

> *Every polynomial equation $P(z) = 0$ has a* solution *in the complex numbers. As Descartes observed, a solution $z = a$ implies that $P(z)$ has a factor $z - a$. The quotient*
>
> $$Q(z) = \frac{P(z)}{z-a} = \frac{P(z)}{a}\left[1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \cdots\right] \tag{3.27}$$
>
> *is then a polynomial of one lower degree. . . . We can go on to factorize $P(z)$ into n linear factors.*
>
> *—Stillwell (2010, p. 285).*

The ultimate expression of this theorem is given by Eq. **??**, which indirectly states that an nth degree polynomial has n roots. We shall use the term *degree* when speaking of polynomials and the term *order* when speaking of differential equations. The accepted rule is that *order* applies in the time domain and *degree* in the frequency domain.

The Laplace transform of a linear differential equation having constant coefficients of order N is a polynomial of degree N in the Laplace frequency domain ($s = \sigma + \jmath\omega$).

Today this theorem is so widely accepted we fail to fully appreciate it. Once you were ready to understand the concept of a polynomial, having N roots, you learned the quadratic formula ($N = 2$). You next learned that the case of the quadratic case may be extended to a higher-degree polynomial. The Octave/Matlab command `roots(`$[1, a_2, a_1, a_0]$`)` provides the roots $[s_1, s_2, s_3]$ of the cubic equation, defined by the coefficient vector $[1, a_2, a_1, a_0]$. The command `poly(`$[s_1, s_2, s_3]$`)` returns the coefficient vector. I don't know the largest degree that can be accurately factored numerically by Matlab/Octave, but I'm sure it's well over $N = 10^3$. Today, finding the roots numerically is a solved problem.

The best way to gain insight into the polynomial factorization problem is through the inverse operation, multiplication of monomials. Given the roots x_k, there is a simple algorithm for computing the coefficients a_k of $P_N(x)$ for any n, no matter how large (Allen (2025)). This method is called *convolution*. Convolution is said to be a *trap-door function*, since it is easy, while the inverse, factoring (deconvolution), is hard and analytically intractable for degree $N \geq 5$ (Stillwell, 2010, p. 102).

3.3.1 Convolution of monomials

As outlined by Eq. **??**, a polynomial has two equivalent descriptions, first as a series with coefficients a_n and second as a product of monomial roots x_r. The question is What is the relationship between the coefficients and the roots? The simple answer is that they are related by convolution.

[5] I believe he refused to publish in English during WWII, was sad but understandable error at that time

[6] University of Illinois Professor 'Mac' Van Valkenburg was arguably more influential in circuit theory during the same period. His book were my first introduction to circuit theory at UIUC, but unfortunately not taught by Mac. His many books are much more readable and insightful, but rarely cited.

[7] `https://en.wikipedia.org/wiki/List_of_films_about_nuclear_issues`

[8] `https://www.warhistoryonline.com/featured/atomic-bomb-movies.html`

Let us start with the quadratic

$$(x+a)(x+b) = x^2 + (a+b)x + ab, \tag{3.28}$$

where in vector notation $[-a, -b]$ are the roots and $[1, a+b, ab]$ are the coefficients.

To see how the result generalizes, we may work out the coefficients for the cubic ($N = 3$). Multiplying the following three factors gives

$$(x-1)(x-2)(x-3) = (x^2-3x+2)(x-3) = x(x^2-3x+2)-3(x^2-3x+2) = x^3-6x^2+11x-6. \tag{3.29}$$

When the roots are $[1, 2, 3]$, the coefficients of the polynomial are $[1, -6, 11, -6]$. To verify, we can substitute the roots into the polynomial and show that they give zero. For example, $r_1 = 1$ is a root, since $P_3(1) = 1 - 6 + 11 - 6 = 0$.

As the degree increases, the algebra becomes more difficult. Imagine trying to work out the coefficients for $N = 100$. What is needed is a simple way of finding the coefficients from the roots. Fortunately, convolution keeps track of the bookkeeping, formalizing the procedure, along with Newton's deconvolution method for finding the roots of polynomials.

Convolution of two vectors: To obtain the coefficients by convolution we may write the monomial roots as vectors $[1, a]$ and $[1, b]$. Convolution is a recursive operation described by $[1, a] \star [1, b]$, where $\star$ denotes convolution. The convolution of $[1, a] \star [1, b]$ is done as follows: Reverse one of the two monomials, padding unused elements with zeros. Next slide one monomial against the other, forming the local scalar product (element-wise multiply and add):

$$\begin{array}{cccc} a & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ = & 0 & & \end{array} \qquad \begin{array}{ccc} a & 1 & 0 \\ 0 & 1 & b \\ = & x^2 & \end{array} \qquad \begin{array}{ccc} a & 1 & 0 \\ 1 & b & 0 \\ = & (a+b)x & \end{array} \qquad \begin{array}{ccc} 0 & a & 1 \\ 1 & b & 0 \\ = & abx^0 & \end{array} \qquad \begin{array}{cccc} 0 & 0 & a & 1 \\ 1 & b & 0 & 0 \\ = & 0 & & \end{array},$$

resulting in coefficients $[\ldots, 0, 0, 1, a+b, ab, 0, 0, \ldots]$.

If we reverse one of the polynomials and then take successive scalar products, all the terms in the sum of the scalar product correspond to the same power of x. This explains why the convolution of the coefficients gives the same answer as the product of the polynomials.

When we convolve one monomial factor at a time, the overlap is always two elements; thus it is never necessary to compute more than two multiplications and one addition for each output coefficient. This greatly simplifies the operations (i.e., they are easily done in your head). Thus the final result is more likely to be correct. Comparing this to the algebraic method, we see that convolution has the clear advantage.

Exercise #*39*

What three nonlinear equations would we need to solve to find the roots of a cubic?

Solution: From our formula for the convolution of three monomials, we may find the nonlinear deconvolution relationships between the roots $[-a, -b, -c]$ and the cubic's coefficients $[1, \alpha, \beta, \gamma]$:[9]

$$\begin{aligned} (x+a) \star (x+b) \star (x+c) &= (x+c) \star (x^2 + (a+b)x + ab) \\ &= x \cdot (x^2 + (a+b)x + ab) + c \cdot (x^2 + (a+b)x + ab) \\ &= x^3 + (a+b+c)x^2 + (ab+ac+cb)x + abc \\ &= [1, a+b+c, ab+ac+cb, abc]. \end{aligned}$$

It follows that the nonlinear equations must be

$$\begin{aligned} \alpha &= a+b+c \\ \beta &= ab+ac+bc \\ \gamma &= abc. \end{aligned}$$

[9]By working with the negative roots, we may avoid an unnecessary and messy alternating sign problem.

These equations may be solved by the classic cubic solution, which therefore is a deconvolution problem, also known as *long division of polynomials*. Therefore the following long division of polynomials must be true:

$$\frac{x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc}{x+a} = x^2 + (b+c)x + bc.$$

The product of a monomial $P_1(x)$ and a polynomial $P_N(x)$ gives $P_{N+1}(x)$: This is another way of stating the fundamental theorem of algebra. Each time we convolve a monomial with a polynomial of degree N, we obtain a polynomial of degree $N+1$. The convolution of two monomials results in a quadratic (degree 2 polynomial). The convolution of three monomials gives a cubic (degree 3). In general, the degree k of the product of two polynomials of degree n, m is the sum of the degrees ($k = n+m$). For example, if the degrees are each 5 ($n = m = 5$), then the resulting degree is 10.

While we all know this theorem from high school algebra class, it is important to explicitly identify the fundamental theorem of algebra.

Note that the degree of a polynomial is one less than the length of the vector of coefficients. Since the leading term of the polynomial cannot be zero, or else the polynomial would not have degree N, when we look for roots, the coefficient can (and should always) be normalized to 1.

In summary, the product of two polynomials of degree m, n having m and n roots is a polynomial of degree $m+n$. This is an analysis process of merging polynomials by coefficient convolution. Multiplying polynomials is a merging process into a single polynomial.

Composition of polynomials: Convolution is not the only important operation between two polynomials. Another is composition $c(z) = f(z) \circ g(z) = f(g(z))$ which is defined for analytic functions $f(z)$ and $g(z)$. For example suppose $f(z) = 1 + z + z^2$ and $g(z) = e^{2z}$. Thus

$$f(z) \circ g(z) = 1 + e^{2z} + (e^{2z})^2 = 1 + e^{2z} + e^{4z}.$$

Note that $f(z) \circ g(z) \neq g(z) \circ f(z)$.

Exercise #*40*

Find $g(z) \circ f(z)$.

Solution: $e^{2f(z)} = e^{2(1+z+z^2)} = e^2 e^{(1+z+z^2)} = e^2 e^z e^{z^2}$

3.3.2 Residue expansions of rational functions

There are eight important Matlab/Octave routines that are closely related: `conv()`, `deconv()`, `poly()`, `polyder()`, `polyval()`, `residue()` and `root()`. Several of these are complements of each other or do a similar operation in a slightly different way. The routines `conv()` and `poly()` build polynomials from the roots, while `root()` solves for the roots given the polynomial coefficients. The operation `residue()` expands the ratio of two polynomials in a partial fraction expansion, as poles and residues.

When lines and planes are defined, the equations are said to be *linear* in the independent variables. In keeping with this definition of *linear*, we say that the equations are *nonlinear* when the equations have degree greater than 1 in the independent variables. The term *bilinear* has a special meaning: Both the domain and codomain are linearly related by lines (or planes). As an example, impedance is defined in frequency as the ratio of the voltage over the current, but it often has a representation as the ratio of two polynomials, $N(s)$ and $D(s)$:

$$Z(s) = \frac{N(s)}{D(s)} = sL_o + R_o + \sum_{k=0}^{K} \frac{K_k}{s - s_k}. \tag{3.30}$$

Here $Z(s)$ is the impedance, V and I are the voltage and current at radian frequency ω, and K_k, s_k are the residues and eigenvalues.[10]

[10]Note that the relationship between the impedance and the residues K_k is a linear one, ideally solved by setting up a linear system of equations in the unknown residues.

Such an impedance is typically specified as a rational or bilinear function—namely, the ratio of two polynomials, $P_N(s) = N(s) = [a_N, a_{n-1}, \ldots, a_o]$ and $P_K(s) = D(s) = [b_K, b_{K-1}, \ldots, b_o]$ of degrees $N, K \in \mathbb{N}$, as functions of complex Laplace frequency $s = \sigma + \jmath\omega$ with simple roots. Most impedances are rational functions, since they may be written as $D(s)V = N(s)I$. Since $D(s)$ and $N(s)$ are both polynomials in s, a rational function is also called a *bilinear transformation*, or in the mathematical literature a *Möbius transformation*, which comes from a corresponding scalar differential equation of the form

$$\sum_{k=0}^{K} b_k \frac{d^k}{dt^k} i(t) = \sum_{n=0}^{N} a_n \frac{d^n}{dt^n} v(t) \quad \leftrightarrow \quad I(\omega) \sum_{k=0}^{K} b_k s^k = V(\omega) \sum_{n=0}^{N} a_n s^n. \tag{3.31}$$

This construction is also known as the ABCD method in the engineering literature (Eq. 3.3. This equation, as well as Eq. 3.30, follows from the Laplace transform of the differential equation (on left) by forming the impedance $Z(s) = V/I = A(s)/B(s)$. This form of the differential equation follows from Kirchhoff's voltage and current laws (KCL, KVL) or from Newton's laws (for the case of mechanics).

Impedance is a very important and general concept. It is typically defined as the ratio of the change in voltage across a device, over the current through the device, which is known as *Ohm's law*. However it applies to many more physical variables than just electricity (see Table 3.3.2), which leads to the concept of a *generalized impedance*.
It began as the real ratio of the voltage drop over the current through, but by at least 1893 it was realized that complex numbers could be used to represent the complex impedance of inductors (mass) and capacitors (springs) (Heaviside, 1892; Kennelly, 1893). As we explore more deeply it is is likely that Maxwell understood this concept as well, since he first formulated his famous equations of electricity using complex analysis (Maxwell, 1865).
Since impedance is the ratio of a force over a flow, it does not directly depend on either the force or the flow. Rather it is the complex, frequency dependent proportionality factor between them:

$$\text{force} = Z(s) \cdot \text{flow} \quad \text{with } s, Z(s) \in \mathbb{C},$$

where $s = \sigma + \omega\jmath$ is the Laplace frequency.

The physical properties of an impedance: Based on d'Alembert's observation that the solution to the wave equation is the sum of forward and backward traveling waves, the impedance may be rewritten in terms of forward and backward traveling waves.

$$Z(s) = \frac{V}{I} = \frac{V^+ + V^-}{I^+ - I^-} = r_o \frac{1 + \Gamma(s)}{1 - \Gamma(s)}, \tag{3.32}$$

where $r_o = V^+/I^+$ is called the *characteristic impedance* of the transmission line (e.g., wire) connected to the load impedance $Z(s)$, and $\Gamma(s) = V^-/V_+ = I^-/I^+$ is the reflection coefficient corresponding to $Z(s)$.

Any impedance of this type is called a *Brune impedance* due to its special properties (Brune, 1931b; Van Valkenburg, 1964a). Like $Z(s)$, $\Gamma(s)$ is causal P_1, linear P_2, passive P_3 time invariant P_4, thus is complex–analytic. The impedance and the reflectance function $\Gamma(s)$ are both complex–analytic, since they are related to the bilinear transformation, which ensures the mutual complex–analytic properties.

Due to the bilinear transformation, the physical properties of $Z(s)$ and $\Gamma(s)$ are very different. Specifically, the real part of the load impedance is non-negative ($\Re\{Z(\omega\jmath)\} \geq 0$) if and only if $|\Gamma(s)| \leq 1$. In the time-domain, the impedance $z(t) \leftrightarrow Z(s)$ must have a value of r_o at $t = 0$. Correspondingly, the time-domain reflectance $\gamma(t) \leftrightarrow \Gamma(s)$ must be zero at $t = 0$.

This is the basis of conservation of energy, which may be traced back to the properties of the reflectance $\Gamma(s)$.

Exercise #*41*

Show that if $\Re\{Z(s)\} \geq 0$, then $|\Gamma(s)| \leq 1$.

Solution: Taking the real part of Eq. 3.32, which must be ≥ 0, we find

$$\Re\{Z(s)\} = \frac{r_o}{2}\left[\frac{1+\Gamma(s)}{1-\Gamma(s)} + \frac{1+\Gamma^*(s)}{1-\Gamma^*(s)}\right] = r_o\frac{1-|\Gamma(s)|^2}{|1+\Gamma(s)|^2} \geq 0.$$

Thus $|\Gamma| \leq 1$.

3.4 Introduction to analytic geometry

Analytic geometry came about as Euclid's geometry merged with algebra. The combination of Euclid's (323 BCE) geometry and al-Khwarizmi's (830 CE) algebra resulted in a totally new and powerful tool, analytic geometry, independently worked out by Descartes and Fermat (Stillwell, 2010). The development of matrix algebra during the 18th century, enabled an analysis in more than three dimensions. Due to modern computation, today this is one of the most powerful tools used in artificial intelligence, data science, and machine learning. The utility and importance of these new tools cannot be overstated. The timeline for this period of development in mathematics is shown in Fig. 1.2.

There are many important relationships between Euclidean geometry and 16th-century algebra. Table 3.4 is an attempt at a detailed comparison. Important similarities include vectors, their Pythagorean lengths $[a, b, c]$,

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \tag{3.33}$$

$a = x_2 - x_1$, and $b = y_2 - y_1$, and the angles. Euclid's geometry had length and angles but no concept of coordinates or thus of vectors. One of the main innovations of analytic geometry is that we could compute with real, and soon after, complex numbers, first observed in the completion of squares, Eq. **??**.

3.4.1 Merging the concepts

Several new concepts came with the development of analytic geometry:

1. Composition of functions: If $y = f(x)$ and $z = g(y)$, then the composition of functions f and g is denoted $z(x) = g \circ f(x) = g(f(x))$.

2. Elimination: Given two functions $f(x, y)$ and $g(x, y)$, elimination removes either x or y. This procedure, well known to the Chinese, is now known as *Gaussian elimination*.

3. Intersection: One may speak of the intersection of two lines to define a point or two planes to define a line. This is a special case of elimination when the functions $f(x, y)$ and $g(x, y)$ are linear in their arguments. The term *intersection* is also an important but very different from the meaning of the term as used in set theory (p. 87).

4. Vectors: Analytic geometry provides the concept of a vector (see Appendix 3.4.1) as a line with length and orientation (i.e., direction). Analytic geometry defines vectors in any number of dimensions as ordered sets of points.

5. Scalar products of vectors: Analytic geometry extends the ideas of Euclidean geometry with the introduction of the scalar (dot) product of two vectors $\mathbf{f} \cdot \mathbf{g}$ and the scalar wedge-product $\mathbf{f} \wedge \mathbf{g}$. The vector wedge-product $\mathbf{f} \wedge \mathbf{g}\hat{\mathbf{z}}$ represents a unit vector $\perp$ to the plane defined by the two vectors, as shown in Fig. 9.

The role of algebra in mathematics allows one to compute with complex numbers. Given geometry, the length of a line (Eq. 3.33) was measured with a compass. The numbers play no role. Once algebra was available, the line's Euclidean length could be computed numerically, directly from the coordinates of the two ends, defined by the 3-vector

$$\mathbf{e} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = [x, y, z]^T,$$

which represents a three dimensional vector. The equivalent matrix is $\mathbf{e} = [x, y, z]^T$. These two notations are alternative ways of representing the $\mathbb{R}^3$ vector. Note that the second method can represent vectors of any length.

Scalar product of two vectors: When we use algebra, many concepts that are obvious in Euclid's geometry are easily expressed. Algebra naturally extends Euclidean geometry, the most basic being the scalar product (dot product) between two vectors $\mathbf{x} \in \mathbb{R}^3$ and $\boldsymbol{\kappa} \in \mathbb{C}^3$:

$$\begin{aligned}\mathbf{x} \cdot \boldsymbol{\kappa} &= (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \cdot (\alpha\hat{\mathbf{x}} + \beta\hat{\mathbf{y}} + \gamma\hat{\mathbf{z}}) \\ &= \alpha x + \beta y + \gamma.\end{aligned}$$

Scalar products play an important role in vector algebra and calculus.

In vector notation the scalar product is written as

$$\mathbf{x} \cdot \boldsymbol{\kappa} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = [x, y, z] \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha x + \beta y + \gamma z. \tag{3.34}$$

If $\boldsymbol{\kappa}(s) \in \mathbb{C}^3$ is a complex function of frequency s, then the dot product is a complex function.

Norm of a vector: The length of a vector

$$||\mathbf{e}|| \equiv +\sqrt{\mathbf{e} \cdot \mathbf{e}} \geq 0$$

is defined as the positive square root of the scalar product of the vector with itself. p. 97). This is a generalization of a vector's length, in any number of dimensions, that forces the sign of the square root to be non-negative. A complex (or negative) length is not physically meaningful. More generally, the Euclidean length of a line is given as the norm of the difference between two real vectors $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}$:

$$\begin{aligned}||\mathbf{e}_1 - \mathbf{e}_2||^2 &= (\mathbf{e}_1 - \mathbf{e}_2) \cdot (\mathbf{e}_1 - \mathbf{e}_2) \\ &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \geq 0. \end{aligned} \tag{3.35}$$

From this formula we see that the norm of the difference of two vectors is a compact expression for the Euclidean length. A zero-length vector is a point, which follows from

$$||\boldsymbol{x} - \boldsymbol{x}||^2 = (\boldsymbol{x} - \boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{x}) = 0.$$

Integral definition of a scalar product: Following Euclid, we only considered a vector to be a set of elements $\{x_n\} \in \mathbb{R}$, index over $n \in \mathbb{N}$. Starting with Fig. 9 we assume the vectors are in $\mathbb{C}$.

An obvious question presents itself: Can we extend our definition of vectors to differentiable functions (i.e., $f(t)$ and $g(t)$) indexed over $t \in \mathbb{R}$ with coefficients labeled by $t \in \mathbb{R}$ rather than by $n \in \mathbb{N}$? Clearly, if the functions are analytic, there is no obvious reason that this should be a problem, since analytic functions may be represented by a convergent series that has Taylor coefficients and thus are integrable term by term.

Specifically, under certain conditions, the function $f(t)$ may be thought of as a vector, defining a normed vector space called a Hilbert space. This intuitive and perhaps obvious idea is powerful. In this case the scalar product can be defined in terms of the integral

$$\begin{aligned}f(t) \cdot g(t) &= \int_t f(t)g(t)dt \\ &= ||f(t)||\,||g(t)|| \cos\theta\end{aligned}$$

summed over $t \in \mathbb{R}$, rather than a sum over $n \in \mathbb{N}$.

This definition of the vector scalar product allows for a significant but straightforward generalization of our vector space, which will turn out to be both useful and an important extension of the concept of a normed vector space. In this space we can define the derivative of a norm with respect to t, which is not possible for the discrete case, indexed over n. The distinction introduces the concept of analytic continuity in the index t, which also fails to exist for the discrete index $n \in \mathbb{N}$.

Pythagorean theorem and the Schwarz inequality: Regarding Fig. 9, suppose we compute the difference between vector $\boldsymbol{A} \in \mathbb{R}$ and $\alpha\boldsymbol{B} \in \mathbb{R}$ as $L = ||\boldsymbol{A} - \alpha\boldsymbol{B}|| \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a scalar that modifies the length of $\boldsymbol{B}$. We seek the value of α, which we denote as α^*, that minimizes the length of L. From simple geometrical considerations, $L(\alpha)$ will be minimum when the difference vector is perpendicular to $\boldsymbol{B}$, as shown in the by the dashed line, from the tip of $\boldsymbol{A} \perp \boldsymbol{B}$.

To show this algebraically, we write the expression for $L(\alpha)$, take the derivative with respect to α, and set it to zero, which gives the formula for α^*. The argument does not change, but the algebra greatly simplifies if we normalize $\boldsymbol{A}$ and $\boldsymbol{B}$ to be unit vectors $\boldsymbol{a} = \boldsymbol{A}/||\boldsymbol{A}||$ and $\boldsymbol{b} = \boldsymbol{B}/||\boldsymbol{B}||$, which each have norm = 1:

$$L^2 = (\boldsymbol{a} - \alpha\boldsymbol{b}) \cdot (\boldsymbol{a} - \alpha\boldsymbol{b}) = 1 - 2\alpha\boldsymbol{a} \cdot \boldsymbol{b} + \alpha^2. \tag{3.36}$$

Thus the length is shortest (vector $\mathbf{C}$ of Fig. 9) when

$$\frac{d}{d\alpha} L_*^2 = -2\boldsymbol{a} \cdot \boldsymbol{b} + 2\alpha^* = 0.$$

To show this, solve for $\alpha^* \in \mathbb{R}$, giving $\alpha^* = \boldsymbol{a} \cdot \boldsymbol{b}$. Since $L_* > 0$ ($\boldsymbol{a} \neq \boldsymbol{b}$), Eq. 3.36 becomes

$$1 - 2|\boldsymbol{a} \cdot \boldsymbol{b}|^2 + |\boldsymbol{a} \cdot \boldsymbol{b}|^2 = 1 - |\boldsymbol{a} \cdot \boldsymbol{b}|^2 > 0.$$

In terms of $\boldsymbol{A}$ and $\boldsymbol{B}$, the *scalar dot product* is $|\boldsymbol{A} \cdot \boldsymbol{B}| < ||\boldsymbol{A}||\,||\boldsymbol{B}|| \cos\theta$, as shown adjacent to $\boldsymbol{B}$ in Fig. 9.

In conclusion, $\cos\theta \equiv |\boldsymbol{a} \cdot \boldsymbol{b}| \leq 1$. Thus the scalar product between two vectors is their direction cosine. Furthermore, since this forms a right triangle, the Pythagorean theorem must hold. The triangle inequality says that the sum of the lengths of the two sides must be greater than the length of the hypotenuse. Note that $\Theta \in \mathbb{R} \not\in \mathbb{C}$. Equality cannot be obtained because in Fourier space the scalar product defines an open set, which gives rise to Gibbs ringing in the time domain (Greenberg, 1988, p. 854). This derivation is

Vector cross ($\times$) and wedge ($\wedge$) products of two vectors: The vector product (cross-product) $\boldsymbol{a} \times \boldsymbol{b}$ and the exterior product (wedge-product) $\boldsymbol{a} \wedge \boldsymbol{b}$ are a second and third type of vector products.

As shown in Fig. 9,

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = (a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}) \times (b_1\hat{\mathbf{x}} + b_2\hat{\mathbf{y}} + b_3\hat{\mathbf{z}}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

$\boldsymbol{c}$ is $\perp$ to the plane defined by $\boldsymbol{a}$ and $\boldsymbol{b}$. The cross-product is strictly limited to two input vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, typically taken from $\mathbb{R}^3$. The exterior (wedge) product generalizes the cross-product, since it may be defined in terms of any two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^2$ taken from n dimensions ($\mathbb{C}^n$) with output in $\mathbb{C}^1$.

From this specific example we see that the absolute value of the wedge-product $|\mathbf{a} \wedge \mathbf{b}| = ||\mathbf{a} \times \mathbf{b}||$; namely,

$$|(a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}) \wedge (b_2\hat{\mathbf{y}} + b_3\hat{\mathbf{z}})| = ||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\;||\mathbf{b}||\;|\sin\theta|.$$

The wedge product is especially useful because it is zero when the two vectors are co-linear, namely when $\hat{\mathbf{x}} \wedge \hat{\mathbf{x}} = 0$ and $\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} = 1$, given unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

As described in Fig. 9,

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}|| \cos\theta \quad \text{and} \quad \mathbf{a} \wedge \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}||\; \sin\theta.$$

Thus it the wedgie $\curlywedge$ of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \curlywedge \mathbf{b} \equiv \mathbf{a} \cdot \mathbf{b} + \jmath\, \mathbf{a} \wedge \mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}||\, e^{\jmath\theta}, \tag{3.37}$$

which may be viewed as a generalized complex scalar product $\in \mathbb{C}$, with the right-hand side the polar form. The wedgie is an complex–analytic representation with important applications.

The key question is "What is the physical meaning of $\jmath\theta$? The obvious possibility is that

$$\jmath\theta \equiv s = \sigma + \jmath\omega \tag{3.38}$$

corresponds to the Laplace frequency s, as in the $\mathcal{LT}$. This interpretation requires that $\mathbf{a}(t), \mathbf{b}(t)$ are functions of time t. Thus

$$\mathbf{a}(t) \curlywedge \mathbf{b}(t) = ||\mathbf{a}(t)||\,||\mathbf{b}(t)||\, e^{st} \leftrightarrow A(s)B(s) \tag{3.39}$$

where $\leftrightarrow$ represents the $\mathcal{LT}$, as discussed in §3.9.

The main advantage of the wedge-product is that it is valid in $n \geq 3$ dimensions since it is defined for any two vectors in any number of dimensions.

Scalar triple product: The triple product $\boldsymbol{c}$ is defined given $\boldsymbol{a} \times \boldsymbol{b} \in \mathbb{R}$ as

$$\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \in \mathbb{R}^3, \tag{3.40}$$

which equals the volume of a parallelepiped.

3.4.2 Generalized wedge product

As discussed in Fig. 9 (p. 108), any two vectors $\boldsymbol{A}, \boldsymbol{B} \in \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ define a plane. There are two types of scalar products:[11] the *scalar dot product* (inner product)

$$\boldsymbol{A} \cdot \boldsymbol{B} = ||\boldsymbol{A}||\;||\boldsymbol{B}|| \cos\theta \in \mathbb{R}, \tag{3.41}$$

and the *scalar wedge product* (exterior product)

$$\boldsymbol{A} \wedge \boldsymbol{B} = ||\boldsymbol{A}||\;||\boldsymbol{B}|| \sin\theta \in \mathbb{R}. \tag{3.42}$$

These two products form a right triangle, when merged, define the *causal complex–analytic wedge product*

$$\boldsymbol{A} \curlywedge \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{B} + \jmath \boldsymbol{A} \wedge \boldsymbol{B} = ||\boldsymbol{A}||\;||\boldsymbol{B}|| e^{\jmath\theta} \quad \in \mathbb{C}. \tag{3.43}$$

3.4.3 Physical examples

Important examples include a generalization in terms of power $[W/m^2]$, of Einstein's 1905 Energy-mass equivalence formula ($E = mc^2$), derived in a few lines directly from Maxwell's equation, based on Poynting's theorem

$$\boldsymbol{\mathcal{P}}(t) = \boldsymbol{e}(t) \curlywedge \boldsymbol{h}(t) = \boldsymbol{e}(t) \cdot \boldsymbol{h}(t) + \jmath \boldsymbol{e}(t) \wedge \boldsymbol{h}(t) \qquad [\mathrm{W/m^2}] \tag{3.44}$$

(Sommerfeld, 1952, p. 26), and the corresponding momentum equation (Johnson et al., 1994)

$$\boldsymbol{\mathcal{M}}(t) = \boldsymbol{d}(t) \curlywedge \boldsymbol{b}(t) = \boldsymbol{d}(t) \cdot \boldsymbol{b}(t) + \jmath \boldsymbol{d}(t) \wedge \boldsymbol{b}(t) = \frac{1}{c_o^2} \boldsymbol{\mathcal{P}}(t) \qquad [\mathrm{J\ s/m^4}]. \tag{3.45}$$

While the *solar constant* $|\boldsymbol{\mathcal{P}}| = 1.3$ [kW/m^2] is large, while solar gravity, which follows directly from Maxwell's equations, is $1/c_o^2 = 1.1 \times 10^{-17}$, thus tiny by comparison.

Example: If we define $\boldsymbol{a} = 3\jmath\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$ and $\boldsymbol{b} = 1\hat{\mathbf{x}} + 1\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$, then the cross-product is

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 3\jmath & -2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = (3\jmath + 2)\hat{\mathbf{z}}.$$

Since $a_1 \in \mathbb{C}$, this example violates the common assumption that $\boldsymbol{a} \in \mathbb{R}^3$. The wedge-product $\boldsymbol{a} \wedge \boldsymbol{b}$ takes two vectors and returns a scalar, which is the magnitude of a vector $\perp$ to the plane defined by the two input vectors (see Fig. 9). It is defined as the vector wedge-product which defines a compact and important causal (P1) complex–analytic algebra (Hestenes, 2003).

Exercise #*42*

[11] `https://en.wikipedia.org/wiki/Bivector`

Expand the above example of $\boldsymbol{a}\times\boldsymbol{b}$

$$
\begin{aligned}
\boldsymbol{a}\wedge\boldsymbol{b} &= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 3\jmath & 1 \\ -2 & 1 \end{vmatrix} \\
&= (3\jmath\hat{\mathbf{x}} - 2\hat{\mathbf{y}}) \wedge (\hat{\mathbf{x}} + \hat{\mathbf{y}}) \\
&= 3\cdot 0\, \cancelto{0}{\boldsymbol{x}\wedge\boldsymbol{x}} - 2\hat{\mathbf{y}}\wedge\hat{\mathbf{x}} + 3\jmath\hat{\mathbf{x}}\wedge\hat{\mathbf{y}} - 2\,\cancelto{0}{\hat{\mathbf{y}}\wedge\hat{\mathbf{y}}} \\
&= (3\jmath + 2)\hat{\mathbf{x}}\wedge\hat{\mathbf{y}} \\
&= (3\jmath + 2)\hat{\mathbf{z}}
\end{aligned}
$$

Impact of Analytic Geometry: The impact of analytic geometry was its detailed analysis of the conic sections using algebra rather than drawings using a compass and ruler. An important example is the composition of the line and circle, a venerable pre-Euclid construction.

Once algebra was invented, analysis could be done using analysis (algebra). With analysis came complex numbers.

The first two mathematicians to appreciate this mixture of Euclid's geometry and the new algebra were Fermat and Descartes (Fig. 1.1. Soon Newton contributed to this effort by adding physics (e.g., calculations in acoustics, orbits of the planets, and the theory of gravity and light, significant concepts for 1687) (Stillwell, 2010, p. 115-117)).

Given these new methods, many new solutions to problems emerged. The complex roots of polynomials continued to appear, without any obvious physical meaning. Newton called them *imaginary*. Complex numbers seem to have been viewed as an inconvenience. Newton's solution to this dilemma was to simply ignore the "imaginary" cases (Stillwell, 2010, p. 115–19).

3.4.4 Development of Analytic Geometry

The first "algebra" (*al-jabr*) is credited to al-Khwarizmi (830 CE). Its invention advanced the theory of polynomial equations in one variable, Taylor series, and composition versus intersections of curves. The solution of the quadratic equation had been worked out thousands of years earlier, but with algebra a general solution could be defined. The Chinese had found the way to solve several equations in several unknowns—for example, finding the values of the intersections of two circles. With the invention of algebra by al-Khwarizmi, a powerful tool became available to solve more difficult problems.

In algebra there are two contrasting operations on functions: composition and elimination (e.g., intersection).

Composition:

Composition is the merging of functions by feeding one into the other. If the two functions are f and g, then their composition is indicated by $f\circ g$, meaning the function $y = f(x)$ is substituted into the function $z = g(y)$, giving $z = g(f(x))$.

Composition is not limited to linear equations, even though that is where it is most frequently applied. That requires solving for that substitution variable, which is not always possible in the case of nonlinear equations. However, many tricks are available that may work around this restriction. For example, if one equation is in x^2 and the other in x^3 or $\sqrt{x}$, it may be possible to multiply the first by x or square the second. The point is that one of the variables must be isolated so that when it is substituted into the other equation, the variable is removed from the mix.

Example: Let $y = f(x) = x^2 - 2$ and $z = g(y) = y + 1$. Then

$$g\circ f = g(f(x)) = (x^2 - 2) + 1 = x^2 - 1. \tag{3.46}$$

In general, composition does not commute (i.e., $f\circ g \neq g\circ f$), as is easily demonstrated. Swapping the order of composition for our example gives

$$f\circ g = f(g(y)) = z^2 - 2 = (y+1)^2 - 2 = y^2 + 2y - 1. \tag{3.47}$$

Intersection:

Complementary to composition is intersection (i.e., decomposition). For example, the intersection of two lines is defined as the point where they meet. This is not to be confused with finding roots. A polynomial of degree N has N roots, but the points where two polynomials intersect has nothing to do with the roots of the polynomials. The intersection is a function (equation) of lower degree, implemented by Gaussian elimination.

A system of linear equations $Ax = y$ has many interpretations, and one should not be biased by the notation. As engineers, we are trained to view x as the input and y as the output. Then $y = Ax$ seems natural, much like the functional relationship $y = f(x)$. But what does the linear relationship $x = Ay$ mean, when x is the input? The answer is $y = A^{-1}x$.

But when we work with systems of equations, there are many uses of equations, and we need to be more flexible in our interpretation. For example, $y = A^2x$ has a useful meaning, and in fact we saw this type of relationship we worked with Pell's equation and the Fibonacci sequence. As another example, consider

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_{1x} & a_{1y} \\ a_{2x} & a_{2y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which is reminiscent of a two-dimensional surface $\mathbf{z} = \mathbf{f}(x, y)$. We shall find that such generalizations are much more than a curiosity.

Intersection of two lines: Unless they are parallel, two lines meet at a point. In terms of linear algebra, this may be written as two linear equations[12] (on the left) along with the intersection point $[x_1, x_2]^T$ given by the inverse of the 2×2 set of equations (on the right):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{3.48}$$

By substituting the expression for the intersection point $[\boldsymbol{x}_1, \boldsymbol{x}_2]^T$ into the original equation, we see that it satisfies the equations. Thus the equation on the right is the solution to the equation on the left.

Elimination:

Note the structure of the inverse: (1) The diagonal values (a, d) are swapped, (2) the off-diagonal values (b, c) are negated, and (3) the 2×2 matrix is divided by the determinant $\Delta = ad - bc$. If $\Delta = 0$, there is no solution. When the determinant is zero ($\Delta = 0$), the slopes of the two lines

$$\text{slope} = \frac{b}{a} = \frac{d}{c}$$

are equal; thus the lines are parallel. Only if the slopes differ can there be a unique solution.

Algebra can give the solution when geometry cannot. When the two curves fail to intersect on the real plane, the solution still exists, but it is complex-valued. In such cases, geometry, which considers only the real solutions, fails. For example, when the coefficients $[a, b, c, d]$ are complex, the solution exists but the determinant can be complex. Thus algebra is much more general than geometry. Geometry fails when the solution has a complex intersection.

3.4.5 Applications of scalar products

Another important example of algebraic expressions in mathematics is Hilbert's generalization of the Pythagorean theorem (Eq. 1.1), known as the *Schwarz* inequality and shown in Fig. 3.2. What is special about this generalization is that it shows that when the vertex is 90°, the Euclidean length of the leg is minimum.

Vectors may be generalized to have ∞ dimensions. For example $\mathbf{U} = [u_1, u_2, \ldots, u_\infty]$, $\mathbf{V} = [v_1, v_2, \ldots, v_\infty]$. The Euclidean inner product (i.e., scalar product) between two such vectors generalizes the finite-dimensional case

$$\mathbf{U} \cdot \mathbf{V} = \sum_{k=1}^{\infty} u_k v_k = \|\mathbf{U}\| \|\mathbf{V}\| \ \cos\theta$$

[12] When we write the equation $Ax = y$ in matrix format, the two equations are $ax_1 + bx_2 = y_1$ and $dx_1 + ex_2 = y_2$ with unknowns (x_1, x_2), whereas in the original equations $ay + bx = c$ and $dy + ex = f$, the unknowns are y and x. Thus in matrix format, the names are changed. The first time you see this scrambling of variables, it can be confusing.

where $\theta \in \mathbb{R}$ is the multi-valued angle between the two normalized (unit) vectors

$$\theta = \cos^{-1}\left(\frac{\mathbf{U}}{||\mathbf{U}||} \cdot \frac{\mathbf{V}}{||\mathbf{U}||}\right).$$

As with the finite case the norm $||\mathbf{U}|| = \sqrt{\mathbf{U}\cdot\mathbf{U}} = \sqrt{\sum u_k^2}$, the scalar product of the vector with itself, defines the length of the infinite component vector. There is an issue of convergence when the norm of the vectors is zero.

It is an arbitrary requirement that $a, b, c \in \mathbb{R}$ for (Eq. 1.1), but may be justified because the sides are lengths. If these lengths are taken from high–dimensionality complex vectors, as for the lossy vector wave equation or the lengths of vectors in the Fourier transform $(\mathcal{FT}) \in \mathbb{C}^n$? Then the equation generalizes to $K \to \infty$ dimensions

$$\mathbf{c}\cdot\mathbf{c} = ||\mathbf{c}||^2 = \sum_{k=1}^{K\to\infty} |c_k|^2.$$

As before, $||\mathbf{c}|| = \sqrt{||\mathbf{c}||^2}$ is the *norm* of vector $\mathbf{c}$, akin to a length, which must be finite (converge). This is simply the important case of complex–analytic functions, which also must converge.

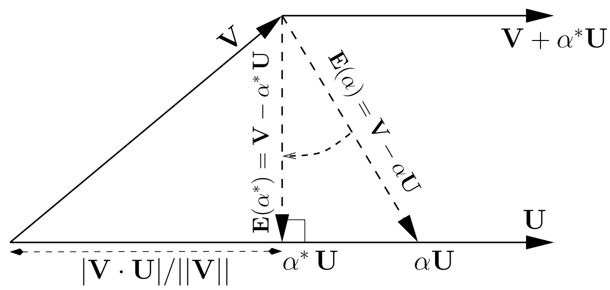


Figure 3.2: *The* Schwarz inequality *is related to the shortest distance (length of a line) between the ends of the two vectors.* $||U|| = \sqrt{U \cdot U}$ *is the scalar product of that vector with itself.*

Schwarz inequality: The Schwarz inequality says that the magnitude of the inner product of two vectors is less than or equal to the product of their lengths:

$$|U \cdot V| \le ||U||\,||V||.$$

This may be simplified by normalizing the vectors to have unit length ($\widehat{U} = U/||U||, \widehat{V} = V/||V||$), in which case $-1 < \widehat{U}\cdot\widehat{V} \le 1$. Another simplification is to define the scalar product in terms of the direction cosine

$$\cos\theta = |\widehat{U}\cdot\widehat{V}| \le 1.$$

From these definitions we may define the minimum difference between the two vectors as the perpendicular from the end of the first to the intersection with the second. As shown in Fig. 9, $U \perp V$ may be found by minimizing the length of the vector difference:

$$\begin{aligned}
\min_\alpha ||V - \alpha U||^2 &= ||V||^2 + 2\alpha V\cdot U + \alpha^2||U||^2 > 0\\
0 &= \partial_\alpha\,(V-\alpha U)\cdot(V-\alpha U)\\
&= V\cdot U - \alpha^*||U||^2\\
\therefore\quad \alpha^* &= V\cdot U/||U||^2.
\end{aligned}$$

The Schwarz inequality follows:

$$I_{\min} = ||V - \alpha^* U||^2 = ||V||^2 - \frac{|U\cdot V|^2}{||U||^2} > 0 \tag{3.49}$$

$$0 \le |U\cdot V| \le ||U||\,||V||.$$

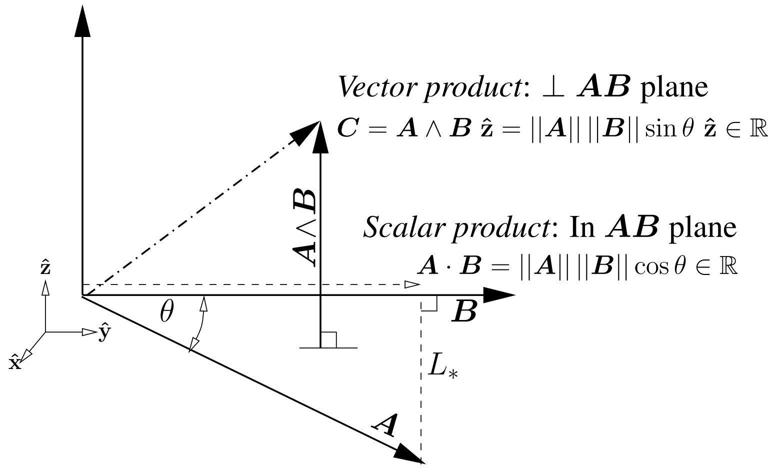


Figure 3.3: *Vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{C}$ are used to define the scalar product $\boldsymbol{A} \cdot \boldsymbol{B} \in \mathbb{C}$ and the scalar wedge-product $\boldsymbol{A} \wedge \boldsymbol{B} \in \mathbb{C}$. The scalar wedge-product is the same as the vector cross-product except the output is a complex scalar. The scalar dot and wedge-products complement each other, since one is proportional to the sine of the angle θ between them, and the other to the cosine. The dot product computes the projection of one vector on the other (the length of the base of the triangle formed by the two vectors), while the vector wedge-product $\boldsymbol{A} \wedge \boldsymbol{B}$ computes the area of the right triangle ($area = base \cdot height = \boldsymbol{A}{\cdot}\boldsymbol{B}\, L_*$) formed by the two vectors. Thus $|\boldsymbol{A} \cdot \boldsymbol{B}|^2 + |\boldsymbol{A} \wedge \boldsymbol{B}|^2 = ||\boldsymbol{A}||^2 ||\boldsymbol{B}||^2$. The* scalar triple product *$\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$ represents the volume of the parallelepiped (i.e., prism) defined by the three vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$. When all the angles are $90°$, the volume becomes a* cuboid.

An important example of such a vector space includes the definition of the $\mathcal{FT}$ where we may set

$$U(\omega) = e^{-\omega_0 \jmath t} \qquad V(\omega) = e^{\omega \jmath t} \qquad U \cdot V = \int_\omega e^{\jmath\omega t} e^{-\jmath\omega_0 t} \frac{d\omega}{2\pi} = \delta(\omega - \omega_0).$$

It seems that the Fourier transform is a result that follows from a minimization, unlike the Laplace transform, which follows from causality. This explains the important differences between the two in terms of their properties (unlike the $\mathcal{LT}$, the $\mathcal{FT}$ is not complex–analytic). Recall that

$$U \cdot V + \jmath U \wedge V = ||U||\ ||V|| e^{\jmath\theta}.$$

3.4.6 Gaussian Elimination

This method, for finding the intersection of equations, is based on the recursive elimination of all the variables but one. This method, well known as *Newton's method* (Allen (2025)). or simply *Gaussian elimination* (p. 100), works across a broad range of cases, and may be defined as a systematic algorithm when the equations are linear (Strang et al., 1993).[13] Rarely do we even attempt to solve problems in several variables of degree greater than 1. Least-square optimization works quite well in this case (Stillwell, 2010, p. 90).

In Appendix 3.5.6 (p. 101) we derive the inverse of a 2×2 linear system of equations. Even for a 2×2 case, the general solution requires a great deal of algebra. Working out a numeric example of Gaussian elimination is more instructive. For example suppose we wish to find the intersection of the two equations

$$x - y = 3$$
$$2x + y = 2.$$

This 2×2 system of equations is so simple that you may immediately visualize the solution: By adding the two equations, y is eliminated, leaving $3x = 5$. But doing it this way takes advantage of the specific example, and we need a method for larger systems of equations. We need a generalized (algorithmic) approach. This general approach is called Gaussian elimination.

First write the equations in matrix form, as:

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}. \tag{3.50}$$

[13] https://en.wikipedia.org/wiki/System_of_linear_equations.

Next we eliminate the lower left term ($2x$) using a scaled version of the upper left term (x). Specifically, we multiply the first equation by -2 and add it to the second equation. This gives

$$\begin{bmatrix}1 & -1\\ 0 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}3\\ 2-3\cdot 2\end{bmatrix} = \begin{bmatrix}3\\ -4\end{bmatrix}. \tag{3.51}$$

Note that the top equation did not change. Once the matrix is "upper triangular" (zero below the diagonal), we have the solution. Starting from the bottom equation, $y = -4/3$. Then the upper equation gives $x - (-4/3) = 3$ or $x = 3 - 4/3 = 5/3$.

In principle, Gaussian elimination is simple, however if you make a calculation mistake along the way, it is difficult to find your error. Best you start over. Suppose the elements are complex numbers, or polynomials in some other variable, such as frequency. Once the equations become more complicated, a seemingly trivial problem becomes corrosive. There is a much better way that is easily verified; it puts all the numerics at the end in a single step.

The above operations may be automated by finding a carefully chosen upper-diagonalized matrix G. For example, we can define the *Gaussian matrix* that zeros the element 2 in the matrix in Eq. 3.50. More generally let

$$G = \begin{bmatrix}1 & 0\\ a & 1\end{bmatrix}. \tag{3.52}$$

Multiplying Eq. 3.50 by G, we find

$$\begin{bmatrix}1 & 0\\ a & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}1 & -1\\ a+2 & 1-a\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}3\\ 3a+2\end{bmatrix}. \tag{3.53}$$

Thus we obtain Eq. 3.51 if we let $a = -2$ (we choose a to force the lower left to be zero). At this point we can either back-substitute and obtain the solution, as we did above, or find a matrix L that finishes the job by removing elements above the diagonal. Note that the determinant of matrix G is 1, thus it will always have an inverse.

Exercise #*43*

Using G and A from the discussion above, show that $\det(G) = \det(GA) = 3$.

Solution: A common convention is to denote $\det(A) = |A|$. The two sides of the identity are

$$|A| = \det\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix} = 1+2 = 3, \qquad |GA| = \det\begin{bmatrix}1 & -1\\ 0 & 3\end{bmatrix} = 3,$$

and $|G| = 1$. Thus $|GA| = |G||A| = 3$.

Matrix inverse: In Appendix 3.5.6 (p. 101), finding the inverse of a general 2×2 matrix takes three steps: (1) swap the diagonal elements, (2) reverse the signs of the off-diagonal elements, and (3) divide by the determinant $\Delta = ab - cd$. Specifically,

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{\Delta}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}. \tag{3.54}$$

There are very few things that you must memorize, but the inverse of a 2×2 matrix is one of them. It needs to be in your mental toolkit, like completing the square (see p. **??**).

While it is difficult to compute the inverse matrix from scratch (see Appendix 5), it takes only a few seconds (four dot products) to verify it (steps 1 and 2):

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix} = \begin{bmatrix}ad-bc & -ab+ab\\ cd-cd & -bc+ad\end{bmatrix} = \begin{bmatrix}\Delta & 0\\ 0 & \Delta\end{bmatrix}. \tag{3.55}$$

Thus dividing by the determinant gives the 2×2 identity matrix. A good strategy (don't trust your memory) is to write down the inverse as best you recall and then verify.

Using the 2×2 matrix inverse on our example (Eq. 3.50), we find

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1+2} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ -6+2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -4/3 \end{bmatrix}. \tag{3.56}$$

If you use this method, you will rarely (never) make a mistake and the solution is easily verified.

Augmented matrix: There is one minor notational improvement. Rather than writing the matrix equation as Eq. 3.50 ($Ax = y$), we place the y vector next to the elements of A to remove the equal sign, which is cumbersome. In this case we write GA_{aug}:

$$GA_{\text{aug}} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & -4 \end{array}\right].$$

3.5 Matrix algebra: Systems 5

3.5.1 Vectors

Vectors as columns of ordered sets of scalars $\in \mathbb{C}$. When we write them out in text, we typically use row notation, with the *transpose* symbol:

$$[a, b, c]^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

This is strictly to save space on the page. The notation for *conjugate transpose is* †, for example

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}^\dagger = \begin{bmatrix} a^* & b^* & c^* \end{bmatrix}.$$

The above example is said to be a *3-dimensional* vector because it has three components.

Row vs. column vectors: With rare exceptions, vectors are columns, denoted *column-major*.[14] To avoid confusion, it is a good rule to make your mental default column-major, in keeping with most signal processing (vectorized) software.[15] Column vectors are the unstated default of Matlab/Octave, only revealed when matrix operations are performed. The need for the column (or row) major is revealed as a consequence of efficiency when accessing long sequences of numbers from computer memory. For example, when forming the sum of many numbers using the Matlab/Octave command `sum(A)`, where `A` is a matrix, Matlab/Octave operates on the columns, returning a row vector of column sums:

$$\texttt{sum} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = [4, 6].$$

If the data were stored in row-major order, the answer would be the column vector $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$. Thus Matlab/Octave is column-major by default.

3.5.2 Vector products

A *scalar product* (aka dot product) is defined to "weight" vector elements before summing them, resulting in a scalar. The transpose of a vector (a *row-vector*) is typically used as a *scale factor* (i.e., weights) on the elements of a vector. For example,

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 2 \cdot 2 - 3 = 2.$$

[14] `https://en.wikipedia.org/wiki/Row-_and_column-major_order`

[15] In contrast, reading words in English is 'row-major.'

A more interesting example defines a polynomial

$$\begin{bmatrix}1\\2\\4\end{bmatrix} \cdot \begin{bmatrix}1\\s\\s^2\end{bmatrix} = \begin{bmatrix}1\\2\\4\end{bmatrix}^T \begin{bmatrix}1\\s\\s^2\end{bmatrix} = [1 \quad 2 \quad 4] \begin{bmatrix}1\\s\\s^2\end{bmatrix} = 1 + 2s + 4s^2.$$

Polar scalar product: The vector-scalar product in polar coordinates is (Fig. 9, p. 108)

$$\boldsymbol{B} \cdot \boldsymbol{C} = \|\boldsymbol{B}\| \, \|\boldsymbol{C}\| \cos\theta \in \mathbb{R}, \tag{3.57}$$

where $\cos\theta \in \mathbb{R}$ is called the *direction-cosine* between $\boldsymbol{B}$ and $\boldsymbol{C}$.

Polar wedge product: The vector wedge product in polar coordinates is (Fig. 9, p. 108)

$$\boldsymbol{B} \wedge \boldsymbol{C} = \|\boldsymbol{B}\| \, \|\boldsymbol{C}\| \sin\theta \in \mathbb{R}, \tag{3.58}$$

where $\sin\theta \in \mathbb{R}$ is therefore the *direction-sine* between $\boldsymbol{B}$ and $\boldsymbol{C}$.

Complex polar scalar product: From these two polar definitions and $e^{\jmath\theta} = \cos\theta + \jmath \sin\theta$,

$$\boldsymbol{B} \cdot \boldsymbol{C} + \jmath \boldsymbol{B} \wedge \boldsymbol{C} = ||\boldsymbol{B}||||\boldsymbol{C}|| \, e^{st}. \tag{3.59}$$

This expression is causal, thus obeys relativity, which follows from causality (P1). It is important to explore applications of this interesting complex–analytic expression.

Taking the norm of both sides,

$$|\boldsymbol{B} \cdot \boldsymbol{C}|^2 + |\boldsymbol{B} \wedge \boldsymbol{C}|^2 = |\|\boldsymbol{B}\|^2 \, \|\boldsymbol{C}\|^2 \cos^2\theta| + |\|\boldsymbol{B}\|^2 \, \|\boldsymbol{C}\|^2 \sin^2\theta| = ||\boldsymbol{B}||^2||\boldsymbol{C}||^2.$$

This relationship holds true for 2, 3 or more dimensions, containing vectors $\boldsymbol{B}$ and $\boldsymbol{C}$. In this case $s = \sigma + \omega\jmath \in \mathbb{C}$ can assumed to be the Laplace frequency. Jaynes (1991) has an relevant discussion about this type of scalar product.

Formula for the non-relativistic energy $E(t)$ of a mass m_o moving at velocity $\nu(t)$. To correct Einstein's famous energy-mass relation $E = m_o c_o^2$, one must express it as a function of its relative velocity $\nu(t)/c_o$,

$$E(t) = m_o \frac{d\nu(t)}{dt}. \tag{3.60}$$

where m_o is the rest mass. This is correct if we use *Lorentz's formulation* of the energy.

$$E(t) = m_o/\sqrt{1 - (v/c)^2} \tag{3.61}$$

Of course neither of these two relations are relativistic. Relativity follows from the relativistic Lorentz expression:

$$E = m \frac{d}{dt} \sqrt{1 - \left(\frac{v(t)}{c_o}\right)^2}. \tag{3.62}$$

3.5.3 Norms of vectors

The norm of a vector is the scalar product of the vector with itself

$$\|\boldsymbol{A}\| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}} \geq 0,$$

resulting in the Euclidean length of the vector.

Euclidean distance between two points in $\mathbb{R}^3$: The scalar product of the difference between two vectors $(\boldsymbol{A}-\boldsymbol{B})\cdot(\boldsymbol{A}-\boldsymbol{B})$ is the Euclidean distance between the points they define

$$||\boldsymbol{A}-\boldsymbol{B}|| = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2}.$$

Triangle inequality

$$||\boldsymbol{A}+\boldsymbol{B}|| = \sqrt{(a_1+b_1)^2+(a_2+b_2)^2+(a_3+b_3)^2} \le ||\boldsymbol{A}|| + ||\boldsymbol{B}||.$$

In terms of a right triangle this says the sum of the lengths of the two sides is greater to the length of the hypotenuse, and equal when the triangle degenerates into a line.

Vector cross product: The *vector product* (aka cross product) $\boldsymbol{A}\times\boldsymbol{B} = ||\boldsymbol{A}||\,||\boldsymbol{B}||\sin\theta$ is defined between the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. In Cartesian coordinates

$$\boldsymbol{A}\times\boldsymbol{B} = \det\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

The triple product: This is defined between three vectors as

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \det\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.$$

This may be indicated without the use of parentheses, since there can be no other meaningful interpretation. However for clarity, parentheses should be used. The triple product is the volume of the parallelepiped (3D-crystal shape) outlined by the three vectors, as shown in Fig. 9 p. 108.

Dialects of vector notation: Physical fields are, by definition, functions of space $\boldsymbol{x}$ [m], and in the most general case, time t [s]. When Laplace transformed, the fields become functions of space and complex frequency (e.g., $\boldsymbol{e}(\boldsymbol{x},t) \leftrightarrow \boldsymbol{e}(\boldsymbol{x},s)$). As before, there are several equivalent vector notations. For example, $\boldsymbol{e}(\boldsymbol{x},t) = [E_x,\ \ E_y,\ \ E_z]^T = E_x(\boldsymbol{x},t)\hat{\mathbf{x}} + E_y(\boldsymbol{x},t)\hat{\mathbf{y}} + E_z(\boldsymbol{x},t)\hat{\mathbf{z}}$ is "in-line," to save space. The same equation may written in "displayed" notation as:

$$\boldsymbol{e}(\boldsymbol{x},t) = \begin{bmatrix}E_x(\boldsymbol{x},t)\\ E_y(\boldsymbol{x},t)\\ E_z(\boldsymbol{x},t)\end{bmatrix} = \begin{bmatrix}E_x\\ E_y\\ E_z\end{bmatrix}(\boldsymbol{x},t) = [E_x,\ \ E_y,\ \ E_z]^T \equiv E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}.$$

Note the four notations for vectors, bold font, element-wise columns, element-wise transposed rows and dyadic format. These are all shorthand notations for expressing the vector. Such usage is similar to a dialect of a language.

Complex elements: When the elements are complex ($\in \mathbb{C}$), the transpose is defined as the complex conjugate of the elements. In such complex cases the transpose conjugate may be denoted with a $\dagger$ rather than T

$$\begin{bmatrix}-2\jmath\\ 3\jmath\\ 1\end{bmatrix}^\dagger = [2\jmath\ \ -3\jmath\ \ 1] \in \mathbb{C}.$$

For this case when the elements are complex, the dot product is a real number (like a length)

$$\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}^\dagger\boldsymbol{b} = [a_1^*\ \ a_2^*\ \ a_3^*]\begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix} = a_1^*b_1 + a_2^*b_2 + a_3^*b_3 \in \mathbb{R}.$$

Norm of a complex vector: The dot product of a vector with itself is called the *norm* of $\boldsymbol{a}$

$$||\boldsymbol{a}|| = \sqrt{\boldsymbol{a}^\dagger \boldsymbol{a}} \geq 0.$$

which is always non-negative, and real.

Such a construction is useful when $\boldsymbol{a}$ and $\boldsymbol{b}$ are related by an impedance matrix

$$\boldsymbol{V}(s) = \boldsymbol{Z}(s)\boldsymbol{I}(s)$$

and we wish to compute the power. For example, the impedance of a mass is ms and a capacitor is $1/sC$. When given a system of equations (a mechanical or electrical circuit) one may define an impedance matrix.

Complex power: In this special case, the *complex power* $\mathcal{P}(s) \in \mathbb{C}(s)$ is defined, in the complex frequency domain (s), as

$$\mathcal{P}(s) = \boldsymbol{I}^\dagger(s)\boldsymbol{V}(s) = \boldsymbol{I}^\dagger(s)\boldsymbol{Z}(s)\boldsymbol{I}(s) \leftrightarrow p(t) \quad \text{[W]}.$$

The real part of the complex power must be positive. The imaginary part corresponds to available stored energy.

3.5.4 Matrices

When working with matrices, the role of the weights and vectors can change, depending on the context. A useful way to view a matrix is as a set of column vectors, weighted by the elements of the column-vector of weights multiplied from the right. For example,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1M} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2M} \\ & & \ddots & & \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NM} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_M \end{bmatrix} = w_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{N1} \end{bmatrix} + w_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{N2} \end{bmatrix} + \cdots + w_M \begin{bmatrix} a_{1M} \\ a_{2M} \\ a_{3M} \\ \vdots \\ a_{NM} \end{bmatrix},$$

where the weights are $[w_1, w_2, \ldots, w_M]^T$. Note that a_{23} is in row 2, column 3, thus is $a_{\text{row,col}}$. Rows are index vertically, according to the column definition of a vector. Think of the matrix as M column vectors with a_{n1} being the first vector. An interesting case is where the weights are the prime numbers, requiring a remapping of the matrix.

Alternatively, the matrix is a set of row vectors of weights, each of which is applied to the column vector on the right ($[w_1, w_2, \ldots, W_M]^T$). Both views are important (and correct). Don't think of a matrix as being just one or the other. It is both, but not at the same time.

The determinant of a matrix is denoted as either $\det \boldsymbol{A}$ or simply $|\boldsymbol{A}|$ (as in the absolute value). The inverse of a square matrix is $\boldsymbol{A}^{-1}$ or inv$\boldsymbol{A}$. If $|A| = 0$, the inverse does not exist. If it does then $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A}$.

Matlab/Octave's notional convention for a row-vector is $[a, b, c]$ and a column-vector as $[a; b; c]$. The prime symbol $'$ operating on a vector takes the complex conjugate–transpose. To suppress the conjugation, place a period before the prime. The ':' argument converts the array into a column vector, without conjugation. A tacit notation in Matlab is that *vectors* are columns and the index to a vector is a row vector. Matlab defines the notation $1\!:\!4$ as the "row-vector" $[1, 2, 3, 4]$, which may be unfortunate as it leads users to assume that the default vector is a row. This can lead to serious confusion later, as Matlab's default vector is a column. I have not found the above convention explicitly stated, and it took me years to figure this out for myself.

3.5.5 $N \times M$ complex matrices

Here are some helpful definitions:

1. *Scalar*: A number – for example $\{a, b, c, \alpha, \beta, \ldots\} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C}\}$

2. *Vector*: A quantity having direction as well as magnitude, often denoted by a bold letter $\mathbf{x}$, or with an arrow over the symbol. In matrix notation, this is typically represented as a single row $[x_1, x_2, x_3, \ldots]$, or single column $[x_1, x_2, x_3 \ldots]^T$, where the superscript T indicates the transpose. In this book the assumed default is a column vector. The vector may also be written out using unit vector notation to indicate direction. For example: $\mathbf{x}_{3,1} = x_1\hat{\mathbf{x}} + x_2\hat{\mathbf{y}} + x_3\hat{\mathbf{z}} = [x_1, x_2, x_3]^T$, where $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are unit vectors in

the x, y, z Cartesian directions. Above the vector's subscript $3, 1$ indicates its dimensions. The specific notation used typically depends on the engineering problem being solving.

3. *Matrix:* $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_M]_{N,M} = \{a_{n,m}\}_{N,M}$ can be a non-square matrix when the number of elements in each of the vectors (N) is not equal to the number of vectors (M). When $M = N$, the matrix is square. It may be inverted if its determinant $|A| = \prod \lambda_k \neq 0$ (where λ_k are the eigenvalues). In this text we typically use 2×2 and 3×3 square matrices.

4. *Linear system of equations*: $A\mathbf{x} = \mathbf{b}$ where $\mathbf{x}$ and $\mathbf{b}$ are vectors and matrix A is square ($M = N$). If this condition does not hold the system is either under or over determined. These cases require careful consideration, as described below.

   (a) *Inverse:* The solution of this system of equations may be found by finding the inverse $\mathbf{x} = A^{-1}\mathbf{b}$.

   (b) *Equivalence:* If two systems of equations $A_0\mathbf{x} = \mathbf{b}_0$ and $A_1\mathbf{x} = \mathbf{b}_1$ have the same solution (i.e., $\mathbf{x} = A_0^{-1}\mathbf{b}_0 = A_1^{-1}\mathbf{b}_1$), they are said to be *equivalent*.

   (c) *Augmented matrix:* The first type of augmented matrix is defined by combining the matrix with the right-hand side. For example, given the linear system of equations of the form $A\mathbf{x} = \mathbf{y}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

   the augmented matrix is

$$[A|y] = \left[\begin{array}{cc|c} a & b & y_1 \\ c & d & y_2 \end{array}\right].$$

   A second type of augmented matrix may be used for finding the inverse of a matrix (rather than solving a specific instance of linear equations $Ax = b$). In this case the *augmented matrix* is

$$[A|I] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right].$$

   On may attempt *Gaussian elimination* the augmented matrix, until the left side becomes the identity matrix, yielding A^{-1}. This is because multiplying both sides by A^{-1} gives $A^{-1}A|A^{-1}I = I|A^{-1}$. When this fails, the system may not have a solution.

5. *Permutation matrix* (P): A matrix that is equivalent to the identity matrix, but with scrambled rows (or columns). Such a matrix has the properties $\det(P) = \pm 1$ and $P^2 = I$. For the 2×2 case, there is only one permutation matrix:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   A permutation matrix P swaps rows or columns of the matrix it operates on. For example, in the 2×2 case, pre-multiplication swaps the rows,

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ a & b \end{bmatrix},$$

   whereas post-multiplication swaps the columns,

$$AP = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ \beta & \alpha \end{bmatrix}.$$

   For the 3×2 case there are $3 \cdot 2/2 = 3$ such matrices (swap a row with the other 2, then swap the remaining two rows).

6. *Gaussian elimination (GE) operations* G_k: There are three types of elementary row operations, which may be performed without fundamentally altering a system of equations (e.g. the resulting system of equations is *equivalent*). These operations are (1) swap rows (e.g. using a permutation matrix), (2) scale

rows, or (3) perform addition/subtraction of two scaled rows. All such operations can be performed using matrices.

For lack of a better term, we'll describe these as 'Gaussian elimination' or 'GE' matrices.[16] We will categorize any matrix that performs only elementary row operations (but any number of them) as a 'GE' matrix. Therefore, a cascade of GE matrices is also a GE matrix.

Consider the GE matrix

$$G = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.$$

(a) This pre-multiplication scales and subtracts row (2) from (1) and returns it to row (2).

$$GA = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} a & b \\ a-\alpha & b-\beta \end{bmatrix}.$$

The shorthand for this Gaussian elimination operation is $(1) \leftarrow (1)$ and $(2) \leftarrow (1) - (2)$.

(b) Post-multiplication adds and scales *columns*.

$$AG = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a-b & b \\ \alpha-\beta & \beta \end{bmatrix}.$$

Here the second column is subtracted from the first, and placed in the first. The second column is untouched. This operation is *not* a Gaussian elimination. Therefore, to put Gaussian elimination operations in matrix form, we form a cascade of pre-multiplication matrices.

Here $\det(G) = 1$, $G^2 = I$, which won't always be true if we scale by a number greater than 1. For instance, if $G = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$ (scale and add), then we have $\det(G) = 1$, $G^n = \begin{bmatrix} 1 & 0 \\ n \cdot m & 1 \end{bmatrix}$.

3.5.6 Inverse of the 2×2 matrix

We shall now apply Gaussian elimination to find the solution $[x_1, x_2]$ for the 2×2 matrix equation $Ax = y$ (Eq. 3.48, left). We assume to know $[a, b, c, d]$ and $[y_1, y_2]$.

Here we wish to show that the left equation (*i*) has an inverse given by the right equation (*ii*):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (i); \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (ii).$$

To take the inverse:
(1) swap the diagonal, (2) change the off-diagonal signs, and (3) divide by the determinant Δ. We wish to show that the intersection (solution) is given by the equation on the right.

Exercise #*1*

Show that the equation on the right is the solution of the equation on the left.

Solution: By direct substitution (composition) of the right equation into the left equation, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -cb+ad \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}, \tag{3.63}$$

which gives the identity matrix.

Start problems AE2

[16] The term 'elementary matrix' may also be used to refer to a matrix that performs an elementary row operation. Typically, each elementary matrix differs from the identity matrix by a single row operation. A cascade of elementary matrices could be used to perform Gaussian elimination.

3.6 Problems AE-2

Topics of this homework:

Linear vs nonlinear systems of equations, Euclid's formula, Gaussian elimination, matrix permutations, Ohm's law, two-port networks,

Deliverables: Answers to problems

Gaussian elimination

Problem # *1: Gaussian elimination*

– *1.1: Find the inverse of*

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution:

$$A^{-1} = \frac{1}{3-8}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}.$$

– *1.2: Verify that* $A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution: Multiply them to show this.

Problem # *2: Find the solution to the following* 3×3 *matrix equation* $Ax = b$ *by GE. Show your intermediate steps. You can check your work at each step using Octave/Matlab.*

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}.$$

– *2.1: Show (i.e., verify) that the first GE matrix* G_1*, which zeros out all entries in the first column is given by*

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Identify the elementary row operations that this matrix performs. **Solution:** Operate with GE matrix on A

$$G_1[A|b] = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & -2 & 5 & 7 \end{array}\right]$$

It scales the first row by -3 and adds it to the second row, and scales the first row by -1 and adds it to the third row.

– *2.2 Find a second GE matrix,* G_2*, to put* G_1A *in upper triangular form. Identify the elementary row operations that this matrix performs.*

Solution:

$$G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

which scales the second row by -1 and adds it to the third row. Thus we have

$$G_2G_1[A|b] = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-3 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\3 & 1 & 1 & 9\\1 & -1 & 4 & 8\end{array}\right]$$
$$= \left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\0 & -2 & 4 & 6\\0 & 0 & 1 & 1\end{array}\right]$$

– *2.3 Find a third GE matrix G_3 that scales each row so that its leading term is 1. Identify the elementary row operations that this matrix performs.*

Solution:

$$G_3 = \begin{bmatrix}1 & 0 & 0\\0 & -1/2 & 0\\0 & 0 & 1\end{bmatrix}$$

which scales the second row by -1/2. Thus we have

$$G_3G_2G_1[A|b] = \begin{bmatrix}1 & 0 & 0\\0 & -1/2 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-3 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\3 & 1 & 1 & 9\\1 & -1 & 4 & 8\end{array}\right]$$
$$= \left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\0 & 1 & -2 & -3\\0 & 0 & 1 & 1\end{array}\right]$$

– *2.4: Find the last GE matrix, G_4, which subtracts a scaled version of row 3 from row 2, and scaled versions of rows 2 and 3 from row 1, such that you are left with the identity matrix ($G_4G_3G_2G_1A = I$).*

Solution:

$$G_4 = \begin{bmatrix}1 & -1 & -1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}$$

Thus we find $G_4G_3G_2G_1[A|b]$ is

$$= \begin{bmatrix}1 & -1 & -1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & -1/2 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-3 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\3 & 1 & 1 & 9\\1 & -1 & 4 & 8\end{array}\right]$$
$$= \left[\begin{array}{ccc|c}1 & 0 & 0 & 3\\0 & 1 & 0 & -1\\0 & 0 & 1 & 1\end{array}\right]$$

– *2.5: Solve for $\{x_1, x_2, x_3\}^T$ using the augmented matrix format $G_4G_3G_2G_1\{A|b\}$ (where $\{A|b\}$ is the augmented matrix). Note that if you've performed the preceding steps correctly, $x = G_4G_3G_2G_1b$.*

Solution: From the preceding problems, we see that $[x_1, x_2, x_3]^T = [3, -1, 1]^T$

– *2.6: Find the pivot matrix G that rescales the second row of the augmented matrix $A|b$ by 1/3.*

Solution:

$$G_1 = \begin{bmatrix}1 & & \\ & 1/3 & \\ & & 1\end{bmatrix}.$$

Proceeding

$$G_1A = \begin{bmatrix}1 & & \\ & 1/3 & \\ & & 1\end{bmatrix}\left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\3 & 1 & 1 & 9\\1 & -1 & 4 & 8\end{array}\right] = \left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\1 & 1/3 & 1/3 & 3\\1 & -1 & 4 & 8\end{array}\right]$$

Two linear equations

Problem # *3 In this problem we transition from a general pair of equations*

$$f(x, y) = 0$$
$$g(x, y) = 0$$

to the important case of two linear equations

$$y = ax + b$$
$$y = \alpha x + \beta.$$

Note that to help keep track of the variables, roman coefficients (a, b) are used for the first equation and Greek (α, β) for the second.

– 3.1: What does it mean, graphically, if these two linear equations have (1) a unique solution, (2) a nonunique solution, or (3) no solution?
Solution: There are three possibilities:

1. When they have different slopes, they meet at one (x,y) point, which is the solution.
2. If the two lines are identical, any point on the line is a solution.
3. If they have the same slope but different intercepts (are parallel to each other) there is no solution.

– 3.2: Assuming the two equations have a unique solution, find the solution for x and y.
Solution: Since there must be one point where the two are equal, we may solve for that by setting the y values equal to each other:

$$ax + b = \alpha x + \beta$$

Thus

$$x = \frac{\beta - b}{a - \alpha}$$
$$y = a\frac{\beta - b}{a - \alpha} + b$$

– 3.3: When will this solution fail to exist (for what conditions on a, b, α, and β)?
Solution: As stated above, if they have the same slope $\alpha = a$ but different intercepts $\beta \neq b$, there is no solution. When $\beta = b$ and $\alpha = a$ every point on the line is a solution.

– 3.4: Write the equations as a 2×2 matrix equation of the form $A\vec{x} = \vec{b}$, where $\vec{x} = \{x, y\}^T$.
Solution:

$$\begin{bmatrix} 1 & -a \\ 1 & -\alpha \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ \beta \end{bmatrix}$$

– 3.5: Find the inverse of the 2×2 matrix, and solve the matrix equation for x and y.
Solution:

$$\begin{bmatrix} y \\ x \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -\alpha & a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b \\ \beta \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -\alpha b + a\beta \\ -b + \beta \end{bmatrix}$$

where the determinant is $\Delta \equiv a - \alpha$.

– 3.6: Discuss the properties of the determinant of the matrix (Δ) in terms of the slopes of the two equations (a and α).
Solution: When the slopes are the same there is no solution and $\Delta = 0$. Thus the matrix solution is consistent with the geometry. This is our first result in analytic geometry.

Problem # *4: The application of* linear functional relationships *between* two variables

We use 2×2 matrices to describe two-port networks, as discussed in §3.7 (p. 113). Transmission lines are a great example: Both voltage and current must be tracked as they travel along the line. The Figure below shows an example segment of a transmission line.

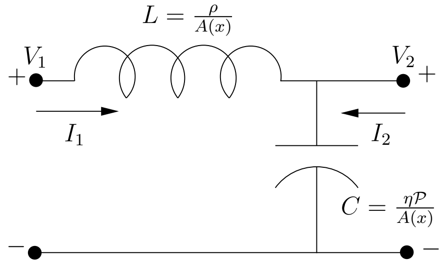


Fig. AE-2.1 *This figure shows a cell from an LC transmission line. The index* 1 *is at the input on the left and* 2 *represents the output, on the right.*

Suppose you are given the following pair of linear relationships between the input (source) variables V_1 and I_1 and the output (load) variables V_2 and I_2 of the transmission line:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

– 4.1: Let the output (the load) be $V_2 = 1$ *and* $I_2 = 2$ *(i.e.,* V_2/I_2 *=1/2* $\{\Omega\}$*). Find the input voltage and current,* V_1 *and* I_1*.*

Solution: This case corresponds to

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\jmath + 2 \\ 1 - 2 \end{bmatrix}$$

Thus $V_1 = 2 + 1\jmath$ and $I_1 = -1$.

– 4.2: Let the input (source) be $V_1 = 1$ *and* $I_1 = 2$*. Find the output voltage and current,* V_2 *and* I_2*.*

Solution: With the input specified the two equations are

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \jmath & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

To find the input we must invert the matrix ($\Delta = -j - 1$)

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \frac{1}{1 + j} \begin{bmatrix} 1 & 1 \\ 1 & -\jmath \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Thus $V_2 = 3/(1 + \jmath) = 3(1 - \jmath)/2$, $I_2 = (1 - 2\jmath)/(1 + 1\jmath) = -(1 + 3\jmath)/2$. The point of this exercise is that the two lines have a complex intersection point, not easily visualized.

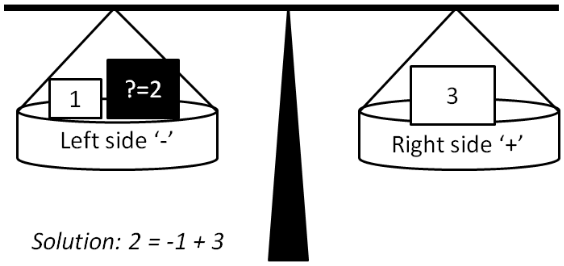
Integer equations: applications and solutions

Any equation for which we seek only integer solutions is called a *Diophantine* equation.

Problem # *5: A practical example of using a Diophantine (Number theory) equation:*

"A merchant had a 40-pound weight that broke into 4 pieces. When the pieces were weighed, it was found that each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds. What were the weights of the pieces?" - *Bachet de Bèziriac (1623)*[17].

[17] Taken from Rotman (1996, p. 50)



– 5.1: Show how the combination of 1-, 3-, 9-, and 27-pound weights can be used to weigh $1, 2, 3, \ldots, 8, 28,$ and 40 pounds of milk. Assuming that the milk is in the left *pan, provide the position of the weights using a negative sign $-$ to indicate the left pan and a positive sign $+$ to indicate the right pan. For example, if the left pan has 1 pound of milk, then 1 pound of milk in the right pan, $+1$ will balance the scales.*

Hint: It is helpful to write the answer in matrix form. Set the vector of values to be weighed equal to a matrix indicating the pan assignments, multiplied by a vector of the weights $[1, 3, 9, 27]^T$. The pan assignments matrix should contain only the values -1 (left pan), $+1$ (right pan), and 0 (leave weight out). We indicate these weights using $-$, $+$, and blanks.

Solution: Any integer between 1 and 40 may be expanded using the weights 1, 3, 9, 27 ($\in \mathbb{Z}$). Here is the problem stated in matrix form:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ \cdots \\ 28 \\ \cdots \\ 40 \end{bmatrix} = \begin{bmatrix} + & & & \\ - & + & & \\ & + & & \\ + & + & & \\ - & - & + & \\ & - & + & \\ + & - & + & \\ - & & + & \\ \cdots & & & \\ + & & & + \\ \cdots & & & \\ + & + & + & + \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \end{bmatrix}$$

The left column is the weight of the milk in integer units. The right-most column is the four weights. Note that these four weights span the integers from 1-40 with binary weights. Each weight may be computed recursively from twice the sum of the previous weights +1, that is

$$W_{n+1} = 2W_n + 1 = 2^{n+1} \quad \text{since} \quad W_n = 2^n.$$

For example, to get 26 we place weights 9+3+1 in the pan. The next weight is 27 = 2*(9+3+1)+1. Recursively, the weights are 3=2*1+1, 9=2*(3+1)+1, 27=2*(9+3+1)+1. To go above 40 a fifth weight (not shown) must be: 81=2*(27+9+3+1)+1 = 2*40+1.

3.6.1 Vector algebra in $\mathbb{R}^3$

Basic setup: Let

$$\boldsymbol{A} = [a_1, a_2, a_3]^T,$$
$$\boldsymbol{B} = [b_1, b_2, b_3]^T,$$
$$\boldsymbol{C} = [c_1, c_2, c_3]^T$$

be composed of three vectors, including $\mathbb{R}^3$. Definitions of the scalar (dot)

$$\boldsymbol{A} \cdot \boldsymbol{B},$$

cross

$$\boldsymbol{A} \times \boldsymbol{B}$$

and triple products

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$$

, are in the Appendix 2.1.1 (p. 20), with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ in $\mathbb{R}^3 \subset \mathbb{C}^3$), as shown in Fig. 9, p. 108.

A fourth "double-cross" ($\mathbf{X}$) vector product is:[18]

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \alpha_o \boldsymbol{B} - \beta_o \boldsymbol{C}.$$

where $\alpha_o = \boldsymbol{A} \cdot \boldsymbol{C}$ and $\beta_o = \boldsymbol{A} \cdot \boldsymbol{B}$ (Note: $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}$).

Problem # 6: Scalar product A · B

– 6.1: If $\boldsymbol{A} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\boldsymbol{B} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$, write out the definition of $\mathrm{A} \cdot \mathrm{B}$.

Solution: See the definition in the above figure. $\boldsymbol{A} \cdot \boldsymbol{B} = a_x b_x + a_y b_y + a_z b_z$. In general: $\boldsymbol{A} \cdot \boldsymbol{B} = \sum_k A_k B_k$.

– 6.2: The dot product is often defined as $||\mathbf{A}||\,||\mathbf{B}|| \cos(\theta)$, where $||\mathbf{A}|| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$ and θ is the angle between $\boldsymbol{A}, \boldsymbol{B}$. If $||\mathbf{A}|| = 1$, describe how the dot product relates to the vector B.

Solution: See the definition in the above figure. The vector product is the portion of B in the direction of A.

Problem # 7: Vector (cross) product A × B

– 7.1: If $\boldsymbol{A} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\boldsymbol{B} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$, write out the definition of $\mathrm{A} \times \mathrm{B}$.

Solution:

$$\boldsymbol{A} \times \boldsymbol{B} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\mathbf{y}} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}.$$

– 7.2: Show that the cross product is equal to the area of the parallelogram formed by $\boldsymbol{A}, \boldsymbol{B}$, namely $||\mathbf{A}||\,||\mathbf{B}|| \sin(\theta)$, where $||\mathbf{A}|| = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}$ and θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

Solution: A parallelogram's area is equal to its base times its height. Therefore, let's say the base is length $||A||$, and the height $||B|| \sin(\theta)$, which is the portion of B that is perpendicular to A.

Problem # 8: *Triple product* $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$

Solution: ?

– 8.1: Starting from the definition of the dot and cross product, explain using a diagram and/or words, how one shows that: $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Solution: Using the determinate-definition of the cross product,

$$\boldsymbol{B} \times \boldsymbol{C} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \hat{\mathbf{x}} \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \hat{\mathbf{y}} \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \hat{\mathbf{z}} \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix}.$$

Let $\boldsymbol{D} = \boldsymbol{B} \times \boldsymbol{C}$ and compute $\boldsymbol{A} \cdot \boldsymbol{D} = \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$. Finally compute the requested right-hand side, and compare the two. It should be clear that they are the same, because the dot product transfers the elements of vector $\boldsymbol{A}$ to cross product and reduces the product to the scalar.

– 8.2: Describe why $|\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})|$ is the volume of parallelepiped generated by $\boldsymbol{A}$,$\boldsymbol{B}$, and $\boldsymbol{C}$.

Solution: Note that the norm of $\boldsymbol{B} \times \boldsymbol{C}$ is the area of the parallelogram generated by $\boldsymbol{C}$ and $\boldsymbol{B}$. Taking the dot product with $\mathbf{A}$ results in the volume of the corresponding parallelepiped (prism). So the absolute value of triple product is volume of parallelepiped.

[18]Greenberg p. 694, Eq. 8.

– 8.3: Explain why three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *are in one plane if and only if the triple product* $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = 0$.

Solution: (triple product is zero) if and only if: (volume is zero), if and only if: (they are in the same plane)

Problem # *9: Given two vectors* $\boldsymbol{A}, \boldsymbol{B}$ *in the* $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ *(p. 108), with* $\boldsymbol{B} = \hat{\mathbf{y}}$ *(i.e.,* $||\boldsymbol{B}|| = 1$*).*

– 9.1: Show that $\boldsymbol{A}$ *may be split into two orthogonal parts, one in the direction of* $\boldsymbol{B}$ *and the other perpendicular (*$\perp$*) to* $\mathbf{B}$*. Hint: Express the Greenberg88 (98?) vector products of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *(dot and cross) in polar coordinates (Greenberg, 1988, 1998).*

$$\begin{aligned}\boldsymbol{A} &= (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{B} + \boldsymbol{B}\times(\boldsymbol{A}\times\boldsymbol{B})\\ &= \boldsymbol{A}_{||} + \jmath\boldsymbol{A}\perp.\end{aligned}$$

Solution: Expressing the vector products in polar form makes this result transparent:

$$\boldsymbol{A}\cdot\boldsymbol{B} = ||\boldsymbol{A}||\cos(\theta) \quad \text{and} \quad \boldsymbol{A}\times\boldsymbol{B} = ||\boldsymbol{A}||\sin(\theta)$$

The first quantity is in the direction of $\boldsymbol{B}$, while the second is in the direction $\boldsymbol{A}\times\boldsymbol{B}$, which is $\perp$ to $\boldsymbol{B}$. Thus

$$\begin{aligned}\boldsymbol{A} &= ||\boldsymbol{A}||\left(\boldsymbol{B}\cos(\theta) + \jmath\boldsymbol{A}\times\boldsymbol{B}\sin(\theta)\right)\\ &= \boldsymbol{A}_{||} + \jmath\boldsymbol{A}_{\perp}.\end{aligned}$$

Ohm's Law

In general, impedance is defined as the ratio of a force to a flow. For electrical circuits, the voltage is the force and the current is the flow. Ohm's law states that the voltage across and the current through a circuit element are related by the impedance of that element (which may be a function of frequency). For resistors, the voltage over the current is called the *resistance* and is a constant (e.g., the simplest case is $V/I = R$). For inductors and capacitors, the voltage over the current is a frequency-dependent impedance (e.g., $V/I = Z(s)$, where s is the complex frequency $s \in \mathbb{C}$).

As shown in Table 3.3.2 (p. 86), the impedance concept also holds in mechanics and acoustics. In mechanics, the force is equal to the mechanical force on an element (e.g., a mass, dashpot, or spring) and the flow is the velocity. In acoustics, the force is pressure and the flow is the volume velocity or particle velocity of air molecules.

Case	Force	Flow	Impedance	units
Electrical	voltage (V)	current (I)	Z_E	Ohms [Ω]
Mechanics	force (F)	velocity (V)	Z_M	Mechanical Ohms [Ω]
Acoustics	pressure (P)	particle velocity (U)	Z_A	Acoustic Ohms [Ω]
Thermal	temperature (T)	heat-flux (J)	Z_T	Thermal Ohms [Ω]
Gravity	Potential (G)	Graviton (g)	Z_G	Gravitional Ohms [Ω]

Problem # *10: The resistance of an incandescent (filament) lightbulb, measured cold, is about 100 ohms. As the bulb lights up, the resistance of the metal filament increases.*

Ohm's law says that the current

$$\frac{V}{I} = R(T),$$

where T is the temperature. In the United States, the voltage is 120 volts (RMS) at 60 [Hz]. Find the current when the light is first switched on. **Solution:** Thus the current is

$$I = 120/R = 120/100 = 1.2. \qquad \text{[Amps]}$$

As the bulb heats up, the current rapidly drops, and the resistance increases. This typically takes less than a milliseconds [ms], which depends on the wattage of the light bulb. Such lightbulbs are *nonlinear.* These rules don't apply to LED bulbs.

Problem # *11: The power in watts is the product of the force and the flow. What is the power of the lightbulb of Problem 10?*
Solution: $P = V \cdot I = 120 \times 1.2 = 120 + 24 = 144$ [W].

Problem # *12: State the impedance $Z(s)$ of each of the following circuit elements: (1) a resistor with resistance R, (2) an inductor with inductance L, and (3) a capacitor with capacitance C.*

Solution: (1) For the resistor, $Z = R$.

(2) For the inductor, $Z = sL$ with $s = \sigma + \omega\jmath$. Note the flux $\psi(t) = Li(t)$. The voltage $v(t)$ is the time derivative of the flux

$$v(t) = \frac{d\psi(t)}{dt} = L\frac{di(t)}{dt}.$$

(3) For the capacitor, $Z = 1/sC$. Note the charge $q(t) = Cv(t)$, thus the current $i(t)$ is the time derivative of the charge

$$i(t) = \frac{d}{dt}q(t) = C\frac{dv(t)}{dt}.$$

Problem # *13: Consider what happens at the triple point of water. As water freezes or thaws, the temperature remains constant at 0 (C°). Once all the water is frozen and more heat is removed, the temperature drops below 0°. As heat is added, water thaws but the temperature remains at 0°.*

– 13.1: Once all the ice has melted, what is the temperature as more heat is added?
Model the triple point using a Zener diode, a resistor, and a capacitor. A Zener diode holds the voltage constant independent of current. For the case of water's triple point, the voltage represents the temperature of water at the triple point, clamped at 0 [C°]. The current represents the heat flux. The latent heat of water at the triple point is 32 Cal/gm. Thus as the temperature rises from below freezing, the water is clamped at 0° once the triple point is reached. At that point, adding more heat flux has no effect on the temperature until all the ice melts. Once the ice has melted, the temperature again begins to rise until it hits the boiling point, where it again stays at 100° until all the water has evaporated.

Solution: Need a figure here showing how to model the triple point of water. The Heat capacity may be modeled by a capacitor, which is fixed at 0°as the capacitor discharges. Once it is empty, the temperature again begins to rise as the heat Q from the sun is added

$$T^\circ = C_w Q,$$

where C_w is the heat capacity of the water mass. Thus the required circuit needs to emulate this temperature behavior due to the latent heat of melting ice and boiling water into steam.

Nonlinear (quadratic) to linear equations

In the following problems we deal with algebraic equations in more than one variable that are not linear equations. For example, the circle $x^2 + y^2 = 1$ may be solved for $y(x) = \pm\sqrt{1 - x^2}$. If we let $z_+ = x + y\jmath = x + \jmath\sqrt{1 - x^2} = e^{\theta\jmath}$, we obtain the equation for half a circle ($y > 0$). The entire circle is described by the magnitude of z as $|z|^2 = (x + y\jmath)(x - y\jmath) = 1$.

Problem # *14: Give the curve defined by the equation:*

$$x^2 + xy + y^2 = 1$$

– 14.1: Find the function $y(x)$.

Solution: Completing the square in y and solve for $y(x)$ to isolate x and y

$$
\begin{aligned}
(y + x/2)^2 - x^2/4 + x^2 &= 1 \\
(y + x/2)^2 &= 1 - \frac{3}{4}x^2 \\
y + x/2 &= \pm\sqrt{\frac{4-3x^2}{4}} \\
y &= \frac{1}{2}\left(\pm\sqrt{4-3x^2} - x\right) \\
(2y)^2 &= \left(\pm\sqrt{4-3x^2} - x\right)^2
\end{aligned}
$$

Expanding the RHS square gives

$$4y^2 = (4 - 3x^2) \mp 2x\sqrt{4-3x^2} + x^2$$

and setting $q = x^2$.

We would like to complete the square for $x = \sqrt{q}$, like this: $(\sqrt{x} \mp x)^2 = x \mp 2x\sqrt{x} + \cancel{(x^2 - x^2)}$ where $x = (4-3q)/2$ gives (this need fixing).

$$4y^2 = (4-3q) \mp 2\sqrt{q}\sqrt{4-3q} + q,$$

gives

$$\mp 4y^2 = \mp q \mp (4-3q) + 2\sqrt{4q-3q^2}$$

$$\mp 4y^2 = \mp q + \left(\mp\sqrt{4q-3q^2} \mp (4-3q)/2\right)^2 - (4-3q)^2/4$$

and verifying:

$$\mp 4y^2 = \mp q + \left(4q - 3q^2 \mp (4-3q)\sqrt{4q-3q^2} + (4-3q)^2/4\right) - (4-3q)^2/4$$

$$
\begin{aligned}
\mp 4y^2 &= \mp q + 4q - 3q^2 \mp (4-3q)\sqrt{4q-3q^2} + \cancelto{0}{\left((4-3q)^2/4 - (4-3q)^2/4\right)} \\
4y^2 &= \mp(4q - 3q^2) + (4-3q)\sqrt{4q-3q^2} + q
\end{aligned}
$$

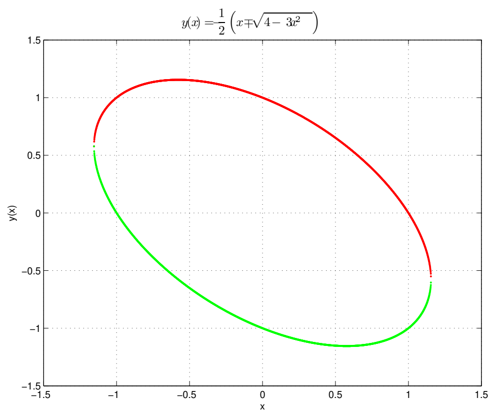
The verification does not check out. Need to try again using

$$(\sqrt{x} \mp x)^2 = x \mp 2x\sqrt{x} + \cancel{(x^2 - x^2)}$$

This last line uses the identity $(\mp)^2 = +$ twice. Was moving the $\mp$ necessary?

– *14.2: Using Matlab/Octave, plot $y(x)$ and describe the graph.*

Solution:



Thus we find the equation is a rotated ellipse.

– *14.3: What is the name of this curve?*
Solution: It is an ellipse, rotated by 45 degrees.

– *14.4: Find the solution (in x, p, and q) to these equations:*

$$\begin{aligned} x + y &= p \\ xy &= q. \end{aligned}$$

Solution: Solve the first equation for y as $y = p - x$, and then substitute it into the second equation

$$x(p - x) = -x^2 + px = q.$$

Thus we find the quadratic

$$x^2 - px + q = 0$$

having roots given by completing the square

$$(x - p/2)^2 = (p/2)^2 - q.$$

resulting in $x = p/2 \pm \sqrt{(p/2)^2 - q}$, $y = p - x$.

Summary: Here we started with one linear and one quadratic (hyperbola). By the use of composition we found the roots.

– *14.5: Find an equation that is linear in y starting from equations that are quadratic (second-degree) in the two unknowns x and y:*

$$x^2 + xy + y^2 = 1 \tag{AE-2.1}$$
$$4x^2 + 3xy + 2y^2 = 3. \tag{AE-2.2}$$

Solution: The goal is to obtain a linear equation in y.
Method 1: remove xy term: Scale the upper equation by 3 and subtract from the lower:

$$\begin{aligned} 4x^2 + 3xy + 2y^2 &= 3 \\ 3x^2 + 3xy + 3y^2 &= 3 \end{aligned}$$

giving $x^2 - y^2 = 0$, or $x = \pm y$.
This results in the two equations

$$\begin{aligned} x^2 \qquad - y^2 &= 0 \\ x^2 + xy + y^2 &= 1 \end{aligned}$$

Adding these gives $2x^2 \pm x^2 = 1$, which is $3x^2 = 1$ and $x^2 = 1$. Thus the final solutions are $x = \pm y = \pm 1/\sqrt{3}$ and $x = \pm y = \pm 1$.

– *14.6: Compose the following two quadratic equations and describe the results.*

$$\begin{aligned} x^2 + xy + y^2 &= 1 \\ 2x^2 + xy \qquad &= 1 \end{aligned}$$

Solution: By isolating y from one of the two equations, we may remove it from the other equation, giving us a single 4^{th} degree equation in x:

$$x^2 + (1 - 2x^2) + (1 - 2x^2)^2/x^2 = 1$$

or

$$x^4 + x^2 - 2x^4 + 1 - 4x^2 + 4x^4 - x^2 = 0$$

Collecting terms

$$3x^4 - 4x^2 + 1 = 0$$

This is a quartic, but quadratic in x^2. Thus it may be solved for x^2 by the completion of squares

$$x^4 - \frac{4}{3}x^2 = -\frac{1}{3}$$
$$\left(x^2 - \frac{2}{3}\right)^2 = \frac{1}{3}\left(\frac{4}{3} - 1\right)$$
$$x^2 = \frac{2}{3} \pm \frac{1}{3}$$
$$x = \pm\frac{\sqrt{2 \pm 1}}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}} \text{ and } \pm\frac{2}{\sqrt{3}}$$

resulting in four roots.

Nonlinear intersection in analytic geometry

Euclid's formula for Pythagorean triplets can be derived by intersecting a circle and a secant line. Consider the nonlinear equation of a unit circle having radius 1, centered at $(x, y) = (0, 0)$,

$$x^2 + y^2 = 1,$$

and the secant line through $(-1, 0)$,

$$y = t(x + 1),$$

a linear equation having slope t and intercept $x = -1$. If the slope $0 < t < 1$, the line intersects the circle at a second point (a, b) in the positive x, y quadrant. The goal is to find $a, b \in \mathbb{N}$ and then show that $c^2 = a^2 + b^2$. Since the construction gives a right triangle with short sides $a, b \in \mathbb{N}$, then it follows that $c \in \mathbb{N}$.

Problem # *15: Derive Euclid's formula*

– 15.1: Draw the circle and the line, given a positive slope $0 < t < 1$.
Solution: The solution is derived in the Figure.

Problem # *16: Substitute* $y = t(x + 1)$ *(the line equation) into the equation for the circle, and solve for* $x(t)$.
Hint: Because the line intersects the circle at two points, you will get two solutions for x. *One of these solutions is the trivial solution* $x = -1$. **Solution:** $x(t) = (1 - t^2)/(1 + t^2)$

– 16.1: Substitute the $x(t)$ *you found back into the line equation, and solve for* $y(t)$.
Solution: $y(t) = 2t/(1 + t^2)$

– 16.2: Let $t = q/p$ *be a rational number, where* p *and* q *are integers. Find* $x(p, q)$ *and* $y(p, q)$.
Solution: $x(p, q) = 2pq/(p^2 + q^2)$ and $y(p, q) = (p^2 - q^2)/(p^2 + q^2)$

– 16.3: Substitute $x(p, q)$ *and* $y(p, q)$ *into the equation for the circle, and show how Euclid's formula for the Pythagorean triples is generated.*
Solution: Multiplying out gives $(p^2 + q^2) = (p^2 - q^2) + 2pq$

For full points you must show that you understand the argument. Explain the meaning of the comment "magic happens" when t^4 cancels.

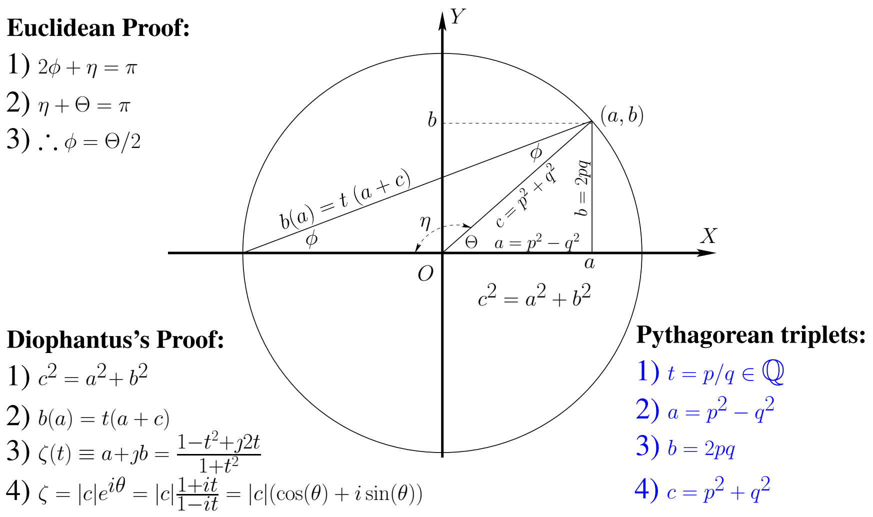


Figure 3.4: *Derivation of Euclid's formula for the Pythagorean triplets (PT) $[a, b, c]$, based on a composition of a line, having a rational slope $t = p/q \in \mathbb{F}$, and a circle $c^2 = a^2 + b^2, [a, b, c] \in \mathbb{N}$. This analysis is attributed to Diophantus (Di·o·phan′·tus) (250 CE), and today such equations are called Diophantine (Di·o·phan′·tine)* equations. *PTs have applications in architecture and scheduling, and many other practical problems. Most interesting is their relation to Rydberg's formula for the eigenstates of the hydrogen atom (Appendix 5.7).*

3.7 Transmission (ABCD) matrix composition method

Matrix composition: Matrix multiplication represents a composition of 2×2 matrices because the input to the second matrix is the output of the first [this follows from the definition of composition: $f(x) \circ g(x) = f(g(x))$]. Thus the ABCD matrix is also known as the *transmission matrix*, or occasionally the *chain matrix*. The general expression for the transmission matrix $\mathcal{T}(s)$ is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \mathcal{T}(s) \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{3.3}$$

The four coefficients $\mathcal{A}(s), \mathcal{B}(s), \mathcal{C}(s), \mathcal{D}(s)$ are all complex–analytic functions of the Laplace frequency $s = \sigma + \jmath\omega$. Typically they are polynomials in s, for example, $\mathcal{C}(s) = s^2 + 1$. A sum and parallel combination of inductors (masses), capacitors (springs), and resistors (dash-pots) results in a *Brune impedance*. Such methods are called lumped-element networks. The convention to define the current (flow) into the node. Using this construction the currents are equivalent.[19]

We have already used 2×2 matrix composition for: (1) representing complex numbers, (2) computing the $\gcd(m, n)$ of $m, n \in \mathbb{N}$ (3) computing Pell's equation and (4) computing the Fibonacci sequence Thus 2×2 complex–analytic matrices are an important tool.

Definitions of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$: By writing the equations that correspond to Eq. 3.3, we see that

$$\mathcal{A}(s) = \left.\frac{V_1}{V_2}\right|_{I_2=0}, \quad \mathcal{B}(s) = -\left.\frac{V_1}{I_2}\right|_{V_2=0}, \quad \mathcal{C}(s) = \left.\frac{I_1}{V_2}\right|_{I_2=0}, \quad \mathcal{D}(s) = -\left.\frac{I_1}{I_2}\right|_{V_2=0}. \tag{3.4}$$

Much of this was first sorted out by Thévenin in about 1883 (Kennelly, 1893; Van Valkenburg, 1964a; Johnson, 2003).

Each equation has a physical interpretation and a corresponding name. Functions $\mathcal{A}$ and $\mathcal{C}$ are said to be *blocked* when either $V_1 = 0$ or $I_1 = 0$, because then the output current (flow) I_2 is zero. Functions $\mathcal{B}$ and $\mathcal{D}$ are said to be *short-circuited* when either $V_1 = 0$ or $I_1 = 0$, because the output voltage (force potential) V_2 is zero. These two terms (blocked vs. short-circuited) are electrical engineering–centric, arbitrary, and fail to generalize to other cases. In mechanics the *isometric force* is defined as the maximum applied force conditioned on zero output velocity (the blocked force), and the *short-circuited* force ($\mathcal{B}$) would correspond to zero force, which is nonsense. Thus these electrical engineering-centric terms do not generalize for mechanical systems.

[19]For a case of more that two ports, it is required we define the flow into the ports, since every port can be either an input or an output, depending how its connected.

$\mathcal{A}$ and $\mathcal{D}$ are called voltage (force) and current (velocity) *transfer functions*, since they are ratios of voltages and currents, whereas $\mathcal{B}$ and $\mathcal{C}$ are known as the *transfer impedance* and *transfer admittance*. For example, the unloaded (blocked) ($I_2 = 0$) output voltage $V_2 = I_1/\mathcal{C}$ corresponds to the isometric force in mechanics. In this way each term expresses an output (port 2) in terms of an input (port 1) for a given load condition.

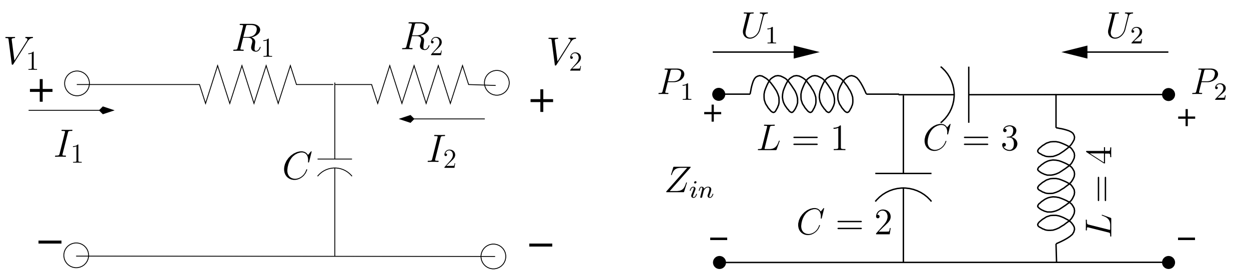


Figure 3.5: **Left:** *A low-pass RC electrical filter. The circuit elements R_1, R_2, and C are defined.* **Right:** *A band-pass acoustic filter. Here, the pressure P is analogous to voltage, and the velocity U is analogous to current. The circuit elements are labeled with their L and C values as integers, to make the algebra simple.*

Example: Figure 3.5 shows two examples of networks that may be analyzed using the ABCD transmission matrix method.

Exercise #*2*

Derive the formula for $\mathcal{C}$ in terms of the input and output currents and voltages. Hint: See Eq. 3.4.

Solution: Writing out the lower equation gives $I_1 = \mathcal{C}V_2 - \mathcal{D}I_2$ and setting $I_2 = 0$, we may obtain the equation for $\mathcal{C} = I_1/V_2|_{I_2=0}$.

Exercise #*3*

Can $\mathcal{C} = 0$?

Solution: Yes, if $I_2 = 0$ and $I_1 = I_2$, then $\mathcal{C} = 0$. In such cases the 2-port is ill-conditioned as shown in Appendix A.3, Eq. A.9. For $\mathcal{C} \neq 0$, there needs to be a finite shunt impedance across V_1, so that $I_1 \neq I_2 = 0$.

3.7.1 Thévenin parameters of a source

An important concept in circuit theory is that of the Thévenin parameters: the blocked force (zero flow) and the blocked flow (zero force). Their ratio defines the Thévenin impedance (Johnson, 2003). The open-circuit voltage is defined as the voltage V_2 when the load current is zero ($I_2 = 0$), which was shown in Eq. 3.4 to be $V_2/I_1|_{I_2=0} = 1/\mathcal{C}$.

Thévenin Voltage: From Eq. 3.3 there are two definitions for the Thévenin voltage $V_{\text{Thev}} = V_2$, conditioned on the source on the left:

$$\left.\frac{V_{\text{Thev}}}{I_1}\right|_{I_2=0} = \frac{1}{\mathcal{C}} \quad \text{and} \quad \left.\frac{V_{\text{Thev}}}{V_1}\right|_{I_2=0} = \frac{1}{\mathcal{A}}. \tag{3.5}$$

A more general expression is needed when the source impedance is mixed.

Thévenin impedance The Thévenin impedance is the impedance looking into port 2 with $V_1 = 0$; thus

$$Z_{\text{Thev}} = \left.\frac{V_2}{I_2}\right|_{V_1=0}. \tag{3.6}$$

From the upper equation of Eq. 3.3, with $V_1 = 0$, we obtain $\mathcal{A}V_2 = \mathcal{B}I_2$; thus

$$Z_{\text{Thev}} = \frac{\mathcal{B}}{\mathcal{A}}. \tag{3.7}$$

3.7.2 The impedance matrix

With a bit of algebra, we can find the impedance matrix in terms of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ (Van Valkenburg, 1964a, p. 310):

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \mathcal{Z}(s) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_\mathcal{T} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ -I_2 \end{bmatrix}. \tag{3.8}$$

The determinate of the transmission matrix is $\Delta_\mathcal{T} = \pm 1$, and if $\mathcal{C} = 0$, the impedance matrix does not exist (see Exercise #42). I key case is when the matrix has a solution know as the eigen function expansion. This will be carefully discussed in Appendix B.

Definitions of $z_{11}(s), z_{12}(s), z_{21}(s), z_{22}(s)$ The definitions of the matrix elements are easily read off of the equation as

$$z_{11} \equiv \left.\frac{V_1}{I_1}\right|_{I_2=0}, \quad z_{12} \equiv -\left.\frac{V_1}{I_2}\right|_{I_1=0}, \quad z_{21} \equiv \left.\frac{V_2}{I_1}\right|_{I_2=0}, \quad z_{22} \equiv -\left.\frac{V_2}{I_2}\right|_{I_1=0}. \tag{3.9}$$

These definitions follow trivially from Eq. 3.8 and each element has a physical interpretation. For example, the unloaded ($I_2 = 0$, also called blocked or isometric) input impedance is $z_{11}(s) = \mathcal{A}(s)/\mathcal{C}(s)$, while the unloaded transfer impedance is $z_{21}(s) = 1/\mathcal{C}(s)$. For reciprocal systems (Postulate P6), $z_{12} = z_{21}$, since $\Delta_\mathcal{T} = 1$. For anti-reciprocal systems, such as dynamic (also called magnetic) loudspeakers and microphones (Kim and Allen, 2013), $\Delta_\mathcal{T} = -1$; thus $z_{21} = -z_{12} = 1/\mathcal{C}$. Finally z_{22} is the impedance looking into port 2 with port 1 open/blocked ($I_1 = 0$).

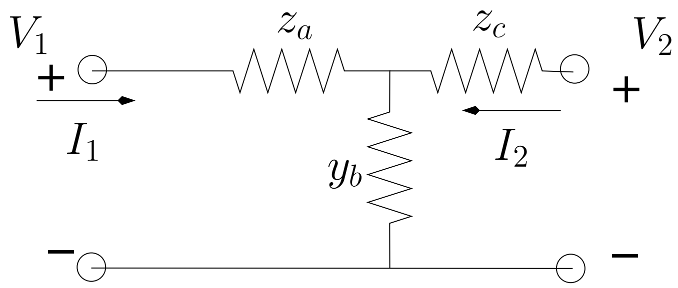


Figure 3.6: *Equivalent circuit for a transmission matrix. This allows us to better visualize the matrix elements in terms of complex impedances* $z_a(s), z_c(s), y_b(s)$*, as defined in this figure.*

To understand the meaning of the four impedance variables we analyze the transmission matrix of Fig. 3.6

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1+z_a y_b & z_c(1+z_a y_b)+z_a \\ y_b & 1+y_b z_c \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{3.10}$$

Note that it is easy to invert the $\mathcal{T}(s)$ matrix because $\Delta_\mathcal{T} = \pm 1$.

From the circuit elements defined in Fig. 3.6 (i.e., z_a, z_c, y_b) we can compute the impedance matrix elements of Eq. 3.8 (i.e., $z_{11}, z_{12}, z_{21}, z_{22}$). For example, the impedance matrix element z_{11}, in terms of z_a and y_b, is easily read off of Fig. 3.6 as the sum of the series and shunt impedances:

$$z_{11}(s)|_{I_2=0} = z_a + 1/y_b = \frac{\mathcal{A}}{\mathcal{C}}.$$

Given the impedance matrix, we can then compute transmission matrix $\mathcal{T}(s)$—namely, from Eq. 3.8,

$$\frac{1}{\mathcal{C}(s)} = z_{21}, \quad \frac{\mathcal{A}(s)}{\mathcal{C}(s)} = z_{11}.$$

The theory is best modeled using the transmission matrix (Eq. 3.3), while experimental data are best modeled using the impedance matrix (Eq. 3.8).

Rayleigh reciprocity: Figure 3.6 is particularly helpful in understanding the Rayleigh reciprocity Postulate P6 ($\mathcal{B}(s) = \pm\mathcal{C}(s)$,

$$\left.\frac{V_2}{I_1}\right|_{I_2=0} = \left.\frac{V_1}{I_2}\right|_{I_1=0}.$$

This says that the unloaded output voltage over the input current is symmetric, which is obvious from Fig. 3.6.

3.7.3 Network power relationships

Impedance is a general concept, closely tied to the definition of power $\mathcal{P}(t)$ (and energy). *Power* is defined as the product of the effort (force) and the flow (current). As described in Table 3.3.2, these concepts are very general, applying to mechanics, electrical circuits, acoustics, thermal circuits, and any other case where conservation of energy applies. Two basic variables are defined, generalized force and generalized flow, also called *conjugate variables*. The product of the conjugate variables is the power, and the ratio is the impedance. For example, for the case of voltage and current,

$$\mathcal{P}(t) \equiv v(t)i(t), \qquad v(t) = z(t) \star i(t), \qquad i(t) = y(t) \star v(t)$$

where $\star$ represents convolution (§4.5.4

$$v(t) = z(t) \star i(t) \equiv \int_{t=0}^{\infty} z(\tau)i(t-\tau)d\tau \leftrightarrow Z(s)I(s).$$

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Power vs. power series, linear vs. nonlinear Another place where second-degree equations appear in physical applications is in energy and power calculations. The electrical power is given by the product of the voltage $v(t)$ and current $i(t)$ (or in mechanics as the force times the velocity). For example, if we define $\mathcal{P} = v(t)i(t)$ to be the power $\mathcal{P}$ [watts], then the total energy [joules] at time t is (Van Valkenburg, 1964a, §14)

$$\mathcal{E}(t) = \int_0^t v(t)i(t)dt.$$

From this observe that the power is the rate of change of the total energy

$$\mathcal{P}(t) = \frac{d}{dt}\mathcal{E}(t),$$

reminiscent of the fundamental theorem of calculus (Eq. 3.7.3).

3.7.4 Ohm's law and impedance

The ratio of voltage to current is called the *impedance* and it has units of [ohms]. For example, given a resistor of $R = 10$ [ohms],

$$v(t) = R\, i(t);$$

namely, 1 [amp] flowing through the resistor would give 10 [volts] across it. Merging the linear relationship due to Ohm's law with the definition of power shows that the instantaneous power in a resistor is quadratic in voltage and current:

$$\mathcal{P}(t) = v(t) \cdot i(t) = v(t)^2/R = i(t)^2 R, \qquad \mathcal{E}(t) = \int_{-\infty}^{t} \mathcal{P}(t)dt. \tag{3.11}$$

Note that Ohm's law is linear in its relationship between voltage and current, whereas power and energy are quadratic nonlinear functions.

Ohm's law generalizes the $I(\omega), V(\omega)$ relation in a very important way, resulting in a linear complex–analytic function of complex frequency $s = \sigma + \omega\jmath$ (Kennelly, 1893; Brune, 1931b). Impedance is a fundamental concept in many fields of engineering. For example:[20] Newton's second law $F = ma$ obeys Ohm's

[20] In acoustics the pressure is a potential, like voltage. The force per unit area is given by $f = -\nabla p$; thus $F = -\int \nabla p\; dS$. Velocity is analogous to a current. In terms of the velocity potential, the velocity per unit area is $v = -\nabla\phi$.

law, with mechanical impedance $Z(s) = sm$. Hooke's law $F = kx$ for a spring is described by a mechanical impedance $Z(s) = k/s$. In mechanics a resistor is called a *dashpot* and its impedance is a positive-real constant.

Kirchhoff's laws: KCL and KVL: The laws of electricity and mechanics may be written using Kirchhoff's current and voltage laws (KCL and KVL), which lead to linear systems of equations in the currents and voltages (velocities and forces) of the system under study, with complex coefficients having positive-real parts.

Transfer functions (transfer matrix): The most common standard reference is a physical system that has an input $x(t)$ and an output $y(t)$. If the system is linear, then it may be represented by its impulse response $h(t)$. In such cases, the system equation is

$$y(t) = h(t) \star x(t) \leftrightarrow Y(\omega) = H(s)|_{\sigma=0} X(\omega),$$

namely, the convolution of the input with the impulse response gives the output. This relationship may be written in the frequency domain as a product of the Laplace transform of the impulse response evaluated on the $\omega\jmath$-axis and the Fourier transform of the input $X(\omega) \leftrightarrow x(t)$ and output $Y(\omega) \leftrightarrow y(t)$.

If the system is nonlinear, then the output is not given by a convolution, and the Fourier and Laplace transforms have no obvious meaning.

The question that must be addressed is why the *power* is nonlinear, whereas a *power series* of $H(s)$ is linear: Both have powers of the underlying variables. This is confusing and rarely, if ever, addressed. The quick answer is that powers of the Laplace frequency s correspond to derivatives, which are linear operations, whereas the product of the voltage $v(t)$ and current $i(t)$ is nonlinear. It is confusing because the word *power* has two different meanings. The important and interesting question will be addressed on page 126 in terms of the system postulates of physical systems.

Ohm's law: *Genaralized impedance*

As shown in Table 3.3.2, the impedance concept also holds for mechanics and acoustics. In mechanics, the force is equal to the mechanical force on an element (e.g., a mass, dashpot, or spring) and the flow is the velocity. In acoustics, the force density is the negative of the gradient of the pressure, and the flow is the volume velocity (or particle velocity) of air molecules.

In this section we shall derive the method of the linear composition of systems, also known as the *ABCD transmission matrix method*, or in the mathematical literature, the *Möbius (bilinear) transformation*. With the method of matrix composition, we can use a linear system of 2×2 matrices to represent a significant family of networks. By the application of Ohm's law to the circuit shown in Fig. 3.7, we can model a cascade of such cells, which characterize transmission lines (Campbell, 1903).

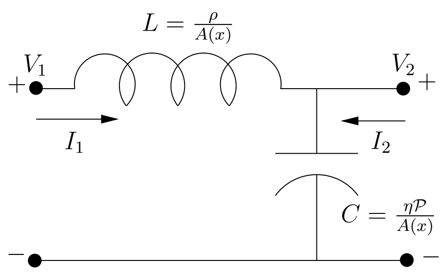

Figure 3.7: *Here we show a single LC cell of the LC transmission line. Every cell of any transmission line may be modeled by the ABCD method as the product of two matrices. For the example shown here, the inductance L of the coil and the capacitance C of the capacitor are in units of [henry/m] and [farad/m]; thus they depend on length Δ_x [m] that the cell represents. Note the flows are always defined as into the $+$ node.*

Example of the use of the ABCD matrix composition: Figure 3.7 shows a network composed of a series inductor (mass) that has an impedance $Z_l = sL$ and a shunt capacitor (compliance) that has an admittance $Y_c = sC \in \mathbb{C}$. As determined by Ohm's law, each equation describes a linear relationship between the current and the voltage. For the inductive impedance, applying Ohm's law gives

$$Z_l(s) = (V_1 - V_2)/I_1,$$

where $Z_l(s) = Ls \in \mathbb{C}$ is the complex impedance of the inductor. For the capacitive impedance, applying Ohm's law gives

$$Y_c(s) = (I_1 + I_2)/V_2,$$

where $Y_c = sC \in \mathbb{C}$ is the complex admittance of the capacitor.

Each of these linear impedance relationships may be written in a 2×2 matrix format. The series inductor ($C = 0$) equation gives ($I_1 = -I_2$)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \tag{3.12}$$

while the shunt capacitor ($L = 0$) equation yields ($V_1 = V_2$)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{3.13}$$

When the second matrix equation for the shunt admittance (Eq. 3.13) is substituted into the series impedance equation (Eq. 3.12), we find that the ABCD matrix composition ($\mathcal{T}_{12} = \mathcal{T}_1 \circ \mathcal{T}_2$) for the cell is the product of two matrices:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1 + Z_l Y_c, & Z_l \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{3.14}$$

Note that the determinant of the matrix $\Delta_{\mathcal{T}} = \mathcal{AD} - \mathcal{BC} = 1$. This is not an accident, since the determinants of the two matrices are each 1; thus the determinant of their product is 1. Every cascade of series and shunt elements will always have $\Delta_{\mathcal{T}} = \pm 1$.

For the case of Fig. 3.7, Eq. 3.14 has $\mathcal{A}(s) = 1 + s^2 LC$, $\mathcal{B}(s) = sL$, $\mathcal{C}(s) = sC$, and $\mathcal{D} = 1$. These equations characterize the four possible relationships of the cell's input and output voltage and current. For example, the ratio of the output to input voltage, with the output unloaded, is

$$\left.\frac{V_2}{V_1}\right|_{I_2=0} = \frac{1}{\mathcal{A}(s)} = \frac{1}{1 + Z_l Y_c} = \frac{1}{1 + s^2 LC}.$$

This is known as the *voltage divider relationship*. To derive the *current divider relationship*, we use the lower equation with $V_2 = 0$:

$$\left.\frac{-I_2}{I_1}\right|_{V_2=0} = 1.$$

Exercise #4

What happens if the order of Z and Y are reversed?

Solution:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y_c & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1 & Z_l \\ Y_c & 1 + Z_l Y_c \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \tag{3.15}$$

This is the same network, reversed in direction.

Exercise #5

What happens if the series element is a capacitor and the shunt an inductor?

Solution:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & 1/Y_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/Z_l & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1 + 1/Z_l Y_c & 1/Y_c \\ 1/Z_l & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \tag{3.16}$$

This circuit is a high-pass filter rather than a low-pass.

Properties of the transmission matrix: The transmission matrix is always constructed from the product of elemental matrices of the form

$$\begin{bmatrix} 1 & Z(s) \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ Y(s) & 1 \end{bmatrix}.$$

Thus for the case of reciprocal systems (Postulate P6),

$$\Delta_{\mathcal{T}} = \det \begin{bmatrix} \mathcal{A}(s) & \mathcal{B}(s) \\ \mathcal{C}(s) & \mathcal{D}(s) \end{bmatrix} = 1,$$

since the determinant of the product of each elemental matrix is 1 and the determinant of their product is 1. An antireciprocal system may be synthesized by the use of a gyrator, and for such cases $\Delta_{\mathcal{T}} = -1$.

The eigenvalue and vector equations for a T matrix are summarized in Appendix A.3 (pp. 261 and 264). The basic postulates of network theory also apply to the matrix elements $\mathcal{A}(s)$, $\mathcal{B}(s)$, $\mathcal{C}(s)$, $\mathcal{D}(s)$, which place restrictions on their functional relationships. places limits on the poles and/or zeros of each function, since the time response must be causal.

3.8 Signals: Fourier transforms

The two most fundamental tools for dealing with differential equations in engineering mathematics are the Fourier and the Laplace transforms, which deal with time-frequency analysis (Papoulis, 1962).

The Fourier transform ($\mathcal{FT}$) takes a time-domain signal $f(t) \in \mathbb{R}$ and transforms it to the frequency domain by taking the scalar product (also called dot product) of $f(t)$ with the complex time vector $e^{-\jmath\omega t}$:

$$f(t) \leftrightarrow F(\omega) = f(t) \cdot e^{-\jmath\omega t},$$

where $F(\omega)$ and $e^{-\jmath\omega t} \in \mathbb{C}$ and $\omega, t \in \mathbb{R}$. Here $f(t)$ and $e^{\jmath\omega t}$ are in a Hilbert space, as discussed in §3.4.1. The scalar product between two vectors results in a scalar (number), as discussed in Appendix 5.

Definition of the Fourier transform: The forward transform takes $f(t)$ to $F(\omega)$, while the inverse transform takes $F(\omega)$ to $\widetilde{f}(t)$. The tilde indicates that, in general, the recovered inverse transform signal can be slightly different from $f(t)$. Examples are presented in Table 3.8.

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-\jmath\omega t} dt & \qquad \widetilde{f}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{\jmath\omega t} d\omega \\ F(\omega) &\leftrightarrow f(t) & \widetilde{f}(t) &\leftrightarrow F(\omega). \end{aligned} \tag{3.17}$$

It is accepted in the engineering and physics literature to use the case of the variable to indicate the type of argument. A time-domain function is $f(t)$, where t has units of seconds [s] and is lowercase. Its Fourier transform is uppercase $F(\omega)$ and is a function of frequency, having units of either hertz [Hz] or radians per second [2π Hz]. This case convention helps the reader parse the variable under consideration. This notation is a helpful but not agree with the notation used in mathematics, where units are rarely cited.

Types of Fourier transforms: As summarized in Table 3.8, each $\mathcal{FT}$ type is determined by symmetries in time and frequency. A time function $f(t)$ may be continuous in time, with $-\infty < t < \infty$, discrete in time, $f_n = f(t_n)$ with $t_k = kT_s$, where T_o is called the Nyquist sample period, or periodic in time, $f((t))_{T_p} = f(t + kT_p)$, where T_p is called the period. Here $k, n \in \mathbb{Z}$ and $T_o, T_p \in \mathbb{R}$. When time is discrete it is commonly represented as either $x[n]$ or $x(t_n)$.

A general rule is that if a function is discrete in one domain (time or frequency), it is periodic in the other domain (frequency or time). For example, the discrete time function f_n must have a periodic frequency response—namely, $f_n \leftrightarrow F((\omega))_{T_p}$. This is the case of the discrete-time Fourier transform (DTFT). Alternatively, when the time function is periodic, the frequencies must be discrete—namely, $f((t))_{T_p} \leftrightarrow F(\omega_k)$. This is the case of the Fourier series (FS). When both the time and frequencies are discrete, both the time and frequencies must be periodic. This is the case of the discrete Fourier transform (DFT). These four cases are summarized in Table 3.8.

Periodic signals: As shown in Table 3.8 there are four variants of the $\mathcal{FT}$ that depend on the symmetry in time and frequency. For example, when the time signal is sampled (discrete in time), the frequency response becomes periodic, leading to the DTFT. When a time response is periodic, the frequency response is sampled (discrete in frequency), leading to the FS. These two symmetries may be simply characterized only as *periodic in time* $\Rightarrow$ discrete in frequency and *periodic in frequency* $\Rightarrow$ discrete in time. When a function is discrete in both time and frequency, it is necessarily periodic in time and frequency, leading to the DFT. The DFT is typically computed with an algorithm called the FFT, which can dramatically speed up the calculation when the data are a power of 2 in length. Typically the transform length N is taken to be a power of 2, such as $N = 1024$ samples. scalar product.

3.8.1 Properties of the Fourier transform

1. Both time t and frequency ω are real.

2. When a function is periodic in one domain (t, f), it must be discrete in the other (Table 3.8).

3. For the forward transform (time to frequency), the sign of the exponential is negative.

4. The limits on the integrals in both the forward and reverse FTs are $[-\infty, \infty]$.

5. When we take the inverse Fourier transform, the scale factor of $1/2\pi$ is required to cancel the 2π in the frequency differential $d\omega = 2\pi df$.

6. The Fourier step function is defined by the use of superposition of 1 and $\text{sgn}(t) = t/|t|$ as

$$\widetilde{u}(t) \equiv \frac{1 + \text{sgn}(t)}{2} = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}.$$

   Taking the FT of a delayed step function, we get

$$\widetilde{u}(t - T_o) \leftrightarrow \frac{1}{2} \int_{-\infty}^{\infty} [1 - \text{sgn}(t - T_o)]\, e^{-j\omega t} dt = \pi \widetilde{\delta}(\omega) + \frac{e^{-j\omega T_o}}{j\omega}.$$

   Thus the FT of the step function has the term $\pi\delta(\omega)$ due to the 1 in the definition of the Fourier step. This term introduces a serious flaw with the FT of the step function: While it appears to be causal, it is not. Compare this to the convolution $u(t) \star u(t)$ in Table 6.

7. The convolution $\widetilde{u}(t) \star \widetilde{u}(t)$ is not defined because both $1 \star 1$ and $\widetilde{\delta}^2(\omega)$ are not defined.

8. The inverse $\mathcal{FT}$ has convergence issues whenever there is a discontinuity in the time response. We indicate this with a hat over the reconstructed time response. The error between the target time function and the reconstructed is zero in the root-mean sense, but not point-wise.

   Specifically, at the discontinuity point for the Fourier step function $(t = 0)$ $\widetilde{u}(t) \neq u(t)$, yet $\int |\widetilde{u}(t) - u(t)|^2 dt = 0$. At the point of the discontinuity, the reconstructed function displays Gibbs ringing (it oscillates around the step and hence does not converge at the jump). The $\mathcal{LT}$ does not exhibit Gibbs ringing and is exact.

9. The $\mathcal{FT}$ is not always analytic in ω, as in this example of the step function. The step function cannot be expanded in a Taylor series about $\omega = 0$ because $\widetilde{\delta}(\omega)$ is not analytic in ω.

10. The Fourier δ function is denoted $\widetilde{\delta}(t)$ to differentiate it from the Laplace delta function $\delta(t)$. They differ because the step functions differ due to the convergence problem.

11. One may define

$$\widetilde{u}(t) = \int_{-\infty}^{t} \widetilde{\delta}(t) dt$$

   and then we obtain

$$\widetilde{\delta}(t) = \frac{d}{dt} \widetilde{u}(t).$$

12. The rec(t) function is defined as

$$\text{rec}(t) = \frac{\widetilde{u}(t) - \widetilde{u}(t - T_o)}{T_o} = \begin{cases} 0 & t < 0 \\ 1/T_o & 0 < t < T_o \\ 0 & t > T_0 \end{cases}.$$

It follows that $\widetilde{\delta}(t) = \lim_{T_o \to 0}$. Like $\widetilde{\delta}(t)$, the rec(t) has unit area.

Table 3.1: *This table of scalar dot products are the four types of Fourier transforms which differ in their support in time and frequency. These four are know as the (1)* Fourier transform *(FT), (2)* Fourier series *(FS), (3)* discrete-time Fourier transform *(DTFT), and (4)* discrete Fourier transform *(DFT). The different support in time and frequency is determined by the kernel of the scalar/inner product. In this way all the various transforms may be reduced to differences in the scalar product, as dictated by the support of the signals in time and frequency. Here $t_n = nT_s$, $f_k = k/T_s$ represent discrete time and frequency samples, where T_s is the sample period. The signal period for the Fourier series (FS) is T [s]. For the* discrete Fourier transform *(DFT), where $N \in \mathbb{N}$ is the number of samples, NT is the signal period and T [s] is the sample period. This column defines the kernel of the transform. The signal is projected onto these vectors by the scalar product via the kernel.*

Name	domain	scalar product	inner product	*kernal*
(1) FT	$-\infty < t \in \mathbb{R} < \infty$	$x(t) \cdot y(t)$	$\int_{-\infty}^{\infty} x(t)y(t)dt$	$e^{-\jmath 2\pi f t}$
	$-\infty < f \in \mathbb{R} < \infty$	$X(f) \cdot Y(f)$	$\int_{-\infty}^{\infty} X(f)Y(f)\frac{d\omega}{2\pi}$	$e^{\jmath 2\pi f t}$
(2) FS	$0 \le t \in \mathbb{R} \le T$	$x((t)) \cdot y((t))$	$\frac{1}{T}\int_{t=0}^{T} x(t)y(t)dt$	$e^{-\jmath 2\pi f_k t}$
	$-\infty < f_k = \frac{k}{T} \in \mathbb{N} < \infty$	$X_k \cdot Y_k$	$\sum_{k=-\infty}^{\infty} X_k Y_k$	$e^{\jmath 2\pi f_k t}$
(3) DTFT	$-\infty < t_n < \infty$	$x_n \cdot y_n$	$\sum_{n=-\infty}^{\infty} x_n y_n$	$e^{-\jmath 2\pi t_n \Omega}$
	$-\pi < \Omega < \pi$	$X((\Omega)) \cdot Y((\Omega))$	$\int_{-\pi}^{\pi} X(e^{\jmath\Omega})Y(e^{\jmath\Omega})\frac{d\Omega}{2\pi}$	$e^{\jmath 2\pi t_n \Omega}$
(4) DFT	$0 \le t_n = nT \le (N-1)T$	$x_n y_n$	$\sum_{n=0}^{N-1} x_n y_n$	$e^{-\jmath 2\pi t_n f_k}$
	$0 \le f_k = \frac{k}{NT} \le \frac{(N-1)}{NT}$	$X_k Y_k$	$\frac{1}{N}\sum_{n=0}^{N-1} X_k Y_k$	$e^{\jmath 2\pi t_n f_k}$

Exercise #6

Consider the Fourier series scalar (dot) product (Eq. 3.34, between "vectors" $f((t))_{T_o}$ and $e^{-\jmath\omega_k t}$:

$$\begin{aligned} F(\omega_k) &= f((t))_{T_o} \cdot e^{-\jmath\omega_k t} \\ &\equiv \frac{1}{T_o}\int_0^{T_o} f(t)e^{-\jmath\omega_k t}dt, \end{aligned}$$

where $\omega_0 = 2\pi/T_o$ and $f(t)$ has period T_o—that is, $f(t) = f(t + nT_o) = e^{\jmath\omega_n t}$ with $n \in \mathbb{N}$ and $\omega_k = k\omega_o$. What is the value of the Fourier series scalar product?

Solution: Evaluating the scalar product, we find

$$\begin{aligned} e^{\jmath\omega_n t} \cdot e^{-\jmath\omega_k t} &= \frac{1}{T_o}\int_0^{T_o} e^{\jmath\omega_n t}e^{-\jmath\omega_k t}dt \\ &= \frac{1}{T_o}\int_0^{T_o} e^{2\pi\jmath(n-k)t/T_o}dt = \begin{cases} 1 & n = k \\ 0 & n \ne k \end{cases}. \end{aligned}$$

The two signals (vectors) are orthogonal.

Exercise #7

Consider the discrete-time $\mathcal{FT}$ (DTFT) as a scalar (dot) product (Eq. 3.34, between "vectors" $f_n = f(t)|_{t_n}$ and $e^{-\jmath\omega t_n}$, where $t_n = nT_s$ and $T_s = 1/2F_{\max}$ is the sample period.

Solution: The scalar product over $n \in \mathbb{Z}$ is

$$\begin{aligned} F((\omega))_{2\pi} &= f_n \cdot e^{-\jmath\omega t_n} \\ &\equiv \sum_{n=-\infty}^{\infty} f_n e^{-\jmath\omega t_n}, \end{aligned}$$

where $\omega_0 = 2\pi/T_o$ and $\omega_k = k\omega_o$ is periodic (i.e., $F(\omega) = F(\omega + k\omega_o)$).

3.9 Systems: Laplace transforms

The Laplace transform $\mathcal{LT}$ takes real causal signals $f(t)u(t) \in \mathbb{R}$, as a function of real time $t \in \mathbb{R}$, that are strictly zero for negative time ($f(t) = 0$ for $t < 0$), and transforms them into complex–analytic functions ($F(s) \in \mathbb{C}$) of complex frequency $s = \sigma + \omega\jmath$. As we did for the Fourier transform, we use the same upper-lower case notation: $f(t) \leftrightarrow F(s)$.

When a signal is zero for negative time $f(t < 0) = 0$, it is said to be *causal*, and the resulting transform $F(s)$ must be complex–analytic over significant regions of the s plane. For a function of time to be causal, time *must* be real ($t \in \mathbb{R}$), since if it were complex, it would lose the order property (thus it could not be causal). It is helpful to emphasize the causal nature of $f(t)u(t)$ to force causality, with the Heaviside step function $u(t)$.

Any restriction on a function (e.g., real, causal, periodic, positive real part, etc.) is called a *symmetry property*. There are many forms of symmetry. The concept of symmetry is very general and widely used in both mathematics and physics, where it is more generally known as *group theory*. One-sided periodic transforms also exist, such as the system shown in Fig. 3.1.

Definition of the Laplace transform: The forward and inverse Laplace transforms are defined in Eq. 3.2. Here $s = \sigma + j\omega \in \mathbb{C}$ [2πHz] is the complex Laplace frequency in radians and $t \in \mathbb{R}$ [s] is the time in seconds.

Forward and inverse Laplace transforms:

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad f(t) = \frac{1}{2\pi\jmath}\int_{\sigma_o-\infty\jmath}^{\sigma_o+\infty\jmath} F(s)e^{st}ds \tag{3.18}$$

$$F(s) \leftrightarrow f(t) \qquad f(t) \leftrightarrow F(s)$$

Tables of functional properties are shown in Table 3.3, while basic transforms are provided in Appendix 5, Table 6. (p. 120). Properties of more advanced $\mathcal{LT}$s are in Table 3.9.

When we deal with engineering problems, it is convenient to separate the *signals* we use from the *systems* that process them. We do this by treating signals, such as speech and music, differently from a system, such as a filter. In general, signals may start and end at any time. The concept of causality has no mathematical meaning in signal space. Systems, on the other hand, obey rigid rules (to ensure that they remain physical). These physical restrictions are described in terms of the system postulates. There is a question as to why postulates are needed and which ones are the best choices. These questions are discussed in lectures by Feynman (1968, 1970a). The original video is also available online in many places, including YouTube.[21] There may be no definitive answers to these questions, but having a set of postulates is a useful way of thinking about physics.

Types of Laplace transforms: As shown in Table 3.2 there are three types of $\mathcal{LT}$s. The function may be continuous in time, in which it is also continuous in the Laplace frequency s. It may be discrete in time and therefore periodic in frequency θ, which is called the z-transform. Or it may be causal-periodic in time and therefore discrete in frequency. This transform has no name ("unknown," as best I know). An example is the Riemann-zeta function (Fig. 3.9).

[21] https://www.youtube.com/watch?v=JXAfEBbaz_4, https://www.youtube.com/watch?v=YaUlqXRPMmY, https://www.youtube.com/watch?v=xnzB_IHGyjg

Table 3.2: *Laplace transforms are complementary to the class of Fourier transforms $\mathcal{FT}$ because the time function must be a causal function. All $\mathcal{LT}$s are* complex–analytic *in the complex frequency $s = \sigma + \omega\jmath$ domain. As an example, a causal function that is continuous but one-sided in time is the* step function *$u(t)$, which has the $\mathcal{LT}$ $u(t) \leftrightarrow 1/s$. When a function is discrete in time but one-sided, it has a Z^{eta}-transform. The discrete-time step function is $u_n = u[n] \leftrightarrow 1/(1-z^{-n})$. It follows that the $\mathcal{LT}$ and z* transform are isomorphic. *The general rule is that if a function is discrete in one domain (time or frequency) it is periodic in the other. Abbreviations: $\mathcal{FT}$: Fourier Transform; FS: Fourier Series; DTFT: Discrete time Fourier transform; DFT: Discrete Fourier transform (the FFT is a "fast" DFT);*

FREQUENCY \ TIME	continuous t	discrete $t[k]$	causal-periodic $((t))_{T_o}$
continuous s	$\mathcal{LT}$	–	–
discrete $\omega[k]$	–	DFT/FFT	FSeries
periodic $\|z\|e^{\theta\jmath}$	–	z-Transform	–

3.9.1 Properties of the Laplace transform

The following is a summary description of the $\mathcal{LT}$:

1. Time $t \in \mathbb{R}$ [s] and the Laplace frequency [rad] are defined as $s = \sigma + \omega\jmath \in \mathbb{C}$.

2. Given a Laplace transform ($\mathcal{LT}$) pair $f(t) \leftrightarrow F(s)$, in the engineering literature, the time domain is always lowercase [$f(t)$] and causal [i.e., $f(t<0) = 0$], and the *frequency domain* is uppercase [$F(s)$]. Maxwell's venerable equations are the unfortunate exception to this otherwise universal rule.

3. The target time function $f(t<0) = 0$ (i.e., it must be causal). The time limits are $0^- < t < \infty$. Thus the integral must start from slightly below $t = 0$ to integrate over a delta function at $t = 0$. For example, if $f(t) = \delta(t)$, the integral must include both sides of the impulse. If we want to include non-causal functions such as $\delta(t+1)$, we must extend the lower time limit. In such cases we simply set the lower limit of the integral to $-\infty$ and let the integrand ($f(t)$) determine the limits.

4. When we take the forward transform ($t \to s$), the sign of the exponential is negative. This is necessary to ensure that the integral converges when the integrand $f(t) \to \infty$ as $t \to \infty$. For example, if $f(t) = e^t u(t)$ (i.e., without the negative σ exponent), the integral does not converge.

5. The limits on the integrals of the reverse $\mathcal{LT}$s are $[\sigma_o - \infty\jmath, \sigma_o + \infty\jmath] \in \mathbb{C}$. These limits are further discussed in Sec. 5.

6. When we take the inverse Laplace transform, the normalization factor of $1/2\pi\jmath$ is required to cancel the $2\pi\jmath$ in the differential ds of the integral.

7. The frequencies for the $\mathcal{LT}$ must be complex, and in general $F(s)$ is complex–analytic for $\sigma > \sigma_o$. It follows that the real and imaginary parts of $F(s)$ are related by the Cauchy-Riemann conditions. Given $\Re\{F(s)\}$, it is possible to find $\Im\{F(s)\}$ (Boas, 1987). Read more on this in Sec. 7.

8. To take the inverse Laplace transform, we must learn how to integrate in the complex s plane.

9. The Laplace Heaviside step function is defined as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt = \begin{cases} 1 & \text{if } t > 0 \\ \text{NaN} & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

   Alternatively, we can define $\delta(t) = du(t)/dt$.

10. It is easily shown that $u(t) \leftrightarrow 1/s$ by direct integration,

$$F(s) = \int_0^\infty u(t)\, e^{-st} dt = -\frac{e^{-st}}{s}\bigg|_o^\infty = \frac{1}{s}.$$

    With the $\mathcal{LT}$ step ($u(t)$), there is no Gibbs ringing effect.

11. The Laplace transform of a Brune impedance takes the form of a ratio of two polynomials. In such cases, the roots of the numerator polynomial are called the *zeros* while the roots of the denominator polynomial are called the *poles*. For example, the $\mathcal{LT}$ of $u(t) \leftrightarrow 1/s$ has a pole at $s = 0$, which represents integration, since
$$u(t) \star f(t) = \int_{-\infty}^{r} f(\tau)d\tau \leftrightarrow \frac{F(s)}{s}.$$

12. The $\mathcal{LT}$ is quite different from the $\mathcal{FT}$ in terms of its analytic properties. For example, the step function $u(t) \leftrightarrow 1/s$ is complex–analytic everywhere except at $s = 0$. The $\mathcal{FT}$ of $1 \leftrightarrow 2\pi\widetilde{\delta}(\omega)$ is not analytic anywhere.

13. The dilated step function ($a \in \mathbb{R}$) is
$$u(at) \leftrightarrow \int_{-\infty}^{\infty} u(at)e^{-st}dt = \frac{1}{a}\int_{-\infty}^{\infty} u(\tau)e^{-(s/a)\tau}d\tau = \frac{a}{|a|}\frac{1}{s} = \pm\frac{1}{s},$$
where we have made the change of variables $\tau = at$. The only effect that a has on $u(at)$ is the sign of t, since $u(t) = u(2t)$. However, $u(-t) \neq u(t)$, since $u(t) \cdot u(-t) = 0$, and $u(t) + u(-t) = 1$, except at $t = 0$, where it is not defined.

Once complex integration in the complex plane has been defined (see Sec. 7. we can justify the definition of the inverse $\mathcal{LT}$ (Eq. 3.2).[22]

Table 3.3: *Functional relationships between function and its $\mathcal{LT}$.*

$f(t)$	$\leftrightarrow$	$F(s)$	identity
$\frac{d}{dt}f(t) = \delta'(t) \star f(t)$	$\leftrightarrow$	$sF(s)$	derivative
$f(t) \star g(t) = \int_{t=0}^{t} f(t-\tau)g(\tau)d\tau$	$\leftrightarrow$	$F(s)G(s)$	causal convolution
$u(t) \star f(t) = \int_{0^-}^{t} f(t)dt$	$\leftrightarrow$	$\frac{F(s)}{s}$	convolution with step
$f(at)u(at)$	$\leftrightarrow$	$\frac{1}{a}F\left(\frac{s}{a}\right)$ with $a \in \mathbb{R} \neq 0$	scaling
$f(t)e^{-at}u(t)$	$\leftrightarrow$	$F(s+a)$	damped
$f(t-T)e^{-a(t-T)}u(t-T)$	$\leftrightarrow$	$e^{-sT}F(s+a)$	damped and delayed
$f(-t)u(-t)$	$\leftrightarrow$	$F(-s)$	reverse time
$f(-t)e^{-at}u(-t)$	$\leftrightarrow$	$F(a-s)$	time-reversed and damped
$\frac{\sin(t)u(t)}{t}$	$\leftrightarrow$	$\tan^{-1}(1/s)$	half-sync

Causal-periodic signals: This is a special symmetry that occurs due to functions that are causal *and* periodic in frequency. The best example is the z-transform, which applies to causal (one-sided in time) discrete-time signals. The harmonic series (Eq. 3.15, is the z-transform of the discrete-time step function and is thus, due to symmetry, analytic within the RoC in the complex frequency (z) domain.

The double brackets on $f((t))_{T_o}$ indicate that $f(t)$ is periodic in t with period T_o—that is, $f(t) = f(t+kT_o)$ for all $k \in \mathbb{N}$. Averaging over one period and dividing by T_o give the average value.

[22] `https://en.wikipedia.org/wiki/Laplace_transform#Table_of_selected_Laplace_transforms`

Inverse $\mathcal{LT}$

To invert the $\mathcal{LT}$ one must use the Cauchy residue theorem (CT-3), which requires closure of the contour $\mathcal{C}$ at $\omega\jmath \to \pm j\infty$,

$$\oint_{\mathcal{C}} = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} + \int_{\subset_\infty},$$

where the path represented by $\subset_\infty$ is a semicircle of infinite radius. For a causal, stable (e.g., doesn't "blow up" in time) signal, all of the poles of $F(s)$ *must* be inside of the Laplace contour, in the left half s-plane.

Example: Hooke's law for a spring states that the force $f(t)$ is proportional to the displacement $x(t)$—that is, $f(t) = Kx(t)$. The formula for a dashpot is $f(t) = Rv(t)$, and Newton's famous formula for mass is $f(t) = d[Mv(t)]/dt$, which for a constant mass M_o is $f(t) = M_o dv/dt$.

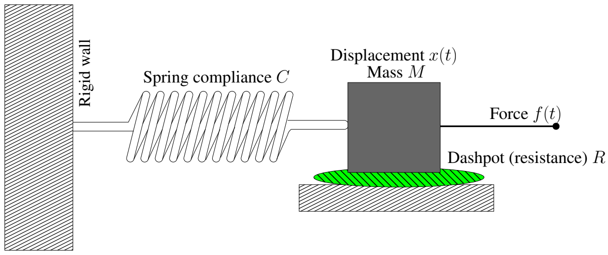

Figure 3.8: *Three-element mechanical resonant circuit consisting of a spring, mass, and dashpot (e.g., viscous fluid).*

The equation of motion for the mechanical oscillator in Fig. 5.7 is given by Newton's second law; the sum of the forces must balance to zero:

$$M_o \frac{d^2}{dt^2} x(t) + R_o \frac{d}{dt} x(t) + K_o x(t) = f(t) \leftrightarrow (M_o s^2 + R_o s + K_o) X(s) = F(s). \tag{3.19}$$

These three constants—mass M_o, resistance R_o, and stiffness K_o ($\in \mathbb{R} \geq 0$)—are real and non-negative. The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$, and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

Newton's second law (ca.1650) is the mechanical equivalent of Kirchhoff's (ca.1850) voltage law (KVL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., $F = q_o E$). The current may be thought of as the flow of charge.

Equation 3.19 may be re-expressed in the frequency domain in terms of an impedance (i.e., Ohm's law), defined as the ratio of the force $F(s)$ to velocity $V(s) = sX(s)$, and the sum of three impedances:

$$Z(s) = \frac{F(s)}{V(s)} = \frac{Ms^2 + Rs + K}{s} = Ms + R + \frac{K}{s}. \tag{3.20}$$

Example: The divergent series

$$e^t u(t) = \sum_0^\infty \frac{1}{n!} t^n \leftrightarrow \frac{1}{s-1}$$

is a valid description of $e^t u(t)$, with an unstable pole at $s = 1$. For values of $|x - x_o| < 1$ ($x \in \mathbb{R}$), the analytic function $P(x)$ is said to have a *region of convergence* (RoC). For cases where the argument is complex ($s \in \mathbb{C}$), this is called the *radius of convergence* (RoC). We will call the region $|s - s_o| > 1$ the *region of divergence* (RoD) and $|s - s_o| = 0$ the *singular circle*. Typically the underlying function $P(s)$, defined by the series, has a pole on the singular circle.

Summary: While the definitions of the $\mathcal{FT}$ and $\mathcal{LT}$ may appear similar, they are not. The key difference is that the time response of the Laplace transform is causal, leading to a complex–analytic frequency response. The frequency response of the Fourier transform is complex but not complex–analytic, since the frequency ω is real. Fourier transforms do not have poles.

The concept of symmetry is helpful in understanding the many different types of time-frequency transforms. The two most fundamental types of symmetry are causality and periodicity.

The $\mathcal{FT}$ characterizes the steady-state response, while the $\mathcal{LT}$ characterizes both the transient and steady-state responses. Given a causal system force response (Eq. 3.20), $F(s) \leftrightarrow f(t)$ with input velocity $V(s) \leftrightarrow v(t)$, the response is

$$f(t) = z(t) \star v(t) \leftrightarrow Z(\omega) = F(s)\Big|_{s=\jmath\omega} V(\omega),$$

which says that the force is the convolution of the mechanical impedance $z(t)$ with the input velocity $v(t)$.

3.9.2 System postulates

Solutions of differential equations, such as the wave equation, are conveniently described in terms of mathematical properties, which we present here in 11 system postulates (see Appendix 3.9.2),

(P1) *Causality* (non-causal/acausal): Causal systems respond when acted upon. All physical systems obey causality. An example of a causal system is an integrator, which has a response of a step function. Filters are also examples of causal systems. Signals represent acausal responses. They do not have a clear beginning or end, such as the sound of the wind in the trees, or traffic noise. A causal linear system is typically complex–analytic and is naturally represented in the complex s plane via Laplace transforms. A nonlinear system may be causal but not complex–analytic.

(P2) *Linearity* (nonlinear): Linear systems obey superposition. Let two signals $x(t)$ and $y(t)$ be the inputs to a linear system, producing outputs $x'(t)$ and $y'(t)$. When the inputs are presented together as $ax(t)+by(t)$ with constant weights $a, b \in \mathbb{C}$, the output is $ax'(t) + by'(t)$. If either a or b is zero, the corresponding signal is removed from the output.

Nonlinear systems mix the two inputs, thereby producing signals that are not present in the input. For example, if the inputs to a nonlinear system are two sine waves, the output contains distortion components that have frequencies not present at the input. One example of a nonlinear system is one that multiplies the two inputs. A second is a diode, which rectifies a signal, letting current flow in only one direction. Most physical systems have some degree of nonlinear response, but this is not always desired. Other systems are designed to be nonlinear, such as the diode example.

(P3) *Passive* (active): An active system has a power source, such as a battery, while a passive system has no power source. Although you may consider a transistor amplifier to be active, it is so only when connected to a power source. Brune impedances satisfy the positive-real condition (Eq. 3.26.

(P4) *Real* (complex) time response: All physical systems are Real in = Real out. They do not naturally have complex responses (real and imaginary parts). While a Fourier transform takes real inputs and produces complex outputs, this is not an example of a complex time response. This postulate is a characterization of the input signal, not its Fourier transform.

(P5) *Time-invariant* (time varying): For a system to be a time-varying system, the output must depend on when the input signal starts or stops. If the output, relative to the input, is independent of the starting time, then the system is said to be *time-invariant*.

(P6) *Reciprocal* (non- or anti-reciprocal): In many ways this is the most difficult postulate to understand. It is best characterized by the ABCD matrix. If $\Delta_\mathcal{T} = 1$, the system is said to be *reciprocal*. If $\Delta_\mathcal{T} = -1$, it is said to be *anti-reciprocal*. The impedance matrix is reciprocal when $z_{12} = z_{21}$ and anti-reciprocal when $z_{12} = -z_{21}$. Dynamic loudspeakers are anti-reciprocal and must be modeled by a gyrator, which may be thought of as a transformer that swaps the force and flow variables (Kim and Allen, 2013). For example, the input impedance of a gyrator terminated by an inductor is a capacitor. This property is best explained by Fig. 3.6. For an extended discussion on reciprocity.

(P7) *Reversibility* (non-reversible): If swapping the input and output of a system leaves the system invariant, it is said to be reversible. When $A = D$, the system is reversible. Note the distinction between reversible and reciprocal.

(P8) *Space-invariant* (space-variant): If a system operates independently as a function of where it physically is in space, then it is space-invariant. When the parameters that characterize the system depend on position, it is space-variant.

(P9) *Deterministic* (random): Given the wave equation along with the boundary conditions, the system's solution may be deterministic, or not, depending on its extent. Consider a radar or sonar wave propagating out into uncharted territory. When the wave hits an object, the reflection can return waves that are not predicted due to unknown objects. This is an example where the boundary condition is not known in advance.

(P10) *Quasi-static* ($ka < 1$): Quasi-statics follows the Nyquist sampling theorem for systems that have dimensions that are small compared to the local wavelength (Nyquist, 1924). This assumption fails when the frequency is raised (the wavelength becomes short). Thus this is also known as the *long-wavelength* approximation. Quasi-statics is typically stated as $ka < 1$, where $k = 2\pi/\lambda = \omega/c_o$ and a is the smallest dimension of the system. See page 127 for a method on how to integrate the transmission matrix and Nyquist sampling.

Postulate P10 is closely related to the Feynman lecture *The "underlying unity" of nature,* where Feynman asks (Feynman, 1970b, Ch. 12-7): "Why do we need to treat the fields as smooth?" His answer is related to the wavelength of the probing signal relative to the dimensions of the object being probed. This raises the fundamental question: Are Maxwell's equations a band-limited approximation to reality? Today we have no definite answer to this question.

The following quote seems relevant:[23]

> *The Lorentz force formula and Maxwell's equations are two distinct physical laws, yet the two methods yield the same results.*

Why the two results coincide was not known. In other words, the flux rule consists of two physically different laws in classical theories. Interestingly, this problem may be the sole motivation behind Einstein's theory of relativity. In 1905, Einstein wrote in the opening paragraph of his first paper on relativity theory, "It is known that Maxwell's electrodynamics when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena." Einstein's argument moved away from this problem to formulated special theory of relativity. Einstein seems to have been following his deep intuition. In my view, relativity follows directly from Maxwell's Equations.

Are photons moving bodies? Are quantum numbers integers, or rational or irrational?

Richard Feynman once described this situation in his famous lecture (The Feynman Lectures on Physics, Vol. II, 1964), "we know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena."

(P11) *Periodic $\leftrightarrow$ discrete:* When a function is discrete in one domain (e.g., time or frequency), it is periodic in the other (frequency or time).

Summary of the 11 system postulates: Each postulate has at least two categories. For example, (P1) is causal, non-causal, or acausal, while (P2) is linear or nonlinear. (P6) and (P9) apply to only two-port algebraic networks (those that have an input and an output). The others apply to both two- and one-port networks (e.g., an impedance is a one-port). An important example of a two-port is the anti-reciprocal transmission matrix of a dynamic (EM) loudspeaker.

Related forms of these postulates may be found in the network theory literature (Van Valkenburg, 1964a,b; Ramo et al., 1965). Postulates P1–P6 were introduced by Carlin and Giordano (1964), and Postulates P7–P9 were added by Kim et al. (2016). While linearity (P2), passivity (P3), realness (P4), and time-invariant (P5) are independent, causality (P1) is a consequence of linearity (P2) and passivity (P3) (Carlin and Giordano, 1964, p. 5). In the more general case, is a multi-ported system. Living systems are all multi-ported.

3.10 Visco-thermal losses

3.10.1 Adiabatic approximation at low frequencies

Newton's early development understandably ignored viscous and thermal losses, by assuming iso-thermal conditions. But starting at very low frequencies, the isothermal assumption breaks down. Modern theory, for audio frequencies, assumes the adiabatic approximation, and is thus described by the scalar wave equation (Pierce, 1981). But it turns out that even at audio frequencies, the adiabatic approximation is invalid. This

[23] `https://www.sciencedaily.com/releases/2017/09/170926085958.htm`

was first shown by Kirchhoff, but was not fully appreciated for more than a century, due to mathematical difficulties, which we now believe can be explained, as discussed below.

Following Helmholtz (1858), and extended by Kirchhoff (1868), visco-thermal loss mechanisms are related. The full theory was first worked out by Kirchhoff (1868, 1974). To understand how the are related is complicated, due to both the history and the mathematics, as briefly discussed by Pierce (1981). Both forms of damping are caused by two different, but coupled, diffusion effects: (1) viscous effects, due to shear at the container walls, and (2) thermal effects, due to deviations from adiabatic expansion (Kirchhoff, 1868, 1974). I believe that Einstein was eventually involved, following his studies on Brownian motion (Einstein, 1905).[24, 25]

These two loss mechanism are restricted to a thin region called the *boundary layer*, the acoustic properties of which critically depend on the square root of the Laplace frequency. The boundary layer is key transition region, where the acoustic wavelength approaches the complex boundary layer thickness. When the radius of the container (a horn) approaches the viscous boundary layer, the theory breaks down.

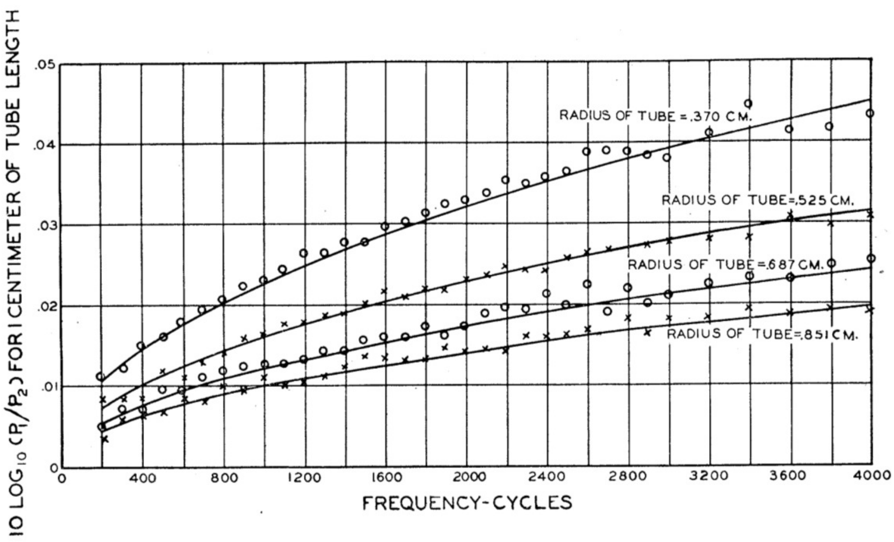

Figure 3.9: *This figure, taken from Mason (1928), compares the Helmholtz-Kirchhoff theory for $|\kappa(f)|$ to Mason's 1928 experimental measurements of the loss. The ratio of two powers (P_1 and P_2) is plotted (see Mason's discussion immediately below his Fig. 4), and as indicated in the label: "$10\log_{10} P_1/P_2$ for 1 [cm] of tube length." This is a plot of the transmission power ratio in [dB/cm] which is $10\log|\Gamma(\omega)|^2$ where $\Gamma(i\omega)$ the reflection coefficient . For a discussion of the reflection coefficient and its properties, refer to §4.3.1, p. 155.*

3.10.2 Lossy wave-guide propagation

The formulation of visco-thermal loss in air transmission was first worked out by Helmholtz (1863a) and then extended by Kirchhoff (1868) to include thermal damping (Rayleigh, 1896, Vol. II, p. 319). These losses are accurately represented by the complex–analytic propagation function $\kappa(s)$ (Eq. 3.25). Following his review of these theories, Crandall (1926, Appendix A), the head of the 1926 Acoustic Research Department at that time, noted that the "Helmholtz-Kirchhoff" theory had never been experimentally verified. Acting on Crandall's suggestion, Physicist Warren Mason set out to experimentally verify Kirchhoff's 60 year old theory. Mason's analysis consumed several years (Mason, 1928).

This was a continued effort. Mason (1928) extended earlier work of Stewart's on acoustic transmission lines, by including viscous and thermal losses. Stewart's acoustic theory (Stewart, 1922; Stewart and Lindsay, 1930) was an acoustic version of the work of George Campbell (1904-1923) on electrical wave filters. If today you design earphones and hearing aids, or do otoacoustic research, the works of Steward and Mason are relevant.

Mason's specification of the propagation function

Mason's results are reproduced in Fig. 3.9 for tubes of radii R between 3.7 and 8.5 [mm] of lengths L, having power reflectance

$$|\Gamma_L(f)|^2 = \left|e^{-\kappa(f)\,L}\right|^2 \text{ [cm}^{-1}\text{]}. \tag{3.21}$$

[24]See Ch. 3 of `https://www.ks.uiuc.edu/Services/Class/PHYS498/`

[25]`https://en.wikipedia.org/wiki/Einstein_relation_(kinetic_theory)`

The complex propagation function $\kappa(\omega)$, as published by (Rayleigh, 1896, p. 319) was reproduced by Mason (1928) as

$$\kappa(\omega) = \frac{P\eta_o'\sqrt{\omega}}{2c_oS\sqrt{2\rho_o}} + \frac{i\omega}{c_o}\left\{1 + \frac{P\eta_o'}{2S\sqrt{2\omega\rho_o}}\right\}, \tag{3.22}$$

and the characteristic impedance as

$$z_o(\omega) = \sqrt{P_o\eta_o\rho_o}\left\{1 + (1-\jmath)\frac{P\eta_o'}{2S\sqrt{2\omega\rho_o}}\right\}, \tag{3.23}$$

Recall that area $S = \pi R^2$ and perimeter $P = 2\pi R$. Solving for the radius we find $R = \frac{2S}{P}$. Thus we may replace $\frac{P}{2S}$ by $\frac{1}{R}$.

Following Mason (1928, Fig. 5), the measured speed of sound

$$c_o'(\omega) = c_o\left\{1 - \frac{\eta_o'}{R\sqrt{2\omega\rho_o}}\right\}, \tag{3.24}$$

which depends on frequency ω, as derived from the imaginary part of Eq. 3.22. From the definitions of P and S, we may replace $P/2S = R/4$, greatly simplifying the expression. Figure 3.10 directly compares Mason's measured sound speed with this equation.

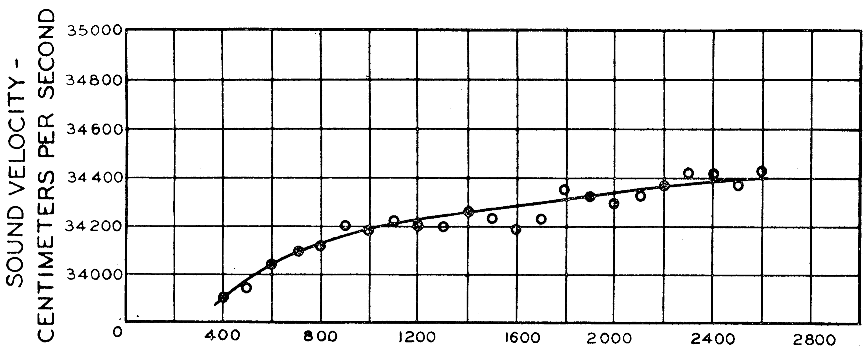

Figure 3.10: *Figure 5 from Mason (1928) showing the velocity of sound as a function of frequency, when visco-thermal losses are included. Note the change in the slope around 0.8 [kHz]. Today the accepted speed of sound is assumed to be 341.57 [m/s]. Clearly it is not constant.*

Reduction of Kirchhoff's equations to complex–analytic form: We may rewrite $\kappa(s)$ and $z_o(s)$ in terms of the Laplace frequency s and a complex constant β_o,

$$\kappa_\pm(s) = \frac{1}{c_o}\left(\beta_o \pm \sqrt{s}\right)^2 \tag{3.25}$$

The boxed analysis below provides the derivation for $c_o\kappa(\omega)$, Mason's simplified complex–analytic formula for (Eq. 3.25). The first step is to define $2\beta_o$ and $c_o\kappa(s) - s$. Finally $c_o\kappa(s)$ is given as a function of the Laplace frequency s. Following a completion of squares in $\sqrt{s}$, one obtains Eq. 3.25, providing a major simplification of these expressions. The inverse Laplace transforms of $\sqrt{s}$ and $1/\sqrt{s}$ are provided in Table 3.9 5 and Table 6 of Sect. 3.9.1.

STARTING FROM MASON (1928) $c_o\kappa(\omega)$:

$$c_o\kappa(\omega) = \frac{P\eta_o'\sqrt{\omega}}{2S\sqrt{2\rho_o}} + \jmath\omega\left\{1 + \frac{P\eta_o'}{2S\sqrt{2\omega\rho_o}}\right\} \tag{3.26}$$

DEFINE VARIABLES $2\beta_o$ AND $s = \jmath\omega$:

$$c_o\kappa(\omega) = s + \cancelto{2\beta_o}{\frac{P\eta_o'}{2S\sqrt{\rho_o}}}\left[\sqrt{\frac{\omega}{2}} + \frac{s}{\sqrt{2\omega}}\right] \tag{3.27}$$

THUS

$$c_o\kappa(\omega) - s = \frac{2\beta_o}{\sqrt{2}}\left[\sqrt{\omega} + \frac{s}{\sqrt{\omega}}\right] \tag{3.28}$$

MULTIPLYING TOP AND BOTTOM ON RIGHT BY $\sqrt{\jmath}$:

$$\begin{aligned}
c_o\kappa(\omega) - s &= \frac{2\beta_o}{\sqrt{2}}\left[\frac{\sqrt{\jmath\omega}}{\sqrt{\jmath}} + \frac{s\sqrt{\jmath}}{\sqrt{\jmath\omega}}\right] && \text{SET } \sqrt{\jmath\omega} = \sqrt{s} \\
&= \frac{2\beta_o}{\sqrt{2}}\left[\frac{\sqrt{s}}{\sqrt{\jmath}} + \frac{s\sqrt{\jmath}}{\sqrt{s}}\right], && \text{CROSS MULTIPLY} \\
&= \frac{2\beta_o}{\sqrt{2}}\left[\frac{s + \jmath s}{\sqrt{\jmath}\cdot\sqrt{s}}\right], && \text{FACTOR OUT } \frac{s}{\sqrt{s}} \\
&= 2\beta_o\left[\cancel{\frac{1+\jmath}{\sqrt{2\jmath}}}\right]\frac{s}{\sqrt{s}} && \text{REPLACE } \frac{s}{\sqrt{s}} = \sqrt{s} \\
&= 2\beta_o\sqrt{s}
\end{aligned}$$

TO SHOW $1 + \jmath = \sqrt{2\jmath}$, SQUARE BOTH SIDES: $\cancel{1-1} + 2\jmath = 2\jmath$.

Acoustic constants for air: Assuming $\eta_o = 1.4$ (ratio of specific heats), $\rho_o = 1.2$ [kgm/m^3] (density), a temperature of 23.5 [°C], and $P_o = 10^5$ [Pa] (atmospheric pressure), the lossless sound velocity is $c_o = \sqrt{P_o\eta_o/\rho_o} = 341.57$ [m/s]. By a comparison of this value of c_o to Fig. 3.10, it is clear that this value does not agree with Mason's measurements. Given the time and trouble Mason went to, I suspect his measurements are superior.

Reduction of the lossy characteristic impedance: Rendering Eq. 3.23 dimensionless, the complex–analytic expression greatly simplifies[26].

$$\beta_o = \beta_1/2 = \frac{\sqrt{s}}{2}\left(\frac{z_o(s)}{\sqrt{P_o\eta_o\rho_o}} - 1\right) \tag{3.29}$$

Thus the lossy normalized characteristic impedance is $z_o(s)/r_o = 1 + \frac{2\beta_o}{\sqrt{s}}$, where $r_o = \sqrt{P_o\eta_o\rho_o}$ is the lossless characteristic resistance.

Case of the cylindrical guide: For the case of a cylindrical wave guide, $P/2S = 1/R$.[27] Thus

$$\beta_0 R = \frac{\eta_o'}{2\sqrt{\rho_o}} = \frac{1.9105 \times 10^{-3}}{2}.$$

It is well documented in the literature that the boundary layer thickness is proportional to $\sqrt{\mu/\rho}$. We may factor Eq. 3.25 to reveal the mathematical impact of the damping on $\kappa(s)$

$$\sqrt{s_o} = -\beta_o \pm \beta_o,$$

namely $\sqrt{s_o} = \{0, -2\beta_0\}$. In general there must be four roots, but since $\beta_0 \in \mathbb{R} > 0$, the roots in this case are degenerate, and in the left half plane.

The smaller the radius the greater the damping ($\beta_0 = 2.2 \times 10^{-3}/R$). Also note that the propagation function $\kappa(\omega)$ has a Helmholtz-Kirchhoff correction for both the real and imaginary parts. Thus both the speed of sound and the damping are dependent on frequency, in the same way.

Pressure Eigen-solutions: The forwarded P_- and backward P_+ pressure waves propagate as

$$P_\pm(s, x) = e^{-\kappa(s)x}, \quad e^{-\overline{\kappa}(s)x}, \tag{3.30}$$

where $\overline{\kappa}(s)$ the complex conjugate of $\kappa(s)$, such that $\Re\kappa(s) > 0$. The term $\beta_o\sqrt{s}$ affects both the real and imaginary parts of $\kappa(s)$. The real part is a frequency-dependent loss, and the imaginary part introduces a frequency-dependent speed of sound (Mason, 1928). These effects are most certainly impact the acoucits and light propagation inside the sun due to the extremely high temperature.

[26]Remove remaining β_1

[27]Recall that P is the primeter and S is the area.

3.10.3 Impact of viscous and thermal losses

Equation 3.22 and the measured data are compared in Fig. 3.9, as reproduced from Mason's Fig. 4, which shows that the wave speed drops from 344 m/s at 2.6 kHz to 339 m/s at 0.4 kHz, a 1.5% reduction. At 1 kHz the loss is 1 dB/m for a 7.5-mm tube, which is more than 3 times the length of the adult human ear canal.

Note how the loss and the speed of sound vary inversely with the radius. As the radius approaches the *boundary layer thickness* (i.e., the radial distance such that the loss is e^{-1}), the effects of damping dominate the propagation.

Cut-off frequency s_o: The frequency where the lossless part equals the lossy part is defined as $\kappa(s_o) = 0$, namely, $\sqrt{s_o} = -\beta_o$, or $s_o = \beta_o^2$.

To get a feeling for the magnitude of s_o, let $R = 0.75/2$ [cm] (i.e., the average radius of the adult ear canal). Then for $R = 3.75 \times 10^{-3}$ [cm]

$$s_o = (1.9 \times 10^{-3}/3.75 \times 10^{-3})^2 = 1/4.$$

The losses are insignificant in the audio range, since for the human ear canal, $f_o = \beta_o^2/\pi \approx 0.25/\pi = 0.08$ Hz.[28] This frequency represents the lower bound of the transition from adiabatic to iso-thermal equilibrium. It should be clear that acoustic frequencies do not actually obey the adiabatic approximation, due to the thin boundary layer. Both the real and imaginary part of propagation function $\kappa(s)$, the characteristic impedance $z_o(s)$ and the speed of sound $c_o'(s)$ all depended on frequency in the auditory range of human hearing.

Summary: The Helmholtz-Kirchhoff theory of viscous and thermal losses results in a frequency-dependent speed of sound that has a frequency dependence proportional to $1/\sqrt{s}$ rather than $1/\sqrt{\omega}$ (Mason, 1928, Eq. 4). This corresponds to a 2% change in the sound velocity over the decade from 0.2 to 2 kHz (Mason, 1928, Fig. 5), in agreement with Mason's experimental results.

3.11 Complex–analytic mappings (domain-coloring)

One of the most difficult aspects of complex functions of a complex variable is visualizing the mappings from the $z = x + y\jmath$ to $w(z) = u + v\jmath$ planes. For example, when $y = 0$, $w(x) = \sin(x)$ seems trivial because it is real. If we let $x = 0$ it is also trivial because

$$\sin(y\jmath) = \frac{e^{-y} - e^{y}}{2\jmath} = -\jmath \sinh(y)$$

is pure imaginary. When $z \in \mathbb{C}$,

$$w(z) = \sin(z) \in \mathbb{C}.$$

When both $z = x + y\jmath \in \mathbb{C}$ and $\sin(x) = u(x, y) + \jmath v(x, y) \in \mathbb{C}$, visualization can become challenging. An even more challenging case is for the real and imaginary parts of the $w(z) = J_0(z)$ Bessel function, namely $u(x, y)$ and $\jmath v(x, y)$. These are easy case of single-valued functions. Much worse yet are the cases of multi-valued functions, which require *branch cuts*. The solution to this visualization problem is solved by the use of *domain-coloring*, as show for example in Fig. 3.11. We shall approach this with specific examples.

Visualizing complex functions: The mapping from $s = \sigma + \omega\jmath$ to $w(s) = u(\sigma, \omega\jmath) + \jmath v(\sigma, \omega\jmath)$ is difficult to visualize because for each point in the domain $s = \sigma + \omega\jmath$, we would like to represent both the magnitude and phase (or real and imaginary parts) of $w(s)$. The best way to visualize these mappings is to use color (hue) to represent the phase, and intensity (dark to light) to represent the magnitude. Fortunately this mapping problem can be solved by adding color to the chart. An Octave/Matlab script is helpful in viewing these relationships[29]

In Fig. 3.11, rather than plotting $u(x, y)$ and $v(x, y)$ separately, domain-coloring allows us to display the entire function on one color chart (i.e., colorized plot). For this visualization we see the complex polar form of $w(s) = |w|e^{\jmath\angle w}$ rather than the 2×2 (four-dimensional) Cartesian graph $w(x + y\jmath) = u(x, y) + v(x, y)\jmath$. On the left is the reference condition, the identity mapping ($w = s$), and on the right the origin has been shifted to the right and up by $\sqrt{2}$.

[28] /home/jba/Mimosa/2C-FindLengths.16/doc.2-c_calib.14/m/MasonKappa.m

[29] https://jontalle.web.engr.illinois.edu/uploads/493/M/

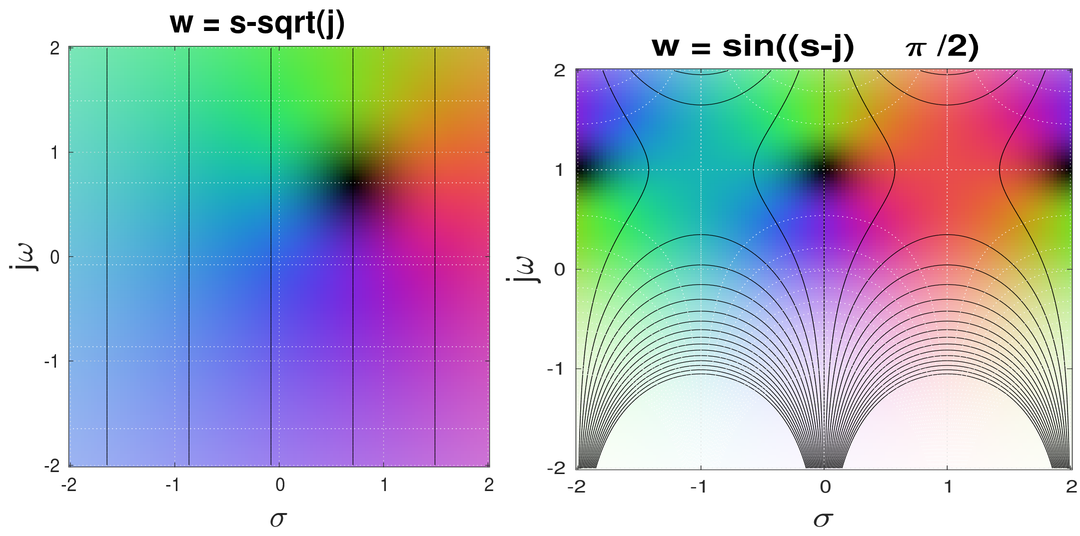

Figure 3.11: *Left: Domain-colorized map showing the complex mapping from the $s = \sigma + \omega\jmath$ plane to the $w(s) = u(\sigma,\omega) + v(\sigma,\omega)\jmath$ plane. This mapping may be visualized by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase) of the mapping, shifted to the right and up by $\sqrt{2}/2 = 0.707$. Right: The $w(s) = sin(s - j)$ plane. The black and white lines are the iso-real and iso-imaginary contours of $u(\sigma,\omega)$ and $v(\sigma,\omega)$.*

Mathematicians typically use the abstract (i.e., nonphysical) notation $w(z)$, where $w = u + v\imath$ and $z = x + y\jmath$. Engineers typically work in terms of a physical complex impedance $Z(s) = R(s) + \jmath X(s)$ that has resistance $R(s)$ and reactance $X(s)$ [ohms], as a function of the complex Laplace radian frequency $s = \sigma + \omega\jmath$ [rad], as used with the Laplace transform (see §3.9.

In Fig. 3.11 we use a mixed notation, with $Z(s) = s$ on the left and $w(s) = s - \sqrt{\jmath}$ on the right, where we show this color code as a 2×2 dimensional domain-coloring graph. Intensity (dark to light) represents the magnitude of the function, while hue (color) represents the phase, where is $0°$, green is $90°$, cyan is $+135°$, and violet is $\pm 90°$.[30]

The function $w = s = |s|e^{\jmath\theta}$ has a dark spot (zero) at $s = 0$ and becomes brighter away from the origin. On the right is $w(s) = s - \sqrt{\jmath}$, which shifts the zero (dark spot) to $s = \sqrt{\jmath}$. Thus domain-coloring gives the full 2×2 complex–analytic function mapping $w(x, y) = u(x, y) + v(x, y)\jmath$ in colorized polar coordinates.

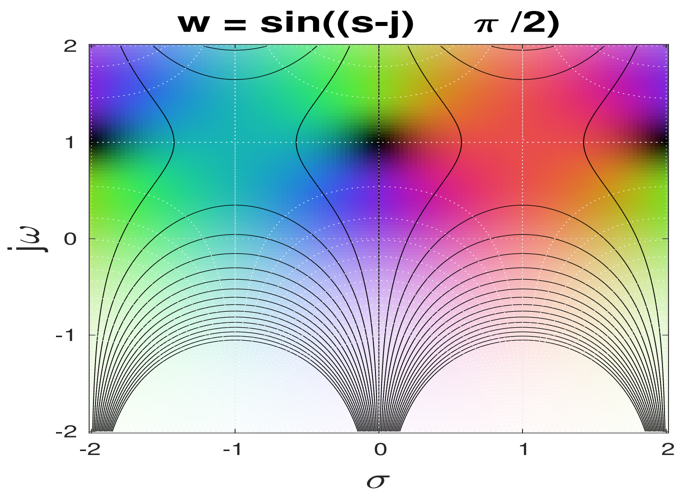

Figure 3.12: **Remove this repeat of above identical figure** *Plot of* $\sin(0.5\pi(z - \jmath))$.

Example: Figure 3.12 shows a colorized plot of $w(z) = \sin(\pi(s - \jmath)/2)$ resulting from the Matlab/Octave command `zviz sin(pi*(s-j)/2)`. The abscissa (horizontal axis) is the real σ-axis and the ordinate (vertical axis) is the complex $\jmath\omega$-axis. The graph is offset along the ordinate axis by $1j$, since the argument $s - \jmath$ causes a shift of the sine function by 1 in the positive imaginary direction.

The visible zeros of $w(s)$ appear as dark regions at $(-2, 1)$, $(0, 1)$, $(2, 1)$. As a function of σ, $w(\sigma + 1\jmath)$ oscillates between red (phase is zero degrees), meaning the function is positive and real, and sea-green (phase is 180°), meaning the function is negative and real.

[30]Hue depends on both the display medium and the eye.

To use the program, we use the syntax (note the period[31] followed by `^2`), namely `zviz s.^2`). This will render a domain-coloring (colorized) version of the function. Examples you can render with `zviz` are given in the comments at the top of the `zviz.m` figure. [32] A good example for testing is `zviz z-sqrt(j)`, which has a dark spot (zero) at $(1+1\jmath)/\sqrt{2} = 0.707(1+1\jmath)$.

Along the vertical axis, the displayed function is either $\cosh(y)$ or $\sinh(y)$, depending on the value of x. The intensity becomes lighter as $|w|$ increases.

What is being plotted? The axes are either $s = \sigma$ and ω, or $z = x$ and y. Superimposed on the s-axis is the function $w(s) = u(\sigma,\omega) + v(\sigma,\omega)\jmath$, represented in polar coordinates by the intensity and color of $w(s)$. The density (dark vs. light) displays the magnitude $|w(s)|$, while the color (hue) displays the angle ($\angle w(s)$) as a function of s. Thus the intensity becomes darker as $|w|$ decreases and lighter as $|w(s)|$ increases. The angle $\angle(w)$ to color map is defined by Fig. 3.11. For example, $0°$ is red, $90°$ is green, $-90°$ is purple, and $180°$ is blue-green.

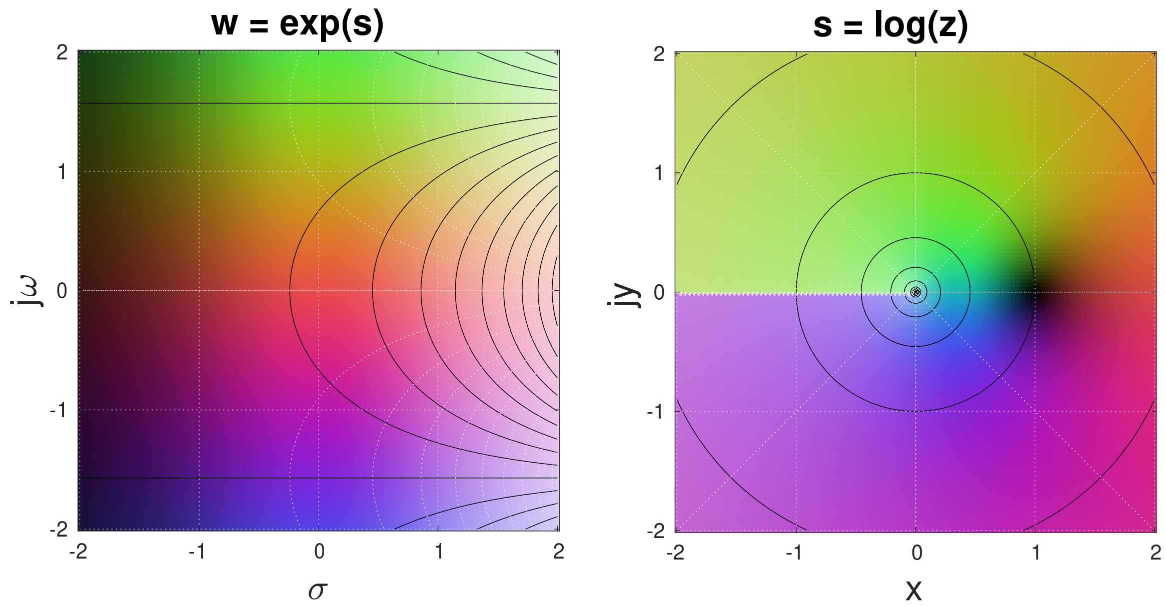

Figure 3.13: *This domain-color map allows us to visualize complex mappings by the use of intensity (light/dark) to indicate magnitude and color (hue) to indicate angle (phase). The white and black lines are the iso-real and iso-imaginary contours of the complex mapping. Left: The domain-color map for the complex mapping from the $s = \sigma + \omega\jmath$ plane to the $w(s) = u + v\jmath = e^{\sigma+\omega\jmath} = e^{\sigma}e^{\omega\jmath}$ plane, which goes to zero as $\sigma \to -\infty$, causing the domain-color map to become dark for $\sigma < -2$. The white and black lines are always perpendicular because e^s is complex–analytic everywhere. Right: The* principal value *of the inverse function $s(z) = u(x,y) + v(x,y)\jmath = \log(z)$, which has a zero (dark) at $x = 1$, since there $\log(1) = 0$ and the imaginary part is zero. Note the branch cut, where the color is discontinuous, from $x = [0, -\infty\jmath)$. On branches other than the one shown, there are no zeros, since the phase ($\angle s = 2\pi n \in \mathbb{Z}$) is not zero.*

Example: Important examples are given in Fig. 3.13 using the notation $w(s) = u(\sigma,\omega) + v(\sigma,\omega)\jmath$, where we show the two complex mappings $w = e^s$ (left) and its inverse $s = \ln(w)$. The exponential is relatively easy to understand, since $w(s) = |e^{\sigma}e^{\omega\jmath}| = e^{\sigma}$.

The red region is where $\omega \approx 0$, in which case $w \approx e^{\sigma}$. As σ becomes large and negative, $w \to 0$; thus the entire field becomes dark on the left. The field is becoming light on the right where $w = e^{\sigma} \to \infty$. If we let $\sigma = 0$ and look along the ω-axis, we see that the function is changing phase: sea-green ($90°$) at the top and violet ($-90°$) at the bottom.

In the right panel note the zero for $s(z) = \ln(z) = \ln|z| + \omega\jmath$ at $z = 1$. The root of the $\log(z)$ function is $\log(z_r) = 0$, $w_r = 1, \angle z = \phi = 0$, since $\log(1) = 0$. More generally, the $\log(z)$ of $z = |z|e^{\phi\jmath}$ is $s(z) = \ln|z| + \phi\jmath$. Thus $s(w)$ can be zero only when the angle of w is zero.

The $\ln(z)$ function has a branch cut along the $\phi(z) = \angle z = 180°$ axis. As one crosses over the cut, the phase goes above $180°$ and the plane changes to the next sheet of the log function. The only sheet with a zero is the principal value, as shown. For all others, the log function is either increasing or decreasing monotonically, and there is no zero.

[31] The period passes the next character to Matlab as an operator.

[32] Matlab/Octave code for ZVIZ.M

3.11.1 The Riemann sphere

Once algebra was formulated, in about 830 CE, mathematicians were able to expand beyond the limits set by geometry on the real plane and the verbose descriptions of each problem in prose (Stillwell, 2010, p. 93). The geometry of Euclid's *Elements* had paved the way, but after 2000 years, the addition of the language of algebra changed everything. The concept of an analytic function was a key development, heavily used by both Newton and Euler. Also Cauchy made important headway with his investigation of complex variables. Of special note were integration and differentiation in the complex plane of complex–analytic functions, which is the topic of Chapter 4.

It was Riemann, working with Gauss in the final years of Gauss's life, who made the breakthrough with the concept of the extended complex plane.[33] This concept was based on the composition of a line with the sphere, similar in concept to the derivation of Euclid's formula for Pythagorean triplets. While the importance of the extended complex plane was unforeseen, it changed analytic mathematics forever, along with the physics it supported. It unified and thus simplified many important integrals to the extreme. The basic idea is captured by the fundamental theorem of complex integral calculus (see Table 4.1).

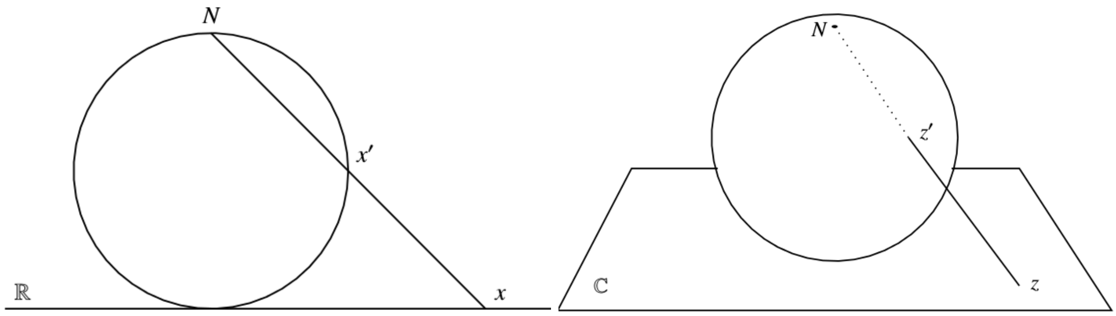

Figure 3.14: *The left panel shows how the real line may be composed with the circle. Each real x value maps to a corresponding point x' on the unit circle. The point $x \to \infty$ maps to the north pole N. This simple idea may be extended with the composition of the complex plane with the unit sphere, thus mapping the plane onto the sphere. As with the circle, the point on the complex plane $z \to \infty$ maps onto the north pole N. This construction is important because, while the plane is open (does not include $z \to \infty$), the sphere is analytic at the north pole. Thus the sphere defines the* closed extended plane. *Figure adapted from Stillwell (2010, pp. 299–300).*

The idea is outlined in Fig. 3.14. On the left is a circle and a line. The difference between this case and the derivation of the Pythagorean triplets is that the line starts at the north pole and ends on the real $x \in \mathbb{R}$ axis at point x. At point x', the line cuts through the circle. Thus the mapping from x to x' takes every point on $\mathbb{R}$ to a point on the circle. For example, the point $x = 0$ maps to the south pole (not indicated). To express x' in terms of x one must compose the line and the circle, similar to the composition used in the derivation of Euclid's formula. The points on the circle, indicated here by x', require a traditional polar coordinate system, with a unit radius and an angle defined between the radius and a vertical line passing through the north pole. When $|x| \to \infty$, the point $x' \to N$, known as the *point at infinity*. But this idea goes much further, as shown on the right of Fig. 3.14.

Here the real tangent line is replaced by a tangent complex plane $z \in \mathbb{C}$ and the complex puncture point $z' \in \mathbb{C}$—in this case on the complex sphere, called the *extended complex plane*. This is a natural extension of the chord/tangent method on the left, but with significant consequences. The main difference between the complex plane z and the extended complex plane, other than the coordinate system, is what happens at the north pole. The point at $|z| = \infty$ is not defined on the plane, whereas on the sphere, the point at the north pole is simply another point, like every other point on the sphere.

Open vs. closed sets: Mathematically the plane is said to be an *open set*, since the limit $z \to \infty$ is not defined, whereas on the sphere, the point z' is a member of a *closed set*, since the north pole *is* defined. The distinction between an open and closed sets is important because the closed set allows the function to be complex–analytic at the north pole, which it cannot be on the plane (since the point at infinity is not defined).

The z plane may be replaced with another tangent plane—say, the $w = F(z) \in \mathbb{C}$ plane, where w is some function F of $z \in \mathbb{C}$. For the moment we shall limit ourselves to complex–analytic functions of z—namely, $w = F(z) = u(x, y) + v(x, y)\jmath = \sum_{n=0}^{\infty} c_n z^n$.

[33] "Gauss did lecture to Riemann but he was only giving elementary courses and there is no evidence that at this time he recognized Riemann's genius." Then "In 1849 he [Riemann] returned to Göttingen and his Ph.D. thesis, supervised by Gauss, was submitted in 1851." Quote from: `https://www-groups.dcs.st-and.ac.uk/~history/Biographies/Riemann.html`.

In summary, given a point $z = x + y\jmath$ on the open complex plane, we map it to $w = F(z) \in \mathbb{C}$, the complex $w = u + v\jmath$ plane, and from there to the closed extended complex plane $w'(z)$. The point of doing this is that it allows the function $w'(z)$ to be analytic at the north pole, meaning it can have a convergent Taylor series at the point at infinity $z \to \infty$.

Since we have not yet defined $dw(z)/dz$, the concept of a complex Taylor series remains undefined.

3.11.2 Bilinear transformation

In mathematics the bilinear transformation has special importance because it is linear in its action on both the input and output variables. Since we are engineers, we shall stick with the engineering terminology. But if you wish to read about this on the internet, be sure to also search for the mathematical term *Möbius transformation*.

When a point on the complex plane $z = x+y\jmath$ is composed with the bilinear transformation $(a, b, c, d \in \mathbb{C})$, the result is $w(z) = u(x, y) + v(x, y)\jmath$ (this is related to the Möbius transformation,

$$w = \frac{az + b}{cz + d}. \tag{3.31}$$

The transformation $z \to w$ is a cascade of four independent compositions:

1. Translation ($w = z + b$: $a = 1, b \in \mathbb{C}, c = 0, d = 1$)
2. Scaling ($w = az$: $a \in \mathbb{R}, b = 0, c = 0, d = 1$)
3. Rotation ($w = az$: $a \in \mathbb{C}, b = 0, c = 0, d = |a|$)
4. Inversion ($w = b/cz$: a=0, b,c$\in \mathbb{C}$, d=0)

Each of these transformations is a special case of Eq. 3.31, with inversion being the most complicated. I highly recommend a video showing the effect of the bilinear (Möbius) transformation on the plane (Arnold, D. and Rogness, J., 2019).[34]

The bilinear transformation is the most general way to move the expansion point in a complex–analytic expansion. For example, when we start from the harmonic series, the bilinear transformation gives

$$\begin{aligned}\frac{1}{1-w} =& \frac{1}{1-\frac{az+b}{cz+d}}\\ =& \frac{cz+d}{(c-a)z+(d-b)}\\ =& \frac{1}{1-\frac{a}{c}}\cdot\frac{z+\frac{d}{c}}{z-\frac{a-b}{c-a}}.\end{aligned} \tag{3.32}$$

The RoC is transformed from $|w| < 1$ to $|(az - b)/(cz - d)| < 1$. An interesting application might be to move the expansion point until it is on top of the nearest pole, so that the RoC goes to zero. This might be a useful way of finding a pole, for example.

When the extended plane (Riemann sphere) is analytic at $z = \infty$, we can take the derivatives there, defining the Taylor series with the expansion point at ∞. When the bilinear transformation rotates the Riemann sphere, the point at infinity is translated to a finite point on the complex plane, revealing the analytic nature at infinity. A second way to transform the point at infinity is by the bilinear transformation $\zeta = 1/z$, mapping a zero (or pole) at $z = \infty$ to a pole (or zero) at $\zeta = 0$. Thus this construction of the Riemann sphere and the Möbius (bilinear) transformation allows us to understand the point at infinity and treat it like any other point. If you felt that you never understood the meaning of the point at ∞ (likely), this should help.

3.12 Probability Analysis

Many things in life follow rules we don't understand, and thus are unpredictable, yet they have structure due to some underlying poorly understood physics (e.g., quantum mechanics). Unlike mathematicians, engineers are taught to deal with uncertainty in terms of random processes using probability theory.

[34] https://www.youtube.com/watch?v=0z1fIsUNhO4

A friend was once told “You’re amazing in how you think outside the box.” He responded “There is no box.”

Some view probability as combinatorics and permutations. In my view probability is much more. Probability is about the signal processing of noise and signals (i.e., not combinatorics), with units of [sec] or [Hz] (Fry, 1928, p. 4) An important goal when working with probability, is to find correlations in observations, such as the relative frequency of observations in sequential observations of events. Hamming (2004) presents an insightful discussion on probability.

1. An *event* is an unpredictable outcome (Papoulis and Pillai, 2002). For example, measuring the temperature $T(\boldsymbol{x}, t) \in \mathbb{R}$ with $\boldsymbol{x} \in \mathbb{R}^3$ at time t [s] is an event. Measuring the temperature every hour gives 24 events per day [degrees/h]. Also, the single toss of a coin, resulting in $\{H, T\}$, is an event.

Exercise #*8*

What are the units of a temperature event?

Solution: Although we might think the answer is degrees, that unit is not the data that are being observed. Rather, the *relative frequency* of temperatures is the observable. For example, how many times was the event between 20° and 21° or between 22° and 27°? Events are dimensionless numbers with no units.

2. A *trial* is N events.

3. An *experiment* $\{M, N\}$ is M trials of N events.

4. We must always keep track of the *number of events* so that we can compute the mean (i.e., average) and the uncertainty of an observable outcome.

5. The *mean* of many trials is the average.

6. A *random variable* X is the outcome from an experiment. A random variable rarely has stated units. For example, flipping a coin $N = 8$ times defines the number of trials.

Exercise #*9*

What are the units of coin flips?

$$X \equiv \{H, H, H, T, H, T, T, T\}$$

Solution: The random variable $\{H, T\}$ has units of [certainty], best measured in terms of *odds*, as the ratio of tails to heads.

Exercise #*10*

How do you identify the meaning of a variable that is dimensionless (has no units)?

Solution: One must get creative. We can let $H = 1$ and $T = -1$ so that the mean can be zero.

Exercise #*11*

What are the mean and standard deviation of the coin toss?

Solution: To compute the mean (or standard deviation) we assign numbers to H and T. For example, we let $H = 1$ and $T = 0$. Then we use the usual formula to compute the numerical values. An important measure is the odds ratio.

7. The *expected value* is the mean of N events.

Exercise #*12*

What is the difference between the mean, the expected value, and the average?

Solution: These terms all mean the same thing. Having several terms that mean the same thing is one of the many things that make probability theory so arbitrary. It is sloppy to have unclear terminology.

Exercise #*13*

How do you assign a numerical mean to random outcomes $\{H, T\}$?

Solution: If we let $H = 1$ and $T = 0$, then the mean is

$$\mu = (1 + 1 + 1 + 0 + 1 + 0 + 1 + 0)/8 = 5/8.$$

The odds are defined as the ratio of P_H/P_T.

It is critically important to keep track of the number of events ($N = 8$ in the Exercise 52). In some sense N is more important than the actual measured sequence. It is helpful to think of N as the independent variable and X as the dependent variable; that is, think of $X(N)$, not $N(X)$.

Example: We define a trial by flipping a coin $N = 10$ times. We form an experiment by M repeated trials ($M = 1000$).

Exercise #*14*

A measure of the quantization in the estimate of the probability density due to the sample size N is defined as the magnitude of *sampling noise*.

Solution: When we compute the average (the mean μ_N) of N samples the error is bounded by $1/N$; thus the variance σ_N^2 from the mean is quantized to $1/N$. It follows that the root-mean-square (RMS) sample error must be bounded by $\sigma_N < \sqrt{2/N}$, independent of frequency (i.e., the $\mathcal{FT}$of the N-sample probability density function).

Two-port network analysis

Problem # 17: *Perform an analysis of electrical two-port networks, shown in Fig. 3.5 (page 114). This can be a mechanical system if the capacitors are taken to be springs and inductors taken as mass, as in the suspension of the wheels of a car. In an acoustical circuit, the low-pass filter could be a car muffler. While the physical representations will be different, the equations and the analysis are exactly the same.*

The definition of the ABCD *transmission matrix* ($\mathcal{T}$) is

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}. \tag{3.33}$$

The *impedance matrix*, where the determinant $\Delta_\mathcal{T} = AD - BC$, is given by

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_\mathcal{T} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{3.34}$$

– 17.1: Derive the formula for the impedance matrix (Eq. 3.34) given the transmission matrix definition (Eq. 3.33). Show your work.

Solution: The formula may be easily derived by re-arranging the equations from the matrix (Eq. 3.34). Begin with

$$V_1 = \mathcal{A}V_2 - \mathcal{B}I_2$$
$$I_1 = \mathcal{C}V_2 - \mathcal{D}I_2$$

From the second equation, we get

$$V_2 = \frac{1}{\mathcal{C}} I_1 + \frac{\mathcal{D}}{C} I_2$$

which gives (upon substitution)

$$V_1 = \frac{\mathcal{A}}{\mathcal{C}} I_1 + \frac{\mathcal{A}\mathcal{D}}{\mathcal{C}} I_2 - \mathcal{B} I_2 = \frac{\mathcal{A}}{\mathcal{C}} I_1 + \left(\frac{\mathcal{A}\mathcal{D}}{\mathcal{C}} - \mathcal{B}\right) I_2$$

which yields the matrix equation

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A}/\mathcal{C} & (\mathcal{A}\mathcal{D}/\mathcal{C} - \mathcal{B}) \\ 1/\mathcal{C} & \mathcal{D}/\mathcal{C} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & \Delta_{\mathcal{T}} \\ 1 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}. \tag{3.35}$$

■

Problem # *18: Consider a single circuit element with impedance $Z(s)$.*

– 18.1: What is the ABCD matrix for this element if it is in series?

Solution:

$$\begin{bmatrix} 1 & Z(s) \\ 0 & 1 \end{bmatrix}$$

■

– 18.2: What is the ABCD matrix for this element if it is in shunt?

Solution:

$$\begin{bmatrix} 1 & 0 \\ 1/Z(s) & 1 \end{bmatrix}$$

■

Problem # *19: Find the ABCD matrix for each of the circuits of Fig. 3.5.*

For each circuit, (i) show the cascade of transmission matrices in terms of the complex frequency $s \in \mathbb{C}$, then (ii) substitute $s = 1\jmath$ and calculate the total transmission matrix at this single frequency.

– 19.1: Left circuit (let $R_1 = R_2 = 10$ kilo-ohms and $C = 10$ nano-farads)

Solution: Write the system in chain matrix form:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y_C & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & R_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

■

– 19.2: Right circuit (use L and C values given in the figure), where the pressure P is analogous to the voltage V, and the velocity U is analogous to the current I.

Solution: Write the system in chain matrix form:

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & sL_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{sL_4} & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix}$$

Now we substitute the given values:

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2j & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{3j} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{4j} & 1 \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3}j \\ \frac{19}{12}j & \frac{5}{3} \end{bmatrix} \begin{bmatrix} P_2 \\ -U_2 \end{bmatrix}$$

I used Matlab/Octave to evaluate this script:

```
a=[1 j;0 1];b=[1 0;2j 1];c=[1 1/3j; 0 1];d=[1 0;1/4j 1]; T=a*b*
```

Finally I found $T(2,1)$ to be 19/12 using the Matlab/Octave command: `rats(1.5833,6)`

■

– 19.3: Convert both transmission (ABCD) matrices to impedance matrices using Eq. 3.34. Do this for the specific frequency $s = 1\jmath$ as in the previous part (feel free to use Matlab/Octave for your computation).

Solution: Left circuit: Using the previous solution, and Matlab:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{\jmath 10^{-8}} \begin{bmatrix} 1+\jmath 10^{-4} & 1 \\ 1 & 1+\jmath 10^{-4} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

■

– 19.4: Right circuit: Repeat the analysis as in question 3.3.

Solution:

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{1.5833\jmath} \begin{bmatrix} -\frac{2}{3} & 1 \\ 1 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

■

Algebra

Problem # *20: Fundamental theorem of algebra (FTA).*

– 20.1: State the fundamental theorem of algebra (FTA).

Solution: There are multiple definitions of the FTA, which of course must be equivalent.
Here are three (equivalent) answers from Wikipedia

1. The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This may then be applied recursively till the degree is zero.
2. Every degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots. The equivalence of the two statements can be proven through the use of successive polynomial division.
3. The field of complex numbers is algebraically closed. Note: this one requires an understanding of the term *algebraically closed*.

Wikipedia warns:

> In spite of its name, there is no purely algebraic proof of the theorem, since any proof must use the completeness of the reals (or some other equivalent formulation of completeness), which is not an algebraic concept.

■

Algebra with complex variables

Problem # *21: Order and complex numbers:*

One can always say that $3 < 4$—namely, that real numbers have order. One way to view this is to take the difference and compare it to zero, as in $4 - 3 > 0$. Here we will explore how complex variables may be ordered. In the following define $\{x, y\} \in \mathbb{R}$ and complex variable $z = x + y\jmath \in \mathbb{C}$.

– 21.1: Explain the meaning of $|z_1| > |z_2|$.

Solution: $|z| = \sqrt{x^2 + y^2}$ is the length of z, so the above expression says that a disk of radius $|z_1|$ contains a second disk of radius $|z_2|$.■

– 21.2: If $x_1, x_2 \in \mathbb{R}$ (are real *numbers), define the meaning of $x_1 > x_2$.*

Solution: This conditions is the same as $x_1 - x_2 > 0$. Order is meaningful on the real line, as a length. ■

– 21.3: Explain the meaning of $z_1 > z_2$.

Solution: It makes no sense to order complex numbers. A complex number has both a length and an angle (it is the same as a vector). The concept of an angle extends the sign of a real number, making order impossible. To show this, place two points on a plane, and ask which is larger than the other. The order of the x and y components, each have order. Thus order cannot be defined. ■

– *21.4: What is the meaning of* $|z_1 + z_2| > 3$?

Solution: Define $z_3 = z_1 + z_2$. Then the problem becomes $|z_3| > 3$, which is a disk of radius $3 > 0$. Thus the solution is all values of $z_1 + z_2$ outside, but not including, a circle of radius 3. ∎

– *21.5: If time were complex, how might the world be different?*

Solution: As best we know, time is real, thus it has the *order* property: the is a past, present and future. If time were complex this would not be the case. Thus if time were complex, the past could be the future.∎

Problem # *22: It is sometimes necessary to consider a function* $w(z) = u + v\jmath$ *in terms of the real functions* $u(x, y)$ *and* $v(x, y)$ *(e.g. separate the real and imaginary parts). Similarly, we can consider the inverse* $z(w) = x + y\jmath$*, where* $x(u, v)$ *and* $y(u, v)$ *are real functions.*

– *22.1: Find* $u(x, y)$ *and* $v(x, y)$ *for* $w(z) = 1/z$.

Solution: Multiply by the complex conjugate $x - y\jmath$

$$w = \frac{1}{x + y\jmath} = \frac{x - y\jmath}{x^2 + y^2}$$

Therefore $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = \frac{-y}{x^2+y^2}$. ∎

Problem # *23: Find* $u(x, y)$ *and* $v(x, y)$ *for* $w(z) = c^z$ *with complex constant* $c \in \mathbb{C}$ *for questions 23.1, 23.2, and 23.3:*

– *23.1:* $c = e$

Solution: Since $u + iv = e^z = e^{x+y\jmath} = e^x(\cos y + \jmath \sin y)$,

$$u = e^x \cos y$$

and

$$v = e^x \sin y.$$

∎

– *23.2:* $c = 1$ *(recall that* $1 = e^{\pm\jmath 2\pi k}$ *for* $k \in \mathbb{Z}$

Solution: From the general formula with $c = 1$

$$1^z = e^{z \log 1}\, e^{\jmath k 2\pi z} = e^0 e^{\jmath k 2\pi z} = e^{-yk2\pi} e^{-\jmath x k 2\pi}$$

where $k \in \mathbb{Z}$ is a signed counting integer. Thus $u = e^{-k2\pi y} \cos k2\pi x$ and $v = e^{-k2\pi y} \sin k2\pi x$. ∎

– *23.3:* $c = \jmath$. Hint: $\jmath = e^{\jmath\pi/2+\jmath 2\pi k}, \quad k \in \mathbb{Z}$.

Solution: $\jmath^\jmath = \left(e^{\jmath\pi/2+\jmath 2\pi m}\right)^\jmath = e^{\jmath\jmath\pi/2+\jmath\jmath 2\pi m} = e^{-\pi/2}\, e^{-2\pi m} = 0.2079\, e^{-2\pi m}$.

Thus for $m = 0$, $\jmath^z = \left(e^{\jmath\pi/2}\right)^z = e^{\jmath z\pi/2} = e^{\jmath(x+\jmath y)\pi/2} = e^{(\jmath x - y)\pi/2} = e^{-\pi y/2}(\cos(x\pi/2) + \jmath \sin(x\pi/2))$. ∎

– *23.4: What is* $\jmath^\jmath$?

Solution: Since $\jmath = e^{\frac{\pi}{2}\jmath}$, then $\jmath^\jmath = e^{\frac{\pi}{2}\jmath\jmath} = e^{-\pi/2} \approx 0.20788$.

Expanding this in a continued-fraction expansion using Matlab's `rat(exp(-pi/2))` function gives $[0; 5, -5, -4, 3, -3, 3, \cdots]$. ∎

Schwarz inequality

Problem # *24: The above figure shows three vectors for an arbitrary value of* $\alpha \in \mathbb{R}$ *and a specific value of* $\alpha = \alpha^*$.

– 24.1: Find the value of $\alpha \in \mathbb{R}$ such that the length (norm) of $\vec{E}$ (i.e., $||\vec{E}|| \geq 0$) is minimum. Show your derivation, not the answer ($\alpha = \alpha^*$).

Solution: In the above figure we see vectors V, U, and for reference, $V + \alpha^* U$. Also shown are scaled values of U, αU and $\alpha^* U$. The setup for the derivation is

$$||\boldsymbol{E}(a)||^2 = \boldsymbol{E} \cdot \boldsymbol{E} = (\vec{V} + \alpha\vec{U}) \cdot (\vec{V} + \alpha\vec{U}) \geq 0. \tag{3.36}$$

Minimize with respect to α.

When U is scaled by α^*, length $||E(\alpha^*)||$ is minimum, and $(V - \alpha^* U) \perp U$, namely vector $E(\alpha^*)$ is $\perp$ to vector U. This follows from $\frac{\partial}{\partial\alpha}||\vec{E}||^2 = \frac{\partial}{\partial\alpha}((\vec{V} + \alpha\vec{U}) \cdot (\vec{V} + \alpha\vec{U})) = 2(\vec{V} + \alpha\vec{U}) \cdot \vec{U} = 0$. Thus

$$\alpha^* = -\frac{\vec{V} \cdot \vec{U}}{||\vec{U}||^2}$$

■

– 24.2: Find the formula for $||\boldsymbol{E}(\alpha^*)||^2 \geq 0$. Hint: Substitute α^* into Eq. 3.49 (p. 93) and show that this results in the Schwarz inequality

$$|\vec{U} \cdot \vec{V}| \leq ||\vec{U}||||\vec{V}||.$$

Solution: From Eq. 3.49

$$||\boldsymbol{V}||^2 + 2\alpha^* \boldsymbol{V} \cdot \boldsymbol{U} + (\alpha^*)^2 \; ||\boldsymbol{U}||^2 \geq 0$$

Substituting α^* gives

$$||\boldsymbol{V}||^2||\boldsymbol{U}||^2 - 2(\boldsymbol{V} \cdot \boldsymbol{U})^2 + |\boldsymbol{U} \cdot \boldsymbol{V}|^2 \geq 0.$$

Simplifying

$$||\boldsymbol{V}||^2||\boldsymbol{U}||^2 \geq |\boldsymbol{U} \cdot \boldsymbol{V}|^2$$

and taking the square root (and swap order), gives the *Schwarz inequality*

$$|\vec{U} \cdot \vec{V}| \leq ||\vec{U}||||\vec{V}||.$$

■

Problem # *25: Geometry and scaler products*

– 25.1: What is the geometrical meaning of the dot product of two vectors?

Solution: The dot product of two vectors is the length of the $\perp$ projection of one vector on the other. According to the Schwarz inequality, this project length must be less than the product of the lengths of the two vectors. ■

– 25.2: Give the formula for the dot product of two vectors. Explain the meaning based on Fig. 9 (page 108).

Solution: $\vec{V} \cdot \vec{U} = ||\vec{V}||||\vec{U}|| \cos\theta_{\vec{V},\vec{U}}$. $\vec{V} \cdot \vec{U} = ||\vec{V}||||\vec{U}|| \cos\theta_{\vec{V},\vec{U}}$. It represents the amount of one vector going in the direction of the other. In a drawing, it is a projection of the one on the other, found by dropping the $\perp$ from the tip of one, on the other.■

– 25.3: Write the formula for the dot product of two vectors $\vec{U} \cdot \vec{V}$ in $\mathbb{R}^n$ in polar form (e.g., assume the angle between the vectors is θ).

Solution: $\vec{U} \cdot \vec{V} = \sum_{i=1}^{n} a_i b_i (= ||\vec{U}|| \; ||\vec{V}|| \cos(\theta))$. This last relationship defines the angle between two vectors. ■

– 25.4: How is the Schwarz inequality related to the Pythagorean theorem?

Solution: It says that for a right triangle, the case when $a = a^*$, the lengths of the two vectors must be greater than the projection of one on the other, unless they are co-linear (i.e., the angle between them is zero).■

– *25.5: Starting from* $||\boldsymbol{U}+\boldsymbol{V}||$, *derive the* triangle inequality

$$||\vec{U}+\vec{V}|| \le ||\vec{U}|| + ||\vec{V}||.$$

Solution: $||\vec{U}+\vec{V}||^2 = (\vec{U}+\vec{V})\cdot(\vec{U}+\vec{V}) = ||\boldsymbol{U}||^2 + ||\boldsymbol{V}||^2 + 2\boldsymbol{U}\cdot\boldsymbol{V} \le ||\boldsymbol{U}||^2 + ||\boldsymbol{V}||^2 + 2|\boldsymbol{U}\cdot\boldsymbol{V}|$ Using the Schwarz inequality we find $||\vec{U}+\vec{V}||^2 \le ||\boldsymbol{U}||^2 + ||\boldsymbol{V}||^2 + 2||\boldsymbol{U}||\ ||\boldsymbol{V}||$. Completing the square on the right gives $||\vec{U}+\vec{V}||^2 \le (||\boldsymbol{U}|| + ||\boldsymbol{V}||)^2$. Final taking the square root gives the *triangle inequality*. ■

– *25.6: The* triangle inequality $||\vec{U}+\vec{V}|| \le ||\vec{U}|| + ||\vec{V}||$ *is true for two and three dimensions: Does it hold for five-dimensional vectors?*

Solution: It is true in any number of dimensions. ■

– *25.7: Show that the wedge product* $\vec{U}\wedge\vec{V} \perp \vec{U}\cdot\vec{V}$.

Solution: $\vec{V}\wedge\vec{U} = ||\vec{V}||||\vec{U}||\sin\theta_{\vec{V},\vec{U}}$ while $\vec{V}\cdot\vec{U} = ||\vec{V}||||\vec{U}||\cos\theta_{\vec{V},\vec{U}}$. Thus they are perpendicular. This is true in any number of dimensions. See the discussion in the text on the wedge product.■

3.13 System Classification

Problem # *26: Complete this system classification problem about physical systems using the system postulates.*

– *26.1: Provide a brief definition of these classificatioms* L/NL : linear (L)/nonlinear (NL) **Solution:** Superposition and scaling hold■

TI/TV : time-invariant (TI)/time-varying (TV) **Solution:** The measurement time is irrelevant■

P/A : passive (P)/active (A) **Solution:** An active system has a power source, a passive system does not.■

C/NC : causal (C)/noncausal (NC) **Solution:** Responds only when driven for $t \ge 0$. Does not anticipate for negative time.■

Re/Clx : real (Re)/complex (Clx) **Solution:** The time function is real (or complex).■

– *26.2: Along the rows of the table, classify each system using the abbreviations L/NL, TI/TV, P/A, C/NC, and Re/Clx:*

					Category	
#	Case	Definition	L/NL	TI/TV	P/A	C/NC
1	Resistor	$v(t) = r_0\, i(t)$	**Solution:** L■	**Solution:** TI■	**Solution:** P■	**Solution**
2	Inductor	$v(t) = L\frac{di}{dt}$	**Solution:** L■	**Solution:** TI■	**Solution:** P■	**Solution**
3	Switch	$v(t) \equiv \begin{cases} 0 & t \le 0 \\ V_0 & t > 0. \end{cases}$	**Solution:** L■	**Solution:** TV■	**Solution:** P■	**Solution**
5	Transistor	$I_{out} = g_m(V_{in})$	**Solution:** NL■	**Solution:** TI■	**Solution:** P■	**Solution**
7	Resistor	$v(t) = r_0\, i(t+3)$	**Solution:** L■	**Solution:** TI■	**Solution:** P■	**Solution:**
8	Modulator	$f(t) = e^{i2\pi t}g(t)$	**Solution:** L■	**Solution:** TV■	**Solution:** P■	**Solution**

Solution: Notes:

1. is a nonlinear system and is active system only when it is connected to a battery, similar to a diode.
2. The current is non-causal since it has a 3 [s] *negative time delay*, specified in the time domain.
3. is 1 Hz complex-modulation, so it is both complex and time-varying (TV)

■

– *26.3: Classify each equation:*

#	Case:	L/NL	TI/TV	P/A	C/NC
1	$A(x)\frac{d^2y(t)}{dt^2} + D(t)y(x,t) = 0$	**Solution: L**■	**Solution: TV**■	**Solution: P**■	**Solution:**
2	$\frac{dy(t)}{dt} + \sqrt{t}\, y(t) = \sin(t)$	**Solution: L**■	**Solution: TV**■	**Solution: ?**■	**Solution**
3	$y^2(t) + y(t) = \sin(t)$	**Solution: NL**■	**Solution: TI**■	**Solution: ?**■	**Solution**
4	$\frac{\partial^2 y}{\partial t^2} + xy(t+1) + x^2 y = 0$	**Solution: L**■	**Solution: TI**■	**Solution: P**■	**Solution:**
5	$\frac{dy(t)}{dt} + (t-1)\, y^2(t) = ie^t$	**Solution: NL**■	**Solution: TV**■	**Solution: A?**■	**Solution**

AE-3: Probability

Problem # *27: Basic terminology of experiments*

– *27.1: What is the* standard deviation *about the mean?*
Solution: This is the expected value of the second moment of the random variable.

– *27.2: What is the definition of* information *of a random variable?*
Solution: The *information* is $\mathcal{I} = 1/P(X_k)$.

– *27.3: How do you combine events? Hint: If the event is the flip of a biased coin, the events are $H = p$, $T = 1 - p$, so the event is $\{p, 1 - p\}$. To solve the problem, you must find the probabilities of two independent events.*
Solution: If one event has probability p it may be captured by a vector $[p, 1-p]^T$. Two uncorrelated (independent) events then have probability $[p, 1-p] \star [p, 1-p] = [p^2, 2p(1-p), (1-p^2)]$. Here $\star$ represents convolution (Section 5, p. 123). Three events have four outcomes $[p, 1-p] \star [p, 1-p] \star [p, -1-p]$. Pascal's triangle is a related structure defined by recursive convolutions of $[1, 1]$, assuming $p = 1/2$.

– *27.4: What does the term* independent *mean in the context of question 27.3? Give an example.*
Solution: This term means that one event (flip of a coin) has no influence on the next (or any other flip) of that same coin. An example of non-independent events might be that upon flipping the coin, it bent. thus changing the probability for any following flips.

– *27.5: Define the* odds *ratio.*
Solution: The *odds* are the ratio of the two outcomes. Namely the odds are $p/(1-p)$, or equivalently $(1-p)/p = 1/p - 1$.

Chapter 4

Stream 3A: Scalar Calculus

Calculus and Differential Equations: Stream 3

Stream 3 is ∞, a concept that typically means unbounded (immeasurably large), but in the case of calculus, ∞ means infinitesimal (immeasurably small), since taking a limit requires small numbers. Taking a limit means you may never reach the target, a concept that the Greeks called *Zeno's paradox* (Stillwell, 2010, p. 76).

When we speak of the class of *ordinary* (versus *vector*) differential equations, the term *scalar* is preferable, since the term ordinary is vague, if not meaningless. For scalar calculus, a special subset of fundamental theorems about integration are summarized in the Table below.

Table 4.1: *The fundamental theorems of integral calculus, each of which deals with integration. At least two main theorems relate to scalar calculus, and three more to vector calculus.*

Name	Mapping	p.	Description
Leibniz (FTC)	$\mathbb{R}^1 \to \mathbb{R}^0$		Area under a real curve
Cauchy (FTCC)	$\mathbb{C}^1 \to \mathbb{R}^0$		Area under a complex curve
Cauchy's theorem	$\mathbb{C}^1 \to \mathbb{C}^0$		Close integral over analytic region is zero
Cauchy's integral formula	$\mathbb{C}^1 \to \mathbb{C}^0$		The fundamental theorem of complex calculus
Residue theorem	$\mathbb{C}^1 \to \mathbb{C}^0$		Residue integration
Helmholtz' decomposition	S.6.87		A Cornerstone of Mathematical Physics

Following our discussion of the integral theorems on scalar calculus are those on vector calculus, without which there can be no understanding of Maxwell's equations. Of these, the fundamental theorem of vector calculus (also known as Helmholtz' decomposition), Gauss's law, and Stokes's theorem form the three cornerstones of modern vector field analysis. These theorems allow us to connect the differential (point) and macroscopic (integral) relationships. For example, we can write Maxwell's equations either as vector differential equations, as shown by Heaviside (along with Gibbs and Hertz), or in integral form. It is helpful to place these two forms side by side to fully appreciate their significance. To understand the differential (microscopic) view, one must fully understand the integral (macroscopic) view.

4.1 The beginning of modern mathematics

As shown in Fig. 1.2, mathematics as we know it today began in the 16th to 18th centuries, arguably starting with Galileo, Descartes, Fermat, Newton, the Bernoulli family, and most important Euler. Galileo was formidable because of his fame, fortune, and "successful" stance against the powerful Catholic establishment. His creativity in scientific circles was certainly well known due to his many skills and accomplishments. Descartes and Fermat were at the forefront of merging algebra and geometry. While Fermat kept meticulous notebooks, he did not publish and tended to be secretive. Thus Descartes's contributions were more widely acknowledged, though not necessarily deeper.

Regarding the development of calculus, much was yet to be developed by Newton and Leibniz using term-by-term integration of functions based on Taylor series representation. This was a powerful technique but, as stated earlier, incomplete because the Taylor series can represent only single-valued functions within the RoC. More important, Newton (and others) failed to recognize (i.e., rejected) the powerful generalization to complex-analytic functions. The first major breakthrough was Newton's publication of *Principia* (1687), and the second was by Riemann (1851), advised by Gauss but possibly more influenced by Cauchy.

Following both Galileo's and Newton's lead, the secretive and introverted behavior of the typical mathematician dramatically changed with the Bernoulli family (Fig. 1.5). The oldest brother Jacob taught his much younger brother Johann, who then taught his son Daniel. But Johann's star pupil was Leonhard Euler. Euler first mastered all the tools and then published with a prolifically previously unknown.

Euler and the circular functions: Euler's first major task was to understand the family of analytic circular functions ($e^{j\omega}, \sin(\omega), \cos(\omega)$, and $\ln(z\jmath)$), a task begun by the Bernoulli family. Euler sought relationships among these many functions, some of which may not be thought of as being related, such as the log and sin functions. The connection that may "easily" be made is through their complex–analytic representation (Eq. 3.14). Using analytic function manipulations,

$$e^{j\omega} = \cos(\omega) + \jmath \sin(\omega) \tag{4.1}$$

and its analytic inverse. Let $z = e^{\jmath\omega}$, and solve for $z(\omega)$.

$$\tan^{-1}(z) = \frac{1}{2\jmath} \ln\left(\frac{1\jmath - z}{1\jmath + z}\right) = \frac{\jmath}{2} \ln\left(\frac{1 - z\jmath}{1 + z\jmath}\right) \tag{4.2}$$

were soon discovered (Greenberg, 1988, p. 1135).

Exercise #*1*

Starting from Eq. 4.1, derive Eq. 4.2.

Solution: We let $z(\omega) = \tan\omega$; then

$$z(\omega) = \tan(\omega) = \frac{\sin\omega}{\cos\omega} = -\jmath\frac{e^{\omega\jmath} - e^{-\omega\jmath}}{e^{\omega\jmath} + e^{-\omega\jmath}} = -\jmath\frac{e^{2\omega\jmath} - 1}{e^{2\omega\jmath} + 1}. \tag{4.3}$$

Solving for $z = e^{-2\omega\jmath}$, we get

$$e^{-2\omega\jmath} = \frac{1 - z\jmath}{1 + z\jmath}. \tag{4.4}$$

Taking $\ln()$ of both sides and using the definition of $z(\omega)$ give Eq. 4.2:

$$\omega = \tan^{-1}(z) = \frac{\jmath}{2} \ln \frac{1 - z\jmath}{1 + z\jmath}.$$

These equations are the basis of transmission lines (TL), and used throughout mathematical physics. They are related to the z (bilinear) transform, and Newton's method for finding roots of complex–analytic functions (Allen, 2025). Here $z(\omega)$ of Eq. 4.3 is the TL's input impedance and Eq. 4.4 is defined as the *reflectance*. The inverse $\mathcal{LT}$of $\exp(-2\omega\jmath)$ is a delay $f(t) = \delta(t - 2T_o)$, where $2T_o$ is the round trip delay.

Although many high school students memorize Euler's relationship, it seems unlikely they appreciate the utility of complex–analytic functions. This possibility could change with a key teaching innovation.[1] While it true that Matlab is more powerful for certain tasks, I prefer Octave over Matlab because it is *open source*, meaning it's a free download. Try your code on both platforms before you switch, and read the license.

History of complex–analytic functions: Newton (ca.1650) famously ignored imaginary numbers and called them imaginary in a disparaging (pejorative) way. Given Newton's prominence, his view must have keenly attenuated interest in complex algebra, even though it had been described by Bombelli in 1526, likely based on his serendipitous finding of Diophantus's book *Arithmetic* in the Vatican library.

According to mathematical evidence, Euler did not appreciate the role of complex–analytic functions. They were first fully appreciated well after his death in 1785, by Augustin-Louis Cauchy (1789–1857), and then extended by Riemann in 1851. Perhaps it should be called the Euler-Cauchy equation. But its been too long to start renaming historical observations.

[1] Google OVERLEAF.COM, then learn how to use its tools.

Euler derived his relationships using real power series (i.e., real–analytic functions). While Euler was fluent with $\jmath = \sqrt{-1}$, he did not consider functions to be complex–analytic. That concept was first explored by Cauchy almost a century later. The missing link to the concept of complex–analytic functions is the definition of the derivative with respect to the complex argument

$$F'(s) = \frac{dF(s)}{ds}, \tag{4.5}$$

where $s = \sigma + \omega\jmath$, without which the complex–analytic Taylor coefficients are not defined.

4.2 Fundamental Theorems of Scalar Calculus

History of scalar calculus: It some sense the story of calculus begins with the *fundamental theorem of calculus* (Table 4.1), also known generically as *Leibniz's formula*. The simplest integral is the length of a line $L = \int_0^L dx$. If we label a point on a line as $x = 0$ and wish to measure the distance to any other point x, we form the line integral between the two points. If the line is straight, this integral is the Euclidean length given by the difference between the two ends (Eq. 3.35.

If $F(\chi) \in \mathbb{R}$ describes a height above the line $\chi \in \mathbb{R}$, then

$$f(x) - f(0) = \int_{x=0}^{x} F(\chi)d\chi \tag{4.6}$$

may be viewed as the *anti-derivative* of $F(\chi)$. Here χ is a dummy variable of integration. Thus the area under $F(\chi)$ depends on only the difference in the area evaluated at the end points.

This property of the area as an integral over an interval, depending on only the end points, has important consequences in physics in terms of conservation of energy, allowing for important generalizations. For example, as long as $\chi \in \mathbb{R}$, we can let $F(\chi) \in \mathbb{C}$ with no loss of generality, due to the linear Postulate P1 of the integral.

4.2.1 The fundamental theorem of real calculus

If $f(x)$ is analytic (Eq. 3.13, then

$$F(x) = \frac{d}{dx}f(x) \tag{4.7}$$

is an exact real differential. It follows that $F(x)$ is real–analytic. This is known as the *fundamental theorem of (real) calculus* (FTC). Thus Eq. 4.7 may be viewed as an exact real differential. This is easily shown by evaluating

$$\frac{d}{dx}f(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} = F(x),$$

starting from the anti-derivative, Eq. 4.6. If $f(x)$ is not analytic then the limit may not exist, so this is a necessary condition.

There are many important variations on this very basic theorem (see Table 4.1). For example, the limits could depend on time $t \in \mathbb{R}$. Also when we take Fourier transforms, the integrand depends on both time $t \in \mathbb{R}$ and frequency $\omega \in \mathbb{R}$ via a complex exponential "kernel" function $e^{\pm\jmath\omega t} \in \mathbb{C}$, which is analytic in both t and ω.

4.2.2 The fundamental theorem of complex calculus

The *fundamental theorem of complex calculus* (FTCC) states (Greenberg, 1988, p. 1197) that for any complex–analytic function $F(s) \in \mathbb{C}$ with $s = \sigma + \omega\jmath \in \mathbb{C}$,

$$f(s) - f(s_o) = \int_{s_o}^{s} F(\zeta)d\zeta. \tag{4.8}$$

Note that the integral is independent of the path since the complex path can change as long as the end points remain the same. A physical example is the amount of energy consumed is independent of the path if you move a mass from point $x \in \mathbb{R}$ to a second point x_o. The amount of energy consumed (think the total power) is independent of how you move the mass.

Equations 4.6 and 4.8 differ due to the integration path, since $F(s)$ is complex–analytic. Thus the line integral is over $s \in \mathbb{C}$ rather than a real integral over $\chi \in \mathbb{R}$. The FTCC states that the integral depends on only the end points, since

$$F(s) = \frac{d}{ds} f(s). \tag{4.9}$$

Comparing exact differentials, Eq. 4.5 (FTCC) and Eq. 3.7.3 (FTC), we see that $f(s) \in \mathbb{C}$ must be complex–analytic and have a Taylor series in powers in $s \in \mathbb{C}$.

Complex–analytic functions: The definition of a *complex–analytic function* $F(s)$ of $s \in \mathbb{C}$ is that the function may be expanded in a Taylor series about an expansion point $s_o \in \mathbb{C}$. Thus we need better understanding (e.g., definition) which follows the same logic as the FTC. This definition leads back to the coefficients $c_n \in \mathbb{C}$, which most naturally follow from Taylor's formula

$$c_n = \frac{1}{n!} \frac{d^n}{ds^n} F(s) \bigg|_{s=s_o}. \tag{4.10}$$

The requirement that $F(s)$ have a Taylor series naturally follows by taking derivatives with respect to s at s_o. The problem is that both integration and differentiation of functions of complex Laplace frequency $s = \sigma + \omega \jmath$ have not yet been defined.

Thus the question: What does it mean to take the derivative of a function $F(s) \in \mathbb{C}$, $s = \sigma + \omega \jmath \in \mathbb{C}$, with respect to s, where s defines a plane rather than a real line? We learned how to form the derivative on the real line. Can the same derivative concept be extended to the complex plane?

The answer is affirmative. The question may be resolved by applying the rules of the real derivative when defining the derivative in the complex plane. However, for the complex case, there is an issue regarding direction. Given any analytic function $F(s)$, is the partial derivative with respect to σ different from the partial derivative with respect to $\omega\jmath$? For complex–analytic functions, the FTCC states that the integral is independent of the path in the s plane. Based on the chain rule, the derivative must also be independent of the direction at s_o. This directly follows from the FTCC. If the integral of a function of a complex variable is to be independent of the path, then the derivative of a function with respect to a complex variable must be independent of the direction. This follows from Taylor's formula for the coefficients of the complex–analytic formula (Eq. 4.10).

4.2.3 Cauchy-Riemann conditions

The Cauchy-Riemann conditions (CRC) should be viewed as the corner-stone of the complex-analytic function. While they were first introduced by Euler, their importance too some years to be appreciated. Such a delay leads naturally lead to confusion. This confusion led to confusion as to who did what, and when. This propriety of a function of a complex variable, such as the Laplace Frequency s,

The FTC defines the area as an integral over a real differential ($dx \in \mathbb{R}$), while the FTCC relates an integral over a complex function $F(s) \in \mathbb{C}$ along a complex interval (i.e., path) ($ds \in \mathbb{C}$). For the FTC the area under the curve depends on only the end points of the anti-derivative $f(x)$. But what is the meaning of an "area" along a complex path? The Cauchy-Riemann conditions provide the answer.

For the integral of $Z(s) = R(\sigma, \omega) + \jmath X(\sigma, \omega)$ to be independent of the path, the derivative of $Z(s)$ must also be independent of the path. This requirement leads to a pair of equations known as the *Cauchy-Riemann conditions*, described next.

To define

$$\frac{d}{ds} Z(s) = \frac{d}{ds} \left[R(\sigma, \omega) + \jmath X(\sigma, \omega) \right],$$

we take partial derivatives of $Z(s)$ with respect to σ and $\jmath\omega$, and equate them:

$$\frac{\partial Z}{\partial \sigma} = \frac{\partial R}{\partial \sigma} + \jmath \frac{\partial X}{\partial \sigma} \qquad \equiv \qquad \frac{\partial Z}{\partial \jmath\omega} = \frac{\partial R}{\partial \jmath\omega} + \jmath \frac{\partial X}{\partial \jmath\omega}.$$

This says that a horizontal derivative, with respect to σ, is equivalent to a vertical derivative, with respect to

$\omega\jmath$. Taking the real and imaginary parts gives the two equations

$$\frac{\partial R(\sigma,\omega)}{\partial \sigma} = \jmath\frac{\partial X(\sigma,\omega)}{\partial \omega\jmath} \quad \text{(CR-1)} \quad \text{and} \quad \frac{\partial R(\sigma,\omega)}{\partial \omega\jmath} = -\jmath\frac{\partial X(\sigma,\omega)}{\partial \sigma} \quad \text{(CR-2)}, \tag{4.11}$$

known as the *Cauchy-Riemann (CR) conditions*. The $\jmath$ cancels in CR-1 but introduces a $\jmath^2 = -1$ in CR-2. They may also be written in polar coordinates ($s = re^{\theta\jmath}$) as

$$\frac{\partial R}{\partial r} = \frac{1}{r}\frac{\partial X}{\partial \theta} \qquad \text{and} \qquad \frac{\partial X}{\partial r} = -\frac{1}{r}\frac{\partial R}{\partial \theta}.$$

The FTCC follows from CR-1 and CR-2.

You may wonder what would happen if we took a derivative along a line at a 45°angle. To do this we only need to multiply the function by $e^{\jmath\pi/4}$. Due to the CR symmetry, this will not change the derivative. Thus we may take the derivative in any direction by multiplying by $e^{\theta\jmath}$, and the CR conditions will not change. I view this as a symmetry condition. Since this symmetry is not obvious, it is likely a source of confusion.

The CR conditions are necessary so that the integral of $Z(s)$, and thus its derivative, is independent of the path, expressed in terms of conditions on the real and imaginary parts of Z. This is a very strong condition on $Z(s)$, which follows assuming that $Z(s)$ may be written as a Taylor series in s

$$Z(s) = Z_0 + Z_1 s + \frac{1}{2}Z_2 s^2 + \cdots, \tag{4.12}$$

where $Z_n \in \mathbb{C}$ are complex constants given by the Taylor series formula (eq. (4.10)). As with the $\mathbb{R}$ (i.e., real) Taylor series, there is the convergence condition that $|s| < 1$, called the *radius of convergence*. Thus the RoC is an important generalization of the region (vs. radius) of convergence.

In summary, every function that may be expressed as a series in $s - s_o$ about point $s_o \in \mathbb{C}$ is defined as *complex–analytic* at s_o. This series is single-valued, and converges within the RoC. This highly restrictive condition has significant physical consequences. For example, every impedance function $Z(s)$ obeys the CR conditions over large regions of the s plane, including the entire *right half s plane* (RHP) ($\sigma > 0$). This condition is summarized by the Brune condition $\Re\{Z(\sigma > 0)\} \geq 0$, or alternatively $\angle Z(s) < \angle s$

It is important to master the above discussion as we explore deeper, with functions having many RoC's, around multiple s_o values, each known as a *pole*. These points are also know as singularities or singular points.

There are three basic types of singularities, poles, zeros and branch cuts. All mature functional engineers understand the concepts. When the CR conditions are generalized to volume integrals, this is called either Gauss's Law or Green's theorem, which is used in the solution of boundary value problems in engineering and physics (Kusse and Westwig, 2010).

Cauchy-Riemann conditions: We may abstract these ideas into a pair of second-order equations by taking a second round of partials. Specifically, eliminating the real part $R(\sigma,\omega)$ of Eq. 4.11 gives

$$\frac{\partial^2 R(\sigma,\omega)}{\partial\sigma\partial\omega} = \frac{\partial^2 X(\sigma,\omega)}{\partial^2\omega} = -\frac{\partial^2 X(\sigma,\omega)}{\partial^2\sigma}, \quad \text{(CR-3)} \tag{4.13}$$

which may be written compactly as $\nabla^2 X(\sigma,\omega) = 0$. Eliminating the imaginary part gives

$$\frac{\partial^2 X(\sigma,\omega)}{\partial\omega\partial\sigma} = \frac{\partial^2 R(\sigma,\omega)}{\partial^2\sigma} = -\frac{\partial^2 R(\sigma,\omega)}{\partial^2\omega}, \quad \text{(CR-4)} \tag{4.14}$$

which may be written as $\nabla^2 R(\sigma,\omega) = 0$. The symbol ∇ is called *nabla* while ∇^2 is called the *Laplacian*.

In summary, for a function $Z(s)$ to be complex–analytic, the derivative dZ/ds must be independent of direction (path), which requires that the real and imaginary parts of the function obey Laplace's equation; that is,

$$\nabla^2 R(\sigma,\omega) = 0 \quad \text{and} \quad \nabla^2 X(\sigma,\omega) = 0. \tag{4.15}$$

Equations CR-1 and CR-2 are easy to work with because they are first-order, but the intuition behind them best follows from the properties of Laplace's equation (Eq. 4.15). Note three important facts:

1. The Laplacian does not hold at singular points.

2. The derivative of a complex–analytic function is independent of direction.

3. The real and imaginary parts of the function obey Laplace's equation.

Such relationships are known as *harmonic functions*. Another arcane term is the *Möbius transformation*, which we shall not need in this book. Only a few terms have survived the trip from the past. Anyone using these terms is likely trying to impress you. Soon we will drop the historical (arcane) terms and step back to minimal terminology. As we shall see in the next few sections, complex–analytic functions must be smooth, since every analytic function may be differentiated an infinite number of times (the complex Taylor series) within the RoC. The magnitude must attain its maximum and minimum on the boundary. For example, when you stretch a rubber sheet over a jagged frame, the height of the rubber sheet obeys Laplace's equation. Nowhere can the height of the sheet rise above or below its value at the boundary.

Harmonic functions define *conservative fields*, which means that energy (like a volume or area) is conserved. The work done in moving a mass from a to b in such a field is conserved. If you return the mass from b back to a, the stored energy is retrieved, thus zero net work is done. This requires no losses or damping (zero viscosity).

4.2.4 Complex Power Series

Problem # *28: In each case derive (e.g., using Taylor's formula) the power series of $w(s)$ about $s = 0$ and give the RoC of your series. If the power series doesn't exist, state why!* Hint: In some cases, you can derive the series by relating the function to another function for which you already know the power series at $s = 0$.

– *28.1:* $1/(1-s)$
Solution: $1/(1-s) = \sum_{n=0}^{\infty} s^n$, which converges for $|s| < 1$ (e.g., the RoC is $|s| < 1$)■

– *28.2:* $1/(1-s^2)$
Solution: $1/(1-s^2) = \sum_{n=0}^{\infty} s^{2n}$, which converges for $|s^2| < 1$. (e.g., the RoC is $|s| < 1$). One can also factor the polynomial, thus write it as: $\frac{1}{(1-s)(1+s)}$. There are two poles, at $s = \pm 1$, and each has an RoC of 1.■

– *28.3:* $1/(1+s^2)$.
Solution: The resulting series is $1/(1+s^2) = 0.5\sum_{n=0}^{\infty} s^n((-i)^n + (i)^n)$. The RoC is $|s| < 1$. We can see this by considering the poles of the function at $s = \pm i$; both poles are 1 from $s = 0$, the point of expansion. An alternative is to write the function as $1/(1-(is)^2) = \sum (is)^n$. ■

– *28.4:* $1/s$
Solution: If you try to do a Taylor expansion at $s = 0$, the first term, $w(0) \to \infty$. Thus, the Taylor series expansion in s does not exist. ■

– *28.5:* $1/(1-|s|^2)$
Solution: The imaginary part is zero. Thus the derivative of the imaginary part is zero. Thus the CR conditions cannot be obeyed. ■

Problem # *29: Consider the function $w(s) = 1/s$*

– *29.1: Expand this function as a power series about $s = 1$.* Hint: Let $1/s = 1/(1-1+s) = 1/(1-(1-s))$. *What is the RoC?*
Solution: The power series is

$$w(s) = \sum_{n=0}^{\infty} (-1)^n (s-1)^n,$$

which converges for $|s-1| < 1$.

To convince you this is correct, use the Matlab/Octave command `syms s; taylor(1/s,s,'ExpansionPo` which is equivalent to the shorthand `syms s; taylor(1/s,s,1)`. What is missing is the logic behind this expansion, given as follows: First move the pole to $z = -1$ via the Möbius "translation" $s = z + 1$, and expand using the Taylor series

$$\frac{1}{s} = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n.$$

Next back-substitute $z = s - 1$ giving

$$\frac{1}{s} = \sum(-1)^n(s-1)^n.$$

It follows that the RoC is $|z| = |s-1| < 1$, as provided by Matlab/Octave. ■

– 29.2: Expand $w(s) = 1/s$ as a power series in $s^{-1} = 1/s$ about $s^{-1} = 1$. What is the RoC?

Solution: Let $z = s^{-1}$ and expand about 1: The solution is $w(z) = z$, which has a zero at 0 thus a pole at ∞. The RoC is $|s| > 0$ or $|z| < \infty$. ■

– 29.3: What is the residue of the pole?

Solution: The pole is at ∞. Since $w(s) = 1/s$ and applying the definition for the residue $c_{-1} = \lim_{s\to\infty} s(1/s) = 1$. Thus residue is 1. Note that it is the amplitude of the pole, which is 1. ■

Problem # *30: Consider the function $w(s) = 1/(2-s)$*

– 30.1: Expand $w(s)$ as a power series in $s^{-1} = 1/s$. State the RoC as a condition on $|s^{-1}|$. Hint: Multiply top and bottom by s^{-1}.

Solution: $1/(2-s) = -s^{-1}/(1-2s^{-1}) = -s^{-1}\sum 2^n s^{-n}$. The RoC is $|2/s| < 1$, or $|s| > 2$.■

– 30.2: Find the inverse function $s(w)$. Where are the poles and zeros of $s(w)$, and where is it analytic?

Solution: Solving for $s(w)$ we find $2 - s = 1/w$ and $s = 2 - 1/w = (2w-1)/w$. This has a pole at 0 and a zero at $w = 1/2$. The RoC is therefore from the expansion point out to, but not including $w = 0$.■

Problem # *31: Summing the series*

The Taylor series of functions have more than one region of convergence.

– 31.1: Given some function $f(x)$, if $a = 0.1$, what is the value of

$$f(a) = 1 + a + a^2 + a^3 + \cdots?$$

Show your work. **Solution:** To sum this series, we may use the fact that

$$f(a) - af(a) = (1 + a + a^2 + a^3 + \cdots) - a(1 + a + a^2) = 1 + a(1-1) + a^2(1-1) + \cdots$$

This gives $(1-a)f(a) = 1$, or $f(a) = 1/(1-a)$. Now since $a = .1$, the sum is $1/(1-0.1) = 1.11$. ■

– 31.2: Let $a = 10$. What is the value of

$$f(a) = 1 + a + a^2 + a^3 + \cdots?$$

Solution: In this case the series clearly does not converge. To make it converge we need to write a formula for $y = 1/x$ rather than for x.

$$f(1/y) - f(1/y)/a = (1 + 1/a + 1/a^2 + 1/a^3 + \cdots) - 1/a(1 + 1/a + a1/^2) = 1 + (1-1)/a + (1-1)/a^2 + \cdots$$

This gives $f(1/a) = -a^{-1}/(1-a^{-1})$. Now since $a = 10$, the series sums to $f(10) = -0.1/(1-0.1) = -1/9$.■

4.2.5 Cauchy-Riemann Equations

Problem # *32: For this problem $\jmath = \sqrt{-1}$, $s = \sigma + \omega\jmath$, and $F(s) = u(\sigma,\omega) + \jmath v(\sigma,\omega)$. According to the fundamental theorem of complex calculus (FTCC), the integration of a complex–analytic function is independent of the path. It follows that the derivative of $F(s)$ is defined as*

$$\frac{dF}{ds} = \frac{d}{ds}\left[u(\sigma,\omega) + \jmath v(\sigma,\omega)\right]. \tag{4.16}$$

If the integral is independent of the path, then the derivative must also be independent of the direction:

$$\frac{dF}{ds} = \frac{\partial F}{\partial \sigma} = \frac{\partial F}{\partial \jmath\omega}. \tag{4.17}$$

The Cauchy-Riemann (CR) conditions

$$\frac{\partial u(\sigma,\omega)}{\partial \sigma} = \frac{\partial v(\sigma,\omega)}{\partial \omega} \quad \text{and} \quad \frac{\partial u(\sigma,\omega)}{\partial \omega} = -\frac{\partial v(\sigma,\omega)}{\partial \sigma}$$

may be used to show where Equation 4.17 holds.

– 32.1: Assuming Equation 4.17 is true, use it to derive the CR equations.

Solution: First form the partial derivatives as indicated and then set the real and imaginary parts equal. This results in the two CR equations. ■

– 32.2: Merge the CR equations to show that u and v obey Laplace's equations.

$$\nabla^2 u(\sigma,\omega) = 0 \quad \text{and} \quad \nabla^2 v(\sigma,\omega) = 0.$$

Solution: Take partial derivatives with respect to σ and ω and solve for one equation in each of u and v.■

– 32.3: What can you conclude?

Solution: We can conclude that the real and imaginary parts of complex–analytic functions must obey these conditions. ■

Problem # *33: Apply the CR equations to the following functions. State for which values of $s = \sigma + i\omega$ the CR conditions do or do not hold (e.g., where the function $F(s)$ is or is not analytic).* Hint: Review where CR-1 and CR-2 hold.

– 33.1: $F(s) = e^s$

Solution: CR conditions hold everywhere.■

– 33.2: $F(s) = 1/s$

Solution: CR conditions are violated at $s = 0$. The function is analytic everywhere except $s = 0$.■

4.2.6 Branch cuts and Riemann sheets

Problem # *34: Consider the function $w^2(z) = z$. This function can also be written as $w_\pm(z) = \sqrt{z_\pm}$. Assume $z = re^{\phi\jmath}$ and $w(z) = \rho e^{\theta\jmath} = \sqrt{r}e^{\phi\jmath/2}$.*

– 34.1: How many Riemann sheets do you need in the domain *(z) and the* range *(w) to fully represent this function as single-valued?*
Solution: There is one sheet for z and two sheet for $w = \pm\sqrt{z}$. When any point in the domain z (being mapped to $w(z)$) crosses the z branch cut, the codomain (range) $w_\pm(z)$ switches from the w_+ sheet to the w_- sheet. $w(z)$ remains analytic on the cut. Look at Fig. 4.3 in Chap. 4 (p. 162) to see how this works. ■

– 34.2: Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.
Solution: Above we show the mapping for the square root function $w(z) = \sqrt{z_\pm} = \sqrt{r}e^{\jmath\phi/2}$.■

– 34.3: Use `zviz.m` *to plot the positive and negative square roots $+\sqrt{z}$ and $-\sqrt{z}$. Describe what you see.*
Solution: The sheet for the positive root is shown in Fig 3.2 (page 106 of the Oct 24 version of the class notes.) Two view the two sheets use Matlab command `zviz sqrt(W) -sqrt(W)`. ■

– *34.4: Where does* `zviz.m` *place the branch cut for this function?*
Solution: Typically the cut is placed along the negative real z axis $\phi = \pm\pi$. This is Matlab's/Octave's default location. In the figure above, it has been placed along the positive real axis, $\phi = 0 = 2\pi$.■

– *34.5: Must the branch cut necessarily be in this location?*
Solution: No, it can be moved, at will. It must start from $z = 0$ and end at $|z| \to \infty$. The cut may be move when using `zviz.m` by multiplying z by $e^{\jmath\phi_o}$. For example, `zviz W = sqrt(j*Z)` rotates the cut by $\pi/2$. The colors of $w(z)$ (angle maps to color) always 'jump' at the branch cut, as you make the transition across the cut.■

Problem # *35: Consider the function* $w(z) = \log(z)$*. As in Problem 34, let* $z = re^{\phi\jmath}$ *and* $w(z) = \rho e^{\theta\jmath}$.

– *35.1: Describe with a sketch and then discuss the* branch cut *for* $f(z)$.
Solution: From the plot of `zviz w(z) = log(z)` of Lecture 18, we see a branch cut going from $w = 0$ to $w = -\infty$. If we express z in polar coordinates ($z = re^{\jmath\phi}$), then

$$w(z) = \log(r) + \phi\jmath = u(x, y) + v(x, y)\jmath,$$

where $r(x, y) = |z| = \sqrt{x^2 + y^2}$ and $\phi = \angle z = \phi(x, y)$. Thus a zero in $w(z)$ appears at $z = 1 + 0\jmath$, and only appears on the principle sheet of z (between $[-\pi < \angle z = \phi < \pi]$), because this is the only place where $\phi = 0$. As the angle ϕ increases, the imaginary part of $w = \angle z$, which increases without bound. Thus w is like a spiral stair case, or cork-screw. If $\rho = 1$ and $\phi \neq 0$, $w(r) = log(1) + \phi\jmath$ is not zero, since the angle is not zero. ■

– *35.2: What is the inverse of the function* $z(f)$*? Does this function have a branch cut? If so, where is it?*
Solution: $z(w) = e^w$ is a single valued function, so a branch cut is not appropriate. Only multi-valued functions require a branch cut.■

– *35.3: Using* `zviz.m`*, show that*

$$\tan^{-1}(z) = -\frac{\jmath}{2}\log\frac{\jmath - z}{\jmath + z}. \tag{4.18}$$

Solution: Use the Matlab commands `zviz atan(Z)` and `zviz -(j/2)*log((j+Z)./(j-Z))`.
■

– *35.4: Algebraically justify Eq. 4.18.* Hint: Let $w(z) = \tan^{-1}(z)$ and $z(w) = \tan w = \sin w/c$ then solve for e^{wj}.
Solution: Following the hint gives

$$z(w) = -\jmath\frac{e^{w\jmath} - e^{-w\jmath}}{e^{w\jmath} + e^{-w\jmath}} = -\jmath\frac{e^{2w\jmath} - 1}{e^{2w\jmath} + 1}.$$

Solving this bilinear equation for $e^{2w\jmath}$ gives

$$e^{2w\jmath} = \frac{1 + z\jmath}{1 - z\jmath} = \frac{\jmath - z}{\jmath + z}$$

Taking the log and using our definition of $w(z)$ we find

$$w(z) = \tan^{-1}(z) = -\frac{\jmath}{2}\log\frac{\jmath - z}{\jmath + z}.$$

■

4.2.7 A Cauer synthesis of any Brune impedance

Problem # *36: One may synthesize a transmission line (ladder network) from a positive real impedance* $Z(s)$ *by using the* continued fraction algorithm *41. To obtain the series and* *shunt impedance values, we can use a residue expansion (p. 154). Here we shall explore this method.*

– 36.1: Starting from the Brune impedance $Z(s) = \frac{1}{s+1}$, find the impedance network as a ladder network.

Solution: Taking the reciprocal we find the sum of two shunt admittances, and capacitor and resistor

$$Y(s) = s + 1.$$

The the impedance is $Z(s) = 1/(s + 1)$. ■

– 36.2: Use a residue expansion in place of the CFA floor function (§2.3.4, p. 41) for polynomial expansions. Find the residue expansion of $H(s) = s^2/(s + 1)$ and express it as a ladder network.

Solution: Verify that

$$Z(s) = s^2/(s + 1) = s - 1 + 1/(s + 1). \tag{4.19}$$

Thus the Cauer synthesis is a series combination $s - 1$ (an inductor $L = 1$ and a resistor $R = -1$ ohms) and a shunt $1||s$ (i.e., $Y(s) = 1 + s$, a resistor of $R = 1$ in parallel with a capacitor $C = 1$.) It would appear that $Z(s)$ is not a positive real impedance. ■

– 36.3: Discuss how the series impedance $Z(s, x)$ and shunt admittance $Y(s, x)$ determine the wave velocity $\kappa(s, x)$ and the characteristic impedance $z_o(s, x)$ when (1) $Z(s)$ and $Y(s)$ are both independent of x; (2) $Y(s)$ is independent of x, $Z(s, x)$ depends on x; (3) $Z(s)$ is independent of x, $Y(s, x)$ depends on x; and (4) both $Y(s, x), Z(s, x)$ depend on x.

Solution: In the most general case

$$z_o(s, x) = \sqrt{Z(s, x)/Y(s, x)}$$

and

$$\kappa(s, x) = \sqrt{Z(s, x)Y(s, x)}.$$

The general equations for $z_o(s, s)$ and $\kappa(s, x)$ are given in Mason (1927), and discussed in Appendix 3.10 (p. 127). When z_o and κ depend on x, the area function $A(x)$ of the WHEN will depend on x. Thus the eigenfunction will critically depend on the characteristic impedance and the propagation function.

For example, $\kappa(s)$ can be independent of the area because it cancels out in the product. This is called the case of *constant* k because the speed of sound is independent of the area function. It follows that the area function only depends on $z_o(s, x)$.

This shows that a Cauer synthesis may be implemented with the residue expansion replacing the floor function in the CFA. This seems to solve Brune's network synthesis problem. ■

4.3 Complex–analytic Brune admittance

It is rarely stated that the variable that we are integrating over, either x (space) or t (time), is real ($x, t \in R$), since that fact is implicit, due to the physical nature of the formulation of the integral. But this intuition must be refined once complex numbers are included with $s \in \mathbb{C}$, where $s = \sigma + \omega \jmath$.

The assumption that time and space are real variables ($\in \mathbb{R}$) is more than an assumption: it is a requirement that follows from the *order property*. Real numbers have order. For example, if $t = 0$ is now (the present), then $t < 0$ is the past and $t > 0$ is the future. Since time and space are real ($t, \boldsymbol{x} \in \mathbb{R}$), they obey the order property. To have time travel, time and space would need to be complex (they are not), since if the space axis were complex, the order property would be violated. Note that time and space are four-dimensional, as noted and appreciated by Einstein.

Interestingly, it was shown by d'Alembert (1747) that time and space are related by the pure delay. To obtain a solution to the governing wave equation, which d'Alembert first proposed for sound waves, $x, t \in \mathbb{R}$ may be functionally combined as

$$\zeta_\pm = t \pm x/c_o,$$

where $c_o \in \mathbb{R}$ [m/s] is the wave phase velocity. The d'Alembert solution to the wave equation, describing waves on a string under tension, is violated

$$u(x,t) = f(t - x/c_o) + g(t + x/c_o), \tag{4.20}$$

which describes the transverse velocity (or displacement) of two independent waves $f(\zeta_-), g(\zeta_+) \in \mathbb{R}$ on the string, which represent forward and backward traveling waves.[2]

For example, starting with a string at rest, if one displaces the left end, at $x = 0$, by a step function $u(t)$, then that step displacement will propagate to the right as $u(t - x/c_o)$, arriving at location x_o [m], at time x_o/c_o [s]. Before this time, the string will not move to the right of the wave-front, at x_o [m], and after t_o [s] it will have a non-zero displacement. Since the wave equation obeys superposition (postulate P2), it follows that the "plane-wave" eigen-function of the wave equation for $\mathbf{x}, \mathbf{k} \in \mathbb{R}^3$ are given by

$$\psi_\pm(\mathbf{x}, t) = \delta(t \mp \mathbf{k} \cdot \mathbf{x}) \leftrightarrow e^{st \pm \jmath \mathbf{k} \cdot \mathbf{x}}, \tag{4.21}$$

where $|\mathbf{k}| = 2\pi/|\boldsymbol{\lambda}| = \omega/c_o$ is the *wave number*, $|\boldsymbol{\lambda}| \in \mathbb{R}$ is the wavelength, and $s = \sigma + \omega\jmath$, is the Laplace frequency.

Complex propagation function $\boldsymbol{\kappa}(s)$: When propagation dispersion and losses are considered, we must replace the *wave number* $\jmath\mathbf{k} \in \mathbb{C}$ having a complex–analytic vector wave number $\boldsymbol{\kappa}(s) = \mathbf{k}_r(s) + \jmath\mathbf{k}(s)$. This is known by several names: (1) the *complex propagation function*, (2) the *dispersion relation*, (3) the *propagation function*. Function $\boldsymbol{\kappa}(s)$ is a subtle and important generalization of the scalar wave number $k = 2\pi/\lambda$.[3]

An interesting example is the exact solution to the acoustic wave equation, including viscous and thermal losses, as discussed in section 3.10, page 127, where it is show that the eigenvalues are

$$\kappa_\pm(s) = \frac{s \pm 2\beta_o\sqrt{s}}{c_o} = \frac{(\beta_o \pm \sqrt{s})^2 - (\beta_o)^2}{c_o},$$

with $\beta_o \in \mathbb{R} \geq 0$.

Forms of energy loss, which include viscosity and radiation, require $\boldsymbol{\kappa}(s) \in \mathbb{C}$. Physical examples include acoustic plane waves, electromagnetic wave propagation, antenna theory, and one of the most difficult cases, that of 3D electron wave propagating in crystals (e.g., silicon), where electrons and electro-magnetic (EM) waves are in a state of quantum mechanical equilibrium.

Even when we cannot solve these more difficult problems, we can still appreciate their qualitative solutions. One of the principles that allows us to do that is the causal nature of $\boldsymbol{\kappa}(s)$. Namely the $\mathcal{LT}^{-1}$ of $\boldsymbol{\kappa}(s)$ must be causal, thus Eq. 4.21 must be causal. The *group delay* then describes the nature of the frequency dependent causal delay. For example, if the group delay is large at some frequency, then the solutions will have the largest causal delay at that frequency (Brillouin, 1953; Papoulis, 1962). Qualitatively this can give a deep insight into the solution, even when we cannot compute it.

Electrons and photons are simply different EM states, and $\boldsymbol{\kappa}(\boldsymbol{x}, s)$ describes the crystal's dispersion relations as functions of both frequency and direction, famously known as *Brillouin zones*. Dispersion is a property of the medium such that the wave velocity is a function of frequency and direction, as in silicon.[4] Highly readable discussions on the history of this topic may be found in Brillouin (1953).

4.3.1 Generalized admittance/impedance

The most elementary examples of Brune admittance and impedance are those made of resistors, capacitors, and inductors. Such discrete element circuits arise not only in electrical networks but in mechanical, acoustical, and thermal networks as well (Table 3.3.2. These lumped-element networks can always be represented by ratios of polynomials in s. This gives them a similar structure, with easily classified properties. Such circuits are called *Brune admittances* (or impedances).[5] An example of a special symmetry is when the degrees of

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[2] d'Alembert's solution is valid for functions that are not differentiable, such as $\delta(t - x/c_o)$.

[3] Recall that for lossless plane waves $\lambda f = c_o$, and $k = 2\pi/\lambda$.

[4] In case you missed it, what I'm suggesting is that photons (propagating waves) and electrons (evanescent waves) are different EM wave "states" (Jaynes, 1991). This difference depends on the medium, which determines the dispersion relation (Papasimakis et al., 2018).

[5] Some texts prefer the term *immittance* to include both admittance and impedance.

the numerator and denominator polynomials cannot differ by more than one. This restriction on the degrees comes about because the real part of the admittance/impedance must be positive, due to physical constraints.

But there is a much broader class of admittances that come from transmission lines and other physical structures, which we refer to as *generalized admittances*. An interesting example is an admittance of the form $1/\sqrt{s}$, called a *semi-capacitor*, and $\sqrt{s}$, called a *semi-inductor*. Generalized admittance/impedance is not the ratio of two polynomials. As a result, they are more difficult to characterize.

When a generalized admittance $Y(s)$ or its impedance $Z(s) = 1/Y(s)$ is transformed into the time domain, it must have a real and positive surge admittance $\mathcal{Y}_r \in \mathbb{R}$ or surge impedance $\mathcal{Z}_r \in \mathbb{R}$, followed by the residual response $\upsilon(t), \zeta(t)$. We define the following notation for the admittance (both frequency and time responses),

$$Y(s) = \mathcal{Y}_r + \Upsilon(s) \leftrightarrow y(t) = \mathcal{Y}_r\delta(t) + \upsilon(t), \tag{4.22}$$

and the impedance,

$$Z(s) = \mathcal{Z}_r + \mathcal{Z}_i(s) \leftrightarrow z(t) = \mathcal{Z}_r\delta(t) + \zeta(t). \tag{4.23}$$

The complexity of the notation is necessary and follows from the fact that $z(t) \leftrightarrow Z(s)$ and $y(t) \leftrightarrow Y(s)$ are positive-real and thus minimum phase.

When we are dealing with a transmission line (i.e., wave guides), the generalized admittance is defined as the ratio of the flow to the force. For an electrical system (voltage Φ, current I), the input admittance looking to the right from location x is

$$Y_{in}^{+}(x > 0, s) = \frac{I^+(x,\omega)}{\Phi^+(x,\omega)},$$

and looking to the left is

$$Y_{in}^{-}(x < 0, s) = \frac{I^-(x,\omega)}{\Phi^-(x,\omega)}.$$

In general these two admittances $Y_{in}^{\pm}(x,s)$ are different.

Generalized reflectance: A function related to the generalized impedance is the *reflectance* $\Gamma(s)$, defined as the ratio of a reflected wave to the incident wave. For the case of acoustics (pressure $\mathcal{P}$, volume velocity $\mathcal{V}$),

$$\begin{aligned} Y_{in}(x,s) &\equiv \frac{\mathcal{V}(\omega)}{\mathcal{P}(\omega)} = \frac{\mathcal{V}^+ - \mathcal{V}^-}{\mathcal{P}^+ + \mathcal{P}^-} && (4.24)\\ &= \frac{\mathcal{V}^+}{\mathcal{P}^+}\,\frac{1 - \mathcal{V}^-/\mathcal{V}_+}{1 + \mathcal{P}^-/\mathcal{P}^+} && (4.25)\\ &= \mathcal{Y}_r^+\,\frac{1-\Gamma(x,s)}{1+\Gamma(x,s)}. && (4.26)\end{aligned}$$

When the physical system is continuous at the measurement point x, $\mathcal{Y}_r^+(x) = \mathcal{Y}_r^-(x) \in \mathbb{R}$. The reflectance $\Gamma(x,s)$ depends on the area function, the boundary conditions, or both.

There is a direct relationship between a transmission line's area function $A(x) \in \mathbb{R}$, its characteristic impedance $\mathcal{Y}_r(x) \in \mathbb{R}$, and its eigen-functions. We shall provide specific examples as they arise in our analysis of transmission lines (e.g., Fig. 6.4. This was previously discussed in §41.

A few papers that deal with the relationship between $Y_{in}(s)$ and the area function $A(x)$ are Youla (1964); Sondhi and Gopinath (1971); Rasetshwane et al. (2012). However, the general theory of this important and interesting problem is beyond the scope of this text,[6] as well as Appendix Section 3.10, Train-mission-line problem **problem #** 53, page page 202.

Complex–analytic $\Gamma(s)$ and $Y_{in}(s) = Z_{in}^{-1}(s)$

When we define the complex reflectance $\Gamma(s)$, we make a key assumption: Even though $\Gamma(s)$ is defined by the ratio of two functions of real (radian) frequency ω, like the impedance, the reflectance must be causal (Postulate P1) That $\gamma(t) \leftrightarrow \Gamma(s)$ and $\zeta(t) \leftrightarrow Z_{in}(s) = 1/Y_{in}(s)$ are causal is required by the physics.

[6] See homework DE-3, Problem # 2

4.3.2 Complex–analytic impedance

Conservation of energy (or power) is a cornerstone of modern physics. It may first have been under consideration by Galileo Galilei (1564–1642) and Marin Mersenne (1588–1648). Today the question is not whether it is true, but why. Specifically, what is the physics behind conservation of energy? Surprisingly, the answer is straightforward, based on its definition and the properties of impedance. Recall that the power is the product of the force and the flow, and impedance is their ratio.

The power is given by the product of two variables, sometimes called *conjugate variables*, the force and the flow. In electrical terms, these are voltage (force) ($v(t) \leftrightarrow V(\omega)$) and current (flow) ($i(t) \leftrightarrow I(\omega)$); thus, the electrical power at any instant of time is[7]

$$\mathcal{P}(t) = v(t)i(t). \tag{4.27}$$

The total energy $\mathcal{E}(t)$ is the integral of the power, since $\mathcal{P}(t) = d\mathcal{E}/dt$. Thus if we start with all the elements at rest (no currents or voltages), then the energy as a function of time is always positive

$$\mathcal{E}(t) = \int_0^t \mathcal{P}(t)dt \geq 0 \tag{4.28}$$

and is simply the total energy applied to the network (Van Valkenburg, 1964a, p. 376). Since the voltage and current are related by either an impedance or an admittance, conservation of energy depends on the property of impedance. From Ohm's law and Postulate P1 (every impedance is causal), and we have

$$\begin{aligned} v(t) &= z(t) \star i(t) \\ &= \int_{\tau=0}^t z(\tau)i(t-\tau)d\tau \\ &\leftrightarrow V(s) = Z(s)I(s). \end{aligned}$$

Example: Let $i(t) = \delta(t)$ (a perfect impulse). Then

$$I_{xx}(t) = \int_{\tau=0}^t z(t-\tau)\delta(\tau)d\tau = \int_0^t z(-\tau)d\tau.$$

Every Brune impedance always has the form $z(t) = r_o\delta(t) + \zeta(t)$. The *characteristic impedance* r_o (also called *surge impedance*) may be defined as (Lundberg et al., 2007)

$$r_o = \int_{0^-}^{\infty} z(t)dt.$$

This definition requires that the integral of $\zeta(t)$ is zero, a conclusion that warrants further investigation.

These ideas are perhaps easier to visualize if we work in the Laplace frequency domain. Then the total energy, equal to the integral of the real part of the power, is

$$\frac{1}{s}\Re VI = \frac{1}{2s}(V^*I + VI^*) = \frac{1}{2s}(Z^*I^*I + ZII^*) = \frac{1}{s}\Re Z(s)|I|^2 \geq 0.$$

Mathematically this is called a *positive definite operator*, since the positive and real resistance is sandwiched between the current, forcing the "definiteness."

In conclusion, conservation of energy is totally dependent on the properties of the impedance. Thus one of the most important and obvious applications of complex–analytic functions of a complex variable is the impedance function. This seems to be the ultimate example of the FTCC applied to $z(t)$.

Every impedance must obey conservation of energy (Postulate P3): The impedance function $Z(s)$ has resistance R and reactance X as a function of complex frequency $s = \sigma + \jmath\omega$. From the causality postulate

[7] The voltage is sometimes called the Electromotive force (EMF). However $v(t)$ is relative to a reference ground. The actual EMF is $-\nabla v(t)$.

P1, $z(t<0)=0$. Every impedance is defined by a Laplace transform pair

$$z(t) \leftrightarrow Z(s) = R(\sigma, \omega) + \jmath X(\sigma, \omega),$$

with $R, X \in \mathbb{R}$.

According to postulate P3 a system is passive if it does not contain a power source. Drawing power from an impedance violates conservation of energy. This property is also called *positive-real*, which was defined by Brune (1931a,b)

$$\Re\{Z(s \geq 0)\} \geq 0. \tag{4.29}$$

Positive-real systems cannot draw more power than is stored in the impedance. The region $\sigma \leq 0$ is called the *left half s plane* (LHP), and the complementary region $\sigma > 0$ is called the *right half s plane* (RHP). According to the Brune condition, the real part of every impedance must be non-negative (the RHP).

It is easy to construct examples of second-order poles or zeros in the RHP such that Postulate P3 is violated. Thus Postulate P3 implies that the impedance may *not* have more than simple (first-order) poles and zeros, strictly in the LHP. But there is more: These poles and zeros in the LHP must meet a *minimum phase condition*, a condition that is easily stated:

$$\angle Z(s) < \angle s \tag{4.30}$$

but difficult to prove. There seems to be no proof that second-order poles and zeros (e.g., second-order roots) are not allowed.[8] However, such roots must violate a requirement that the poles and zeros must alternate on the $\sigma = 0$ axis, which follows from Postulate P3. In the complex plane the concept of "alternate" is not defined (complex numbers cannot be ordered). What *has* been proved (i.e., Foster's reactance theorem), that says that if the poles are on the real or imaginary axis, they must alternate, leading to simple poles and zeros (Van Valkenburg, 1964a). The restriction on poles is sufficient but not necessary, as $Z(s) = 1/\sqrt{s}$ is a positive-real impedance but is less than a first-degree pole (Kim and Allen, 2013). The corresponding condition in the LHP, and its proof, remains elusive (Van Valkenburg, 1964a).

For example, a series resistor R_o and capacitor C_o have an impedance given by Table 6,

$$Z(s) = R_o + 1/sC_o \leftrightarrow R_o\delta(t) + \frac{1}{C_o}u(t) = z(t), \tag{4.31}$$

with constants $R_o, C_o \in \mathbb{R} \geq 0$. In mechanics, an impedance composed of a dashpot (damper) and a spring has the same mathematical form.

A full 2d order resonant system has an inductor, resistor, and capacitor, with an impedance given by

$$Z(s) = \frac{sC_o}{1 + sC_oR_o + s^2C_oM_o} \leftrightarrow C_o\frac{d}{dt}\left(c_+e^{s_+t} + c_-e^{s^-t}\right) = z(t), \tag{4.32}$$

which is a second-degree polynomial with two complex resonant frequencies $s_\pm$ and their residues $c_\pm$. When $R_o > 0$, these roots are in the LHP, with $z(t) \leftrightarrow Z(s)$.

Systems (networks) that contain many elements and transmission lines can be much more complicated, yet still have a simple frequency-domain representation. This is the key to understanding how these physical systems work, as we describe next.

Poles and zeros of positive-real functions must be first-degree: The definition of *positive-real* (PR) functions requires that the poles and zeros of the impedance function be simple (only first-degree). Second-degree poles would have a reactive "secular" response of the form $h(t) = t\sin(\omega_k t + \phi)u(t)$, and these terms would not average to zero, depending on the phase, as is required of an impedance. As a result, only single-degree poles are possible.[9] I believe that no one has ever reported an impedance that has second-degree poles and zeros. Network analysis books never report second-degree poles and zeros in their impedance functions. Nor has there ever been any guidance about where the poles and zeros might lie in the LHP. Understanding the exact relationships between pairs of poles and zeros, to assure that the real part of the impedance is real, would resolve this longstanding unsolved problem (Van Valkenburg, 1964b). It is the residues that determine the LHP simple pole degree.

Calculus on Complex–analytic functions: To solve a differential equation or integrate a function, Newton used the Taylor series to integrate one term at a time. However, he used only real functions of a real variable

[8] As best I know, this is an open problem in network theory (Brune, 1931b; Van Valkenburg, 1964a).

[9] Secular terms result from second-degree poles, since $u(t) \star u(t) = tu(t) \leftrightarrow 1/s^2$.

due to his fundamental lack of appreciation of the complex analysis. This same method is how one finds solutions to scalar differential equations today, but using an approach that makes the solution method less obvious. Rather than working directly with the Taylor series, today we use the complex exponential, since the complex exponential is an eigen-function of the derivative

$$\frac{d}{dt}e^{st} = se^{st}.$$

Since e^{st} may be expressed as a Taylor series that has coefficients $c_n = 1/n!$, in some real sense the modern approach is a compact way of doing what Newton did. Thus every linear constant coefficient differential equation in time may be simply transformed into a polynomial in complex Laplace frequency s, by looking for solutions of the form $F(s)e^{st}$ and transforming the differential equation into polynomial $F(s)$ in complex frequency.

For example

$$\frac{d}{dt}f(t) + af(t) = \delta(t) \leftrightarrow (s+a)F(s) = 1.$$

The pole of $F(s_r)$ is $s_r + a = 0$ is the eigenvalue of the differential equation. Thus a powerful tool for understanding the solutions of differential equations, both scalar and vector, is to work in the Laplace frequency domain using their eigenvalues (i.e., $s_r = -a$) and their eigen-functions.

The Taylor series may be replaced by e^{st}, which transforms Newton's real Taylor series into the complex exponential eigen-function. In some sense, these are the same methods, since

$$e^{s_r t} = \sum_{n=0}^{\infty} \frac{(s_r t)^n}{n!}. \tag{4.33}$$

Taking the derivative with respect to time gives

$$\frac{d}{dt}e^{st} = se^{st} = s\sum_{n=0}^{\infty} \frac{(st)^n}{n!}, \tag{4.34}$$

which is also complex–analytic. Thus if the series for $F(s)$ is valid (i.e., it converges), then its derivative is also valid. This was a very powerful concept, exploited by Newton for real functions of a real variable, and later by Cauchy and Riemann for complex functions of a complex variable. The key question is "Where does the series fail to converge?" This is the main concept behind the FTCC (Eq. 4.8).

The FTCC (Eq. 4.8) is formally the same as the FTC (Eq. 3.7.3) (Leibniz's formula), the key (and significant) difference being that the argument of the integrand $s \in \mathbb{C}$. Thus this integration is a line integral in the complex plane. One would naturally assume that the value of the integral depends on the path of integration. And it does, but in a subtle way, as quantified by Cauchy's various theorems. If the path stays in the same RoC region, then the integral is independent of that path. If a path includes a different pole, then the integral depends on the path, as quantified by the Cauchy residue theorem. The test is to deform the path from the first to the second. If in that deformation the path crosses a pole, then the integral will change (namely it will depend on the path). All of this is follows from causality.

The FTC and FTCC are clearly distinguishable yet the reasoning is the same. If $F(s) = df(s)/ds$ is complex–analytic [i.e., has a power series $f(s) = \sum_k c_k s^k$, with $f(s), c_k, s \in \mathbb{C}$], then it may be integrated, term by term, and yet the integral does not depend on the path. This seems a bit amazing. The key is that $F(s)$ and $f(s)$ must be complex–analytic, which means they are differentiable. This all follows from the Taylor series formula Eq. 4.10. for the coefficients of the complex–analytic series. For Eq. 4.8 to hold, the derivatives must be independent of the direction (the path). The concept of a complex–analytic function therefore has eminent consequences in the form of several key theorems on complex integration, as first discovered by Cauchy in about 1820.

Role of the Complex Taylor series: The complex Taylor series generalizes the functions it describes, with unpredictable consequences, as nicely shown by the domain-coloring diagrams in Fig. 3.11, where a simple translation of the s plane by a complex number can void the positive-real property ($s - \sqrt{j}$) cannot be a physical impedance).

Cauchy's of complex integration tools were first exploited in physics by Sommerfeld (1952), to explain the onset (e.g., causal) transients in waves, as he explained in detail in Brillouin (1960, Chap. 3).

The importance of causality: Up to 1910, when Sommerfeld first published his results using complex–analytic signals and saddle point integration in the complex plane, the implications of the causal wave-front were poorly understood. It would be reasonable to say that Sommerfeld's insight changed our understanding of wave propagation for both light and sound. Unfortunately, in my view, his insight has never been fully appreciated, perhaps even to this day. If you question this summary, please read Brillouin (1960, Chap. 1).

The full power of the complex–analytic function was first appreciated by Bernard Riemann (1826–1866) in his University of Göttingen PhD thesis of 1851, under the tutelage of Gauss (1777–1855), which drew heavily on the work of Cauchy.

The key definition of a complex–analytic function is that it has a Taylor series representation, over a region of the complex frequency plane $s = \sigma + j\omega$, that converges in its RoC about the expansion point, with a radius determined by the nearest pole of the function. A further surprising feature of all analytic functions is that within the RoC, the inverse of that function also has a complex–analytic expansion. Thus given $w(s)$, in theory, we can determine $s(w)$ to any desired accuracy, critically depending on the RoC. As an example if $w(s) = e^s$ then its inverse is $s(w) = log(w)$. Given the right software (e.g., `zviz.m`), this relationship may be made precise for every complex–analytic function.

4.3.3 Multi-valued functions

In the field of mathematics there seems to have been a tug-of-war regarding the basic definition of a function. The accepted definition today is a single-valued (i.e., complex–analytic) mapping from the domain to the codomain (or range). This makes the discussion of complex–analytic multi-valued functions awkward. In 1851 Riemann (working with Gauss) resolved this problem for the complex–analytic set of multi-valued functions by introducing the geometric concept of single-valued sheets, delineated by branch cuts.

Two simple yet important examples of multi-valued functions are the circle $z^2 = x^2 + y^2$ and $w = \log(z)$. For example, if we assume $\rho = |z|$ is the radius of the circle, then solving for $y(x)$ gives the double-valued function

$$y(x) = \pm\sqrt{\rho^2 - x^2}.$$

The related function $z = \pm\sqrt{x}$, with $x \in \mathbb{C}$, is shown in Fig. 4.1 as a three-dimensional display in polar coordinates, with $z(r)$ as the vertical axis, as a function of the angle and radius of $x \in \mathbb{C}$.

If we accept the modern injective definition of a function, as the mapping from one set to a second set, then $y(x)$ is not a function, or even two functions. For example, what if $x > z$? Or worse, what if $z = 2\jmath$ with $|x| < 1$? Riemann's construction, using branch cuts for multi-valued function, resolves all these difficulties (as best I know).

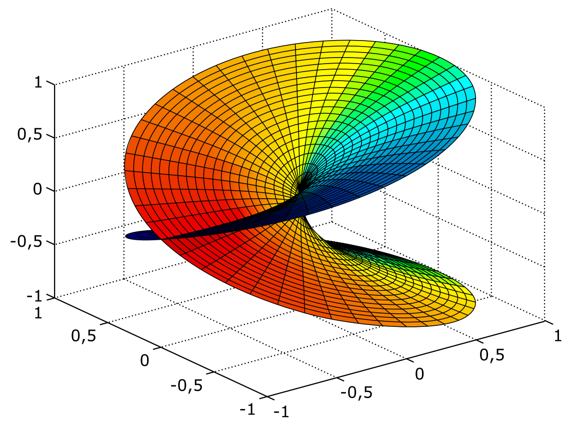


Figure 4.1: *The mapping for the square root function This function has two single-valued sheets of the x plane corresponding to the two signs of the square root. The best way to view this function is in polar coordinates, with $x = |x|e^{\jmath\phi}$ and $z = \sqrt{|x|}e^{\jmath\phi/2}$. For a great video, see* `https://en.wikipedia.org/wiki/Riemann_surface`.

To proceed, we need definitions and classifications of the various types of complex singularities:

1. Poles of degree 1 are called *simple poles*.

2. When the numerator and denominator of a rational function (i.e., ratio of two polynomials) have a common root (i.e., factor), that root is said to be *removable*.

3. A singularity that is not removable, a pole, or a branch point is called *essential*.

4. A complex–analytic function (except for isolated poles) is called *meromorphic* (Boas, 1987). Meromorphic functions can have any number of poles, even an infinite number. The poles need not be simple.

5. When the first derivative of a function $Z(s)$ has a simple pole at a, then a is said to be a *branch point* of $Z(s)$. An important example is the logarithmic derivative:

$$d\ln(s-a)^\alpha/ds = \alpha/(s-a),\ \alpha \in \mathbb{I}.$$

However, the converse does not necessarily hold.

More complex typologies are being researched today, and progress is expected to accelerate due to modern computing technology.[10] It is helpful to identify the physical meaning of these more complex surfaces, to guide us in their interpretation and possible applications.[11]

Exercise #2

Find x for[12] $y(x) \equiv x^{\sqrt{x}} = 3$. Hint: Use Newton's method. **Solution:** According to Matlab/Octave command using *syms x y* followed by *diff(y), gives* $\frac{dy}{dx} = 1 + \log(\sqrt{x})$. *Newton's method is then* $x_{n+1} = x_n - \eta \frac{x^{\sqrt{x}}}{1+\log\sqrt{x_n}}$, *where* $0 < \eta < 1$ *is the step size to improve convergence Allen (2025).*

Branch cuts: Up to this point we have considered only poles of degree $\alpha \in \mathbb{N}$ of the form $1/s^\alpha$. The concept of a branch cut allows us to manipulate (and visualize) multi-valued functions for which $\alpha \in \mathbb{F}$. This is done by breaking each region into single-valued sheets, as shown in Fig. 4.2 (right). The branch cut is a curve $\in \mathbb{C}$ that separates the various single-valued sheets of a multi-valued function. The concepts of branch cuts, sheets, and the extended plane were first devised by Riemann, as described in his thesis of 1851. It was these three mathematical and geometrical constructions that provided deep insight into complex–analytic functions, greatly extending the important earlier work of Cauchy (1789–1857) on the calculus of complex–analytic functions. For an alternative helpful discussion of Riemann sheets and branch cuts, see Boas (1987, pp. 221–25), Kusse and Westwig (2010), and Greenberg (1988).

To study the properties of multi-valued functions and branch cuts, we look at $w(s) = \sqrt{s}$ and $w(s) = \log(s)$, along with their inverse functions $w(s) = s^2$ and $w(s) = e^s$. For uniformity we refer to the *complex abscissa* ($s = \sigma + \omega\jmath$) and the *complex ordinate* ($w(s) = u + v\jmath$). When the complex domain and range are swapped, by taking the inverse of a function, multi-valued functions are a common consequence. For example, $f(t) = \sin(t)$ is single-valued and analytic in t and thus has a Taylor series. The inverse function $t(f)$ is multi-valued.

The best way to explore the complex mapping from the complex planes $s \to w(s)$ is to master the single-valued function $s = w^2(s)$ and its double-valued inverse $w(s) = \sqrt{s}$.

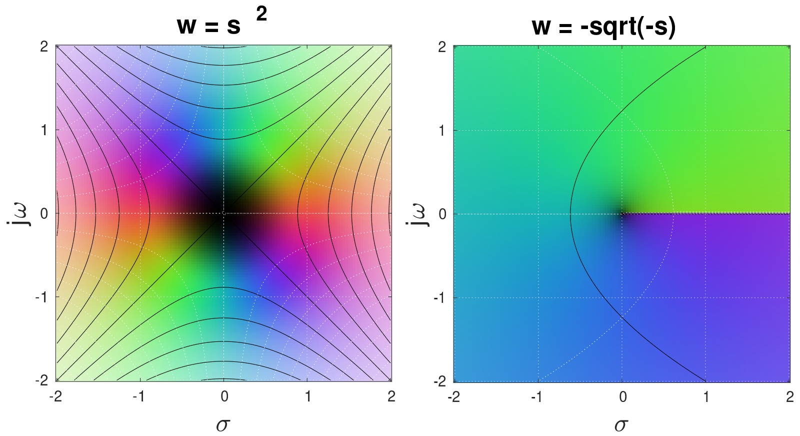


Figure 4.2: *Here we use Cartesian coordinates in the domain $s = \sigma + \omega\jmath$ and polar coordinates for the range $w(\sigma, \omega) = |w|e^{\psi\jmath}$. The color intensity indicates the magnitude $|s|$, with black being $|s = 0|$ and bright (eventually white) indicating $|s| \to \infty$.* **Left:** *Mapping: $w(s) = s^2$.* **Right:** *Mapping of the principal branch from $-s$ to $w(s) = -\sqrt{-s}$ (i.e., the rotated inverse of s^2). This sheet was generated by rotating w by 180°. The branch cut is on the $\psi = 0$ axis, with branch points at $|w| = 0$ and ∞. Neither of these functions are Brune impedances, since they violate the positive-real condition, Eq. 3.26.*

Figure 4.2 (left) shows the single-valued function $w(s) = s^2$, and (right) one sheet of its inverse, the double-valued mapping of $w(s) = -\sqrt{-s}$. Single-valued functions such as $w(s) = s^2$ are relatively straightforward. Multi-valued functions require the concept of a branch cut, defined in the image plane (also called the codomain or range). This is a technique to render the multiple values as single-valued on each of several sheets. Each sheet is labeled in the domain (s) plane by a sheet index $k \in \mathbb{Z}$, while branch points and cuts are defined in the image (w) plane.

The sheets are indexed by a sheet index, and separated by the branch cut. It is important to understand that the path of every branch cut is not unique and may be moved. However, branch points are unique and thus not movable.

A function may be multi-valued in both the domain and image planes. As an example consider $w(s) = s^{3/2}$.

The multi-valued nature of $w(s) = \sqrt{s}$ is best understood by working with the function in polar coordinates. We let

$$s_k = re^{\theta\jmath}e^{2\pi k\jmath}, \tag{4.35}$$

[10] https://www.maths.ox.ac.uk/about-us/departmental-art/theory

[11] https://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201

[12] https://www.youtube.com/watch?v=lI8DAgFfh9c

where $r = |s|, \theta = \angle s, \in \mathbb{R}$, and $k \in \mathbb{Z}$ is the sheet index.

This concept of analytic inverses becomes important only when the function is multi-valued. For example, since $w(s) = s^2$ has a period of 2, $s(w) = \pm\sqrt{w}$ is multi-valued. Riemann dealt with such extensions using the concept of a branch cut with multiple sheets labeled by sheet numbers. Each sheet describes an analytic function (Taylor series) that converges within some RoC that has a radius out to the nearest pole. Thus Riemann's branch cuts and sheets explicitly deal with the need to define unique single-valued inverses of multi-valued functions. Since the square root function has two overlapping regions corresponding to the $\pm$ due to the radical, there must be two connected region—sort of like mathematical Siamese twins: distinct, yet the same.

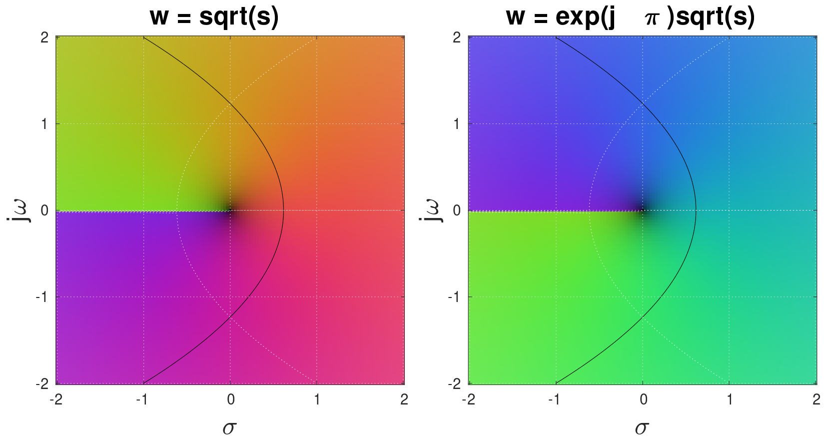


Figure 4.3: *Colorized plots of $s_k = |s|e^{\theta\jmath}e^{2\pi k\jmath}$ and $w_k(s) = \sqrt{s_k} = \sqrt{|s|}e^{\theta\jmath/2}e^{\pi k\jmath}$, as defined in polar coordinates by Eqs. 4.35 and 4.36.* **Left:** *Color map hue reference plane $s = |s|e^{\theta\jmath}e^{2\pi k\jmath}$. This function is a Brune impedance (it represents an inductor).* **Center:** *Sheet index $k = 0$, $w(s) = \sqrt{|s|}e^{\jmath\theta/2}$ for $-\pi < \theta < +\pi$ and $\psi = \angle w(s) = \theta/2$ between $\pm\pi/2$ [rad]. This function is positive-real (Eq. 3.26.* **Right:** *Sheet index $k = 1$, $w(s) = e^{\jmath\theta/2}e^{\jmath\pi}\sqrt{|s|}$ for $\pi < \theta < 3\pi$ and $\pi/2 < \psi < 3\pi/2$ [rad]. The branch cut is at $\theta = \angle s = \pm 180°$ (π [rad]) where the hue of w changes abruptly from green to purple (center) and from purple back to green (right). Note how the hue matches between the center and right panels at the branch cut: in the center panel purple runs along -180° and along +180° on the right. Likewise, green runs along +180° in the center and along -180° on the right. Thus $w(s) = \sqrt{s}$ is analytic on the branch cut connecting the two sheets ($k = 0 \to k = 1$). This function is not an impedance, since it is negative in the RHP.*

Hue: By studying Fig. 4.3, we can appreciate domain-coloring. The angle-to-hue map is shown in the left panel of Fig. 4.3. The domain angles $\angle s$ go from $-90° < \theta < 90°$ (purple to red to green). For $w(s) = \sqrt{s}$, the $\angle s$ is *compressed* by a factor of 2 ($\psi = \theta/2$) with purple being $-180°$, and green being $+180°$.

Thus for Fig. 4.3 the principal ($k = 0$) angle $\angle s$ ($-180° < \theta < 180°$) maps to $w(s) = \sqrt{s}$ (middle panel) to half the w plane ($-90° < \psi < 90°$ (from purple to red to green).

The $k = 1$ branch, of $\angle s$ ($+180° < \theta < 180 + 360 = 520°$) maps to $\angle w$ (on the right) to green to blue to purple ($\psi = \theta/2$). Note how the panel on the left matches the right half of s (green = $+90°$, purple = $-90°$). The center panel $\angle w$ is green where $\angle s = 180°$. Thus $\angle w = \frac{1}{2}\angle s$. Going around the s plane one more time gives the right most figure. $w(s)$ is analytic everywhere except at the branch points $s = 0$ and $s = \infty$.

Moving the branch cut: Furthermore we can change $\angle s$ by $180°$ to move the branch cut

$$w = \rho e^{\jmath\psi} = \sqrt{r}e^{\jmath\theta/2}e^{\jmath\pi k}, \tag{4.36}$$

where $\rho = |w|, \psi \angle w, \in \mathbb{R}$. The generic Cartesian coordinates are $s = \sigma + \omega\jmath$ and $w(s) = u(\sigma,\omega) + v(\sigma,\omega)\jmath$. For single-valued functions such as $w(s) = s^2$ on the left in Fig. 4.2 there is no branch cut, since $\psi = 2\theta$. Note how the red color ($\theta = 0°$) appears twice in this mapping. For multi-valued functions, a branch cut is required, typically along the negative $v(\sigma,\omega)$ axis (i.e, $\psi = \pi$), but it may be freely distorted, as seen by comparing the right panel of Fig. 4.2 with the right panel of Fig. 4.3.

Properties of the branch cut: It is important to understand that the function is analytic on the branch cut but not at the branch points. One is free to move the branch cut (at will). It does not need to be on a line: it could be cut in almost any connected manner, such as a spiral. The only rule is that it must start and stop at the matching branch points, or at ∞, which must have the same degree.

The location of the branch cut may be moved by rotating the s coordinate system of Fig. 4.1. For example, $w(s) = \pm \jmath\sqrt{s}$ and $w(s) = \pm\sqrt{-s}$ have different branch cuts, as may be verified using the Matlab/Octave commands `j*zviz(s)` and `zviz(-s)`. Every function is analytic on the branch cut (since moving it does not change the function). If a Taylor series is formed on the branch cut, it will describe the function on the two different sheets. Thus the complex–analytic series (i.e., the Taylor formula, Eq. 4.10) does not depend on the location of a branch cut, as it only describes the function uniquely (as a single-valued function), valid in its local region of convergence.

The second sheet ($k = 1$) in Fig. 4.3 picks up at $\theta = \pi$ [rads] and continues on to $\pi + 2\pi = 3\pi$. The first sheet maps the angle of w (i.e., $\phi = \angle w = \theta/2$) from $-\pi/2 < \phi < \pi/2$ ($w = \sqrt{r}e^{\jmath\theta/2}$). This corresponds to $u = \Re\{w(s)\} > 0$. The second sheet maps $\pi/2 < \psi < 3\pi/2$ (i.e., 90° to 270°), which is $\Re\{w\} = u < 0$. In summary, twice around the s plane is once around the $w(s)$ plane because the angle is half due to the $\sqrt{s}$.

Branch cuts emanate and terminate at *branch points*, defined as singularities (poles) that can even have fractional degree, as for example $1/\sqrt{s}$, and terminate at one of the matching roots, which includes the possibility of ∞.[13] For example, suppose that in the neighborhood of the pole, at s_o the function is

$$f(s) = \frac{w(s)}{(s - s_o)^k}, \tag{4.37}$$

where $w, s, s_o \in \mathbb{C}$ and $k \in \mathbb{Q}$. When $k = 1$, $s_o = \sigma_o + \omega_o\jmath$ is a first-degree "simple pole," having degree 1 in the s plane, with residue $w(s_o)$. Typically the order and degree are positive integers, but fractional degrees and orders are common in modern engineering applications (Kirchhoff, 1868; Lighthill, 1978). Here we allow both the degree and the order to be fractional ($\in \mathbb{F}$). When $k \in \mathbb{F} \subset \mathbb{R}$, $k = n/m$ is a real reduced fraction—namely, when `GCD(n,m)=1`, $n \perp m$). This defines the *degree* of a fractional pole. In such cases there must be two sets of branch cuts of degrees n and m. For example, if $k = 1/2$, the singularity (branch cut) is of degree 1/2 and there are two Riemann sheets, as shown in Fig. 4.2.

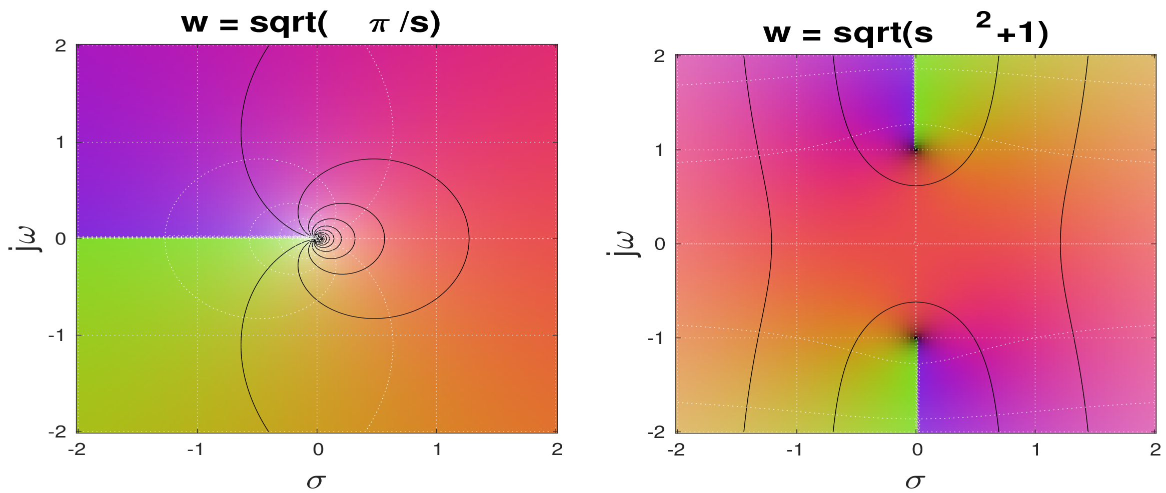


Figure 4.4: *Colorized plots of two $\mathcal{LT}$ pairs: Left: $\sqrt{\pi/s} \leftrightarrow u(t)/\sqrt{t}$. Right: $\sqrt{s^2+1} \leftrightarrow \delta(t) + \frac{1}{t}J_1(t)u(t)$.*

Fractional-order Bessel function: An important example is the Bessel function and its Laplace transform ($\mathcal{LT}$)

$$\delta(t) + \frac{1}{t}J_1(t)u(t) \leftrightarrow \sqrt{s^2+1},$$

as shown in Fig. 4.4, which is related to the solution to the wave equation in two-dimensional cylindrical coordinates (see Table 3.9. Bessel functions are the solutions (i.e., eigen-functions) of guided acoustic waves in round pipes, or surface waves on the earth (seismic waves), or waves on the surface of a pond.

There are a limited number of possibilities for the degree, $k \in \mathbb{Z}$ or $\in \mathbb{F}$ of Eq. 4.37. If the degree is drawn from $\mathbb{R} \notin \mathbb{F}$, the pole can not have a residue. According to the definition of the residue, $k \in \mathbb{F}$ has no residue. But there remains open the possibility of generalizing the concept of the Riemann integral theorem to include $k \in \mathbb{F}$. One way to do this is to use the logarithmic derivative, which transforms fractional poles to simple poles with fractional residues.

[13]This presumes that poles and zeros appear in pairs, one of which may be at ∞.

If the singularity has an irrational degree ($k \in \mathbb{I}$), the branch point has the same irrational degree. Accordingly there are an infinite number of Riemann sheets, as in the case of the log function. An example is $k = \pi$, for which

$$F(s) = \frac{1}{s^\pi} = e^{-\log(s^\pi)} = e^{-\pi\log(s)} = e^{-\pi\log(\rho)}e^{-\pi\theta\jmath},$$

where the domain is expressed in polar coordinates $s = \rho e^{\theta\jmath}$. When $k \in \mathbb{F}$, it may be close (e.g., $k = \pi_{152}/\pi_{153} = \pi_{152}/(\pi_{152}+2) = 881/883 \approx 0.99883$, or its reciprocal ≈ 1.0023). The branch cut could be subtle if unnoticed, but would have a significant impact on its inverse Laplace transform. Consider the difference in the definition of a constant in one dimension, versus 2 dimensions. There is a problem of negative numbers when the set is in one dimension. What is -1 if the sent doesn't allow negative numbers. One cannot take-away 1 from the null set, if the set is assumed to be $\in \mathbb{R}$.

Exercise #*3*

Find the poles, zeros, and residues of $F(s)$.

1.
$$F(s) = \frac{d}{ds}\ln\frac{s+e}{s+\pi}$$

Solution:

$$F(s) = \frac{d}{ds}\left[\ln(s+e) - \ln(s+\pi)\right] = \left(\frac{1}{s+e} - \frac{1}{s+\pi}\right)$$

The poles are at $s_1 = -e$ and $s_2 = -\pi$ with respective residues of ± 1.

2.
$$F(s) = \frac{d}{ds}\ln\frac{(s+3)^e}{(s+\jmath)^{-\pi}}$$

Solution:

$$F(s) = \frac{d}{ds}\left(e\ln(s+3) + \pi\ln(s+\jmath)\right) = \frac{e}{s+3} + \frac{\pi}{s+\jmath}.$$

There is a very important take-home message here regarding the utility of the logarithmic derivative, which "linearizes" the fractional pole.

3.
$$F(s) = e^{\pi\ln s}$$

Solution: To simplify this expression take the log

$$\ln F(s) = \ln e^{\pi\ln s} = \pi\ln s = \ln s^\pi$$

Thus $F(s) = s^\pi$. The only pole is $s \to \infty$. Thus the definition of the residue is to multiply by the pole and take the limit as $s \to \infty$

$$c_{-1} = \lim_{s\to\infty}\frac{s^\pi}{s} = 0^{\pi-1} = 0.$$

4.
$$F(s) = \pi^{-s}$$

Solution: Converting to exponential format: $F(s) = e^{-s\ln\pi} \leftrightarrow \delta(t - \ln\pi)$. I don't think the pure time delay has poles. I'm not sure what this tells us about the residue.

Log function: Next we discuss the multi-valued nature of the log function. In this case there are an infinite number of Riemann sheets, not captured by Fig. 3.13. which displays only the principal sheet. However, if we look at the formula for the log function, the nature is easily discerned. The abscissa s may be defined as multi-valued, since

$$s_k = re^{2\pi k \jmath} e^{\theta \jmath}.$$

Here we have extended the angle of s by $2\pi k$, where k is the sheet index $\in \mathbb{Z}$. Now we take the log:

$$\log(s) = \log(r) + (\theta + 2\pi k)\jmath.$$

When $k = 0$, we have the principal value sheet, which is zero when $s = 1$. For any other value of k, $w(s) \neq 0$, even when $r = 1$, since the angle is not zero, except for the $k = 0$ sheet.

4.4 Three Cauchy integral theorems

4.4.1 Cauchy's theorems for integration in the complex plane

There are three basic definitions related to Cauchy's integral formula. They are closely related and can greatly simplify integration in the complex plane. The choice of names is unfortunate, if not totally confusing. Hence I call them CT-1, CT-2 and CT-3.

1. **Cauchy's (integral) theorem (CT-1):**
$$\oint_{\mathcal{C}} F(s)ds = 0 \tag{4.38}$$
if and only if $F(s)$ is complex–analytic inside of a simple closed curve $\mathcal{C}$ (Stillwell, 2010; Boas, 1987, p. 45). The FTCC (Eq. 4.8) says that the integral depends on only the end points if $F(s)$ is complex–analytic. With the path (contour $\mathcal{C}$) closed, the end points are the same and thus the integral must be zero as long as $F(s)$ is complex–analytic.

2. **Cauchy's integral formula (CT-2):**
$$\frac{1}{2\pi j}\oint_{\mathcal{B}} \frac{F(s)}{s-s_o}ds = \begin{cases} F(s_o), s_o \in \mathbb{C} < \mathcal{B} \text{ (inside)} \\ \quad 0 \quad , s_o \in \mathbb{C} > \mathcal{B} \text{ (outside)}. \end{cases} \tag{4.39}$$
Here $F(s)$ is required to be analytic everywhere within (and on) the boundary $\mathcal{B}$ of integration (Greenberg, 1988, p. 1200); (Boas, 1987, p. 51); (Stillwell, 2010, p. 220). When the point $s_o \in \mathbb{C}$ is within the boundary, the value $F(s_o) \in \mathbb{C}$ is the residue of the pole s_o of $F(s)/(s - s_o)$. When the point s_o lies outside the boundary, the integral is zero.

3. **The (Cauchy) residue theorem (CT-3):** (Greenberg, 1988, p. 1241); (Boas, 1987, p. 73)
$$\oint_{\mathcal{C}} f(s)ds = 2\pi j \sum_{k=1}^{K} c_k = \sum_{k=1}^{K} \oint \frac{F(s)}{s-s_k}ds, \tag{4.40}$$
where the residues $c_k \in \mathbb{C}$ correspond to the kth pole of $f(s)$ enclosed by the contour $\mathcal{C}$ [(Greenberg, 1988, p. 1241); (Boas, 1987, p. 73)]. Cauchy's integral formula (CT-2) is a special case of the residue theorem (CT-3).

How to calculate the residue: The case of first-degree poles has special significance because the Brune impedance allows only simple poles and zeros, thus increasing its utility. The residues for simple poles are $F(s_k)$, which is complex–analytic in the neighborhood of the pole, but not at the pole.

Consider the function $f(s) = F(s)/(s - s_k)$, where we have factored $f(s)$ to isolate the first-order pole at $s = s_k$, with $F(s)$ analytic at s_k. Then the residue of the poles at $c_k = F(s_k)$. This coefficient is computed by removing the singularity, placing a zero at the pole frequency, and taking the limit as $s \to s_k$—namely,

$$c_k = \lim_{s \to s_k} [(s - s_k)f(s)] \tag{4.41}$$

(Greenberg, 1988; Boas, 1987, p. 72).

When the pole is an Nth degree, the procedure is much more complicated and requires taking $N-1$ order derivatives of $f(s)$ followed by the limit process (Greenberg, 1988, p. 1242). Higher-degree poles are rarely encountered; thus it is good to know that this formula exists, but perhaps it is not worth the effort to learn (i.e., memorize) it.

Some open questions: Without the use of CT-3 it is hard to evaluate the inverse Laplace transform of $1/s$ directly. For example, how do we show that the integral (Eq. 4.39) is zero for negative time (or 1 for positive time)? CT-3 resolves this difficult problem by the convergence of the integral for negative and positive times. Clearly the convergence of the integral at $\omega \to \infty$ plays an important role.

4.4.2 Cauchy Integral Formula and Residue Theorem

CT-2 is an important extension of CT-1, in that a pole has been explicitly represented in the integrand at $s = s_o$. If the pole location is outside the curve $\mathcal{C}$, the result of the integral is zero, in keeping with CT-1. When the pole is inside $\mathcal{C}$, the integrand is no longer complex–analytic at the enclosed pole. When this pole is simple, the residue theorem applies. For the related CT-3 the same result holds, except it is assumed that there are K simple poles in the function $F(s)$. This requires K repeated applications of CT-2. Thus it represents a minor extension of CT-2. When the integrand is $f(s)/P_K(s)$ where $f(s)$ is analytic in $\mathcal{C}$ and $P_K(s)$ is a polynomial of degree K, with all of its roots $s_k \in \mathcal{C}$, then CT-3 applies.

Non-integral degree singularities: A key point is that this theorem applies when $k \in \mathbb{I}$, including fractionals $k \in \mathbb{F}$, but for these cases the residue is always zero, since by definition the residue is the amplitude of the $1/s$ term (Boas, 1987, p. 73). Below are examples:

1. When $k \in \mathbb{F}$ (e.g., $k = 2/3$), the residue of s^k is zero, by definition.
2. The function $1/\sqrt{s}$ has a zero residue (we apply the definition of the residue, Eq. 3.2.3).
3. When $k \neq 1 \in \mathbb{I}$, the residue is, by definition, zero.
4. When $k = 1$, the residue is given by Eq. 3.2.3.
5. CT-1, CT-2, and CT-3 are essential when computing the inverse Laplace transform.

Summary and examples: These three CT theorems, all attributed to Cauchy, collectively are related to the fundamental theorems of complex calculus. The general principles are:

1. In general it makes no sense (nor is there any need) to integrate *through* a pole; thus the poles (or other singularities) must not lie on $\mathcal{C}$.
2. CT-1 (Eq. 4.38) follows trivially from the fundamental theorem of complex calculus (Eq. 4.8, since if the integral is independent of the path, and the path returns to the starting point, the closed integral must be zero. Thus Eq. 4.38 holds when $F(s)$ is complex–analytic within C.
3. Since the real and imaginary parts of every complex–analytic function obey Laplace's equation (Eq. 4.13, it follows that every closed integral over a Laplace field—that is, those defined by Laplace's equation—must be zero. This is the property of a conservative system, corresponding to many physical systems. If a closed box has fixed potentials on the walls, with any distribution whatsoever, and a point charge (i.e., an electron) is placed in the box, then a force equal to $F = qE$ is required to move that charge, and thus work is done. However, if the point is returned to its starting location, the net work done is zero.
4. Work is done in charging a capacitor, and energy is stored. However, when the capacitor is discharged, all of the energy is returned to the load.
5. Soap bubbles and rubber sheets on a wire frame obey Laplace's equation.
6. These are all cases where the fields are Laplacian, thus closed line integrals must be zero. Laplacian fields are commonly seen because they are so basic.

7. We have presented the impedance as the primary example of a complex–analytic function. Physically, every impedance has an associated stored energy, and every system having stored energy has an associated impedance. This impedance is usually defined in the frequency s domain, as a force *over* a flow (i.e., voltage over current). The power $\mathcal{P}(t)$ is defined as the force *times* the flow and the energy $\mathcal{E}(t)$ as the time integral of the power

$$\mathcal{E}(t) = \int_{-\infty}^{t} \mathcal{P}(t)dt, \tag{4.42}$$

which is similar to Eq. 4.6 [see Eq. 3.11.]. In summary, impedance and power and energy are all fundamentally related.

Two fundamental theorems of calculus

According to the *Fundamental Theorem of (real) Calculus (FTC)*,

$$f(x) = f(a) + \int_{a}^{x} F(\xi)d\xi, \tag{4.43}$$

where $x, a, \xi, F, f \in \mathbb{R}$. This is an indefinite integral (since the upper limit is unspecified). It follows that

$$\frac{df(x)}{dx} = \frac{d}{dx}\int_{a}^{x} F(x)dx = F(x).$$

This justifies also calling the indefinite integral the *antiderivative*.

For a closed interval $[a, b]$, the FTC is

$$\int_{a}^{b} F(x)dx = f(b) - f(a), \tag{4.44}$$

thus the integral is independent of the path from $x = a$ to $x = b$.

Fundamental Theorem of Complex Calculus: According to the fundamental theorem of complex calculus (FTCC),

$$f(z) = f(z_0) + \int_{z_0}^{z} F(\zeta)d\zeta, \tag{4.45}$$

where $z_0, z, \zeta, f, F \in \mathbb{C}$. It follows that

$$\frac{df(z)}{dz} = \frac{d}{dz}\int_{z_0}^{z} F(\zeta)d\zeta = F(z). \tag{4.46}$$

For a closed interval $[s, s_o]$, the FTCC is

$$\int_{s_o}^{s} F(\zeta)d\zeta = f(s) - f(s_o), \tag{4.47}$$

thus the integral is independent of the path from $x = a$ to $x = b$. ck eq numbe

Problem # *37*

– 37.1: Consider Equation 4.43. What is the condition on $F(x)$ for which this formula is true?

Solution: The sufficient condition is that the integrand $F(x)$ is be analytic, namely $F(x) = \sum_{x=0}^{\infty} a_n x^n$. This assures that $F(x)$ is single valued and may be integrated, since it may integrated term by term. It follows that as long as $x <$ ROC, this integral exists. Thus the integral equals $F(x) - F(a)$. Note that if the integrand has a Taylor series, all of its derivatives exist within the ROC, because the coefficients depend on derivatives of $F(x)$. ∎

– 37.2: Consider Equation 4.45. What is the condition on $F(z)$ for which this formula is true?

Solution: The sufficient condition is that the integrand $F(z)$ must be complex analytic, namely $F(z) = \sum_{z=0}^{\infty} c_n z^n$, with $c \in \mathbb{C}$.■

– 37.3: Let $F(z) = \sum_{k=0}^{\infty} c_k z^k$.

Solution: Applying term by term integration gives

$$\begin{aligned} I = \int_{\mathcal{C}} \sum_{k=0}^{\infty} c_k z^k dz &= \sum_{k=0}^{\infty} c_k \int_{\mathcal{C}} z^k dz \\ &= \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1} \end{aligned}$$

■

– 37.4: Let

$$F(z) = \frac{\sum_{k=0}^{\infty} c_k z^k}{z - \jmath}.$$

Solution: Applying term by term integration, and using CT-3, gives

$$\begin{aligned} I = \int_{\mathcal{C}} \frac{\sum_{k=0}^{\infty} c_k z^k}{z-\jmath} dz &= \sum_{k=0}^{\infty} c_k \int_{\mathcal{C}} \frac{z^k}{z-\jmath} dz \\ &= \begin{cases} 0 & z = \jmath \notin \mathcal{C} \\ \frac{1}{2\pi\jmath} \sum_{k=0}^{\infty} c_k \jmath^k & z = \jmath \in \mathcal{C} \end{cases} \end{aligned}$$

Note:

$$\int_{\mathcal{C}} \frac{z^k}{z-\jmath} dz = \sum_{m=1}^{k} \frac{(\jmath z)^m}{m} + \ln(z - \jmath),$$

but the residue is $\jmath^k$ which saves the day. ■

Problem # *38: In the following problems, solve the integral*

$$I = \int_{\mathcal{C}} F(z) dz$$

for a given path $\mathcal{C} \in \mathbb{C}$.

– 38.1: Perform the following integrals ($z = x + iy \in \mathbb{C}$):
$I = \int_0^{1+\jmath} z dz$

Solution: $I = \frac{1}{2} z^2 \Big|_0^{1+\jmath} = \frac{1}{2}(1+\jmath)^2 = \frac{1}{2}(1 - 1 + 2\jmath) = \jmath$ ■

– 38.2: $I = \int_0^{1+j} z dz$, but this time make the path explicit: from 0 to 1, with $y = 0$, and then to $y = 1$, with $x = 1$.

Solution: Some text to start the ball rolling

$$\begin{aligned}
I &= \int_{x=0}^{1} (x + 0\jmath)\, dx + \int_{y=0}^{1} (1 + y\jmath)\, dy\, \jmath \\
I &= \frac{1}{2}\, x^2\Big|_0^1 + \int_{y=0}^{1} (\jmath - y)\, dy \\
&= \frac{1}{2} + \left(y\jmath - \frac{1}{2}y^2\right)\Big|_0^1 \\
&= \frac{1}{2} + j - \frac{1}{2} \\
&= j
\end{aligned}$$

We conclude that the integration of z is independent of the path. This is true for any integrand z^n with $n \in \mathbb{Z}$. ■

– 38.3: Discuss whether your results agree with Eq. 4.46?

Solution: Yes the two integrals must agree, because the function is analytic, and the integral must be the same, independent of the path.■

Problem # *39: Perform the following integrals on the closed path $\mathcal{C}$, which we define to be the unit circle. You should substitute $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$, and integrate from $\{-\pi, \pi\}$ to go once around the unit circle.*

Discuss whether your results agree with Eq. 4.46?

– 39.1: $\int_{\mathcal{C}} zdz$

Solution: $\int_{\mathcal{C}} zdz = \int_{-\pi}^{\pi} e^{i\theta} de^{i\theta} = \int_{-\pi}^{\pi} e^{i2\theta} id\theta = e^{i2\theta}|_{-\pi}^{\pi} = 0.$
This example obeys the FTCC because $f(z) = z$ is analytic everywhere; ■

– 39.2: $\int_{\mathcal{C}} \frac{1}{z} dz$

Solution: $\int_{-\pi}^{\pi} id\theta = 2\pi i.$
This example does not obey FTCC because $f(z) = 1/z$ is not analytic at $z = 0$ (inside C), instead it satisfies CT-2; ■

– 39.3: $\int_{\mathcal{C}} \frac{1}{z^2} dz$

Solution: $\int_{\mathcal{C}} \frac{dz}{z^2} = \int_{-\pi}^{\pi} e^{-i2\theta} de^{i\theta} = \int_{-\pi}^{\pi} e^{-i\theta} id\theta = -\frac{e^{-i\theta}}{\theta}|_{-\pi}^{\pi} = -\frac{e^{-i\pi}}{\pi} + \frac{e^{i\pi}}{\pi} = \frac{1-1}{\pi} = 0.$
This example obeys the FTCC because the residue is $-e^{-i\theta}/\theta$, and the loop is closed (starting and ending points are the same). ■

– 39.4: $I = \int_{\mathcal{C}} \frac{1}{(z+2\jmath)^2} dz.$

Recall that the path of integration is the unit circle, starting and ending at -1.

Solution: Let $\zeta = z + 2\jmath$, then the limits become $[-1 + 2\jmath, 1 + 2\jmath] = [2\jmath + e^{-\jmath\pi}, 2\jmath + e^{\jmath\pi + 2\jmath}]$.

$$I = \int_{\mathcal{C}} \frac{d\zeta}{\zeta^2} = \int_{-\pi}^{\pi} e^{-i2\theta} de^{i\theta} = \int_{-\pi}^{\pi} e^{-i\theta} id\theta = -\frac{e^{-i\theta}}{\theta}|_{-\pi}^{\pi} = -\frac{e^{-i\pi}}{\pi} + \frac{e^{i\pi}}{\pi} = \frac{1-1}{\pi} = 0.$$

This example reduces to the case of (3), and therefore must have the same conclusion as (3). But in this case the reasoning is different because the second order pole (singular point) is outside the unit circle, thus the function is analytic inside $\mathcal{C}$, so CT-1 applies. ■

Problem # *40: FTCC and integration in the complex plane*
Let the function $F(z) = c^z$, where $c \in \mathbb{C}$ is given for each question. *Hint: Can you apply the FTCC?*

– 40.1: For the function $f(z) = c^z$, where $c \in \mathbb{C}$ is an arbitrary complex constant, use the Cauchy-Riemann (CR) equations to show that $f(z)$ is analytic for all $z \in \mathbb{C}$.

Solution: We may rewrite this function as $f(z) = e^{\ln(c)z}$, where $z = x + iy$ and $f = u + iv$. Thus

$$u(x, y) = e^{\ln(c)x} \cos(\ln(c)y),$$
$$v(x, y) = e^{\ln(c)x} \sin(\ln(c)y)$$
$$\frac{\partial u}{\partial x} = \ln(c)\, e^{\ln(c)x} \cos(\ln(c)y) = \frac{\partial v}{\partial y} = \ln(c)\, e^{\ln(c)x} \cos(\ln(c)y)$$

$$\frac{\partial u}{\partial y} = -\ln(c)\, e^{\ln(c)x} \sin(\ln(c)y) = -\frac{\partial v}{\partial x} = -\ln(c)\, e^{\ln(c)x} \sin(\ln(c)y)$$

Thus the CR conditions are satisfied everywhere and the function is analytic for all $z \in \mathbb{C}$. ■

– 40.2: Find the antiderivative of $F(z)$.

Solution: Since $c^z = e^{\ln(c)z}$, the indefinite integral (anti-derivative) is

$$I(z) = \frac{1}{\ln c} e^{\ln(c)z} \quad \text{since} \quad \frac{d}{dz} I(z) = \frac{d}{dz} \frac{1}{\ln c} e^{\ln(c)z} = e^{\ln(c)z} = F(z).$$

■

– 40.3: $c = 1/e = 1/2.7183, \ldots$ where $\mathcal{C}$ is $\zeta = 0 \rightarrow i \rightarrow z$

Solution: The integrand is $F(z) = e^{-z}$, which is *entire*. Thus the the integral is independent of the path (i.e., $\mathcal{C}$ is not relevant to the final answer).

$$I(z) = \int_0^i e^{-\zeta} d\zeta + \int_i^z e^{-\zeta} d\zeta = F(z) - F(i) + F(i) - F(0)$$
$$= \int_0^z e^{-\zeta} d\zeta = -e^z \Big|_0^z = -(e^{-z} - 1)$$

■

– 40.4: $c = 2$, where $\mathcal{C}$ is $\zeta = 0 \rightarrow (1 + i) \rightarrow z$

Solution: The integrand is $F(z) = 2^z$, where $2 = e^{\ln 2}$. The path $\mathcal{C}$ is not relevant to the final answer.

$$I(z) = \int_0^z 2^\zeta d\zeta = \int_0^z e^{\zeta \ln 2} d\zeta = \frac{e^{\zeta \ln 2}}{\ln 2}\Big|_0^z = (e^{z \ln 2} - 1)/\ln 2 \approx 1.443(e^{0.693z} - 1)$$

■

– 40.5: $c = i$, where the path $\mathcal{C}$ is an inward spiral described by $z(t) = 0.99^t e^{i2\pi t}$ for $t = 0 \rightarrow t_0 \rightarrow \infty$

Solution: $i = e^{i\pi/2} e^{i2\pi n}$. We have already proved that the path doesn't matter for any $F(z) = c^z$, so we just need to evaluated $z(t)$ for $t = 0$ and $t \rightarrow \infty$. This gives $z(0) = 1$ and $z(t \rightarrow \infty) = 0$.

$$I = \int_{z(0)}^{z(t\rightarrow\infty)} i^z dz = \int_{z(0)}^{z(t\rightarrow\infty)} e^{i\pi z/2} dz = \frac{2e^{i\pi z/2}}{i\pi}\Big|_1^0 = \frac{2}{i\pi}(1 - e^{i\pi/2}) = \frac{-2(i + 1)}{\pi}$$

■

– 40.6: $c = e^{t-\tau_0}$, where $\tau_0 > 0$ is a real number and $\mathcal{C}$ is $z = (1 - i\infty) \rightarrow (1 + i\infty)$.
Hint: Do you recognize this integral? If you do not, please do not spend a lot of time trying to solve it via the "brute force" method.

Solution: This is the basically the inverse Laplace transform of $e^{-\tau_0 z}$, we are just missing the scale factor $\frac{1}{2\pi i}$.

$$I(t) = \int_{1-i\infty}^{1+i\infty} e^{(t-\tau_0)z} dz = \int_{1-i\infty}^{1+i\infty} e^{-\tau_0 z} e^{zt} dz = 2\pi i \delta(t - \tau_0)$$

■

4.4.3 Cauchy's theorems CT-1, CT-2, CT-3

There are three basic definitions related to Cauchy's integral formula. They are all related and can greatly simplify integration in the complex plane. When a function depends on a complex variable, we use uppercase notation, consistent with the engineering literature for the Laplace transform.

Problem # *41: Describe the relationships between the theorems:*

– *41.1: CT-1 and CT-2*
Solution: When z_0 falls outside of $\mathcal{C}$, CT-2 reduces to CT-1.■

– *41.2: CT-1 and CT-3*
Solution: When there are no poles inside $\mathcal{C}$, all the residues are zero, and CT-3 reduces to CT-1.■

– *41.3: CT-2 and CT-3*
Solution: Case CT-2 has only one induced pole at $z = z_0$, having residue $F(z_0)$. Thus CT-3 is the same as CT-2 when $K = 1$, the pole at z_0 is within contour $\mathcal{C}$, and the single residue is $F(z_0)$. ■

– *41.4: Consider the function with poles at $z = \pm j$,*

$$F(z) = \frac{1}{1+z^2} = \frac{1}{(z-j)(z+j)}.$$

Find the residue expansion.
Solution:

$$F(z) = \frac{j}{2}\Big(\frac{1}{z+j} - \frac{1}{z-j}\Big).$$

■

Problem # *42: Apply Cauchy's theorems to solve the following integrals. State which theorem(s) you used and show your work.*

– *42.1: $\oint_{\mathcal{C}} F(z)dz$, where $\mathcal{C}$ is a circle centered at $z = 0$ with a radius of $\frac{1}{2}$*

Solution: Because the contour $\mathcal{C}$ does not include the poles, $F(z)$ is analytic everywhere inside $\mathcal{C}$). Using *Cauchy's integral theorem*, the integral is 0. ■

– *42.2: $\oint_{\mathcal{C}} F(z)dz$, where $\mathcal{C}$ is a circle centered at $z = j$ with a radius of 1*

Solution: Since we only enclose the pole at $z = j$, use the *integral formula* with $F(z) = 1/(z + j)$:

$$\oint_{\mathcal{C}} \frac{F(z)}{z-j} dz = 2\pi j \text{Res}_j = 2\pi j \Big[\frac{1}{z+j}\Big]_{z=j} = 2\pi j \frac{1}{2j} = \pi$$

■

– *42.3: $\oint_{\mathcal{C}} F(z)dz$, where $\mathcal{C}$ is a circle centered at $z = 0$ with a radius of 2*

Solution: Since we enclose both poles, using the *residue theorem*:

$$\oint_{\mathcal{C}} F(z)dz = 2\pi j(\text{Res}_j + \text{Res}_{-j}) = 2\pi j\Big(\frac{1}{2j} - \frac{1}{2j}\Big) = 0$$

As a side note, the inverse Laplace transform for $F(z)$ is $\sin(t)$, which is zero for $t = 0$, consistent with this result. ■

Integration of analytic functions

Problem # 43: *In the following questions, you'll be asked to integrate $F(s) = u(\sigma, \omega) + iv(\sigma, \omega)$ around the contour $\mathcal{C}$ for complex $s = \sigma + i\omega$,*

$$\oint_{\mathcal{C}} F(s) ds. \tag{4.48}$$

Follow the directions carefully for each question. When asked to state where the function is and is not analytic, you are not required to use the Cauchy-Riemann equations

– *43.1: $F(s) = \sin(s)$*

Solution: Analytic everywhere. $-\cos(s) = \int_{\theta=0}^{2\pi} \sin(s) ds = 0$. This function is *entire* (i.e., has no poles) so the integral must be zero. ∎

– *43.2: Given function $F(s) = \frac{1}{s}$ State where the function is and is not analytic.*

Solution: Analytic everywhere except at $s = 0$, where it has a pole.∎

– *43.3: Explicitly evaluate the integral when $\mathcal{C}$ is the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$.*

Solution:

$$\oint_{\mathcal{C}} F(s) ds = \int_0^{2\pi} \frac{1}{e^{i\theta}} i e^{i\theta} d\theta = \int_0^{2\pi} i d\theta = 2\pi i$$

∎

– *43.4: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.*

Solution: The residue is 1 so the integral is $2\pi i$. ∎

– *43.5: $F(s) = \frac{1}{s^2}$ State where the function is and is not analytic.*

Solution: Analytic everywhere except at $s = 0$, where it has a 2^{nd} order pole.∎

– *43.6: Explicitly evaluate the integral when $\mathcal{C}$ is the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$.*

Solution:

$$\oint_{\mathcal{C}} F(s) ds = \int_0^{2\pi} \frac{1}{e^{i2\theta}} i e^{i\theta} d\theta = \int_0^{2\pi} i e^{-i\theta} d\theta = i \frac{1}{-i} e^{-i\theta} \Big|_0^{2\pi} = 1(e^{-i2\pi} - e^0) = 0$$

∎

– *43.7: What does your result imply about the residue of the second-order pole at $s = 0$?*

Solution: The residue is 0. ∎

– *43.8: $F(s) = e^{st}$: State where the function is and is not analytic.*

Solution: Analytic everywhere.∎

– *43.9: Explicitly evaluate the integral when $\mathcal{C}$ is the square $(\sigma, \omega) = (1, 1) \to (-1, 1) \to (-1, -1) \to (1, -1) \to (1, 1)$.*

Solution: When you perform this integral piece-wise, you will find that all terms cancel out and the result is 0. ∎

– *43.10: Evaluate the same integral using Cauchy's theorem and/or the residue theorem.*

Solution: The function is analytic everywhere, so the integral is 0 by Cauchy's theorem.∎

– *43.11: $F(s) = \frac{1}{s+2}$: State where the function is and is not analytic.*

Solution: Analytic everywhere except at $s = -2$, where it has a pole.∎

– *43.12: Let $\mathcal{C}$ be the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.*

Solution: The function is analytic everywhere inside $\mathcal{C}$, so the integral is 0 by Cauchy's theorem. ∎

– 43.13: Let $\mathcal{C}$ be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Solution: This contour contains the pole. The residue is 1, therefore the integral is equal to $2\pi i$. ∎

– 43.14: $F(s) = \frac{1}{2\pi i}\frac{e^{st}}{(s+4)}$ State where the function is and is not analytic.
Solution: Analytic everywhere except at $s = -4$, where it has a pole.∎

– 43.15: Let $\mathcal{C}$ be a circle of radius 3, defined as $s = 3e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate the integral using Cauchy's theorem and/or the residue theorem.

Solution: This contour contains the pole. The residue is $\frac{1}{2\pi i}e^{-2t}$, therefore the integral is equal to e^{-2t}. ∎

– 43.16: Let $\mathcal{C}$ contain the entire left half s plane. Evaluate the integral using Cauchy's theorem and/or the residue theorem. Do you recognize this integral?
Solution: This contour contains the pole. The residue is $\frac{1}{2\pi i}e^{-2t}$, therefore the integral is equal to e^{-2t}. This contour is the inverse Laplace transform. ∎

– 43.17: $F(s) = \pm\frac{1}{\sqrt{s}}$ (e.g., $F^2 = \frac{1}{s}$) State where the function is and is not analytic.
Solution: Analytic everywhere except $s = 0$, where there is a pole.∎

– 43.18: This function is multivalued. How many Riemann sheets do you need in the domain (s) and the range (f) to fully represent this function? Indicate (e.g., using a sketch) how the sheet(s) in the domain map to the sheet(s) in the range.

Solution: There are 2 sheets in the domain (for the $\pm$ square root) which map to 1 sheet in the range.∎

– 43.19: Explicitly evaluate the integral $\int_{\mathcal{C}} \frac{1}{\sqrt{z}}dz$ when $\mathcal{C}$ is the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 2\pi$. Is this contour closed? State why or why not.

Solution: The solution is

$$2\sqrt{z}\Big|_{\theta=0}^{2\pi} = 2e^{j\theta/2}\Big|_{0}^{2\pi} = 2(e^{j\pi} - e^0) = -4.$$

In polar coordinates

$$\begin{aligned}\int_0^{2\pi}\frac{ds}{\sqrt{s}} &= \int_0^{2\pi}\frac{de^{i\theta}}{e^{i\theta/2}}\\ &= i\int_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta/2}}d\theta\\ &= i\int_0^{2\pi} e^{i\theta/2}d\theta\\ &= 2\; e^{i\theta/2}\Big|_0^{2\pi}\\ &= 2[e^{i\pi} - 1] = 2(-2) = -4.\end{aligned}$$

This contour is *not* closed. One way to determine this is to see if going once around the unit circle returns $F(s)$ to its original value.

$$F(e^{i0}) = 1 \ne F(e^{i2\pi}) = e^{-i\pi} = -1.$$

∎

– 43.20: Explicitly evaluate the integral $\int_{\mathcal{C}} \frac{1}{\sqrt{z}}dz$ when $\mathcal{C}$ is twice *around the unit circle, defined as $s = e^{i\theta}$, $0 \le \theta \le 4\pi$. Is this contour closed? State why or why not. Hint: Note that $\sqrt{e^{i(\theta+2\pi)}} = \sqrt{e^{i2\pi}e^{i\theta}} = e^{i\pi}\sqrt{e^{i\theta}} = -1\sqrt{e^{i\theta}}$.*

Solution:

$$\begin{aligned}\int_0^{4\pi} \frac{ds}{\sqrt{s}} &= \int_0^{4\pi} \frac{de^{i\theta}}{e^{i\theta/2}} \\ &= i\int_0^{4\pi} \frac{e^{i\theta}}{e^{i\theta/2}} d\theta \\ &= i\int_0^{4\pi} e^{i\theta/2} d\theta \\ &= 2\ e^{i\theta/2}\Big|_0^{4\pi} \\ &= 2[e^{i2\pi} - 1] = 2(0) = 0.\end{aligned}$$

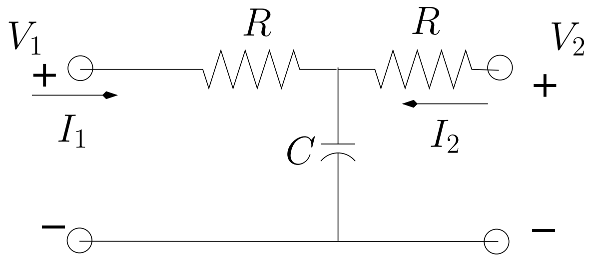
This contour *is* closed. One way to determine this is to see if going twice around the unit circle returns $F(s)$ to its original value. $F(e^{i0}) = 1 = F(e^{i4\pi}) = e^{-i2\pi} = 1$. ∎

– 43.21: What does your result imply about the residue of the (twice-around $\frac{1}{2}$ order) pole at $s = 0$?
Solution: The residue is 0. ∎

– 43.22: Show that the residue is zero. Hint: Apply the definition of the residue.
Solution: $c_{-1} = \lim_{z\to z_k} z/\sqrt{z} = \lim_{z\to z_k} \sqrt{z} = 0$. ∎

4.4.4 Laplace transform applications

Problem # *44: A two-port network application for the Laplace transform*



This three-element electrical circuit is a system that acts to low-pass filter the signal voltage $V_1(\omega)$, to produce signal $V_2(\omega)$. It is convenient to define the dimensionless ratio $s/s_c = RCs$ in terms of a time constant $\tau = RC$ and cutoff frequency $s_c = 1/\tau$.

4.4.5 Computer exercises with Matlab/Octave

Problem # *45: With the help of a computer*
Now we look at a few important concepts using Matlab/Octave's `syms` commands or Wolfram Alpha's symbolic math toolbox.[14]

For example, to find the Taylor series expansion about $s = 0$ of

$$F(s) = -\log(1-s),$$

we first consider the derivative and its Taylor series (about $s = 0$)

$$F'(s) = \frac{1}{1-s} = \sum_{n=0}^{\infty} s^n.$$

Then, we integrate this series term by term:

$$F(s) = -\log(1-s) = \int^s F'(s)ds = \sum_{n=0}^{\infty} \frac{s^n}{n}.$$

[14] `https://www.wolframalpha.com/`

Alternatively we can use Matlab/Octave commands:

```
syms s
taylor(-log(1-s),'order',7)
```

– 45.1: Use Octave's `taylor(-log(1-s))` to the seventh order, as in the example above. Try the above Matlab/Octave commands. Give the first seven terms of the Taylor series (confirm that Matlab/Octave agrees with the formula derived above).

Solution:

$$F(s) = \cdots + \frac{s^7}{7} + \frac{s^6}{6} + \frac{s^5}{5} + \frac{s^4}{4} + \frac{s^3}{3} + \frac{s^2}{2} + s$$

■

– 45.2: What is the inverse Laplace transform of this series? Consider the series term by term.

Solution: $f(t) = \sum \delta^{(n)}/n$■

– 45.3: The function $1/\sqrt{z}$ has a branch point at $z = 0$; thus it is singular there. Can you apply Cauchy's integral theorem when integrating around the unit circle?

Solution: No, one cannot apply the Cauchy Theorem since it is not analytic at $z = 0$. But the integral may be evaluated. ■

– 45.4: This Matlab/Octave code computes $\int_0^{4\pi} \frac{dz}{\sqrt{z}}$ using Matlab's/Octave's symbolic analysis package.

Run the following script:

```
syms  z
I=int(1/sqrt(z))
J = int(1/sqrt(z),exp(-j*pi),exp(j*pi))
eval(J)
```

What answers do you get for I and J?

Solution: This script returns the answers $I = 2 * \sqrt{z}$ and $J = 2.4493e - 16$, which is numerically the same as zero.■

– 45.5: Modify this code to integrate $f(z) = 1/z^2$ once *around the unit circle. What answers do you get for I and J?*

Solution: This function has a 2d order pole at $s = 0$. Thus from the CIT, the integral evaluates to zero.

Proof:

$$I = \oint \frac{ds}{s^2} = -\left.\frac{1}{s}\right|_0^{2\pi} = -\left. e^{-i\theta}\right|_0^{2\pi} = -(1-1) = 0$$

More generally $I = \oint \frac{ds}{s^n} = 0$ for $n \neq 1$. As best I know, this holds for any $n \in \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{R}, \mathbb{C}$. For $n = 1$ it has a value of $2\pi j$. ■

– 45.6: Bessel functions can describe waves in a cylindrical geometry.

The Bessel function has a Laplace transform with a branch cut

$$J_0(t)u(t) \leftrightarrow \frac{1}{\sqrt{1+s^2}}.$$

Draw a hand sketch showing the nature of the branch cut. Hint: Use `zviz`.

Solution: The roots are given by $s_\pm = \pm \jmath$. The branch cut connects the two roots, or can go from each root to ∞. Either choice is valid.■

Problem # *46: Matlab/Octave exercises*

– 46.1: Try the following Matlab/Octave commands, and then comment on your findings.

```
syms s
I=laplace{\frac{1}{\sqrt{1+s^2}}};
disp(I)
```

Solution: $I = J_o(t)u(t)$.■
Find the Taylor series of the LT

```
T = taylor(1/\sqrt(1+s^2),10); disp(T);
```

Solution:

$$T = \cdots + \frac{3s^4}{8} - \frac{s^2}{2} + 1$$

■

Verify the following:

```
syms t
J=laplace(besselj(0,t));
disp(J);
```

Solution: $I = \frac{1}{\sqrt{1+s^2}}$.■

Plot the Bessel function and Verify

```
t=0:0.1:10*pi;
b=besselj(0,t);
plot(t/pi,b);
grid on;
```

Solution: Plot of $J_o(t)u(t)$.■

– 46.2: When did Friedrich Bessel live?
Solution: 1784–1846, in Königsberg, Germany.■

– 46.3: What did he use Bessel functions for?
Solution: Solving the Bessel equation, which is the wave equation in 2D. Bessel functions were first introduced by the Daniel Bernoulli.■

Problem # *47: Colorized plots of analytic functions. Use* `zviz` *for each of the following.*

– 47.1: Describe the plot generated by `zviz Z=s`.
Solution: It is a polar plot of the function, with intensity coding the magnitude and color coding the phase. Red is a positive real number while and blue is a negative real number. ■

– 47.2: Describe the plot generated by `zviz` $1./\sqrt{1+s^2}$.
Solution: No. The RHP has blue near the branch cut, in the RHP.■

– 47.3: Describe the plot generated by `zviz` $1./\sqrt{1-s^2}$. *Is this function a Brune impedance (i.e., does this function obey*
Solution: NO, there is a branch cut in the RHP.■

– 47.4: `zviz` $1./(1+\sqrt{s})$
Solution: Yes, it's red almost everywhere even though it has a branch cut from $[-\infty < \sigma \le -10]$. Since $1/\sqrt{s}$ has an $\mathcal{LT}^{-1}$, this function must as well. Matlab found

$$\frac{1}{\sqrt{1+s}} \leftrightarrow \frac{e^{-t}}{\sqrt{\pi}\sqrt{t}} u(t),$$

however Octave failed to find the inverse transform, (but was able to find the forward transform). ■

4.4.6 Inverse of Riemann $\zeta(s)$ function

Problem # *48: Inverse zeta function (This problem is for extra credit).*

– 48.1: Find the $\mathcal{LT}^{-1}$ of one factor of the Riemann zeta function $\zeta_p(s)$, where $\zeta_p(s) \leftrightarrow z_p(t)$. Describe your results in words. Hint: Consider the geometric series representation

$$\zeta_p(s) = \frac{1}{1-e^{-sT_p}} = \sum_{k=0}^{\infty} e^{-skT_p}, \tag{4.49}$$

for which you can look up the $\mathcal{LT}^{-1}$ of each term.
Solution: Since each term in the series is a pure delay[15]

$$z_p(t) = \delta(t))_{T_p} \equiv \sum_{k=0}^{\infty} \delta(t-kT_p) \leftrightarrow \frac{1}{1-e^{-sT_p}}. \tag{4.50}$$

■

Problem # *49: Inverse transform of products:*
The time-domain version of Eq. 4.49 may be written as the convolution of all the $z_k(t)$ factors:

$$z(t) \equiv z_2(t) \star z_3(t) \star z_5(t) \star z_7(t) \star \cdots \star z_p(t) \star \cdots, \tag{4.51}$$

where $\star$ represents time convolution.
Solution: In terms of the physics, these transmission line equations are telling us that $\zeta(s)$ may be decomposed into an infinite cascade of transmission lines, each having a delay given by $T_p = \ln \pi_p$. ■

---
[15]Here we use a shorthand double-parentheses notation to define the infinite (one-sided) sum $f(t))_T \equiv \sum_{k=0}^{\infty} f(t-kT)$.

Physical interpretation: Such functions may be generated in the time domain, as shown in Fig. 3.1, using a feedback delay of T_p seconds, described by the two equations in the Fig. 3.1 with a unity feedback gain $\alpha = -1$. Taking the Laplace transform of the system equation, we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $\zeta_p(s)$, which is an all-pole function, since

$$Q(s) = e^{-sT_n}Q(s) + V(s), \text{ or } \zeta_p(s) \equiv \frac{Q(s)}{V(s)} = \frac{1}{1 - e^{-sT_p}}. \tag{4.52}$$

Closing the feed-forward path gives a second transfer function $Y(s) = I(s)/V(s)$, namely

$$Y(s) \equiv \frac{I(s)}{V(s)} = \frac{1 - e^{-sT_p}}{1 + e^{-sT_p}}. \tag{4.53}$$

If we take $i(t)$ as the current and $v(t)$ as the voltage at the input to the transmission line, then $y_p(t) \leftrightarrow \zeta_p(s)$ represents the input impedance at the input to the line. The poles and zeros of the impedance interleave along the $j\omega$ axis. By a slight modification, $\zeta_p(s)$ may alternatively be written as

$$Y_p(s) = \frac{e^{sT_p/2} + e^{-sT_p/2}}{e^{sT_p/2} - e^{-sT_p/2}} = j\tan(sT_p/2). \tag{4.54}$$

Every impedance $Z(s)$ has a corresponding *reflectance* function given by a Möbius transformation, which may be read off of Eq. 4.54 as

$$\Gamma(s) \equiv \frac{1 + Z(s)}{1 - Z(s)} = e^{-sT_p}, \tag{4.55}$$

since impedance is also related to the round-trip delay T_p on the line. The inverse Laplace transform of $\Gamma(s)$ is the round-trip delay T_p on the line

$$\gamma(t) = \delta(t - T_p) \leftrightarrow e^{-sT_p}. \tag{4.56}$$

Working in the time domain provides a key insight, as it allows us to parse out the best analytic continuation of the infinity of possible continuations that are not obvious in the frequency domain. Transforming to the time domain is a form of analytic continuation of $\zeta(s)$ that depends on the assumption that $Z^{eta}(t) \leftrightarrow \zeta(s)$ is one-sided in time (causal).

4.4.7 Quadratic forms

A matrix that has positive eigenvalues is said to be positive-definite. The eigenvalues are real if the matrix is symmetric, so this is a necessary condition for the matrix to be positive-definite. This condition is related to conservation of energy, since the power is the voltage times the current. Given an impedance matrix

$$\mathbf{V} = \mathcal{Z}\mathbf{I},$$

the power $\mathcal{P}$ is

$$\mathcal{P} = \mathbf{I} \cdot \mathbf{V} = \mathbf{I} \cdot \mathcal{Z}\mathbf{I},$$

which must be positive-definite for the system to obey conservation of energy.

Problem # *50: In this problem, consider the* 2×2 *impedance matrix*

$$\mathcal{Z} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

– 50.1: Solve for the power $\mathcal{P}(i_1, i_2)$ *by multiplying out this matrix equation (which is a* quadratic form*):*

$$\mathcal{P}(i_1, i_2) = \mathbf{I}^T \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{I}.$$

Solution:

$$\mathcal{P}(i_1, i_2) = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} 2i_1 + i_2 \\ i_1 + 4i_2 \end{bmatrix} = 2i_1^2 + 2i_1 i_2 + 4i_2^2.$$

■

– 50.2: Is the impedance matrix positive-definite? Show your work by finding the eigenvalues of the matrix $\mathbf{Z}$.

Solution: Yes, as it is positive-definite if the eigenvalues are both positive. You need to show that the eigenvalues are positive (not zero or negative). They are, so it is.

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 3 \pm \sqrt{2} > 0$$

■

– 50.3: Should an impedance matrix always be positive-definite? Explain.

Solution: Yes.■

4.5 The $\mathcal{LT}$ and its inverse $\mathcal{LT}^{-1}$

The Laplace transform $\mathcal{LT}$ take causal time functions into the complex–analytic frequency domain s. The inverse Laplace transform $\mathcal{LT}^{-1}$ (Eq. 3.2), transforms a function of complex frequency $F(s)$ and returns a causal function of time $f(t)$,

$$f(t)u(t) \leftrightarrow F(s),$$

where $u(t) = 0$ for $t < 0$. Examples are provided in Table 6. The forward transform is typically a relatively simple set of integrals to find $F(s)$. However the inverse transform is the key to understanding this powerful tool. Here we discuss the details of finding the inverse transform by using CT-3, and we see how the causal requirement $f(t < 0) = 0$ comes about.

As shown in Fig. 4.5, the integrand of the inverse transform is $F(s)e^{st}$ and the limits of integration are $\sigma_o \mp \omega \jmath$ with $\sigma_o > 0$. To find the inverse using CT-3 we must close the curve at $\omega \to \infty$, and specify that the integral converges. There are two ways to close the integral: to the right $\sigma > 0$ (RHP), and to the left $\sigma \le 0$ (LHP). But there must be some logical reason for this choice. That logic is determined by the sign of t. For the integral to converge, the term $|e^{st}| = e^{\sigma t}$ must go to zero as $\omega \to \infty$. Note that both t and ω go to ∞. Thus it is the interaction between these two limits that determines how we pick the closure, RHP or LHP.

4.5.1 Case for negative time ($t < 0$) and causality:

Let us first consider negative time, including $t \to -\infty$. If we were to close $\mathcal{C}$ in the LHP ($\sigma < 0$), then the product σt is positive ($\sigma < 0, t < 0$, thus $\sigma t > 0$). In this case, as $\omega \to \infty$, the closure integral $|s| \to \infty$ will diverge. If we close in the RHP ($\sigma > 0$), then the product $\sigma t < 0$ and e^{st} will go to zero as $\omega \to \infty$. Thus we may not close in the LHP for negative time. This then justifies closing the contour, allowing for the use of Cauchy CT-3.

4.5.2 Case for zero time ($t = 0$):

When the time is zero, the integral does not, in general, converge, which leaves $f(t)$ undefined. This is most obvious in the case of the step function $u(t) \leftrightarrow 1/s$, where the integral may not be closed because the convergence factor $e^{st} = 1$ fails for $t = 0$.

The fact that $u(t)$ does not exist at $t = 0$ helps to explain the-Gibbs phenomenon in the inverse Fourier transform. At the time where a jump occurs, the derivative of the function does not exist, and thus the time response function is not analytic. The Fourier expansion does not point-wise converge where the function is not analytic. A low-pass filter may be used to smooth the function, but at the cost of temporal resolution.

4.5.3 Case for positive time ($t > 0$)

Next we investigate the convergence of the integral for positive time $t > 0$. In this case we must close the integral in the LHP ($\sigma < 0$) for convergence, so that $st < 0$ ($\sigma \le 0$ and $t > 0$). When there are poles on the

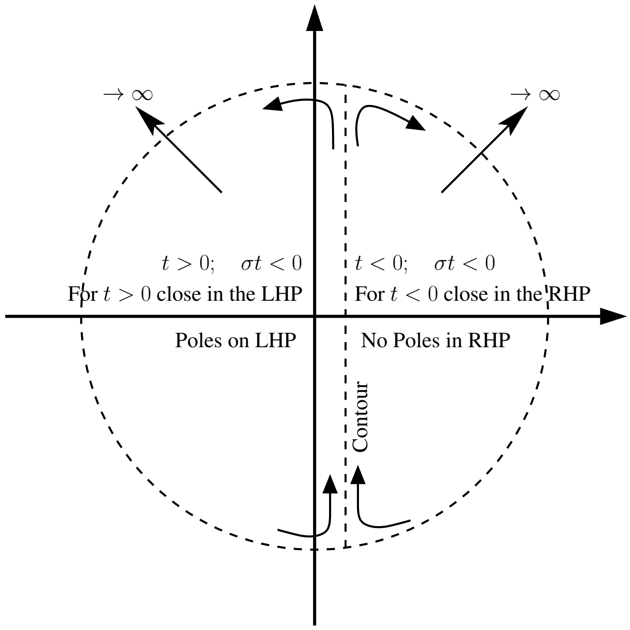


Figure 4.5: *When computing the $\mathcal{LT}^{-1}$ we must take advantage of the powerful Cauchy residue integral theorem (CT-3). To use CT-3 we must close the integral at $s \to \infty$. Furthermore this closure integral must go to zero as $\{t, \sigma\} \to \infty$. For $t < 0$, the convergence of the residue $e^{-\sigma t}$ depends on $\sigma t < 0$, since the product st must have a negative real part. For convergence for $t < 0$, $\sigma > 0$ so that $\sigma t < 0$. Thus the integral must be close in in the RHP. Likewise, for $t > 0$, $\sigma < 0$, so that $\sigma t < 0$. Thus the integral must be close in the LHP. Following these guidelines based on CT-3, poles in the LHP lead to causal stable solutions, since there $\sigma < 0$ and $t \to \infty$, while poles in the RHP will lead to causal unstable solutions, each the form $e^{\sigma t}u(t)$.*

$\omega\jmath = 0$ axis, $\sigma_o > 0$ assures convergence by keeping the on-axis poles inside the contour. At this point, CT-3 is relevant. If we restrict ourselves to simple poles (as required for a Brune impedance), the residue theorem may be directly applied.

Unstable poles: An important but subtle point arises: If $F(s)$ has a pole in the RHP, then the above argument still applies if we pick σ_o to be to the right of the RHP pole. This means that the inverse transform may still be applied to unstable poles (those in the RHP). This then explains the need for the σ_o in the limits. If $F(s)$ has no RHP poles in the extended RHP ($\sigma \geq 0$), we may take $\sigma_o = 0$.

The simplest example is the step function, for which $F(s) = 1/s$, thus

$$u(t) = \oint_{\text{LHP}} \frac{e^{st}}{s} \frac{ds}{2\pi\jmath} \leftrightarrow \frac{1}{s},$$

which is a direct application of CT-3. The forward transform of $u(t)$ is straightforward, as discussed in Appendix 5. This is true of most of the elementary forward Laplace transforms. In these cases, causality may be built into the integral by the limits. An interesting problem is showing that $u(t)$ is not defined at $t = 0$.

The inverse Laplace transform of $F(s) = 1/(s + 1)$ has a residue of 1 at $s = -1$, thus that is the only contribution to the integral. More demanding cases are Laplace transform pairs

$$\frac{1}{\sqrt{t}}u(t) \leftrightarrow \sqrt{\frac{\pi}{s}} \quad \text{and} \quad J_o(t)u(t) \leftrightarrow \frac{1}{\sqrt{s^2 + 1}},$$

as shown in Fig 4.6 (right), and more in Table 3.9. Many of these are easily proved in the forward direction but are much more difficult in the inverse direction due to the properties at $t = 0$, unless CT-3 is invoked.

Along the x-axis of Fig. 4.6, $\cos(\pi x)$ is periodic with a period of π. The dark spots are at the zeros at $\pm\pi/2, \pm 3\pi/2, \ldots$. Along the $\jmath y$-axis, the function goes to either zero (black) or ∞ (white). This behavior carries the same π periodicity as on the $x = 0$ line.

The last $\mathcal{LT}$ example of Fig. 4.7 gives an important insight into the properties of the Hankel function $H_0^{(1)}(\pi z/2)$, which has a branch cut along the negative real axis. On the right is the Hankel function $H_0^{(1)}(\pi z/2)$, which is a mixed and distorted version of $\cos(\pi z)$ with the zeros pushed downward, and $e^{\pi z}$. Note how the white and black contour lines of the colorized maps are always perpendicular where they cross.

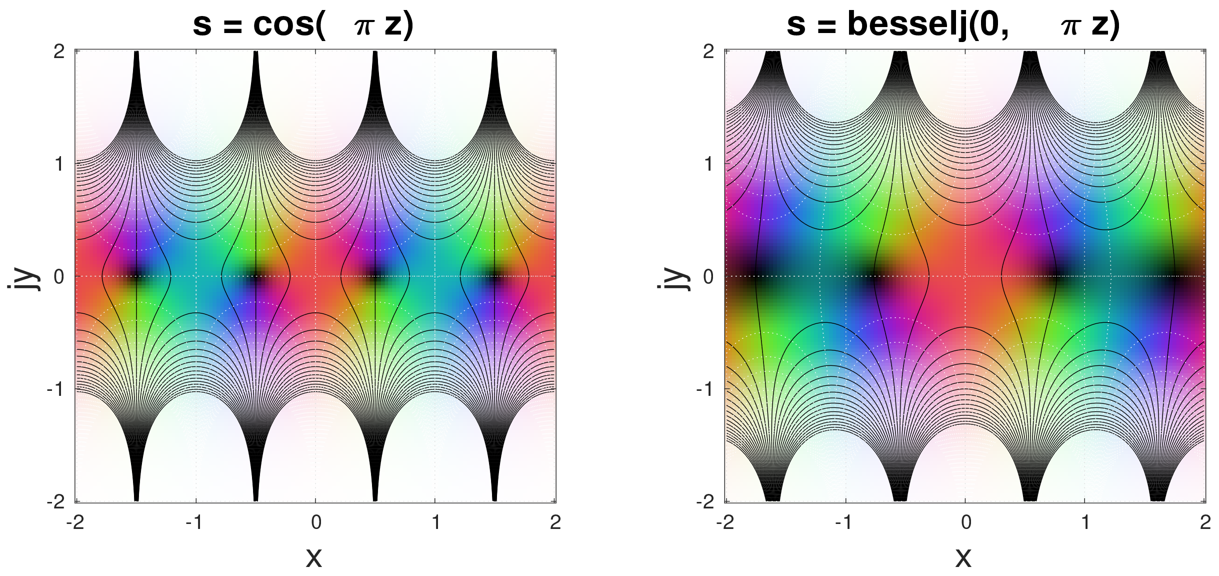


Figure 4.6: *Left: Colorized plot of $w(z) = sin(z)$. Right: Colorized plot of $w(z) = J_o(\pi z)$. Note the similarity of the two functions. The first Bessel zero is at 2.405 and thus appears at $0.7655 = 2.405/\pi$, about 1.53 times larger than the root of $\cos(\pi z)$ at $1/2$. Other than this minor distortion of the first few roots, the two functions are basically identical. It follows that their $\mathcal{LT}$s must have similar characteristics, as documented in Table 3.9. These colorized plots show that these two functions become the same for $x = \Re z > 0$. The black lines indicate where the function has a constant real part.*

4.5.4 Properties of the $\mathcal{LT}$

As shown in Table 3.3 of Laplace transforms, there are integral (i.e., integration, not integer) relationships, or properties, that are helpful to identify. The first of these is a definition, not a property:

$$f(t) \leftrightarrow F(s).$$

Causality: When we take the $\mathcal{LT}$, the time-domain response is in lowercase (e.g., $f(t)$) and the frequency-domain transform is in uppercase (e.g., $F(s)$). It is required, but not always explicitly specified, that $f(t < 0) = 0$; that is, the time function must be causal, as stated by Postulate P1.

Linearity: The most basic property is the linearity (superposition) property of the $\mathcal{LT}$, stated by Postulate P2.

Convolution property: The product of two $\mathcal{LT}$s in frequency results in convolution in time:

$$F(s)G(s) \leftrightarrow f(t) \star g(t) = \int_0^t f(\tau)g(t-\tau)d\tau,$$

where we use $\star$ to indicate the convolution of two time functions.

A key application of convolution is filtering, which takes many forms. The most basic filter is the moving average, the moving sum of data samples, normalized by the number of samples. Such a filter has very poor performance. It also introduces a delay of half the length of the average, which may or may not constitute a problem, depending on the application. Other important examples are a low-pass filter that removes high-frequency noise and a notch filter that removes line noise (i.e., 60 [Hz] in the United States, and its second and third harmonics, 120 and 180 [Hz]). Such noise is typically a result of poor grounding and ground loops. It is better to solve the problem at its root than to remove it with a notch filter. Still, filters are very important in engineering.

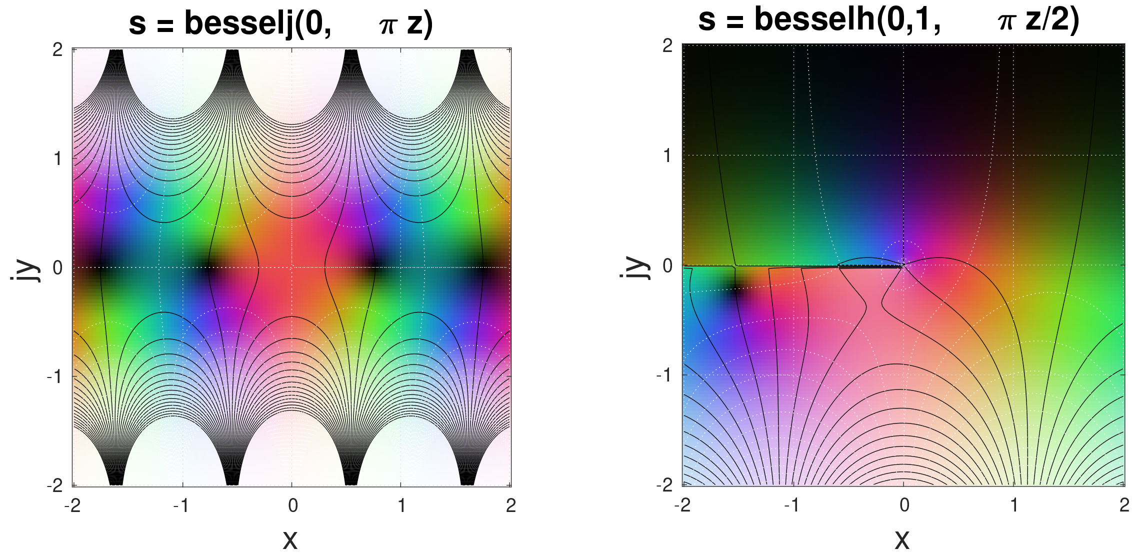


Figure 4.7: *Left: The Bessel function $J_0(\pi z)$, which is similar to $\cos(\pi z)$, except the zeros are distorted away from $s = 0$ by a small amount, due to the cylindrical geometry. Right: The related Hankel function $H_0^{(1)}(\pi z/2)$. The Hankel function $H_0^{(1)}(\pi z/2)$ has a branch cut and a complex zero at $z_{0,1} 2/\pi = -1.5 - 0.1\jmath$, as may be seen in the plot.*

By taking the $\mathcal{LT}$ of the convolution we can derive this relationship:

$$\begin{aligned}
\int_0^\infty [f(t) \star g(t)] e^{-st} dt &= \int_{t=0}^{\infty} \left[\int_0^t f(\tau) g(t-\tau) d\tau \right] e^{-st} dt \\
&= \int_0^t f(\tau) \left(\int_{t=0}^{\infty} g(t-\tau) e^{-st} dt \right) d\tau \\
&= \int_0^t f(\tau) \left(e^{-s\tau} \int_{t'=0}^{\infty} g(t') e^{-st'} dt' \right) d\tau \\
&= G(s) \int_0^t f(\tau) e^{-s\tau} d\tau \\
&= G(s) F(s).
\end{aligned}$$

We first encountered this relationship in the context of multiplying polynomials, which is the same as convolving their coefficients. The parallel should be obvious. In the case of polynomials, the convolution is discrete in the coefficients, and here it is continuous in time. But the relationships are the same.

Time-shift property: When a function is time-shifted by time T_o, the $\mathcal{LT}$ is modified by e^{-sT_o}, leading to the property

$$f(t - T_o) \leftrightarrow e^{-sT_o} F(s).$$

This is easily shown by applying the definition of the $\mathcal{LT}$ to a delayed time function.

Time derivative: The key to the eigen-function analysis provided by the $\mathcal{LT}$ is the transformation of a time derivative on a time function—that is,

$$\frac{d}{dt} f(t) \leftrightarrow sF(s).$$

Here s is the eigenvalue corresponding to the time derivative of e^{st}. Given the definition of the derivative of e^{st} with respect to time, this definition seems trivial. Yet that definition was not obvious to Euler. It needed to be extended to the space of the complex–analytic function e^{st}, which did not happen before Cauchy's key results.

Given a differential equation of order K, the $\mathcal{LT}$ results in a polynomial in s of degree K. It follows that this $\mathcal{LT}$ property is the cornerstone of why the $\mathcal{LT}$ is so important to scalar differential equations, as it was to the early analysis of Pell's equation and the Fibonacci sequence, presented in chapter 2. While the relation $e^{\jmath\theta} = \cos\theta + \jmath \sin\theta$ was first uncovered by Euler. By the time of his death the formula's significance would have been clear to him. Who first coined the terms *eigenvalue* and *eigen-function*? The word *eigen* is a German word meaning *of one*.

Initial and final value theorems: There are much more subtle relationships between $f(t)$ and $F(s)$ that characterize $f(0^+)$ and $f(t \to \infty)$, which are known as initial value theorems. If the system under investigation has potential energy at $t = 0$, then the voltage (velocity) need not be zero for negative time. An example is a charged capacitor or a moving mass. These are important concepts, but best explored in a more in-depth treatment. They are not violations of causality.

Chapter 5

Laplace transforms $\mathcal{LT}$

Laplace transforms are discussed in Sec. 3.9 (p. 122), with the definition of the $\mathcal{LT}$ in Eq. 3.2 (p. 123). Level-I (basic) $\mathcal{LT}$s are listed in Table 3.8 (p. 119).

5.1 Tables of Laplace transforms

The following tables of $\mathcal{LT}$ and $\mathcal{LT}^{-1}$ are a convenient summary of their properties and evaluations for many different functions. Table 3.3 gives basic function properties such as convolution and the properties of step functions and frequency scaling. Table 5.1 provides the commands for doing symbolic (computer algebra and calculus) transformations, which includes some unusual $\mathcal{LT}$s and Taylor series of the $\Gamma(s)$ function (Graham et al., 1994), a complex–analytic extension of the factorial. Table 6 gives the basic transforms typically used for more common calculations. Table 3.9 provides extended less common transforms, such as the half-derivative and integration and Bessel functions.

These tables are available in most books on differential equations and remain a core technology for analytic methods for solving differential equations.

Table 5.1: *Symbolic relationships among Laplace transforms. K_3 is a constant.*

syms	command	result
syms t s p	laplace$\left(t^{(p-1)}\right)$	$\Gamma(p)s^{-p}$
syms s	ilaplace(gamma(s))	$e^{e^{-t}}$
syms s t a	ilaplace(exp(-a*s)/s,s,t)	Heaviside$(t-a)$
syms Gamma s t	taylor(Gamma,s,t)	$\frac{1}{s}-\gamma+s\left(\frac{\gamma^2}{2}+\frac{\pi^2}{12}\right)+$ $s^2\left(+\frac{1}{6}\,\text{polygamma}\,(2,1)-\frac{\gamma\pi^2}{12}-\frac{\gamma^3}{6}\right)+$ $s^3K_3+\cdots$

5.1.1 $\mathcal{LT}^{-1}$ of the Riemann zeta function

The analytic properties of the zeta function $\zeta(s)$ have been a holy grail for mathematicians, starting with Euler, all of whom have made their reputation on that function. For the neophyte, $\zeta(s)$ is important because it is an analytic extension of the sieve, which is the prime identification method. Analytic continuation of the $\zeta(s)$ function was first stated by Riemann, as described in his 1851 paper. But I question if his argument is correct? Review
In the case of the geometric series the analytic continuation is the closed form expression $f(s) = 1/(1-s)$ which is valid for all $s \neq 1$. This is not Riemann's definition of analytic continuation.

This section is a beginners review of $\zeta(s)$, building on the developments of analytic functions from Chapter 3, especially in Secs. 3.2.5, 5.4, and Fig. 5.3 p. 256. Well understood are the locations of the poles of zeta, which depend on the prime numbers. Not so well understood are the remaining analytic properties over the entire plane, such as the zeros of $\zeta(x)$, namely the poles of $1/\zeta(s)$. A key function is $\ln\zeta(s)$.

Consider $z \equiv e^{sT}$, where $T \in \mathbb{I}$ is the sample period, meaning samples are taken every T seconds. For example, if $T = 22.676 = 10^6/44,100$ [μs], then the data is sampled at 44.10 [kHz]. This is precisely how

Table 5.2: *An extended table of Laplace transforms. Bessel functions J_0, K_1 are of the first and second kind. The case of the error function erfc$\sqrt{\alpha t}$ may be found in Morse and Feshback (1953), p. 1582.*

$f(t)$	$\leftrightarrow$	$F(s)$	Name
$\delta(at)$	$\leftrightarrow$	$\frac{1}{a}$	$a \neq 0$; time-scaled Dirac
$\delta(t+t_0)$	$\leftrightarrow$	e^{st_0}	negative delay
$u(at)$	$\leftrightarrow$	$\frac{a}{s}$	$a \neq 0$; dilate
$u(-t)$	$\leftrightarrow$	$-\frac{1}{s}$	anticausal step
$\frac{1}{\sqrt{\pi t}}u(t)$	$\leftrightarrow$	$\frac{1}{\sqrt{s}}$	semi-capacitor
$\frac{d^{1/2}}{dt^{1/2}}f(t)u(t)$	$\leftrightarrow$	$\sqrt{s}F(s)$	half derivative (5.1)
$\frac{d^{1/2}}{dt^{1/2}}u(t)$	$\leftrightarrow$	$\sqrt{s}$	viscous damping
$\frac{d}{dt}\frac{1}{\sqrt{\pi t}}u(t)$	$\leftrightarrow$	$\frac{s}{\sqrt{s}} = \sqrt{s}$	semi-inductor
$\frac{1}{\sqrt{\pi t}}u(t)$	$\leftrightarrow$	$\frac{1}{\sqrt{s}}$	semi-capacitor
$\mathrm{erfc}(\alpha\sqrt{t})$	$\leftrightarrow$	$\frac{1}{s}e^{-2\alpha\sqrt{s}}$	$\alpha > 0$; erfc
$J_0(at)u(t)$	$\leftrightarrow$	$\frac{1}{\sqrt{s^2+a^2}}$	J-Bessel
$J_n(\omega_o t)\,u(t)$	$\leftrightarrow$	$\frac{\left(\sqrt{s^2+\omega_o^2}-s\right)^n}{\omega_o^n\sqrt{s^2+\omega_o^2}}$	

modern digital audio works, for CD-quality music. The unit-time delay time operator z^{-1} is

$$\delta(t-T) \leftrightarrow e^{-sT}.$$

When we deal with the Euler and Riemann zeta functions, the only sampling period that makes sense is $T = 1$ [s] or 1 [Hz] (i.e., $n \in \mathbb{Z}$). In this case, the samples of interest are $\mathrm{mod}(n, \pi_k)$. Starting from the sieve of Eratosthenes, Euler showed that the counting numbers $n \in \mathbb{Z}$, presented at a rate of one per second [1-Hz], may be uniquely reduced to multiples of the primes. This is the basis for the fundamental theorem of arithmetic, the theorem of the concept of the prime number, which states that every integer may be uniquely factored into a product of prime numbers.

The zeta function The poles of the zeta function depend explicitly on the primes, which makes it a very special function. In 1737 Euler proposed the real-valued function $\zeta(\sigma) \in \mathbb{R}$, and $\sigma \in \mathbb{R}$ to showed that the number of primes is infinite (Goldstein, 1973). Euler's definition of $\zeta(\sigma) \in \mathbb{R}$ is given by the analytic power series,

$$\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \qquad \text{for } \sigma > 1 \in \mathbb{R}. \tag{5.2}$$

This series converges for $\sigma > 0$, since $R = n^{-\sigma} < 1, n > 1 \in \mathbb{N}$.[1]

In 1860 Riemann extended the zeta function into the complex plane, resulting in $\zeta(s)$, defined by the complex–analytic power series, identical to the Euler formula except $x \in \mathbb{R}$ has been replaced by $s \in \mathbb{C}$;

$$Z^{eta}(t) \leftrightarrow \zeta(s) \equiv \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s} \quad \text{for } \Re\{s\} = \sigma > 1. \tag{5.3}$$

This formula converges for $\Re\{s\} > 1$ (Goldstein, 1973). To determine the formula in other regions of the s plane, we need to extend the series via analytic continuation. As it turns out, Euler's formulation provided

[1]Sanity check: For example, let $n = 2$ and $\sigma > 0$. Then $R = 2^{-\epsilon} < 1$, where $\epsilon \equiv \lim \sigma \to 0^+$. Taking the log gives $\log_2 R = -\epsilon \log_2 2 = -\epsilon < 0$. Since $\log R < 0$, $R < 1$.

detailed information about the structure of primes, going far beyond his original goal.

Euler product formula

As first published by Euler in 1737, we can recursively factor out the leading prime term, which results in Euler's product formula. Euler's procedure is an algebraic implementation of the sieve of Eratosthenes (Fig. 2.4, p. 36).

Multiplying $\zeta(s)$ by the factor $1/2^s$ and subtracting from $\zeta(s)$ remove all the powers of 2: $1/2^0 + 1/2^s + 1/2^{2s} + 1/2^{3s} + \cdots$

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \frac{1}{16^s} + \cdots\right), \tag{5.4}$$

which results in RoC?

$$\zeta_1(s) = \left(1 - 2^{-s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots. \tag{5.5}$$

Repeating this with a lead factor $1/3^s$ applied to Eq. 5.5 gives[2]

$$\frac{1}{3^s}\left(1 - 2^{-s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \cdots. \tag{5.6}$$

Subtracting Eq. 5.6 from Eq. 5.5 cancels the terms on the right-hand side of Eq. 5.5, giving

$$\zeta_2(s) = \left(1 - 3^{-s}\right)\left(1 - 2^{-s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} + \cdots.$$

If we express this in terms of the primes π_k, we can better visualize the structure:

$$\zeta_2(s) = \left(1 - \pi_2^{-s}\right)\left(1 - \pi_1^{-s}\right)\zeta(s) = 1 + \frac{1}{\pi_3^s} + \frac{1}{\pi_4^s} + \frac{1}{\pi_5^s} + \frac{1}{\pi_6^s} + \frac{1}{\pi_7^s} + \frac{1}{\pi_8^s} + \cdots.$$

Thus ζ_2 has removed primes π_1, π_2, leaving π_3 as the lead term in the series on the right-hand side.

This leads to a recursion in ζ_k,

$$\zeta_k(s) = \zeta(s)\prod_{l=1}^{k}\zeta_l(s) = 1 + \sum_{l=k+1}^{\infty}\pi_l^{-s}.$$

The series on the right-hand side converges rapidly to 1 as each prime is removed, because the RoC is becoming much larger with each recursion. Each recursive step in this construction ensures that the lead term, along with all of its multiplicative factors, is subtracted out, just like the cancellations with the sieve of Eratosthenes. It is instructive to compare each iteration with that of the sieve (see Fig. 2.4, p. 36).

Repeating this process with the remaining primes removes all the terms on the right-hand side but the first (leaving 1), which results in *Euler's analytic product formula* ($s = x \in \mathbb{R}$), or *Riemann's complex–analytic product formula* ($s \in \mathbb{C}$):

$$\begin{aligned}1 &= \zeta(s)(1 - 2^{-s}) \cdot (1 - 3^{-s}) \cdot (1 - 5^{-s}) \cdot (1 - 7^{-s}) \cdot \;\cdots\; \cdot (1 - \pi_n^{-s}) \cdot \;\cdots. \\ &= \zeta(s)\prod_{k=1}^{\infty}(1 - \pi_k^{-s}) \end{aligned} \tag{5.7}$$

$$\zeta(s) = \frac{1}{\prod_k \mathcal{P}_k(s)}, \quad \Re\{s\} = \sigma > 0, \tag{5.8}$$

where the zeros of $\mathcal{P}_k(s) = 1 - \pi_k^{-s}$ define the poles of $\zeta(s)$ for prime π_k.

[2] This is known as *Euler's sieve*, as distinguished from the *Eratosthenes sieve*.

Finding the RoC of the product formula: It would be interesting to find the RoC for $\mathcal{P}_k(s)$, and for rigor, this question demands further investigation. To find the RoC, we need to evaluate

$$|\pi_k^{-s}| = |e^{-sT_k}| = |e^{-\sigma T_k}| = \left(\frac{1}{\pi_k}\right)^{\sigma} < 1 \quad \text{for} \quad \sigma > 0,$$

where $T_k = \ln \pi_k$. For example,

$$\frac{1}{\mathcal{P}_5(s)} = \frac{1}{1 - \left(\frac{1}{5}\right)^s} = 1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} \cdots , \quad \Re\{s\} = \sigma > 0.$$

Since $1/\pi_k < 1$ for all $k \in \mathbb{N}$, the Taylor series of $\zeta_k(x)$ is entire except at its poles. Note that the RoC of a Taylor series in powers of π_k^{-s} increases with k.

Exercise #*1*

Work out the RoC for $k = 2$.

Solution: The formula for the RoC is given above, which for $\pi_2 = 3$ is

$$|\pi_k^{-s_r}| = \begin{cases} \left(\frac{1}{3}\right)^{\sigma_r} < 1 & \text{for} \quad \sigma_r > 0, \\ \left(\frac{1}{3}\right)^{-\sigma_r} < 1 & \text{for} \quad \sigma_r < 0, \end{cases}$$

where σ_r is the boundary of the RoC.

■

Exercise #*2*

Show how to construct $Z_2^{eta}(t) \leftrightarrow \zeta_2(s)$ by working in the time domain.

Solution: The basic procedure for building a sieve is to sum the integers

$$S_1 = \sum_{n=1}^{\infty} n 2^{n-1} = 1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + \cdots ,$$

while the sieve for the kth prime π_k is

$$S_k = \sum_{n=1}^{\infty} n \pi_k^{n-1} = 1 \cdot \pi_k^0 + \pi_k \cdot 2^1 + \pi_k \cdot 2^2 + \cdots .$$

This sum may be written in terms of the convolution with the Heaviside step function u_k, since

$$u_k \star u_k = n u_k = 0 \cdot u_0 + 1 \cdot u_1 + 2u_2 + \cdots + k u_k + \cdots .$$

■

Poles of $\zeta_k(s)$

Riemann proposed that Euler's zeta function $\zeta(s) \in \mathbb{C}$ has a complex argument [first explored by Chebyshev in 1850 (Bombieri, 2000)] that extends $\zeta(s)$ into the complex plane ($s \in \mathbb{C}$), thus making it a complex–analytic function. Thus we might presume that $\zeta(s)$ has an inverse Laplace transform. There seems to be very little written on this topic (Hill, 2007). We explore this question further here.

We can now identify the poles of $\zeta_k(s)$ ($p \in \mathbb{N}$), which are required to determine the RoC. For example, the kth factor of Eq. 5.8 expressed as an exponential, is

$$\zeta_k(s) \equiv \frac{1}{1 - \pi_k^{-s}} = \frac{1}{1 - e^{-sT_k}} = \sum_{k=0}^{\infty} e^{-skT_k}, \tag{5.9}$$

where $T_k \equiv \ln \pi_k$. Thus $\zeta_p(s)$ has poles at $-s_n T_p = 2\pi n \jmath$ (when $e^{-sT_p} = 1$), and

$$\omega_n = \frac{2\pi n}{T_k},$$

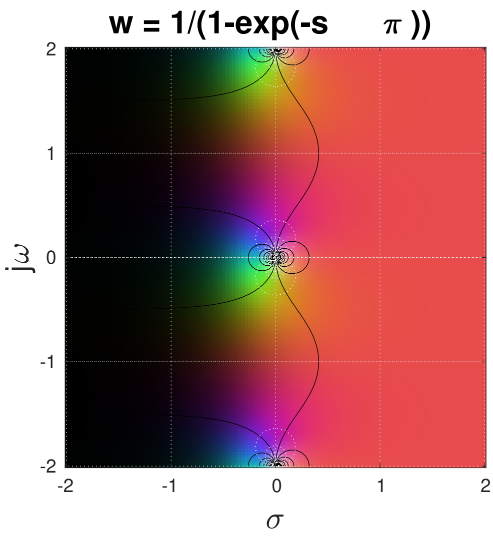


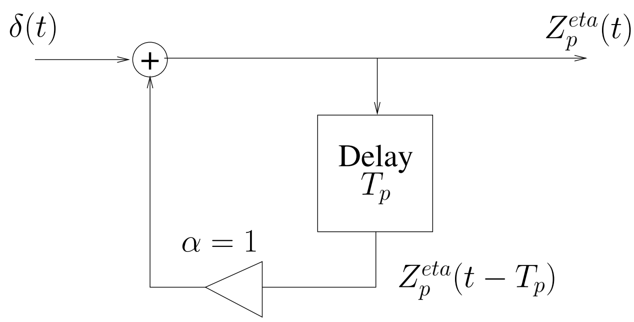
Figure 5.1: *Plot of $w(s) = \frac{1}{1-e^{-s\pi}}$. Here $w(s)$ has poles where $e^{s_n \ln 2} = 1$—namely, where $\omega_n \ln 2 = n2\pi$, as seen in the colorized map ($s = \sigma + \omega\jmath$) is the Laplace frequency [rad]).*

with $-\infty < n \in \mathbb{Z} < \infty$. These poles are the eigenmodes of the zeta function. Fig. 3.9 is a domain-colorized plot of this function. It is clear that the RoC of ζ_k is > 0. It would be helpful to determine why $\zeta(s)$ has a more restrictive RoC than each of its factors.

Inverse Laplace transform

The inverse Laplace transform of Eq. 5.9 is an infinite series of delays T_p (Table 6)[3]

$$Z_p^{eta}(t) = \delta(t))_{T_p} \equiv \sum_{k=0}^{\infty} \delta(t - kT_p) \leftrightarrow \frac{1}{1 - e^{-sT_p}}. \tag{5.10}$$



$$Z_p^{eta}(t) = \alpha Z_p^{eta}(n - T_p) + \delta(t)$$

Figure 5.2: *This feedback network is described by a time-domain difference equation with delay $T_p = \ln \pi_k$. It has an all-pole transfer causal function given by Eq. 5.12. Physically this delay corresponds to T_p [s].*

Inverse transform of the product of factors

The time-domain version of Eq. 5.8 may be written as the convolution of all the $Z_k^{eta}(t)$ factors

$$Z^{eta}(t) \equiv Z_2^{eta} \star Z_3^{eta}(t) \star Z_5^{eta}(t) \star Z_7^{eta}(t) \star \cdots \star Z_p^{eta}(t) \star \cdots , \tag{5.11}$$

where $\star$ represents time convolution (Table 3.3).

Such functions may be generated in the time domain as shown in Fig. 5.2, using a feedback delay of T_p [s] as described by the equation in the figure, with a unity feedback gain $\alpha = 1$,

$$Z^{eta}(t) = Z^{eta}(t - T_p) + \delta(t).$$

[3]Here we use a shorthand double-parentheses notation $f(t))_T \equiv \sum_{k=0}^{\infty} f(t - kT)$ to define the one-sided infinite sum.

Taking the Laplace transform of the system equation, we see that the transfer function between the state variable $q(t)$ and the input $x(t)$ is given by $Z_p^{eta}(t)$. Taking the $\mathcal{LT}$, we see that $\zeta(s)$ is a causal all-pole function,

$$\zeta_p(s) = e^{-sT_p}\zeta_p(s) + 1(t) \text{ or } \zeta_p(s) = \frac{1}{1-e^{-sT_p}}, \tag{5.12}$$

which is a key expression in the theory of Black Body radiation (Allen, 2025)

$$S(s) = \frac{-s^3}{1-e^{-\hbar s/kT}}.$$

In terms of the physics, these transmission line equations are telling us that $\zeta(s)$ may be decomposed into an infinite cascade of transmission lines (Eq. 5.11), each having a unique delay given by $T_k = \ln \pi_k$, $\pi_k \in \mathbb{P}$, the log of the primes. The input admittance of this cascade may be interpreted as an analytic continuation of $\zeta(s)$ that defines the eigenmodes of that cascaded impedance function.

Working in the time domain provides a key insight, as it allows us to determine the analytic continuation of the infinity of possible continuations, which may not be obvious in the frequency domain. Transforming to the time domain is a variant of analytic continuation of a function $Z(s) \leftrightarrow Z^{eta}(t)$ that depends on the assumption that $Z^{eta}(t)$ is one-sided in time (causal). It may be helpful to compare this variant to Euler's continuation of $\zeta(s)$, and later Riemann's classic 1851 definition of complex–analytic continuation $Z^{eta}(s)$.

What is the RoC of $\zeta(s)$? It is commonly stated that Euler's and thus Riemann's product formulas are valid only for $\Re s > 1$. I believe this to be a long-held misunderstanding. Here I argue that the product formula is entire.

Starting from the product formula (Eq. 5.8, p. 187), we form the log-derivative and study the poles and residues:

$$D(s) \equiv \frac{d}{ds}\ln \Pi_k \frac{1}{1-e^{-sT_k}} = -\sum_k \frac{T_k e^{-sT_k}}{1-e^{-sT_k}} \leftrightarrow \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\delta(t-nT_k). \tag{5.13}$$

Here $T_k = \ln \pi_k$, as previously defined, and $\leftrightarrow$ is the Laplace transform. Since $d(t)$ is causal, $D(s)$ is complex analytic.

Zeros of $\zeta(s)$ We are still left with the most important question: Where are the zeros of $\zeta(s)$? Equation 5.12 has no zeros; it is an all-pole system. The cascade of many such systems is also all-pole. As I see it, the issue is: What is the actual formula for $\zeta(s)$? To answer this question, we need to study the properties of the reflectance function $\Gamma(s)$. Frequency-domain transfer functions having unity magnitude on the $j\omega$ axis are called *all-pass filters* (Robinson et al. (2016)).

When the reflectance is loss-less, it is therefore all-pass since $|\Gamma(j\omega| = 1$. An important property of all-pass filters is that they may be accurately approximated by pole-zero pairs straddling the $j\omega$ axis, with the poles to the left (as required by causality) and the zeros to the right. Given this placement, the phases of the poles and zeros add. The group delay gives the net delay of the all-pass filter, which is twice the delay of the poles alone. It would seem that this careful placement of the zeros exactly across from the poles provides the requirement that the zeros all line up parallel to the $j\omega$ axis, as deemed by the Riemann hypothesis. Could this be the resolution of this long-standing mystery?

Example: Given the function
$$F(s) = \frac{(s+1)(s-1)}{(s+2)},$$

1. Find the minimum phase $M(s)$ and all-pass $A(s)$ parts. The minimum phase part has all of its poles and zeros in the left half-plane (LHP), while the all-pass part has its poles in the LHP and mirrored zeros in the RHP. Thus we place a removable pole zero pair symmetrically across from the RHP zero, and then write the expression as the product, that is $F(s) = M(s) \cdot A(s)$:

$$F(s) = \frac{(s+1)(s-1)}{(s+2)} \cdot \frac{s+1}{s+1} = \frac{(s+1)^2}{s+2} \cdot \frac{s-1}{s+1}$$

Thus $M(s) \equiv \frac{(s+1)^2}{s+2}$ and $A(s) \equiv \frac{s-1}{s+1}$

2. Find the magnitude of $M(s)$ Take the real part of the log of M and then the anti-log. Thus $|M| = e^{\Re \ln M(s)}$

3. Find the phase of $M(s)$ In this case we use the imaginary part: $\angle M = \Im \ln M(s)$

4. Find the magnitude of $A(s)$ 1, by definition.

5. Find the phase of $A(s)$ $\angle A = \Im \ln(A)$

More questions

There are a number of question to be addressed:

1. Can we interpret the zeta function as a frequency domain quantity, and then inverse transform it into the time domain?

   The answer to this is yes, and the results are quite interesting.

2. Make a histogram of the entropy for the first million integers.

   This is a 5 minute job in Matlab/Octave.

```
K=1e5; N=1:K; F=zeros(K,10);
 for n=1:K;
  f=factor(n);
  F(n,1:length(f))=f;
 end;
hist(F);
```

5.1.2 Solving differential equations:

Many differential equations may be solved by assuming a power series (i.e., Taylor series) solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n, \tag{5.14}$$

with $r \in \mathbb{Z}$ and coefficients $c_n \in \mathbb{C}$. This method of Frobenius is quite general (Greenberg, 1988, p. 193).

Example: When a solution of this form is substituted into the differential equation, a recursion relationship in the coefficients results. For example, if the equation is

$$y''(x) = \lambda^2 y(x),$$

the recursion is $c_n = c_{n-1}/n$. The resulting equation is

$$y(x) = e^{\lambda x} = x^0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

namely, $c_n = 1/n!$, thus $nc_n = 1/(n-1)! = c_{n-1}$.

Exercise #*3*

Find the recursion relationship for $y(x) = J_\nu(x)$ of order ν that satisfies Bessel's equation

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2) y(x) = 0.$$

Solution: If we assume a complex–analytic solution of the form of Eq. 5.14, we find the Bessel recursion relationship for coefficients c_k (Greenberg, 1988, p. 231):

$$c_k = -\frac{1}{k(k+2\nu)} c_{k-2}.$$

5.2 Quantum Mechanics and the WHEN

ler's benefit, the conflict!

While it is clear that both Schrödinger's equation and Dirac's equations are highly accurate, after about 100 years, it is not clear why. Both of these theories seem to violate classical electromagnetics (EM), such as Ohm's law, since they are built on energy principles rather than electric and magnetic fields. The point I find most disturbing is that QM defines a probabilistic wave function. What are the units for such a function?

Here we delve into this question, by providing a classical (i.e., EM-based) derivation for the hydrogen atom, one of the most important and obvious successes of quantum mechanics (QM). The problem with QM is not that it fails—rather, it succeeds, without obvious basis. It seems to be in contradiction with basic principles of a physical theory, which was Heisenberg's view. (See footnote 13, p.100).

Based on the Rydberg formula, we determine the reflection coefficient, and thus the radiation impedance seen by the electron, in a radial coordinate system centered on the proton. Since the electron and proton both have spin $\frac{1}{2}$, their magnetic fields must attractively align, accounting for the near-field vector potential, and complementing the far-field attraction due to their opposite signs. As the electron and proton approach each other, due to their far-field potential attraction, the magnetic near field becomes more attractive at close range, due to the magnetic dipoles of the two "particles," causing them to merge with neutral net magnetic moment and neutral charge, giving a highly stable hydrogen atom. However, given a sufficiently strong distorting field, this highly symmetric state could be disturbed, leading to photon radiation, constrained by the radial eigenstates. It seems more clear than ever that photons and electrons are in a state of equilibrium at the outskirts of very large Rydberg atoms.[4]

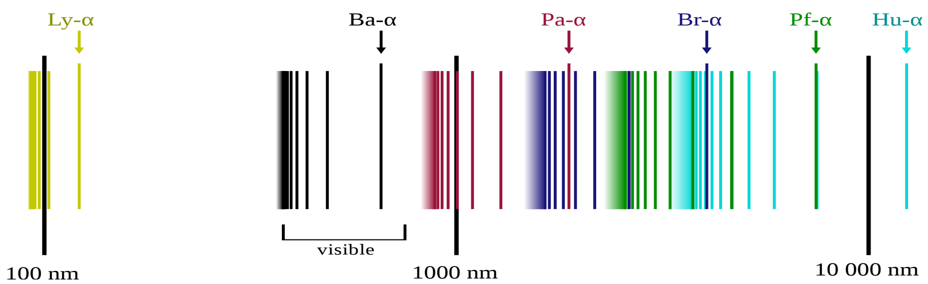


Figure 5.3: *Diagram of the wavelength spectrum of hydrogen for the Lyman, Balmer, and Paschen series, as a function of each line's wavelength. The notation "Ly-α" indicates the longest wavelength $\lambda_{11} = 122$ [nm] (i.e., lowest frequency of 2.46 [GHz]) for the Lyman series. Figure citation:* `https://en.wikipedia.org/wiki/Hydrogen_spectral_series`

5.3 Equation for Rydberg eigenmodes

Like every tuned resonant circuit, atoms have well-defined resonant frequencies, or eigenmodes, which must be labled using integers, as required by quantum mechanics (e.g., eigenmodes must have integer labels, not irrational numbers). Integer quantum numbers might serve as the definition of quantum mechanics. One uncertainty is the spin quantum numbers, which are all multiples of 1/2.

Figure 5.9 shows the observed radiation spectra for hydrogen. From the very beginning, it has been clear that there is a pattern to these spectral lines. In 1880 Rydberg fitted a formula that quantifies the observed eigen spectral lines of hydrogen in terms of the reciprocals of the radiated wavelengths:

$$\frac{1}{\lambda_{nm}} = R_\infty \left(\frac{1}{n^2} - \frac{1}{m^2} \right), \qquad \frac{f_{nm}}{c_o R_\infty} = \frac{1}{n^2} - \frac{1}{m^2}, \tag{5.15}$$

[4] `https://physics.aps.org/synopsis-for/10.1103/PhysRevLett.121.193401`

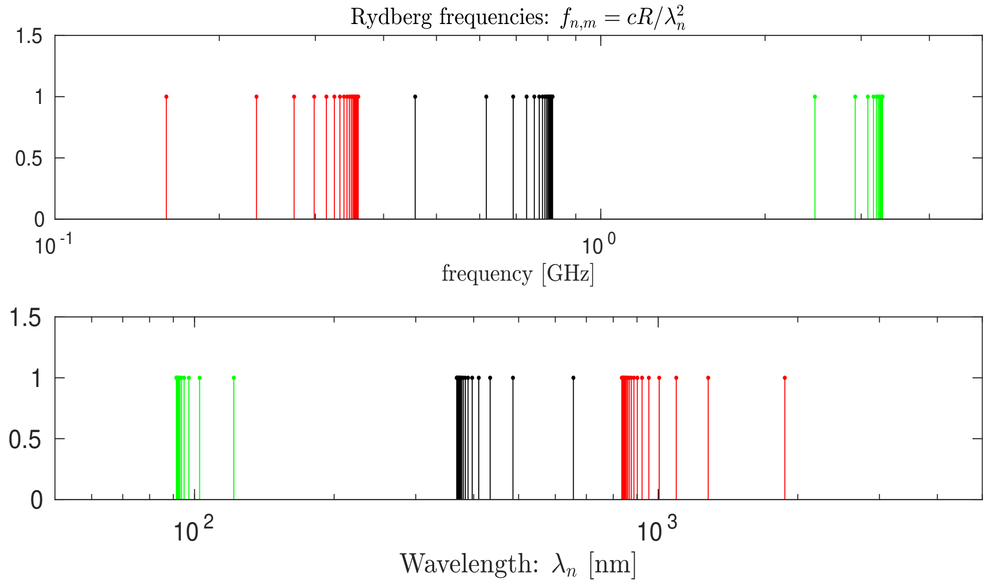


Figure 5.4: *Rydberg frequencies in [GHz] and the corresponding wavelengths [nm], computed from the Rydberg formula (Eq. 5.15). where integer n defines the series (Lyman: $n = 1$, Balmer: $n = 2$, Paschen: $n = 3$, etc.) and integer $m > n$ defines the outer transition line (see Fig. 5.11). For example, according to the lower panel (green series), the Lyman series line $\lambda_{1,2} = 122$ [nm] ($n = 1$ and $m = 2$), in agreement with the lower panel of this figure, Figs. 5.11 and 5.9. The frequency of the Paschen series line (3,6) is at 1.094 [μm] (0.3 [GHz]) (upper panel)*

all based on these simple observations. Here $R_\infty = 1.097 \times 10^7$ [m^{-1}] is the Rydberg constant, $c_o = 3 \times 10^8$ [m/s] is the speed of light, f_{nm} are the dimensionless *Rydberg integer frequencies*, where $n, m \in \mathbb{N}$ are positive integers. Here n labels the series and $m > n$ describes the transition from orbit m to orbit n, as described in the caption of Fig. 5.11. During every transition, the electrons go from waves to particles, having mass and energy, and both the mass and energy are modified, known as *wave-partical duality*.[5] Both constants are attempts to estimate the size of a Hydrogen atom.

One would expect a close relation between the Bohr radius and λ_{nm} (see footnote 19 on p. 301.)

5.3.1 The Rydberg atom model

In 1909 Rutherford demonstrated that the atom consisted of a dense core (the proton) surrounded by electrons. This view was supported by the spectrum of the atom, which allows for a radiation spectrum caused by electrons jumping from one energy level to another. It was then noted by Bohr in 1913 (Bohr, 1954) and others that the wavelengths of hydrogen, as described by Eq. 5.41, are consistent with Fig. 5.10, where the reciprocal wavelength [m^{-1}] is given by Eq. 5.41, having frequencies $f_{nm} = c/\lambda_{nm}$ [Hz]. The challenge of the 1920s was to explain these intuitive and simple models of hydrogen. This gave rise to the birth of quantum mechanics, the history of which is carefully review in Condon and Morse (1929).

It was clear from the days of Bohr that the Rydberg formula did not follow the typical rules of eigenspectra, so much so that Arnold Sommerfeld wrote (Sommerfeld, 1949, p. 201):

> The lines of this spectrum cumulate at the limit given by the Rydberg constant R. The adjoining *continuum* lies in the near ultraviolet range. Both the discrete and the continuous spectrum are given by the Schrödinger equation. This equation reduces to a simple mathematical formula the enigma of the spectral lines, with their finite cumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

5.3.2 Rydberg wave equation

The objective of this analysis is to demonstrate that one can define a classical Sturm-Liouville model of the *enigmatic* Rydberg atom, by the use of the Webster horn equation

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\psi(r,t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\psi(r,t), \tag{5.16}$$

[5] `https://en.wikipedia.org/wiki/Wave\%e2\%80\%93particle_duality`

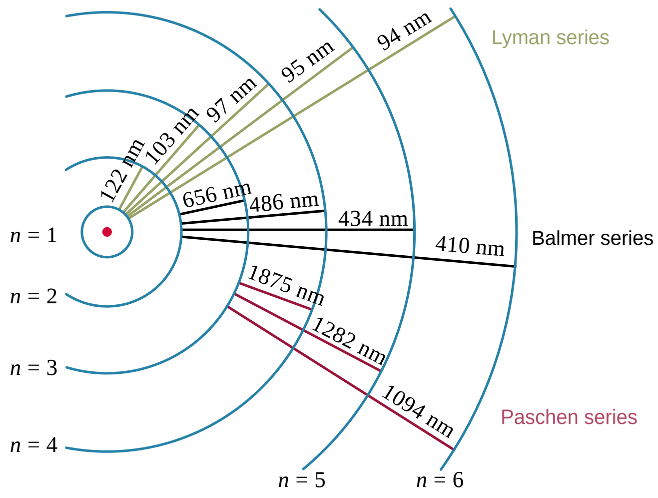


Figure 5.5: *This diagram defines hydrogen's allowed electron transitions, defining the Lyman ($n = 1$), Balmer ($n = 2$), and Paschen ($n = 3$) series. The numbers represent the wavelengths λ [nm] of the photons having frequencies $f_{nm} = c_o/\lambda_{nm}$, following an electron transition from level n to m (taken from:* `https://en.wikipedia.org/wiki/Hydrogen_spectral_series`*).*

which is a one-dimensional wave equation for the electric potential $\psi(r, t)$ propagating in a wave guide having area $A(r)$ as a function of the range, where r is the *range* variable (the axis of wave propagation).

We shall show that given the Rydberg spectrum (Eq. 5.41), we may accurately estimate the electric reflectance $\Gamma(s)$ looking out from the origin (i.e., the proton location, as indicated by the small red dot in Fig. 5.11). The radiation impedance $Z_{\text{rad}}(s)$ seen by the proton is related to the reflectance $\Gamma(s)$ by the relation

$$Z_{\text{rad}}(s) = r_o \frac{1 + \Gamma(s)}{1 - \Gamma(s)}. \tag{5.17}$$

This formula is the basis of the *Smith chart* used in both physics and engineering studies. It follows that once $\Gamma(s)$ is known (i.e., evaluated given Eq. 5.41), the radiation impedance may be computed. It has been shown that the area function $A(r)$ may be found given the radiation impedance (Sondhi and Gopinath, 1971; Youla, 1964).

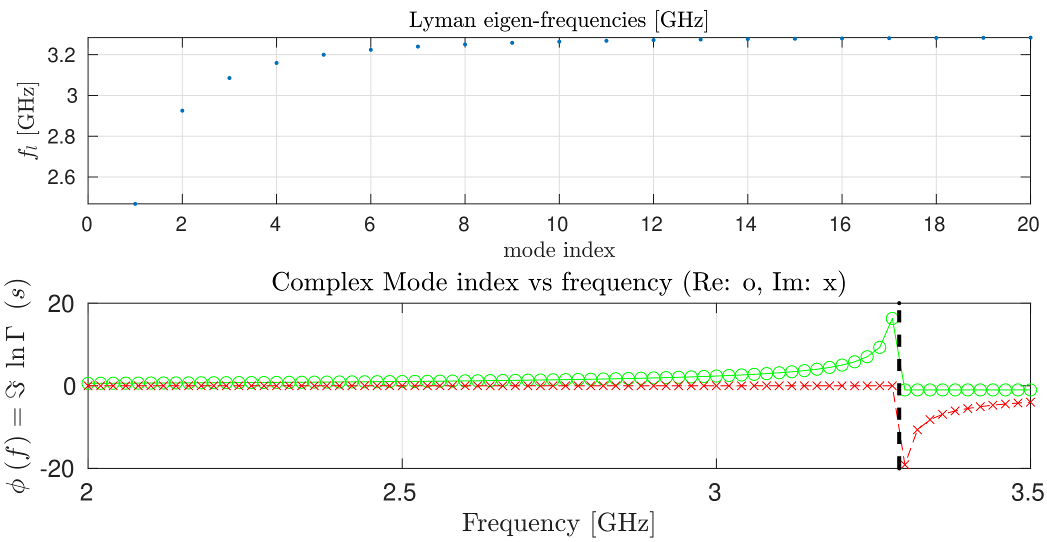


Figure 5.6: *The top panel is a plot of Eq. 5.41, showing how the eigenmode frequencies f_l depend on the eigen-number index l. As the mode number increases, the frequency reaches an asymptote at f_o = 3.29 [GHz], with a wavelength limit near $1/R \approx 91.2$ [nm]. The lower panel shows the inverse mapping from frequency to the mode index number $\phi(f)$. This figure is for the Lyman series ($n = 1$ and $m = 1, \ldots, 20$). The inverse of this relationship is $l = \phi^{-1}(f_l)$ may be derived from Eq. 5.41, which provides the pole frequencies required to satisfy Eq. 5.43. Note that for frequencies greater than c_o/R the phase switches from purely real to imaginary, accounting for free electrons above 3.29 [GHz].*

5.4 Rydberg solution methods

The basic idea behind the method is to use Eq. 5.43, by noting that the poles of the impedance are determined by the roots of the denominator of Z_{rad}. Specifically, if s_p is an impedance pole, then it must satisfy $\Gamma(s_p) \approx 1$. Except for losses due to radiation, the atom is lossless; thus $|\Gamma(s)| = 1$. Namely, it must be of the form

$$\Gamma(s) = e^{-\jmath\phi(f)}, \tag{5.18}$$

where the *phase* $\phi(f) \in \mathbb{R}$ and $s = \sigma + \omega\jmath$ is the complex Laplace radian frequency, with $\omega = 2\pi f$ [Hz]. Since we know the eigenmode frequencies, which obey $\phi(f_{n_o,m}) = 2\pi m$, we may find $\phi(f)$, as follows: For a given series index n_o, and given the eigenfrequencies f_m, we seek the phase mode function $\phi_{n_o}(f)$ that maps the eigenfrequencies to their mode index m, i.e.,

$$\phi_{n_o}(f_m) = 2\pi m.$$

5.4.1 Group delay $\tau(s)$

The phase $\phi(\omega)$ is related to the group delay $\tau(\omega)$ by the relation

$$\tau(\omega) = -\frac{\partial}{\partial\omega}\phi(\omega).$$

Here one may assume that the phase is complex–analytic,[6] thus allowing a causal damping term into the reflectance phase Eq. 5.44. This follows naturally because the reflectance must be causal (Postulate 3.9.2, p. 126). In the time domain the delay may be written in terms of the inverse $\mathcal{LT}$ of the group delay,

$$\Gamma(s) = e^{-\jmath\int_o^s \tau(s)ds}.$$

Typically one uses the reflectance phase $2\pi\phi(f)$; thus the group delay is $\tau(f) = -\partial\phi(f)/\partial f$, which is physically interpreted here as the frequency-dependent delay from the proton to the radius of the electron's orbit. Thus this delay is given by

$$\tau(f) = n\frac{\partial}{\partial f}\left(1 - \frac{n^2}{c_oR}\,f\right)^{-1/2} = \frac{n^3}{2c_oR}\left(1 - \frac{n^2}{c_oR}\,f\right)^{-3/2},$$

which is constant for low frequencies and then rises to ∞ as frequency approaches the Rydberg frequency ($f \to c_oR/n^2$).

One may solve Eq. 5.41 for m, for the case of the Lyman series ($n_o = 1$), by the use of the following identity for the Rydberg eigenfrequencies f_{nm}, which follow directly from Eq. 5.41, with $m = n_o + l$ (with $n_o, m, l \in \mathbb{N}$)

$$\begin{aligned} f_{nm} = \frac{c_o}{\lambda_{nm}} &= c_oR\left(\frac{1}{n_o^2} - \frac{1}{(n_o+l)^2}\right) \\ &= \frac{c_oR}{n_o^2}\left(1 - \frac{1}{(1+l/n_o)^2}\right). \end{aligned} \tag{5.19}$$

Note that as $l \to \infty$, $f_{n_o,l} \to= c_oR/n_o^2$, which is Sommerfeld's "finite cumulation point" [Hz] $f_{n_o,\infty}$ for the Lyman series ($n_o = 1$).

[6] It follows that these relationships are related by a Hilbert transform.

We can solve Eq. 5.45 for the mode number $l/n < 1$ as a function of mode frequency:

$$
\begin{aligned}
n^2 \frac{f_{nl}}{c_o R} &= 1 - \frac{1}{(1+l/n)^2} && \text{Starting from Eq. 5.45}\\
\frac{1}{(1+l/n)^2} &= 1 - n^2 \frac{f_{nl}}{c_o R} && \text{Solving for } l/n\\
(1+l/n)^2 &= \frac{1}{1 - n^2 \frac{f_{nl}}{c_o R}} &&\\
\frac{l}{n} &= \pm \frac{1}{\sqrt{1 - n^2 \frac{f_{nl}}{c_o R}}} - 1 && \phi(f_{nl})/2\pi = l = m - n_o \in \mathbb{N}, \qquad (5.20)
\end{aligned}
$$

as summarized in the lower panel of Fig. 5.12. The square root term is the relativistic *Lorentz Transformation* applied to a rest mass traveling at speeds close to the speed of light This is quite different from the *Lorentz force*. The *Lorentz transformation* was first introduced in Einstein's theory of relativity, One must be careful to distinguished the Lorentz transformation from the *Lorentz force*.

5.4.2 Finding the area function

Once the phase has been determined, we can compute the impedance using Eq. 5.43. We may then decompose the impedance by using the analytic continued fraction algorithm (or Cauer synthesis), discussed in Sec. 3.7, p. 113.

5.5 Euclid's formula and the Rydberg atom model

Fundamental to quantum mechanics is the Rydberg formula, which describes the quantized energy levels of atoms[7]

$$\nu_{n,m} = c_o R_\infty Z_n^2 \left(\frac{1}{n^2} - \frac{1}{m^2}\right), \tag{5.21}$$

where $\nu_{n,m}$ are the possible eigenfrequencies, $c_o \approx 3{\times}10^8$ [m/s] is the speed of light, $R_\infty \approx 10.97{\times}10^{-6}$ $[m^{-1}]$ is the Rydberg constant Z_n is the atomic number, along with positive integers $m > n \in \mathbb{N}$, which represent the *principal quantum numbers* that label all possible allowed atomic eigenstates.$(5.29177210544 \times 10^{-}9)$. Integer n indicates the lowest (rest) atomic eigenstate while m labels the higher (excited) state.[8] When $n = 1$, the series is the Lyman series corresponding to hydrogen ($Z_1 = 1$). When $n = 1$, $m = 2$, and $Z_1 = 1$, the frequency is

$$\nu_{1,2} = c_o R_\infty \left(\cancelto{\frac{3}{4}}{\frac{1}{1^2} - \frac{1}{2^2}}\right) = \frac{3}{2} \times 10^{15} \text{ [Hz]}. \tag{5.22}$$

An open question in this model is: *Why are states either empty or filled?* The amplitudes of the modes of a string or organ pipe are never empty. What is it about the atom that forces the energy state to be empty? Can it be true that an eigenstate is either empty or full? Perhaps the answer is due to the Lorentzian term, which is relativistic. Classical physics, by definition, is never relativistic. What does the experimental, if any, evidence say? I dont't have an answer to this interesting question.

5.5.1 Solving for the area function

Given observed frequencies $\nu_{n,m}$ it is possible to determine the area function that traps the photons into the Rydberg eigenstates. Eq. 5.47 may be rewritten as

$$\nu_{n,m} = c_o R Z_n^2 4 \left(\frac{m^2 - n^2}{(2nm)^2}\right).$$

---

[7] `https://www.youtube.com/watch?v=e0IWPEhmMho`

[8] `http://en.wikipedia.org/w/index.php?title=Rydberg_formula`

It is interesting to compare Eq. 5.47 to Euclid's formula

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2, \tag{5.23}$$

where $m > n \in \mathbb{N}$. Euclid's formula is equivalent to the Pythagorean theorem for integers, since

$$c^2 = a^2 + b^2, \tag{5.24}$$

with $\{a, b, c\} \in \mathbb{N}$. Here $a < b < c$.

If we interpret the quantum numbers as multiples of a quarter wavelength, then the Rydberg formula is congruent to the Pythagorean theorem. Given the symmetry, this cannot be an accident.

In terms of the lengths of the right triangle $\{a, b, c\}$, Rydberg's formula becomes

$$\nu_{n,m} = c_o R Z_n^2 4 \left(\frac{a}{b^2} \right).$$

But since $b^2 = c^2 - a^2$,

$$\begin{aligned}
\nu_{n,m} &= c_o \frac{RZ_n^2}{a} 4 \left(\frac{a^2}{c^2 - a^2} \right) \\
&= c_o \frac{RZ_n^2}{a} 4 \frac{a^2}{c^2} \left(\frac{1}{1 - (a/c)^2} \right).
\end{aligned}$$

In terms of quantized (discrete) angles, $\sin(\theta_{n,m}) = a/c$,

$$\begin{aligned}
\nu_{n,m} &= c_o \frac{RZ_n^2}{a} 4 \left(\frac{\sin^2\theta}{1 - \sin^2\theta} \right) \\
&= c_o \frac{RZ_n^2}{a} 4 \left(\frac{\sin^2\theta}{\cos^2\theta} \right) \\
&= c_o \frac{RZ_n^2}{a} 4 \tan^2\theta_{n,m}.
\end{aligned}$$

5.5.2 Eigenmodes of the Rydberg atom

One way to think of eigenmodes is to make an analogy to a piano string or an organ pipe. In these much simpler systems, there is an almost constant delay, say τ, due to a characteristic length, say $L = \tau c_o$, such that the eigenmodes of a string are given by integer multiples of a half wavelength $\nu_n = nc_o/2L$, while the eigenmodes of the organ pipe are multiples of a quarter wavelength. The distinction is the boundary conditions. For the string the endpoint boundary conditions are pinned displacement (i.e., zero velocity). The organ pipe is closed at one end and open at the other, resulting in multiples of a quarter wavelength $\nu_n = nc_o/4L$. In each case $\nu = n/\tau$, where $\tau = 2L/c_o$ is the round-trip delay; thus $\nu = nc_o/2L$. We suggest looking at the Rydberg formula in the same way, but with very different eigenfrequencies (Eq. 5.47). Sommerfeld (1949, p. 201) makes an interesting comment regarding Eq. 5.47:

> This equation reduces to a simple mathematical formula the enigma of the spectral lines, with their finite cumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

5.5.3 Discussion

The Rydberg frequencies f_{nl} ($n = 1, l = 1, \ldots, \infty$) has poles in the radiation impedance (Eq. 5.43) when $\phi_l(f_{nl}) \in \mathbb{N}$. Working backwards from the Rydberg formula (Eq. 5.44), we have solved for $\phi(f_{nl})$ indicating where this condition is valid (Eq. 5.46). Since the reflectance and the impedance must be causal complex–analytic functions of Laplace frequency s, we must replace the discrete frequency f_{nl} with s:

$$\jmath 2\pi f_{nl} \rightarrow s = \sigma + \omega\jmath,$$

thereby forcing $l(s)$ to be a complex–analytic function of s. Then the poles of the radiation impedance must satisfy

$$\Gamma(s_{nl}) = e^{\jmath 2\pi l(f_{nl})} = 1,$$

resulting in eigenfrequencies at f_{nl}.

The next step in this analysis is to determine the area function $A(r)$ given Z_{rad} (Eq. 5.43). To do this we must solve an integral equation for $A(r)$, as discussed by Sondhi and Gopinath (1971) and by Youla (1964).

Perhaps this could be done using an analytic representation for the area function,

$$A(r) = \sum_k a_k r^k.$$

5.6 Relations between Sturm-Liouville and quantum mechanics

If we compare the Schrödinger equation (SE) for hydrogen with the corresponding Sturm-Liouville equation we can begin to appreciate their differences. The QM equation for hydrogen is

$$\begin{aligned}
\imath\hbar\frac{\partial}{\partial t}\psi(x,t) &= -\frac{\hbar^2}{2m_o}\nabla_r^2\psi(x,t) + V(r)\psi(x,t) \\
&= -\frac{\hbar^2}{2m_o}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\psi(x,t) + V(r)\;\psi(x,t) && (5.25)\\
&= -\frac{\hbar^2}{2m_o}\left[\frac{2}{r}\frac{\partial}{\partial r}\psi(x,t) + \frac{\partial^2}{\partial r^2}\psi(x,t)\right] + V(r)\;\psi(x,t), && (5.26)
\end{aligned}$$

whereas the horn equation is given by Eq. 5.42.

There are several obvious and disturbing differences between these two equations. First, the SE is, of course, first-order in time. Diffusion equations have no delay and thus cannot have traditional eigenmodes, which result from standing waves in a wave equation, due to boundary conditions. Second, the EM horn equation is of Sturm-Liouville (SL) form, which is a true wave equation (vs. the SE, which is a diffusion equation). The obvious question arises: Is there a potential V that would allow these two formulations to be equivalent? If so, then this would provide an explanation as to why the SE is successful in explaining the properties of Rydberg atoms.

To explore this possibility we may expand the two differential equations and directly compare them. Expanding Eq. 5.42 gives

$$\begin{aligned}
\frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\psi(r,t) &= \frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\psi(r,t) && (5.27)\\
&= \frac{\partial^2}{\partial r^2}\psi(r,t) + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\psi(r,t). && (5.28)
\end{aligned}$$

Between these two equations we may remove ψ'':

$$\imath\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m_o}\left[\frac{2}{r}\frac{\partial}{\partial r}\psi(x,t) + \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\psi(r,t) - \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\psi(r,t)\right] + V(r)\;\psi(x,t). \quad (5.29)$$

It seems that this may isolate the time and spatial parts (as in separation of variables).

5.6.1 The exponential horn

A relevant and motivational example is the solution of the exponential horn, having area function $A(r) = A_o e^{2mr}$. This case is interesting because it has a closed-form solution, which seems relevant and perhaps even related to the hydrogen atom.

Assuming that $\varrho(r,t) \leftrightarrow \mathcal{P}(r,\omega)$ are a Fourier transform pair, with $A(r) = A_o e^{2mr}$, Eq. 5.42 reduces to

$$\frac{\partial^2\,\mathcal{P}(r,\omega)}{\partial r^2} + 2m\frac{\partial\,\mathcal{P}(r,\omega)}{\partial r} = \kappa^2\,\mathcal{P}(r,\omega) \leftrightarrow \frac{1}{c_o^2}\frac{\partial^2\varrho}{\partial t^2}, \quad (5.30)$$

with $\kappa(s) = s/c_o$.

Exercise #*4*

Show that Eq. 5.56 follows from Eq. 5.42.

Solution: Starting from Eq. 5.42 with area $A(r) = A_o e^{2mr}$

$$\frac{1}{\cancel{A_o}e^{2mr}}\frac{\partial}{\partial r}\left(\cancel{A_o}e^{2mr}\frac{\partial \varrho}{\partial r}\right) = \frac{1}{c_o^2}\frac{\partial^2 \varrho}{\partial t^2}$$
$$\varrho_{rr}(r,t) + 2m\varrho_r(r,t) = \frac{1}{c_o^2}\frac{\partial^2 \varrho}{\partial t^2} \leftrightarrow \kappa^2\,\mathcal{P}(r,\omega),$$

which is the time-domain version of Eq. 5.56. ■

Since this equation is second-order in time with constant coefficients, it has two closed-form solutions:

$$\begin{aligned}\mathcal{P}_c^{\pm}(r) &= \mathcal{P}_o^{\pm}(\omega)\,e^{-mr}e^{\mp\sqrt{m^2+\kappa^2}\,r}\\ &= \mathcal{P}_o^{\pm}(\omega)\,e^{-mr}e^{\mp j\frac{r}{c_o}\sqrt{\omega^2-\omega_c^2}},\end{aligned}$$

with $\omega_c = mc_o$. The two wave amplitudes $\mathcal{P}_0^{\pm}(\omega)$ must be determined from the boundary conditions.

Exercise #*5*

Shown that $\mathcal{P}^{\pm}(r,\omega)$ satisfy Eq. 5.56.

Solution: Taking partials with respect to r,

$$\begin{aligned}\partial_r\,\mathcal{P}^{\pm}(r,\omega) &= \left(-m \mp \sqrt{m^2+\kappa^2}\right)\,\mathcal{P}^{\pm}(r,\omega)\\ \partial_{rr}\,\mathcal{P}^{\pm}(r,\omega) &= \left(-m \mp \sqrt{m^2+\kappa^2}\right)^2\,\mathcal{P}^{\pm}(r,\omega)\\ &= \left(2m^2+\kappa^2 \pm 2m\sqrt{m^2+\kappa^2}\right)\,\mathcal{P}^{\pm}(r,\omega).\end{aligned}$$

Thus Eq. 5.56 reduces to

$$\left(2m^2+\kappa^2 \pm 2m\sqrt{m^2+\kappa^2}\right) + 2m\left(-m \mp \sqrt{m^2+\kappa^2}\right) = \kappa^2,$$

which is an identity. ■

Next consider the Fourier series (or Fourier transform) of the area function,

$$A(r) = \sum_k a_k e^{2m_k r}.$$

It follows from the linearity of the wave equation that the general solution of Eq. 5.56 is

$$\mathcal{P}^{\pm}(r,\omega) = \sum_k a_k^{\pm}(\omega)\,e^{-m_k r}\,e^{\mp\sqrt{m_k^2+\kappa^2}\,r}.$$

Here we have combined $\mathcal{P}^{\pm}(\omega)$ and a_k as coefficients $a_k^{\pm}(\omega)$.

5.6.2 Brune Impedance

Problem # *51: Residue form*

A Brune impedance is defined as the ratio of the force $F(s)$ to the flow $V(s)$ and may be expressed in residue form as

$$Z(s) = c_0 + \sum_{k=1}^{K}\frac{c_k}{s-s_k} = \frac{N(s)}{D(s)} \tag{5.31}$$

with

$$D(s) = \prod_{k=1}^{K}(s-s_k) \qquad \text{and} \qquad c_k = \lim_{s\to s_k}(s-s_k)D(s) = \prod_{n'=1}^{K-1}(s-s_n).$$

The prime on the index n' means that $n = k$ is not included in the product.

– 51.1: Find the Laplace transform ($\mathcal{LT}$) of a (1) spring, (2) dashpot, and (3) mass. Express these in terms of the force $F(s)$ and the velocity $V(s)$, along with the electrical equivalent impedance: (1) Hooke's law $f(t) = Kx(t)$, (2) dashpot resistance $f(t) = Rv(t)$, and (3) Newton's law for mass $f(t) = Mdv(t)/dt$. **Solution:**

1. Hooke's Law $f(t) = Kx(t)$. Taking the $\mathcal{LT}$ gives

$$F(s) = KX(s) = KV(s)/s \leftrightarrow f(t) = Ku(t) \star v(t) = K\int^t v(t),$$

since

$$v(t) = \frac{d}{dt}x(t) \leftrightarrow V(s) = sX(s).$$

Thus the impedance of the spring is

$$Z_s(s) = \frac{K}{s} \leftrightarrow z(t) = Ku(t),$$

which is analogous to the impedance of an electrical capacitor. The relationship may be made tighter by specifying the compliance of the spring as $C = 1/K$.

2. Dashpot resistance $f(t) = Rv(t)$. From the $\mathcal{LT}$ this becomes

$$F(s) = RV(s)$$

and the impedance of the dashpot is then

$$Z_r = R \leftrightarrow R\delta(t),$$

analogous to that of an electrical resistor.

3. Newton's law for mass $f(t) = Mdv(t)/dt$. Taking the $\mathcal{LT}$ gives

$$f(t) = M\frac{d}{dt}v(t) \leftrightarrow F(s) = M\,sV(s),$$

thus

$$Z_m(s) = sM \leftrightarrow M\frac{d}{dt},$$

analogous to an electrical inductor.

■

– 51.2: Take the Laplace transform ($\mathcal{LT}$) of Eq. 5.32 and find the total impedance $Z(s)$ of the mechanical circuit.

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t) \leftrightarrow (Ms^2 + Rs + K)X(s) = F(s). \tag{5.32}$$

Solution: From the properties of the $\mathcal{LT}$ that $dx/dt \leftrightarrow sX(s)$, we find

$$f(t) \leftrightarrow F(s) = Ms^2X(s) + RsX(s) + KX(s).$$

In terms of velocity this is $(Ms + R + K/s)V(s) = F(s)$. Thus the circuit impedance is

$$z(t) \leftrightarrow Z(s) = \frac{F}{V} = \frac{K + Rs + Ms^2}{s}.$$

■

– 51.3: What are $N(s)$ and $D(s)$ (see Eq. 5.31)?
Solution: $D(s) = s$ and $N(s) = K + Rs + Ms^2$. ■

– 51.4: Assume that $M = R = K = 1$ and find the residue form of the admittance $Y(s) = 1/Z(s)$ (see Eq. 5.31) in terms of the roots $s_\pm$. Hint: Check your answer with Octave's/Matlab's `residue` command.
Solution: First find the roots of the numerator of $Z(s)$ (the denominator of $Y(s)$):

$$s_\pm^2 + s_\pm + 1 = (s_\pm + 1/2)^2 + 3/4 = 0,$$

which is

$$s_\pm = \frac{-1 \pm \jmath\sqrt{3}}{2}.$$

Second form a partial fraction expansion

$$\frac{s}{1+s+s^2} = c_0 + \frac{c_+}{s-s_+} + \frac{c_-}{s-s_-} = \frac{s(c_+ + c_-) - (c_+ s_- + c_- s_+)}{1+s+s^2}.$$

Comparing the two sides shows that $c_0 = 0$. We also have two equations for the residues $c_+ + c_- = 1$ and $c_+ s_- + c_- s_+ = 0$. The best way to solve this is to set up a matrix relation and take the inverse

$$\begin{bmatrix} 1 & 1 \\ s_- & s_+ \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{thus:} \quad \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{s_+ - s_-} \begin{bmatrix} s_+ & -1 \\ -s_- & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which gives $c_\pm = \pm\frac{s_\pm}{s_+ - s_-}$ The denominator is $s_+ - s_- = j\sqrt{3}$ and the numerator is $\pm 1 + \jmath\sqrt{3}$. Thus

$$c_\pm = \pm\frac{s_\pm}{s_+ - s_-} = \frac{1}{2}\Big(1 \pm \frac{\jmath}{\sqrt{3}}\Big).$$

As always, finding the coefficients is always the most difficult part. Using 2x2 matrix algebra automates the process. Always check your final result as correct. ■

– 51.5: By applying Eq. 3.2.3 (page 77), find the inverse Laplace transform ($\mathcal{LT}^{-1}$). Use the residue form of the expression that you derived in question 51.4.
Solution:

$$z(t) = \frac{1}{2\pi j} \oint_{\mathcal{C}} Z(s) e^{st} ds.$$

were $\mathcal{C}$ is the Laplace contour which encloses the entire left-half s plane. Applying the CRT

$$z(t) = c_+ e^{s_+ t} + c_- e^{s_- t}.$$

where $s_\pm = -1/2 \pm \jmath\sqrt{3}/2$ and $c_\pm = 1/2 \pm \jmath/(2\sqrt{3})$. ■

5.6.3 Laplace transforms

Problem # *52: Laplace transform pairs*

In this problem you are given a *Laplace transform* ($\mathcal{LT}$) pair $f(t) \leftrightarrow F(s)$. The frequency domain function will always be upper-case (e.g. $F(s)$) and the time domain lower case ($f(t)$). Time domain functions are always causal (i.e., $f(t < 0) = 0$). The definition of the forward transform ($f(t) \to F(s)$) is

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt,$$

where $s = \sigma + j\omega$ is the complex Laplace frequency in [radians] and t is time in [seconds].

The *inverse Laplace transform* ($\mathcal{LT}^{-1}$), $F(s) \to f(t)$ is defined as

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s) e^{st} ds = \frac{1}{2\pi j} \oint_{\mathcal{C}} F(s) e^{st} ds$$

with $\sigma_0 > 0 \in \mathbb{R}$ is a positive constant.

As discussed in the lecture notes (Section 1.4.7, p. 72) we may use the Cauchy Residue Theorem (CRT), to evaluate the $\mathcal{LT}^{-1}$, by requiring closure of the contour $\mathcal{C}$ at $\omega \jmath \to \pm j\infty$

$$\oint_{\mathcal{C}} = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} + \int_{\subset_\infty},$$

where the path represented by '$\subset_\infty$' is a semicircle of infinite radius. For a causal, 'stable' (e.g. doesn't "blow up" in time) signal, all of the poles of $F(s)$ *must* be inside of the Laplace contour, in the full (closed) left-half s-plane ($\sigma \leq 0$).

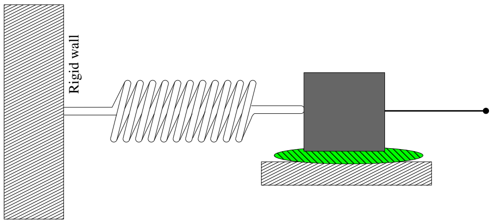


Figure 5.7: *Three-element mechanical resonant circuit consisting of a spring, mass and dash-pot (e.g., viscous fluid).*

Hooke's law for a spring states that the force $f(t)$ is proportional to the displacement $x(t)$, i.e., $f(t) = Kx(t)$. The formula for a dash-pot is $f(t) = Rv(t)$, and Newton's famous formula for mass is $f(t) = d[Mv(t)]/dt$, which for constant M is $f(t) = Mdv/dt$.

The equation of motion for the mechanical oscillator in Fig. 5.7 is given by Newton's second law; the sum of the forces must balance to zero

$$M\frac{d^2}{dt^2}x(t) + R\frac{d}{dt}x(t) + Kx(t) = f(t). \tag{5.33}$$

These three constants – the mass M, resistance R and stiffness K – are all real and positive. The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$ and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

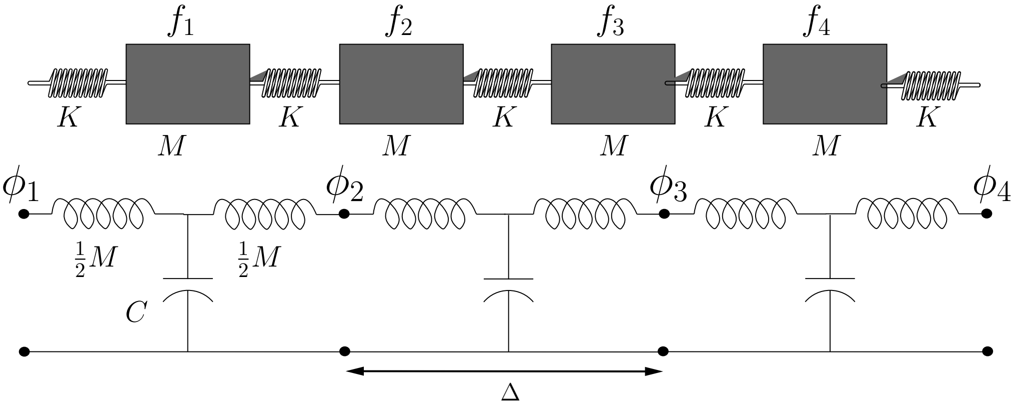


Figure 5.8: *Depiction of a train consisting of cars treated as masses M and linkages treated as springs of stiffness K or compliance $C = 1/K$. Below it is the electrical equivalent circuit for comparison. The masses are modeled as inductors and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is Δ [m]. The train may be accurately modeled as a transmission line (TL), since the equivalent electrical circuit is a lumped model of a TL. This method, called a* Cauer synthesis, *is based on the ABCD transmission line method of Sec. 3.7 (p. 113). Note: Δ is the symbol for the length of a cell.*

5.6.4 Transmission-line analysis

Problem # *53:* **Train-mission-line** *We wish to model the dynamics of a freight train that has N such cars and study the* velocity or displacement transfer function *under various load conditions for $c_o = 3 \times 10^8$ (m/s) and $\lambda = L/4$ (m).*

As shown in Fig. 5.8, the train model consists of masses connected by springs.

Problem # *54: Transfer functions*

Use the ABCD method (see the discussion pages 148-151) to find the matrix representation of the system of the above figure. Define the force on the nth train car $f_n(t) \leftrightarrow F_n(\omega)$ and the velocity $v_n(t) \leftrightarrow V_n(\omega)$.

Break the model into cells consisting of three elements: a series inductor representing half the mass ($M/2$), a shunt capacitor representing the spring ($C = 1/K$), and another series inductor representing half the mass ($L = M/2$), transforming the model into a cascade of symmetric ($\mathcal{A} = \mathcal{D}$) identical cell matrices $\mathcal{T}(s)$.

– 54.1: Find the elements of the ABCD matrix $\mathcal{T}$ for the single cell that relate the input node 1 to output node 2

$$\begin{bmatrix} F \\ V \end{bmatrix}_1 = \mathcal{T} \begin{bmatrix} F(\omega) \\ -V(\omega) \end{bmatrix}_2 . \tag{5.34}$$

Solution:

$$\begin{aligned} \mathcal{T} &= \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 1 & sM/2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + s^2MC/2 & (sM)(1 + s^2MC/4) \\ sC & 1 + s^2MC/2 \end{bmatrix} \end{aligned} \tag{5.35a}$$

■

– 54.2: Express each element of $\mathcal{T}(s)$ in terms of the complex Nyquist ratio $s/s_c < 1$ ($s = 2\pi jf$, $s_c = 2\pi jf_c$). The Nyquist wavelength sampling condition is $\lambda_c > 2\Delta$. It says the critical wavelength $\lambda_c > 2\Delta$. condition is $\lambda_c > 2\Delta$.[9] It says the critical wavelength $\lambda_c > 2\Delta$. Namely it is defined in terms the minimum number of cells 2Δ, per minimum wavelength λ_c.

The Nyquist wavelength sampling theorem says that there are at least two cars per wavelength.

Proof: From the figure, the distance between cars $\Delta = c_o T_o$ [m], where

$$c_o = \frac{1}{\sqrt{MC}} \quad \text{[m/s]}.$$

The cutoff frequency obeys $f_c\lambda_c = c_o$. The Nyquist critical wavelength is $\lambda_c = c_o/f_c > 2\Delta$. Therefore the Nyquist sampling condition is

$$f < f_c \equiv \frac{c_o}{\lambda_c} = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}} \quad \text{[rad/sec]}. \tag{5.36}$$

Finally, the Laplace frequency is $s_c = \jmath 2\pi f_c$ with $f_c = 3 \times 10^8 \times (2/L)$

Solution: The solution is summarized above: the system in Fig. 5.8 represents a transmission line having a wave speed of $c_o = 1/\sqrt{MC}$ and characteristic impedance $r_o = \sqrt{M/C}$. Each cell, composed of 2 masses M connected by one spring K, has length Δ.

We wish to define the *Nyquist frequency* f_c such that the wavelength $\lambda > 2\Delta$, where Δ is the cell length. Using the formula for the wavelength in terms of the wave velocity and frequency we find

$$\lambda = c_o/f_c = 2\Delta,$$

thus we conclude that

$$f < f_c = \frac{c_o}{2\Delta} = \frac{1}{2\Delta\sqrt{MC}}. \tag{5.37}$$

If we wish to have the system be accurate for a given frequency we may make the cell length Δ smaller, while keeping the velocity constant (MC is held constant). Thus the characteristic resistance [ohms/unit length] r_o must change as $f_c \to \infty$ and $\Delta \to 0$. We can either let $M \to \infty$ and $C \to 0$ (their product remains constant), or the other way around. In one case $r_o \to \infty$ and in the other case it goes to 0. ■

– 54.3: Use the property of the Nyquist sampling frequency *$\omega < \omega_c$ (Eq. 5.36) to remove higher order powers of frequency*

[9] The history of this relation has been traced back to 1841, as discussed by (Brillouin, 1953, Chap. I,II, Eq. 4.7).

$$1 + \cancel{\left(\frac{s}{s_c}\right)^2}^{\,0} \approx 1 \tag{5.38}$$

to determine a band-limited approximation of $\mathcal{T}(s)$.

Solution:

$$\begin{aligned}\mathcal{T} &= \begin{bmatrix} 1+2(s/s_c)^2 & sM(1+(s/s_c)^2) \\ sC & 1+2(s/s_c)^2 \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & sM \\ sC & 1 \end{bmatrix}\end{aligned}$$

The approximation is highly accurate below the Nyquist cutoff frequency $s < s_c$. Given any desired frequency f, we can always make the cell size Δ smaller by decreasing M and C, while keeping $f < f_c$ and the cell velocity constant ($c_o = 1/\sqrt{MC}$). Thus the Nyquist condition represents a computational bound, not a physical limitation. ■

Problem # *55: Now consider the cascade of N such $\mathcal{T}(s)$ matrices and perform an eigen-analysis.*

– 55.1: Find the eigenvalues and eigenvectors of $\mathcal{T}(s)$ as functions of s/s_c.

Solution: Matrix $\mathcal{T}(s)$ has eigenvalues

$$\lambda_\pm = 1 \mp 2s/s_c \approx e^{\pm 2s/s_c} = e^{\mp sT_c}.$$

From this we can interpret the eigenvalues as the cell delay $T_c = 2/s_c$.

The corresponding unnormalized eigenvectors are

$$\boldsymbol{e}_\pm = \begin{bmatrix} \mp\sqrt{M/C} \\ 1 \end{bmatrix},$$

where the characteristic impedance defined is $r_o = \sqrt{M/C}$. ■

Problem # *56: Find the velocity transferfunction $H_{12}(s) = V_2/V_1|_{F_2=0}$.*

– 56.1: Assuming that $N = 2$ and $F_2 = 0$ (two half-mass problem), find the transfer function $H(s) \equiv V_2/V_1$. From the results of the $\mathcal{T}$ matrix, find

$$H_{21}(s) = \left.\frac{V_2}{V_1}\right|_{F_2=0}$$

Express H_{12} in terms of a residue expansion.

Solution: From Eq. 5.35a, $V_1 = sCF_2 - (s^2MC/2+1)V_2$. Since $F_2 = 0$

$$\frac{V_2}{V_1} = \frac{-1}{s^2MC/2+1} = \left(\frac{c_+}{s-s_+} + \frac{c_-}{s-s_-}\right)$$

having eigenfrequencies $s_\pm = \pm\jmath\sqrt{\frac{2}{2MC}} = \pm s_c$ and residues $c_\pm = \pm\jmath/\sqrt{2MC} = \pm s_c$. ■

– 56.2: Find $h_{21}(t) \leftrightarrow H_{21}(s)$.

Solution:

$$h(t) = \oint_{\sigma_0-j\infty}^{\sigma_0+j\infty} \frac{e^{st}}{s^2MC/2+1}\frac{ds}{2\pi j} = c_+e^{-s_+t}u(t) + c_-e^{-s_-t}u(t).$$

The integral follows from the Cauchy Residue theorem (CRT). ■

– *56.3: What is the input impedance $Z_2 = F_2/V_2$, assuming $F_3 = -r_0V_3$?*
Solution: Starting from Eq. 5.35a find Z_2

$$Z_2(s) = \frac{F_2}{V_2} = T\begin{bmatrix} F \\ -V \end{bmatrix}_2 = \frac{-(1+s^2CM/2)r_0\cancel{V_2} - sM(1+s^2CM/4)\cancel{V_2}}{-sCr_0\cancel{V_2} - (1+s^2CM/2)\cancel{V_2}}$$

■

– *56.4: Simplify the expression for Z_2 as follows:*

1. Assuming the *characteristic impedance* $r_0 = \sqrt{M/C}$,
2. terminate the system in r_0: $F_2 = -r_0V_2$ (i.e., $-V_2$ cancels).
3. Assume higher-order frequency terms are less than 1 ($|s/s_c| < 1$).
4. Let the number of cells $N \to \infty$. Thus $|s/s_c|^N = 0$.

When a transmission line is terminated in its characteristic impedance r_0, the input impedance $Z_1(s) = r_0$. Thus, when we simplify the expression for $\mathcal{T}(s)$, it should be equal to r_0. Show that this is true for this setup.

Solution: Applying the Nyquist approximation (i.e., ignore second order frequency terms $(s/s_c)^2 \approx 0$)

$$\begin{aligned} Z_1(s) &= \frac{r_o(1+\cancelto{0}{s^2CM/2}) + sM(1+\cancelto{0}{s^2CM/4})}{r_osC + (1+\cancelto{0}{s^2CM/2})} \\ &\approx \frac{r_o + sM}{1+r_osC} = \frac{MC}{MC}\cdot\frac{r_o+sM}{1+r_osC} = \frac{M}{C}\cdot\frac{r_oC+sMC}{M+r_osMC} = r_o^2\,\frac{r_oC+s/s_c}{M+r_os/s_c} \\ &\approx r_o^2\,\frac{r_oC+\cancelto{0}{s/s_c}}{M+r_o\cancelto{0}{s/s_c}} = r_o^3\,\frac{C}{M} \\ &= r_o. \end{aligned}$$

We conclude that below the Nyquist cutoff frequency, as $N \to \infty$ the system equals a transmission line terminated by its characteristic impedance thus $Z_1(s) = r_o$. ■

– *56.5: State the ABCD matrix relationship between the first and Nth nodes in terms of the cell matrix. Write out the transfer function for one cell, H_{21}.*
Solution:

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$$

Now use the formulae for the eigenvalues and vectors to obtain $\mathcal{T}$ for $N = 1$:

$$\mathcal{T} = E\Lambda E^{-1} = E\begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} E^{-1}.$$

■

– *56.6: What is the velocity transfer function $H_{N1} = \frac{V_N}{V_1}$?*
Solution:

$$\begin{bmatrix} F_1 \\ V_1 \end{bmatrix} = \mathcal{T}^N \begin{bmatrix} F_N(\omega) \\ -V_N(\omega) \end{bmatrix}$$

along with the eigenvalue expansion

$$\mathcal{T}^N = E\Lambda^N E^{-1} = E\begin{bmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{bmatrix} E^{-1}.$$

where $\lambda_\pm^N = e^{\mp sNT_o}$. Recall that NT_o is the one way delay.

We conclude that as we add more cells, the delay linearly increases with N, since each eigenvalue represents the delay of one cell, and delay adds. ■

5.7 Quantum Mechanics and the WHEN

ler's benefit,
the conflict!

While it is clear that both Schrödinger's equation and Dirac's equations are highly accurate, after about 100 years, it is not clear why. Both of these theories seem to violate classical electromagnetics (EM), such as Ohm's law, since they are built on energy principles rather than electric and magnetic fields. The point I find most disturbing is that QM defines a probabilistic wave function. What are the units for such a function?

Here we delve into this question, by providing a classical (i.e., EM-based) derivation for the hydrogen atom, one of the most important and obvious successes of quantum mechanics (QM). The problem with QM is not that it fails—rather, it succeeds, without obvious basis. It seems to be in contradiction with basic principles of a physical theory, which was Heisenberg's view. (See footnote 13, p.100).

Based on the Rydberg formula, we determine the reflection coefficient, and thus the radiation impedance seen by the electron, in a radial coordinate system centered on the proton. Since the electron and proton both have spin $\frac{1}{2}$, their magnetic fields must attractively align, accounting for the near-field vector potential, and complementing the far-field attraction due to their opposite signs. As the electron and proton approach each other, due to their far-field potential attraction, the magnetic near field becomes more attractive at close range, due to the magnetic dipoles of the two "particles," causing them to merge with neutral net magnetic moment and neutral charge, giving a highly stable hydrogen atom. However, given a sufficiently strong distorting field, this highly symmetric state could be disturbed, leading to photon radiation, constrained by the radial eigenstates. It seems more clear than ever that photons and electrons are in a state of equilibrium at the outskirts of very large Rydberg atoms.[10]

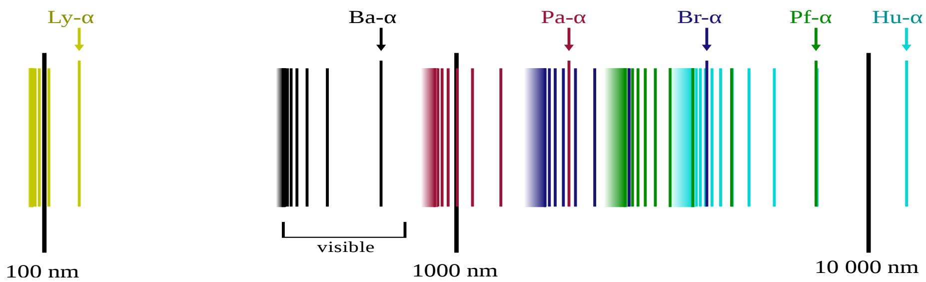


Figure 5.9: *Diagram of the wavelength spectrum of hydrogen for the Lyman, Balmer, and Paschen series, as a function of each line's wavelength. The notation "Ly-α" indicates the longest wavelength $\lambda_{11} = 122$ [nm] (i.e., lowest frequency of 2.46 [GHz]) for the Lyman series. Figure citation:* `https://en.wikipedia.org/wiki/Hydrogen_spectral_series`

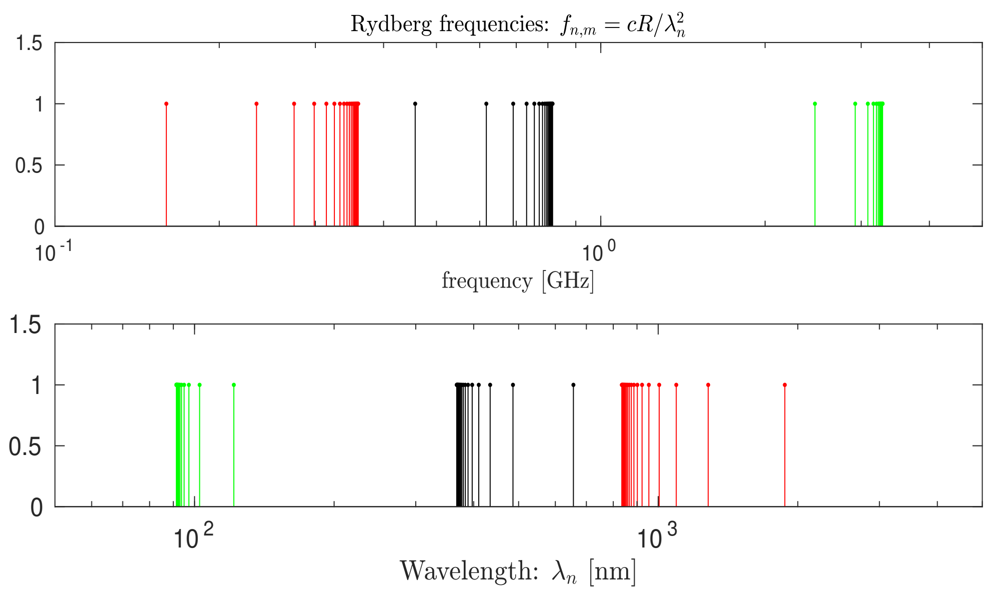


Figure 5.10: *Rydberg frequencies in [GHz] and the corresponding wavelengths [nm], computed from the Rydberg formula (Eq. 5.15). where integer n defines the series (Lyman: $n = 1$, Balmer: $n = 2$, Paschen: $n = 3$, etc.) and integer $m > n$ defines the outer transition line (see Fig. 5.11). For example, according to the lower panel (green series), the Lyman series line $\lambda_{1,2} = 122$ [nm] ($n = 1$ and $m = 2$), in agreement with the lower panel of this figure, Figs. 5.11 and 5.9. The frequency of the Paschen series line (3,6) is at 1.094 [μm] (0.3 [GHz]) (upper panel)*

[10] `https://physics.aps.org/synopsis-for/10.1103/PhysRevLett.121.193401`

5.8 Equation for Rydberg eigenmodes

Like every tuned resonant circuit, atoms have well-defined resonant frequencies, or eigenmodes, which must be labled using integers, as required by quantum mechanics (e.g., eigenmodes must have integer labels, not irrational numbers). Integer quantum numbers might serve as the definition of quantum mechanics. One uncertainty is the spin quantum numbers, which are all multiples of 1/2.

Figure 5.9 shows the observed radiation spectra for hydrogen. From the very beginning, it has been clear that there is a pattern to these spectral lines. In 1880 Rydberg fitted a formula that quantifies the observed eigen spectral lines of hydrogen in terms of the reciprocals of the radiated wavelengths:

$$\frac{1}{\lambda_{nm}} = R_\infty \left(\frac{1}{n^2} - \frac{1}{m^2}\right), \qquad \frac{f_{nm}}{c_o R_\infty} = \frac{1}{n^2} - \frac{1}{m^2}, \tag{5.41}$$

all based on these simple observations. Here $R_\infty = 1.097 \times 10^7$ [m^{-1}] is the Rydberg constant, $c_o = 3 \times 10^8$ [m/s] is the speed of light, f_{nm} are the dimensionless *Rydberg integer frequencies*, where $n, m \in \mathbb{N}$ are positive integers. Here n labels the series and $m > n$ describes the transition from orbit m to orbit n, as described in the caption of Fig. 5.11. During every transition, the electrons go from waves to particles, having mass and energy, and both the mass and energy are modified, known as *wave-partical duality*.[11] Both constants are attempts to estimate the size of a Hydrogen atom.

One would expect a close relation between the Bohr radius and λ_{nm} (see footnote 19 on p. 301.)

5.8.1 The Rydberg atom model

In 1909 Rutherford demonstrated that the atom consisted of a dense core (the proton) surrounded by electrons. This view was supported by the spectrum of the atom, which allows for a radiation spectrum caused by electrons jumping from one energy level to another. It was then noted by Bohr in 1913 (Bohr, 1954) and others that the wavelengths of hydrogen, as described by Eq. 5.41, are consistent with Fig. 5.10, where the reciprocal wavelength [m^{-1}] is given by Eq. 5.41, having frequencies $f_{nm} = c/\lambda_{nm}$ [Hz]. The challenge of the 1920s was to explain these intuitive and simple models of hydrogen. This gave rise to the birth of quantum mechanics, the history of which is carefully review in Condon and Morse (1929).

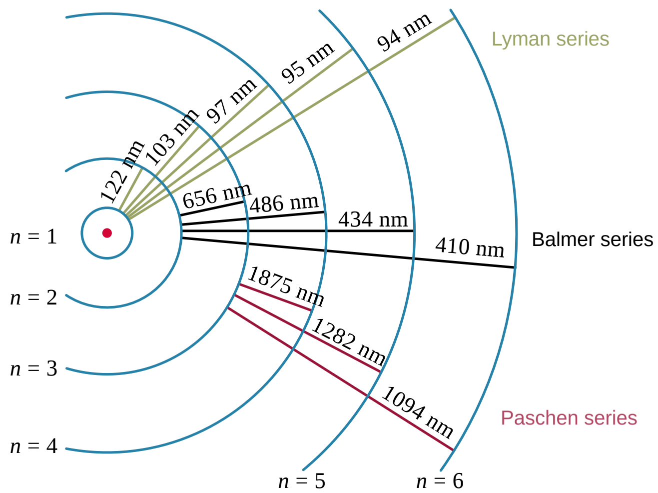


Figure 5.11: *This diagram defines hydrogen's allowed electron transitions, defining the Lyman ($n = 1$), Balmer ($n = 2$), and Paschen ($n = 3$) series. The numbers represent the wavelengths λ [nm] of the photons having frequencies $f_{nm} = c_o/\lambda_{nm}$, following an electron transition from level n to m (taken from:* `https://en.wikipedia.org/wiki/Hydrogen_spectral_series`*).*

It was clear from the days of Bohr that the Rydberg formula did not follow the typical rules of eigenspectra, so much so that Arnold Sommerfeld wrote (Sommerfeld, 1949, p. 201):

> The lines of this spectrum cumulate at the limit given by the Rydberg constant R. The adjoining *continuum* lies in the near ultraviolet range. Both the discrete and the continuous spectrum are given by the Schrödinger equation. This equation reduces to a simple mathematical formula the enigma of the spectral lines, with their finite cumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

[11] `https://en.wikipedia.org/wiki/Wave\%e2\%80\%93particle_duality`

5.8.2 Rydberg wave equation

The objective of this analysis is to demonstrate that one can define a classical Sturm-Liouville model of the *enigmatic* Rydberg atom, by the use of the Webster horn equation

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\psi(r,t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\psi(r,t), \tag{5.42}$$

which is a one-dimensional wave equation for the electric potential $\psi(r,t)$ propagating in a wave guide having area $A(r)$ as a function of the range, where r is the *range* variable (the axis of wave propagation).

We shall show that given the Rydberg spectrum (Eq. 5.41), we may accurately estimate the electric reflectance $\Gamma(s)$ looking out from the origin (i.e., the proton location, as indicated by the small red dot in Fig. 5.11). The radiation impedance $Z_{\text{rad}}(s)$ seen by the proton is related to the reflectance $\Gamma(s)$ by the relation

$$Z_{\text{rad}}(s) = r_o\frac{1+\Gamma(s)}{1-\Gamma(s)}. \tag{5.43}$$

This formula is the basis of the *Smith chart* used in both physics and engineering studies. It follows that once $\Gamma(s)$ is known (i.e., evaluated given Eq. 5.41), the radiation impedance may be computed. It has been shown that the area function $A(r)$ may be found given the radiation impedance (Sondhi and Gopinath, 1971; Youla, 1964).

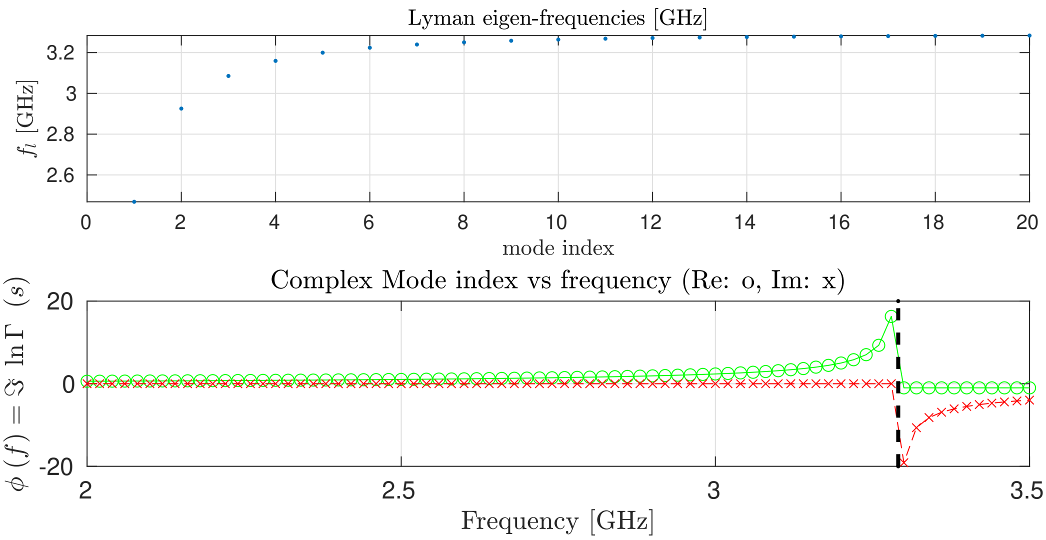


Figure 5.12: *The top panel is a plot of Eq. 5.41, showing how the eigenmode frequencies f_l depend on the eigen-number index l. As the mode number increases, the frequency reaches an asymptote at f_o = 3.29 [GHz], with a wavelength limit near $1/R \approx 91.2$ [nm]. The lower panel shows the inverse mapping from frequency to the mode index number $\phi(f)$. This figure is for the Lyman series ($n = 1$ and $m = 1, \ldots, 20$). The inverse of this relationship is $l = \phi^{-1}(f_l)$ may be derived from Eq. 5.41, which provides the pole frequencies required to satisfy Eq. 5.43. Note that for frequencies greater than c_o/R the phase switches from purely real to imaginary, accounting for free electrons above 3.29 [GHz].*

5.9 Rydberg solution methods

The basic idea behind the method is to use Eq. 5.43, by noting that the poles of the impedance are determined by the roots of the denominator of Z_{rad}. Specifically, if s_p is an impedance pole, then it must satisfy $\Gamma(s_p) \approx 1$. Except for losses due to radiation, the atom is lossless; thus $|\Gamma(s)| = 1$. Namely, it must be of the form

$$\Gamma(s) = e^{-\jmath\phi(f)}, \tag{5.44}$$

where the *phase* $\phi(f) \in \mathbb{R}$ and $s = \sigma + \omega\jmath$ is the complex Laplace radian frequency, with $\omega = 2\pi f$ [Hz]. Since we know the eigenmode frequencies, which obey $\phi(f_{n_o,m}) = 2\pi m$, we may find $\phi(f)$, as follows: For a given series index n_o, and given the eigenfrequencies f_m, we seek the phase mode function $\phi_{n_o}(f)$ that maps the eigenfrequencies to their mode index m, i.e.,

$$\phi_{n_o}(f_m) = 2\pi m.$$

5.9.1 Group delay $\tau(s)$

The phase $\phi(\omega)$ is related to the group delay $\tau(\omega)$ by the relation

$$\tau(\omega) = -\frac{\partial}{\partial\omega}\phi(\omega).$$

Here one may assume that the phase is complex–analytic,[12] thus allowing a causal damping term into the reflectance phase Eq. 5.44. This follows naturally because the reflectance must be causal (Postulate 3.9.2, p. 126). In the time domain the delay may be written in terms of the inverse $\mathcal{LT}$of the group delay,

$$\Gamma(s) = e^{-\jmath \int_o^s \tau(s)ds}.$$

Typically one uses the reflectance phase $2\pi\phi(f)$; thus the group delay is $\tau(f) = -\partial\phi(f)/\partial f$, which is physically interpreted here as the frequency-dependent delay from the proton to the radius of the electron's orbit. Thus this delay is given by

$$\tau(f) = n\frac{\partial}{\partial f}\left(1 - \frac{n^2}{c_oR}f\right)^{-1/2} = \frac{n^3}{2c_oR}\left(1 - \frac{n^2}{c_oR}f\right)^{-3/2},$$

which is constant for low frequencies and then rises to ∞ as frequency approaches the Rydberg frequency ($f \to c_oR/n^2$).

One may solve Eq. 5.41 for m, for the case of the Lyman series ($n_o = 1$), by the use of the following identity for the Rydberg eigenfrequencies f_{nm}, which follow directly from Eq. 5.41, with $m = n_o + l$ (with $n_o, m, l \in \mathbb{N}$)

$$\begin{aligned} f_{nm} = \frac{c_o}{\lambda_{nm}} &= c_oR\left(\frac{1}{n_o^2} - \frac{1}{(n_o+l)^2}\right) \\ &= \frac{c_oR}{n_o^2}\left(1 - \frac{1}{(1+l/n_o)^2}\right). \end{aligned} \tag{5.45}$$

Note that as $l \to \infty$, $f_{n_o,l} \to= c_oR/n_o^2$, which is Sommerfeld's "finite cumulation point" [Hz] $f_{n_o,\infty}$ for the Lyman series ($n_o = 1$).

We can solve Eq. 5.45 for the mode number $l/n < 1$ as a function of mode frequency:

$$\begin{aligned} n^2\frac{f_{nl}}{c_oR} &= 1 - \frac{1}{(1+l/n)^2} && \text{Starting from Eq. 5.45} \\ \frac{1}{(1+l/n)^2} &= 1 - n^2\frac{f_{nl}}{c_oR} && \text{Solving for } l/n \\ (1+l/n)^2 &= \frac{1}{1-n^2\frac{f_{nl}}{c_oR}} \\ \frac{l}{n} &= \pm\frac{1}{\sqrt{1-n^2\frac{f_{nl}}{c_oR}}} - 1 && \phi(f_{nl})/2\pi = l = m - n_o \in \mathbb{N}, \end{aligned} \tag{5.46}$$

as summarized in the lower panel of Fig. 5.12. The square root term is the relativistic *Lorentz Transformation* applied to a rest mass traveling at speeds close to the speed of light This is quite different from the *Lorentz force*. The *Lorentz transformation* was first introduced in Einstein's theory of relativity, One must be careful to distinguished the Lorentz transformation from the *Lorentz force*.

5.9.2 Finding the area function

Once the phase has been determined, we can compute the impedance using Eq. 5.43. We may then decompose the impedance by using the analytic continued fraction algorithm (or Cauer synthesis), discussed in Sec. 3.7, p. 113.

[12]It follows that these relationships are related by a Hilbert transform.

5.10 Euclid's formula and the Rydberg atom model

Fundamental to quantum mechanics is the Rydberg formula, which describes the quantized energy levels of atoms[13]

$$\nu_{n,m} = c_o R_\infty Z_n^2 \left(\frac{1}{n^2} - \frac{1}{m^2}\right), \tag{5.47}$$

where $\nu_{n,m}$ are the possible eigenfrequencies, $c_o \approx 3{\times}10^8$ [m/s] is the speed of light, $R_\infty \approx 10.97{\times}10^{-6}$ $[m^{-1}]$ is the Rydberg constant Z_n is the atomic number, along with positive integers $m > n \in \mathbb{N}$, which represent the *principal quantum numbers* that label all possible allowed atomic eigenstates.$(5.29177210544 \times 10^{-}9)$. Integer n indicates the lowest (rest) atomic eigenstate while m labels the higher (excited) state.[14] When $n = 1$, the series is the Lyman series corresponding to hydrogen ($Z_1 = 1$). When $n = 1$, $m = 2$, and $Z_1 = 1$, the frequency is

$$\nu_{1,2} = c_o R_\infty \left(\cancelto{\frac{3}{4}}{\frac{1}{1^2} - \frac{1}{2^2}}\right) = \frac{3}{2} \times 10^{15} \text{ [Hz]}. \tag{5.48}$$

An open question in this model is: *Why are states either empty or filled?* The amplitudes of the modes of a string or organ pipe are never empty. What is it about the atom that forces the energy state to be empty? Can it be true that an eigenstate is either empty or full? Perhaps the answer is due to the Lorentzian term, which is relativistic. Classical physics, by definition, is never relativistic. What does the experimental, if any, evidence say? I dont't have an answer to this interesting question.

5.10.1 Solving for the area function

Given observed frequencies $\nu_{n,m}$ it is possible to determine the area function that traps the photons into the Rydberg eigenstates. Eq. 5.47 may be rewritten as

$$\nu_{n,m} = c_o R Z_n^2 4 \left(\frac{m^2 - n^2}{(2nm)^2}\right).$$

It is interesting to compare Eq. 5.47 to Euclid's formula

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2, \tag{5.49}$$

where $m > n \in \mathbb{N}$. Euclid's formula is equivalent to the Pythagorean theorem for integers, since

$$c^2 = a^2 + b^2, \tag{5.50}$$

with $\{a, b, c\} \in \mathbb{N}$. Here $a < b < c$.

If we interpret the quantum numbers as multiples of a quarter wavelength, then the Rydberg formula is congruent to the Pythagorean theorem. Given the symmetry, this cannot be an accident.

In terms of the lengths of the right triangle $\{a, b, c\}$, Rydberg's formula becomes

$$\nu_{n,m} = c_o R Z_n^2 4 \left(\frac{a}{b^2}\right).$$

But since $b^2 = c^2 - a^2$,

$$\begin{aligned}\nu_{n,m} &= c_o \frac{R Z_n^2}{a} 4 \left(\frac{a^2}{c^2 - a^2}\right) \\ &= c_o \frac{R Z_n^2}{a} 4 \frac{a^2}{c^2} \left(\frac{1}{1 - (a/c)^2}\right).\end{aligned}$$

[13] `https://www.youtube.com/watch?v=e0IWPEhmMho`

[14] `http://en.wikipedia.org/w/index.php?title=Rydberg_formula`

In terms of quantized (discrete) angles, $\sin(\theta_{n,m}) = a/c$,

$$\begin{aligned}
\nu_{n,m} &= c_o \frac{RZ_n^2}{a} 4 \left(\frac{\sin^2\theta}{1-\sin^2\theta} \right) \\
&= c_o \frac{RZ_n^2}{a} 4 \left(\frac{\sin^2\theta}{\cos^2\theta} \right) \\
&= c_o \frac{RZ_n^2}{a} 4 \tan^2\theta_{n,m}.
\end{aligned}$$

5.10.2 Eigenmodes of the Rydberg atom

One way to think of eigenmodes is to make an analogy to a piano string or an organ pipe. In these much simpler systems, there is an almost constant delay, say τ, due to a characteristic length, say $L = \tau c_o$, such that the eigenmodes of a string are given by integer multiples of a half wavelength $\nu_n = nc_o/2L$, while the eigenmodes of the organ pipe are multiples of a quarter wavelength. The distinction is the boundary conditions. For the string the endpoint boundary conditions are pinned displacement (i.e., zero velocity). The organ pipe is closed at one end and open at the other, resulting in multiples of a quarter wavelength $\nu_n = nc_o/4L$. In each case $\nu = n/\tau$, where $\tau = 2L/c_o$ is the round-trip delay; thus $\nu = nc_o/2L$. We suggest looking at the Rydberg formula in the same way, but with very different eigenfrequencies (Eq. 5.47). Sommerfeld (1949, p. 201) makes an interesting comment regarding Eq. 5.47:

> This equation reduces to a simple mathematical formula the enigma of the spectral lines, with their finite cumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

5.10.3 Discussion

The Rydberg frequencies f_{nl} ($n = 1, l = 1, \ldots, \infty$) has poles in the radiation impedance (Eq. 5.43) when $\phi_l(f_{nl}) \in \mathbb{N}$. Working backwards from the Rydberg formula (Eq. 5.44), we have solved for $\phi(f_{nl})$ indicating where this condition is valid (Eq. 5.46). Since the reflectance and the impedance must be causal complex–analytic functions of Laplace frequency s, we must replace the discrete frequency f_{nl} with s:

$$\jmath 2\pi f_{nl} \to s = \sigma + \omega\jmath,$$

thereby forcing $l(s)$ to be a complex–analytic function of s. Then the poles of the radiation impedance must satisfy

$$\Gamma(s_{nl}) = e^{\jmath 2\pi l(f_{nl})} = 1,$$

resulting in eigenfrequencies at f_{nl}.

The next step in this analysis is to determine the area function $A(r)$ given Z_{rad} (Eq. 5.43). To do this we must solve an integral equation for $A(r)$, as discussed by Sondhi and Gopinath (1971) and by Youla (1964).

Perhaps this could be done using an analytic representation for the area function,

$$A(r) = \sum_k a_k r^k.$$

5.11 Relations between Sturm-Liouville and quantum mechanics

If we compare the Schrödinger equation (SE) for hydrogen with the corresponding Sturm-Liouville equation we can begin to appreciate their differences. The QM equation for hydrogen is

$$\begin{aligned}
\imath\hbar \frac{\partial}{\partial t}\psi(x,t) &= -\frac{\hbar^2}{2m_o}\nabla_r^2\psi(x,t) + V(r)\psi(x,t) \\
&= -\frac{\hbar^2}{2m_o}\frac{1}{r^2}\frac{\partial}{\partial r} r^2\psi(x,t) + V(r)\,\psi(x,t) \qquad (5.51) \\
&= -\frac{\hbar^2}{2m_o}\left[\frac{2}{r}\frac{\partial}{\partial r}\psi(x,t) + \frac{\partial^2}{\partial r^2}\psi(x,t)\right] + V(r)\,\psi(x,t), \qquad (5.52)
\end{aligned}$$

whereas the horn equation is given by Eq. 5.42.

There are several obvious and disturbing differences between these two equations. First, the SE is, of course, first-order in time. Diffusion equations have no delay and thus cannot have traditional eigenmodes, which result from standing waves in a wave equation, due to boundary conditions. Second, the EM horn equation is of Sturm-Liouville (SL) form, which is a true wave equation (vs. the SE, which is a diffusion equation). The obvious question arises: Is there a potential V that would allow these two formulations to be equivalent? If so, then this would provide an explanation as to why the SE is successful in explaining the properties of Rydberg atoms.

To explore this possibility we may expand the two differential equations and directly compare them. Expanding Eq. 5.42 gives

$$\begin{aligned}\frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\psi(r,t) &= \frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\psi(r,t) && (5.53)\\ &= \frac{\partial^2}{\partial r^2}\psi(r,t) + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\psi(r,t). && (5.54)\end{aligned}$$

Between these two equations we may remove ψ'':

$$\imath\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m_o}\left[\frac{2}{r}\frac{\partial}{\partial r}\psi(x,t) + \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\psi(r,t) - \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\psi(r,t)\right] + V(r)\,\psi(x,t). \tag{5.55}$$

It seems that this may isolate the time and spatial parts (as in separation of variables).

5.11.1 The exponential horn

A relevant and motivational example is the solution of the exponential horn, having area function $A(r) = A_o e^{2mr}$. This case is interesting because it has a closed-form solution, which seems relevant and perhaps even related to the hydrogen atom.

Assuming that $\varrho(r,t) \leftrightarrow \mathcal{P}(r,\omega)$ are a Fourier transform pair, with $A(r) = A_o e^{2mr}$, Eq. 5.42 reduces to

$$\frac{\partial^2\mathcal{P}(r,\omega)}{\partial r^2} + 2m\frac{\partial\mathcal{P}(r,\omega)}{\partial r} = \kappa^2\mathcal{P}(r,\omega) \leftrightarrow \frac{1}{c_o^2}\frac{\partial^2\varrho}{\partial t^2}, \tag{5.56}$$

with $\kappa(s) = s/c_o$.

Exercise #6

Show that Eq. 5.56 follows from Eq. 5.42.

Solution: Starting from Eq. 5.42 with area $A(r) = A_o e^{2mr}$

$$\begin{aligned}\frac{1}{\cancel{A_o}e^{2mr}}\frac{\partial}{\partial r}\left(\cancel{A_o}e^{2mr}\frac{\partial\varrho}{\partial r}\right) &= \frac{1}{c_o^2}\frac{\partial^2\varrho}{\partial t^2}\\ \varrho_{rr}(r,t) + 2m\varrho_r(r,t) &= \frac{1}{c_o^2}\frac{\partial^2\varrho}{\partial t^2} \leftrightarrow \kappa^2\mathcal{P}(r,\omega),\end{aligned}$$

which is the time-domain version of Eq. 5.56. ■

Since this equation is second-order in time with constant coefficients, it has two closed-form solutions:

$$\begin{aligned}\mathcal{P}_c^{\pm}(r) &= \mathcal{P}_o^{\pm}(\omega)\,e^{-mr}e^{\mp\sqrt{m^2+\kappa^2}\,r}\\ &= \mathcal{P}_o^{\pm}(\omega)\,e^{-mr}e^{\mp j\frac{r}{c_o}\sqrt{\omega^2-\omega_c^2}},\end{aligned}$$

with $\omega_c = mc_o$. The two wave amplitudes $\mathcal{P}_0^{\pm}(\omega)$ must be determined from the boundary conditions.

Exercise #7

Shown that $\mathcal{P}^{\pm}(r,\omega)$ satisfy Eq. 5.56.

Solution: Taking partials with respect to r,

$$\begin{aligned}
\partial_r \mathcal{P}^{\pm}(r,\omega) &= \left(-m \mp \sqrt{m^2+\kappa^2}\right) \mathcal{P}^{\pm}(r,\omega) \\
\partial_{rr} \mathcal{P}^{\pm}(r,\omega) &= \left(-m \mp \sqrt{m^2+\kappa^2}\right)^2 \mathcal{P}^{\pm}(r,\omega) \\
&= \left(2m^2+\kappa^2 \pm 2m\sqrt{m^2+\kappa^2}\right) \mathcal{P}^{\pm}(r,\omega).
\end{aligned}$$

Thus Eq. 5.56 reduces to

$$\left(2m^2+\kappa^2 \pm 2m\sqrt{m^2+\kappa^2}\right) + 2m\left(-m \mp \sqrt{m^2+\kappa^2}\right) = \kappa^2,$$

which is an identity. ∎

Next consider the Fourier series (or Fourier transform) of the area function,

$$A(r) = \sum_k a_k e^{2m_k r}.$$

It follows from the linearity of the wave equation that the general solution of Eq. 5.56 is

$$\mathcal{P}^{\pm}(r,\omega) = \sum_k a_k^{\pm}(\omega)\, e^{-m_k r}\, e^{\mp\sqrt{m_k^2+\kappa^2}\, r}.$$

Here we have combined $\mathcal{P}^{\pm}(\omega)$ and a_k as coefficients $a_k^{\pm}(\omega)$.
from Ch4: line 2252 just before DE3

Chapter 6

Stream 3B: Vector Calculus

6.1 Properties of fields and potentials

Before we can define the vector operations $\nabla(\ ), \nabla\cdot(\ ), \nabla\times(\ ),$ and $\nabla^2(\ )$, we must define the objects they operate on: scalar and vector fields. The word *field* has two very different meanings: a mathematical one, which defines an algebraic structure, and a physical one, discussed next.

Integration is quantified by several fundamental theorems of calculus, each about integration. Ultimately we wish to integrate in $\in \mathbb{R}^3, \mathbb{R}^n$ and $\mathbb{C}^n$.

6.1.1 Scalar and vector fields: notation

Scalar fields:

We use the term *scalar field* interchangeably with *analytic* in a smooth connected region of the spatial vector $\boldsymbol{x} = [x, y, z]^T \in \mathbb{R}^3$. In mathematics, functions that are piece-wise differentiable are called *smooth*, which is different from *analytic*. A smooth function has at least one or more derivatives. Every analytic function is single-valued and is an infinitely differentiable power series (§ 3.2.4).

Vector fields: A vector field is composed of three scalar fields. For example, the electric field used in Maxwell's equations, $\boldsymbol{e}(\boldsymbol{x}, t) = [E_x, E_y, E_z]^T$ [V/m], has three components, each a scalar field. When the magnetic flux vector $\boldsymbol{a}(\boldsymbol{x})$ is static (Postulate P5), the potential $\phi(\boldsymbol{x}, t)$ [V] uniquely defines $\boldsymbol{e}(\boldsymbol{x}, t)$. via the gradient,

$$\boldsymbol{e}(\boldsymbol{x}, t) = -\nabla\phi(\boldsymbol{x}, t) + \cancelto{0}{\partial_t \boldsymbol{a}(\boldsymbol{x})} \text{[V/m]}. \tag{6.1}$$

The electric force on a charge q is $\boldsymbol{F} = q\boldsymbol{e}$; thus $\boldsymbol{e}$ is proportional to the force, and when the medium is conductive, the current density (a flow) is $\boldsymbol{j_m} = \sigma_o \boldsymbol{e}$ [A/m^2]. The ratio of the potential to the flow is an impedance, so σ_o is called the *conductance*.

To verify that a field is a potential, we may check the units. For a proper mathematical definition, the potential must be an analytic function of $\boldsymbol{x}$ and the Laplace frequency s.

Example: Suppose we are given the vector field in $\mathbb{R}^3$

$$\boldsymbol{a}(\boldsymbol{x}) = [\phi(\boldsymbol{x}), \psi(\boldsymbol{x}), \theta(\boldsymbol{x})]^T \quad \text{[Wb/m]},$$

where each of the three functions is a scalar field. Then $\boldsymbol{a}(\boldsymbol{x}) = [x, xy, xyz]^T$ is analytic in $\boldsymbol{x}$.

Example: From Maxwell's equations in the Laplace frequency domain, the magnetic flux vector is a function of the Laplace frequency s,

$$\boldsymbol{b}(\boldsymbol{x}, s) = \nabla \times \boldsymbol{a}(\boldsymbol{x}, s) \quad \text{[Wb/m}^2\text{]}. \tag{6.2}$$

We shall see that this is always true because the magnetic charge $\nabla\cdot\boldsymbol{b}(\boldsymbol{x}, s)$ must be 0, which is true in-vacuo. Working in the Laplace frequency domain s greatly simplifies the analysis. Feynman (1970b, pp. 14-1 to 14-3) provides an extended and helpful tutorial on the vector potential, with many examples. Since it is a function of the Laplace frequency, s, it must be *causal*.

Scalar potentials: The above discussion describes the utility of potentials for defining vector fields (e.g., 6.2). The key distinction between a potential and a scalar field is that potentials have units and thus have a physical meaning. Scalar potentials (i.e., voltage $\phi(\boldsymbol{x}, s)$ [V], temperature $T(\boldsymbol{x}, s)$ [°C], and pressure $\varrho(\boldsymbol{x}, s)$ [pascals]) are physical scalar fields. All potentials are composed of scalar fields, but not all scalar fields are potentials.

For example, the $\hat{\mathbf{y}}$ component of $\boldsymbol{e}$, $E_y(\boldsymbol{x}, s) = \hat{\mathbf{y}} \cdot \boldsymbol{e}(\boldsymbol{x}, s)$ [V/m], is not a potential. While $\nabla \boldsymbol{e}_y$ is mathematically defined as the gradient of one component of a vector field, it has no physical meaning (as best I know).

Vector potentials: Vector potentials, like scalar potentials, are vector fields with physically meaningful units. They are more complicated than scalar potentials because they are composed of three scalar fields. Vector fields are composed of laminar and rotational flow, which are mathematically described by the fundamental theorem of vector calculus (also called Helmholtz's decomposition theorem). One helpful comparison is the momentum of a mass, which may be decomposed into its forward (linear) and rotational momentum.

Since we find it helpful (more physical) to analyze problems using potentials (e.g., voltage) so we may take the gradient to find the flow (i.e., current density $\boldsymbol{j} = \sigma \boldsymbol{e}(\boldsymbol{x}, s)$ $[A/m^2]$), where σ $[\mho/m^2]$ is the conductivity. The same logic and utility apply when we use the vector potential to describe the magnetic flux (flow) $\boldsymbol{b}(\boldsymbol{x}, s)$ (Feynman, 1970c). When operating on a scalar potential using the gradient, whereas for the vector potential, we must operate with the curl (Eq. 6.2).

If we assume the magnetic flux vector $\boldsymbol{b}(\boldsymbol{x})$ is static, $\boldsymbol{e}(\boldsymbol{x}, s)$ is simply the gradient of the time-dependent voltage $\phi(\boldsymbol{x}, s)$. However, when the magnetic field is dynamic (*not* static), Eq. 6.1 is not valid, due to magnetic induction: A voltage is induced into a loop of wire is proportional to the time-varying flux cutting across that loop of wire. This is known as the *Ampere-Maxwell law*. In the static case the induced voltage is zero.

In summary, the electric field strength includes both scalar potential $\phi(\boldsymbol{x}, s)$ and magnetic flux vector potential ($\boldsymbol{a}(\boldsymbol{x}, s)$), while the magnetic field strength depends solely on the magnetic potential.

6.1.2 Gradient ∇, divergence $\nabla\cdot$, curl $\nabla\times$, and Laplacian ∇^2

Three key vector differential operators are used in linear partial differential equations, such as the wave and diffusion equations. All of these begin with the ∇ operator:

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.$$

As outlined in Table 6.1, the accepted name of ∇ is *nabla*. It has three basic uses: (1) the gradient of a scalar field, (2) the divergence of a vector field, and (3) the curl of a vector field. The shorthand notation $\nabla\phi(\boldsymbol{x}, s) = (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_x + \hat{\mathbf{z}}\partial_x)\phi(\boldsymbol{x}, s)$ is convenient.[1]

Table 6.1: *The three vector operators manipulate scalar and vector fields. The gradient converts scalar fields into vector fields. The divergence maps vector fields to scalar fields. The curl maps vector fields to vector fields. Four second-order operators (for example DoG and* **gOd***) are defined in §2.2.4. Bold mnemonics are reserved for vector-in, vector-out operators, with the CoW being an interesting exception.*

Name	Input	Output	Operator	Mnemonic
Gradient	Scalar	**Vector**	$\nabla(\,)$	grad
Divergence	**Vector**	Scalar	$\nabla\cdot(\,)$	div
Laplacian	Scalar	Scalar	$\nabla\cdot\nabla = \nabla^2(\,)$	DoG
Curl	**Vector**	**Vector**	$\nabla\times(\,)$	curl
God	**Vector**	**Vector**	$\boldsymbol{\nabla}^2(\,) = \nabla(\nabla\cdot(\,))$	**gOd**
Bull-DoG	**Vector**	**Vector**	$\boldsymbol{\nabla}^2(\,) = \nabla\cdot\nabla(\,)$	**DoG**
Curl of Curl	**Vector**	**Vector**	$\nabla\times\nabla\times = \boldsymbol{\nabla}^2(\,) - \boldsymbol{\nabla}^2(\,)$	**CoC**
Curl of Wedge	**Vector**	Scalar	$\approx$**CoC**	CoW
Div of Curl	**Vector**	0	$\nabla\cdot\nabla\times(\,)$	**DoC**
Curl of Grad	Scalar	0	$\nabla\times\nabla(\,)$	**CoG**

[1] https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates

6.1.3 Gradient $\nabla(\,)$:

As shown in Fig. 6.1, the gradient transforms a complex scalar field $\Phi(\boldsymbol{x}, s) \in \mathbb{C}$ into a vector field ($\mathbb{C}^3$)

$$\nabla\Phi(\boldsymbol{x}, s) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\Phi(\boldsymbol{x}, s)$$
$$= \hat{\mathbf{x}}\frac{\partial\Phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\Phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\Phi}{\partial z}.$$

The gradient may also be factored into a unit vector $\hat{\mathbf{n}}$, as defined in Fig. 6.1, that gives the direction of the gradient, and the gradient's length $||\nabla(\,)||$, defined in terms of the norm of the gradient. Thus the gradient of $\Phi(\boldsymbol{x})$ may be written in "polar coordinates" as $\nabla\Phi(\boldsymbol{x}) = ||\nabla\Phi||\,\hat{\mathbf{n}}$, which leads to the unit vector

$$\hat{\mathbf{n}} = \frac{\nabla(\Phi(\boldsymbol{x}))}{||\nabla\Phi||}.$$

Consider the paraboloid $z = 1 - (x^2 + y^2)$ as the potential, with iso-potential circles of constant z having a radius of zero at $z = 1$, and unit radius at $z = 0$. The negative gradient

$$\boldsymbol{e}(\boldsymbol{x}) = -\nabla z(x, y) = 2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + 0\hat{\mathbf{z}})$$

is $\perp$ to the circles of constant radius (constant z) and thus points in the direction of the radius.

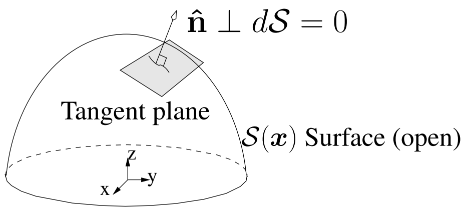


Figure 6.1: *Definition of the unit vector $\hat{\mathbf{n}}$ defined by the gradient $\nabla\mathcal{S} \perp$ to the tangent plane. A bifurcated (split) volume defines surface $\mathcal{S}(\boldsymbol{x})$. At one point a tangent plane (shaded) touches the surface. At that point the gradient $\nabla\mathcal{S}(\boldsymbol{x})$ is normalized to unit length, defined as $\hat{\mathbf{n}}$, which is perpendicular ($\perp$) to the shaded tangent plane.*

A skier in free fall on this surface would be the first one down the hill. Normally skiers try to stay close to the isoclines (not in the direction of the gradient) so they can stay in control. If you ski on an isocline, you must walk, since there is no pull due to gravity. The gravitational potential at the surface of a spherical earth is

$$\phi = G\frac{mM}{r}.$$

Of course the earth is *not* a perfect sphere, due to the rotation about its axial tilt.

Divergence $\nabla\cdot(\,)$:

The divergence of a vector field results in a scalar field. For example, the divergence of the electric field flux vector $\boldsymbol{d}(x)$ [C/m^2] equals the scalar field charge density $\rho(\boldsymbol{x})$ [C/m^3]:

$$\nabla\cdot\boldsymbol{d}(\boldsymbol{x}) \equiv \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot\boldsymbol{d}(\boldsymbol{x}) = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho(\boldsymbol{x}). \tag{6.3}$$

Thus the divergence is analogous to the scalar (dot) product (e.g., $\boldsymbol{a}\cdot\boldsymbol{b}$) between two vectors.

Recall that the voltage is the line integral of the electric field,

$$V(a) - V(b) = \int_a^b \boldsymbol{e}(\boldsymbol{x})\cdot d\boldsymbol{x} = -\int_a^b \nabla V(\boldsymbol{x})\cdot d\boldsymbol{x} = -\int_a^b \frac{dV}{dx}dx, \tag{6.4}$$

which is simply the fundamental theorem of calculus. In a charge-free region, this integral is independent of the path from a to b, which is the critical property of a conservative system.

When we work with guided waves (narrow tubes of flux) having rigid walls that block the flow, such that the diameter is small compared with the wavelength (Postulate P10), the divergence simplifies to

$$\nabla\cdot\boldsymbol{d}(\boldsymbol{x}) = \nabla_r D_r = \frac{1}{A(r)}\frac{\partial}{\partial r}A(r)D_r(r), \tag{6.5}$$

where r is the distance down the horn (range variable), $A(r)$ is the area of the iso response surface as a function of the range r, and $D_r(r)$ is the radial component of vector $\boldsymbol{d}(r)$, as a function of the range r. In spherical, cylindrical, and rectangular coordinates, Eq. 6.5 provides the correct expression (Table 6.2).

Properties of the divergence: The divergence is a direct measure of the flux density of the vector field. A vector field is said to be *in-compressible* if the divergence of that field is zero, thus it is *compressible* when the divergence is nonzero [e.g., $\nabla\cdot\boldsymbol{d}(x, s) = \rho(\boldsymbol{x}, s)$] (see Table 6.3).

For example, compared to air, water is considered to be in-compressible. The stiffness of a fluid (i.e., the bulk modulus) is a measure of its compressibility. At very low frequencies, air may be treated as in-compressible (like water), as $s \to 0$,

$$-\nabla\cdot\boldsymbol{u}(\boldsymbol{x}) = \frac{s \to 0}{\eta_o P_o}\mathcal{P}(\boldsymbol{x}, s \to 0) \to 0.$$

The definition of *compressible* depends on the wavelength in the medium, so the term must be used with an awareness of the frequencies (a wavelength $\gg$ the radius). If the wavelength $\lambda = c_o/f$ is much larger than the size of the system, the medium may be treated as an in-compressible fluid.

Curl $\nabla\times(\,)$:

The curl $\nabla\times(\,)$ takes a vector in $\mathbb{C}^3$ into a second vector in $\mathbb{C}^3$. For example, in the case of fluids, the vorticity is defined as $\boldsymbol{\omega} = \nabla\times\nu$ and *rotation* as $\boldsymbol{\Omega} = \boldsymbol{\omega}/\mathbf{2}$. The curl is a measure of the rotation of a vector field in a plane about the axis perpendicular to that plane. In the case of a liquid, it corresponds to the angular momentum, such as in a whirlpool in water, or a tornado in air. A massive top falls over when not spinning. But once spinning, it can stably stand on its pointed tip. These systems are stable due to conservation of angular momentum.

The curl and the divergence are both critical operations when we working with Maxwell's equations. For example, the curl transforms the vector field $\boldsymbol{h}(\boldsymbol{x}, s) \in \mathbb{C}^3$ [A/m] into a complex vector current density $\boldsymbol{c}(\boldsymbol{x}, s) \in \mathbb{C}^2$ [A/m^2]:

$$\nabla\times\boldsymbol{h}(\boldsymbol{x}, s) \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{vmatrix} = \boldsymbol{c}(\boldsymbol{x}, s) \qquad [\text{A/m}^2]. \tag{6.6}$$

The notation $|\cdot|$ indicates the determinant ∂_x is shorthand for $\partial/\partial x$, and $\boldsymbol{h} = [H_x, H_y, H_z]^T$.

Exercise #*1*

If we let $\boldsymbol{h} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$, $\nabla\times\boldsymbol{h} = 2\hat{\mathbf{z}}$, thus $\boldsymbol{h}$ has a constant rotation; when $\boldsymbol{h} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + z^2\hat{\mathbf{z}}$, $\nabla\times\boldsymbol{h} = \mathbf{0}$ has a curl of zero and thus is irrotational.

There are precisely rules that govern when a vector field is rotational versus irrotational, and compressible versus in-compressible. These classifications are dictated by Helmholtz's theorem, the fundamental theorem of vector calculus (Eq. 6.70).

CoW (wedge) $\nabla\wedge(\,)$:

A special case of the curl is the two-dimensional differential wedge products[2]

$$\nabla_x \wedge \boldsymbol{h}(\boldsymbol{x}, s) = \begin{vmatrix} \partial_y & \partial_z \\ H_y & H_z \end{vmatrix} = C_x(\boldsymbol{x}, s) \qquad [\text{A/m}^2].$$

The curl is made up of three such differential wedge products.[3]

Laplacian $\nabla^2(m)$:

The Laplacian operator $\nabla^2 \equiv \nabla\cdot\nabla$ (Table 6.1,) is defined as the divergence of the gradient

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{6.7}$$

[2] `https://en.wikipedia.org/wiki/Triple_product#As_an_exterior_product`

[3] This notation suggests that $||\nabla\cdot\boldsymbol{e} + \imath\wedge\boldsymbol{e}||^2 = ||\nabla\cdot\boldsymbol{e}||^2 + ||\wedge\boldsymbol{e}||^2$ is related to Helmholtz's theorem. Here $\imath = +\sqrt{-1}$.

Since the Laplacian does so much common work, we nickname it DoG (Div of Grad).

Starting from a scalar field, the gradient produces a vector, which is then operated on by the divergence to take the output of the gradient back to a scalar field. Thus the Laplacian transforms a scalar field back to a scalar field. We have seen the Laplacian before when we defined complex–analytic functions (Eq. 4.13).

A classic example of the Laplacian is a voltage scalar field $\Phi(\boldsymbol{x})$ [V], which results in the electric field vector

$$\boldsymbol{e}(\boldsymbol{x}) = [E_x(\boldsymbol{x}), E_y(\boldsymbol{x}), E_z(\boldsymbol{x})]^T = -\nabla\Phi(\boldsymbol{x}) \quad \text{[V/m]}.$$

When this is scaled by the permittivity, we obtain the electric flux $\boldsymbol{d} = \epsilon_o \boldsymbol{e}$ [C/m^2], the charge density per unit area. Here ϵ_o [F/m] is the vacuum permittivity $\epsilon_o = 1/c_o r_o \approx 8.85 \times 10^{-12}$ [F/m].

Taking the divergence of $\boldsymbol{d}$ results in the charge density $\rho(\boldsymbol{x})$ [C/m^3] at $\boldsymbol{x}$:

$$\nabla\cdot\boldsymbol{d} = \nabla^2\Phi(\boldsymbol{x}) = \rho(\boldsymbol{x}).$$

Thus the Laplacian of the voltage, scaled by ϵ_o, results in a local constant charge density.

A second classic example of the Laplacian is an acoustic pressure field $\varrho(\boldsymbol{x}, s)$ [Pa], which defines a vector force density $\mathbf{f}(\boldsymbol{x}, s) = -\nabla\varrho(\boldsymbol{x}, s)$ [N/m^2] (Eq. 6.20. When this force density [N/m^2] is integrated over an area, the net radial force [N] is

$$F_r = -\int_{\mathcal{S}} \nabla\varrho(\boldsymbol{x})d\boldsymbol{x} \quad \text{[N]}. \tag{6.8}$$

An inflated balloon with a static internal pressure of 3 [atm], in an ambient pressure of 1 [atm] (sea level), forms a sphere due to the elastic nature of the rubber, which acts as a stretched spring under its surface tension. The net normal force on the surface of the balloon is its area times the pressure drop of 2 [atm] across the surface. Thus the static pressure is

$$\varrho(\boldsymbol{x}) = 3u(r_o - r) + 1 \quad \text{[Pa]},$$

where $u(r)$ is a step function of the radius $r = ||\boldsymbol{x}|| > 0$, centered at the center of the balloon, having radius r_o.

Taking the gradient gives the negative[4] of the radial force density (i.e., perpendicular to the surface of the balloon):

$$-f_r(r) = \nabla\varrho(\boldsymbol{x}) = \frac{\partial}{\partial r}3u(r_o - r) + 1 = -2\delta(r_o - r) \quad \text{[Pa]}.$$

This equation describes a static pressure that is 1 [atm] (10^5 [Pa]) outside the balloon and 3 [atm] inside. The net positive force density is the negative of the gradient of the static pressure.

Finally, taking the divergence of the force produces a double delta function at the balloon's surface. Specifically, $\nabla^2\varrho(\boldsymbol{x}) = -2\delta^{(1)}(r_o - r)$, where 2 is the pressure drop across the balloon. If we take the thickness of the rubber (l [m]) into account, then $\nabla^2\varrho = -2(\delta(r_o) - \delta(r_o - l))$.

If the elasticity of the rubber is not constant, the balloon may assume a very different shape.

Vector Laplacian $\boldsymbol{\nabla}^2()$

A second form of the Laplacian is the *vector Laplacian* $\boldsymbol{\nabla}^2()$ defined as the divergence of the gradient $\boldsymbol{\nabla}^2() \equiv \nabla\cdot\nabla()$ thus nicknamed **Bull-Dog**, operates on a vector to produce a vector (Table 6.1). We shall need this operator when working with Maxwell's equations.

6.1.4 Scalar Laplacian operator in N dimensions

In general, it may be shown that for $N = 1, 2, 3$ dimensions (Sommerfeld, 1949, p 227),

$$\nabla_r^2\mathcal{P} \equiv \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial\mathcal{P}}{\partial r}\right). \tag{6.9}$$

For each value of N, the area $A(r) = A_o r^{N-1}$. This result will turn out to be useful when we work with the Laplacian in one, two, and three dimensions. This naturally follows from

$$\frac{1}{A(r)}\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]\varrho(r,t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(r,t) \leftrightarrow \frac{s^2}{c_o^2}\mathcal{P}(r,s).$$

[4]The force is pointing out, stretching the balloon.

Example: Here $N = 3$ (i.e., spherical geometry) with $A(r) = A_o r^2$. Thus

$$\nabla_r^2 \mathcal{P} \equiv \frac{1}{r^2}\partial_r r^2 \partial_r \mathcal{P} \tag{6.10}$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2} r\,\mathcal{P}, \tag{6.11}$$

resulting in the general d'Alembert solutions (Eq. 4.20) for the spherical wave equation,

$$\mathcal{P}^{\pm}(r,s) = \frac{1}{r}e^{\mp\kappa(s)r},$$

where $\kappa(s) = s/c_o$.

Exercise #2

Find the result of the above example by expanding Eqs. 6.10 and 6.11 using the chain rule.

Solution: Expanding Eq. 6.10:

$$\frac{1}{r^2}\partial_r r^2 \partial_r \mathcal{P} = \frac{1}{r^2}\left(2r + r^2\partial_r\right)\partial_r \mathcal{P}$$
$$= \frac{2}{r}\mathcal{P}_r + \mathcal{P}_{rr}.$$

Expanding Eq. 6.11, we obtain

$$\frac{1}{r}\partial_{rr} r\,\mathcal{P} = \frac{1}{r}\partial_r\left(\mathcal{P} + r\,\mathcal{P}_r\right)$$
$$= \frac{1}{r}\left(\mathcal{P}_r + \mathcal{P}_r + r\,\mathcal{P}_{rr}\right)$$
$$= \frac{2}{r}\mathcal{P}_r + \mathcal{P}_{rr}.$$

Thus the two definitions have identical solutions.

Summary

The radial component of the Laplacian in spherical coordinates (Eq. 6.10) simplifies to

$$\nabla_r^2 \varrho(\boldsymbol{x}) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\varrho(\boldsymbol{x}) = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\,\varrho(\boldsymbol{x}).$$

Since DoG is $\nabla^2 = \nabla\cdot\nabla$, it follows that the net force $\boldsymbol{f}(\boldsymbol{x}) = [F_r, 0, 0]^T$, Eq. 6.8 in spherical coordinates has a radial component F_r and angular component of zero. Thus the force across a balloon may be approximated by a delta function across the thin sheet of the stretched rubber.

We can extended the preceding example in an interesting way, to the case of a rigid hose (a rigid tube), that terminates at the end in an elastic medium (the above example of a balloon). Think of as an automobile tire. On the far left let's assume there is a pump injecting the fluid into the rigid hose. Consider two different fluids: air and water. Air is treated as a compressible fluid, whereas water is in-compressible. However, such a classification is relative, determined by the relative compliance of the balloon (i.e., tire) at the relatively rigid pump and hose.

Next is a special case of a more general situation: When a fluid is treated as incompressible (rigid), the sound speed becomes infinite, and the wave equation is an invalid description. In this case the system is best approximated by Laplace's equation. This represents the transition from short to long wavelengths, from wave propagation having delay, to quasistatics, having no delay.

The above example may be modeled as either an electrical or a mechanical system. While the two systems are very different in their physical realization, they are mathematically equivalent, forming a perfect analog. If we take the electrical analog, the pump is a current source, injecting charge Q_o into the hose, which being rigid, cannot expand (it has a fixed volume). The hose may be modeled as a resistor and the tire as a capacitor C_o, which fills with charge as it is delivered via the resistor, from the pump. The capacitor obeys the same equation as Hooke's law for a spring, $F = K_o\Delta$, where K_o is the stiffness of the spring, $C_o = 1/K_o$ is the spring's compliance, and Δ is the displacement. In electrical terms, $Q_o = C_o\Phi$ where Φ is the voltage, which acts like a force F; Q_o is the mass of the fluid. The mass is conserved, just as the mass of the fluid is conserved (it cannot be created or destroyed).

The flow of the fluid (charge or mass) is called the *flux*, which is the general term for the mass flow, heat or electrical charge, or current flow. The two equations may be rewritten directly in terms of the force and flow, which physically are the same as momentum and charge flux. The electrical model is

$$I = C_o\frac{d}{dt}\Phi \quad \text{[A]} \tag{6.12}$$

and mechanical

$$J = C_o\frac{d}{dt}F \quad \text{[kgm-s/m]} \tag{6.13}$$

impedance/admittance relationship.

The stiffness of the balloon, which acts as a spring, has a compliance $C_o = 1/K_o$, in which case the impedance Z is defined in the frequency domain, as the ratio of the generalized force over the generalized flow

$$Z(s) = \frac{1}{sC_o} \quad \text{[ohms]}.$$

In the case of mechanical systems $Z_m(s) \equiv F/J$, while for the electrical system, $Z_e(s) \equiv \Phi/I$. In thermodynamics the thermal compliance is $C_o = \mathcal{S}/T$, thus $Z_{themo} = T/s\,\mathcal{S} = 1/sC_o$. It is customary to use the unit [ohms] when working with any impedance, allowing for a uniform terminology for the different physical situations and many forms of impedance. This greatly simplifies the notation.

In the time domain, Ohm's law becomes Eq. 6.13 for the case of a mechanical compliance $C_o = 1/K_o$ and Eq. 6.12 for the electrical capacitor C. As shown in Table 6.1.4, the formula for the generalized impedance is typically expressed in terms of the Laplace frequency s, which is the $\mathcal{LT}$ of the time variables.

The final solution is expressed in the $\mathcal{LT}$domain. If the impedance Z seen by the source is the sum of the source's internal resistance R and a capacitive load,

$$Z = R + \frac{1}{sC}.$$

This results in a simple relationship between the force and the flow, as determined by the action of the source on the load $Z(s)$. The results may be given in terms of the voltage across the compliance in terms of the voltage Φ_s (or current I_s) supplied by the source. Given some algebra, the voltage across the compliance Φ_c, divided by the voltage of the source, is

$$\frac{\Phi_c}{\Phi_{\text{source}}} = \frac{R}{R + 1/sC}.$$

Thus the calculus reduces to some algebra in the frequency domain, which in this case has a simple pole at $s_p = -1/RC$. The time domain response is then found by taking the inverse $\mathcal{LT}$.

Cauchy's residue theorem gives the final answer, which describes how the voltage across the compliance builds exponentially with time, from zero to the final value. Given the voltage, we can also compute the current, as a function of time. This then represents the entire process of either blowing up a balloon, charging a capacitor, or heating water on a stove, the difference being the physical notation. The math is identical.

The differential equation is first-order in time, which in frequency means the impedance has a single pole. Thus the equation for charging a capacitor or pumping up a balloon describes a diffusion process. If we had taken the impedance of the mass of the fluid in the hose into account, we would have a lumped-parameter model of the wave equation with a second-order system. This is mathematically related to the homework assignment about train cars (masses) connected by springs (Fig. 6.5, Homework DE-3, problem 53).

Example: The voltage

$$\phi(\boldsymbol{x}, t) = e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}}u(t - x/c) \leftrightarrow \frac{1}{s}e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} \quad \text{[V]} \tag{6.14}$$

represents one of d'Alembert's solution of the wave equation as an eigen-function of the gradient operator ∇.

From the definition of the scalar (dot) product of two vectors

$$\boldsymbol{\kappa} \cdot \boldsymbol{x} = \kappa_x x + \kappa_y y + \kappa_z z = ||\boldsymbol{\kappa}||\, ||\boldsymbol{x}|| \cos\theta_{\kappa x},$$

where $||\boldsymbol{\kappa}|| = \sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}$ and $||\boldsymbol{x}|| = \sqrt{x^2 + y^2 + z^2}$ are the lengths of vectors $\boldsymbol{\kappa}$ and $\boldsymbol{x}$, and $\theta_{\kappa x}$ is the angle between them.

To keep things simple, we let $\boldsymbol{\kappa} = [\kappa_x, 0, 0]^T$ so that $\boldsymbol{\kappa} \cdot \boldsymbol{x} = \kappa_x x\hat{\mathbf{x}}$. We shall soon see that $||\boldsymbol{\kappa}|| = 2\pi/\lambda$ follows from the basic relationship between a wave's radian frequency $\omega = 2\pi f$ and its wavelength λ:

$$\omega\lambda = c_o. \tag{6.15}$$

As the frequency increases, the wavelength becomes shorter. This key relationship may have been first researched by Galileo in about 1564, followed by Mersenne[5] in about 1627 (Fig. 1.1), When we use the $\mathcal{LT}$, the complex frequency is $s = \jmath\omega$.

Exercise #*3*

Show that Eq. 6.14 is an eigen-function of the gradient operator ∇.

Solution: Taking the gradient of $\phi(\boldsymbol{x}, t)$ gives

$$\begin{aligned}\nabla e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t) &= -\nabla\boldsymbol{\kappa} \cdot \boldsymbol{x}\, e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t)\\ &= -\boldsymbol{\kappa}\, e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t),\end{aligned}$$

or in terms of $\phi(\boldsymbol{x}, t)$,

$$\nabla\phi(\boldsymbol{x}, t) = -\boldsymbol{\kappa}\,\phi(\boldsymbol{x}, t) \leftrightarrow -\frac{s}{c} e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}}.$$

Thus $\phi(\boldsymbol{x}, t)$ is an eigen-function of ∇, having the vector eigenvalue $\boldsymbol{\kappa}$. As before, $\nabla\phi$ is proportional to the current, since ϕ is a voltage, and the ratio (i.e., the eigenvalue) may be thought of as a mass, analogous to the impedance of a mass (or inductor). In general, the units provide the physical interpretation of the eigenvalues and their spectra.

Exercise #*4*

Compute $\hat{\mathbf{n}}$ for $\phi(\boldsymbol{x}, s)$ as given by Eq. 6.14.

Solution: $\hat{\mathrm{n}} = \boldsymbol{\kappa}/||\boldsymbol{\kappa}||$ represents a unit vector in the direction of the wave propagation.

Exercise #*5*

If the sign of $\boldsymbol{\kappa}$ is negative, what are the eigenvectors and eigenvalues of $\nabla\phi(\boldsymbol{x}, t)$?

Solution:

$$\begin{aligned}\nabla e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t) &= -\boldsymbol{\kappa} \cdot \nabla(\boldsymbol{x}) e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t)\\ &= -\boldsymbol{\kappa}\ e^{-\boldsymbol{\kappa}\cdot\boldsymbol{x}} u(t).\end{aligned}$$

Nothing changes other than the sign of $\boldsymbol{\kappa}$. Physically this means the wave is traveling in the opposite direction, corresponding to the forward and retrograde d'Alembert waves.

Prior to this section, we had considered the Taylor series in only one variable, such as for polynomials $P_N(x), x \in \mathbb{R}$ (Eq. **??**. and $P_N(s), s \in \mathbb{C}$ (Eq. 3.22. The generalization from real to complex–analytic functions led to the $\mathcal{LT}$ and the host of integration theorems (FTCC, Cauchy CT-1, CT-2, CT-3). What is in store when we generalize from one spatial variable ($\mathbb{R}$) to three ($\mathbb{R}^3$)?

---

[5] See `https://www-history.mcs.st-and.ac.uk/Biographies/Mersenne.html`;

> *"In the early 1620s, Mersenne listed Galileo among the innovators in natural philosophy whose views should be rejected. However, by the early 1630s, less than a decade later, Mersenne had become one of Galileo's most ardent supporters." (Garber, 2004)*

Exercise #6

Find the velocity $v(t)$ of an electron in a field $\boldsymbol{e}$.

Solution: From Newton's second law, $-qE = m_e\dot{v}(t)$ [N], where m_e is the mass of the electron. Thus we must solve this first-order differential equation to find $v(t)$. As before, this is best done in the frequency domain $v(t) \leftrightarrow V(s)$.

Role of potentials: Note that the scalar fields (e.g., temperature, pressure, voltage) are all scalar potentials, as summarized in Table 3.3.2. In each case the gradient of the potential results in a vector force field, just as in the electric case above (Eq. 6.1).

Table 3.3.2 is helpful in understanding the physical meaning of the gradient of a potential, which is typically a generalized force (electric field, acoustic force density, temperature flux), that in turn generates a flow (current, velocity, heat flux (entropy)). Four examples of impedance are provided in the following list:

1. The voltage drop across a resistor causes a current to flow, as described by Ohm's law. The difference in voltage between two points is a crude form of gradient when the frequency f [Hz] is low, such that the wavelength is much larger than the distance between the two points. This is the essence of the quasistatic approximation (Postulate P10).

2. The gradient of the pressure gives rise to a force density in the fluid medium (air, water, oil, etc.), that causes a flow (velocity vector) in the medium.

3. The gradient of the temperature also causes a flow of heat that is proportional to the thermal resistance, given Ohm's law for heat (Feynman, 1970b, p 3–7).

4. When a solution contains ions, it defines an *electro–chemical Nernst potential* $N(\boldsymbol{x}, t)$ (Fermi, 1936; Scott, 2002). This electro-chemical potential is similar to a voltage or temperature field, the gradient of which defines a virtual force on the ions, resulting in an ionic current.

In the above examples there is a potential, the gradient of which is a force, which when applied to the medium (impedance) causes a flow (flux). The electrical impedance [Ohms] is the ratio of the gradient of the potential [Volts] over the current [Amps]. The product of the force and flow is the power [Watts].

Exercise #7

Show that the integral of Eq. 6.1 is an anti-derivative.

Solution: We use the definition of the anti-derivative given by the FTC (Eq. 3.7.3):

$$
\begin{aligned}
\phi(\boldsymbol{x}, t) - \phi(\boldsymbol{x}_o, t) = & \int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \boldsymbol{e}(\boldsymbol{x}, t) \cdot d\boldsymbol{x} \\
&= -\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \nabla\phi(\boldsymbol{x}, t) \cdot d\boldsymbol{x} \\
&= -\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\phi(\boldsymbol{x}, t) \cdot d\boldsymbol{x} \\
&= -\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \left(\hat{\mathbf{x}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\phi}{\partial z}\right) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz) \\
&= -\int_{x_o}^{x} \frac{\partial\phi}{\partial x}dx - \int_{y_o}^{y} \frac{\partial\phi}{\partial y}dy - \int_{z_o}^{z} \frac{\partial\phi}{\partial z}dz \\
&= -\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} d\phi(\boldsymbol{x}, t) \\
&= -\Big(\phi(\boldsymbol{x}, t) - \phi(\boldsymbol{x}_o, t)\Big).
\end{aligned}
$$

This may be verified by taking the gradient of both sides:

$$
\nabla\phi(\boldsymbol{x}, t) - \cancelto{0}{\nabla\phi(\boldsymbol{x}_o, t)} = -\nabla\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \boldsymbol{e}(\boldsymbol{x}, t) \cdot d\boldsymbol{x} = \boldsymbol{e}(\boldsymbol{x}, t).
$$

If we apply the FTC, the anti-derivative must be $\phi(\boldsymbol{x},t) = E_x x\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$. This same point is made by Feynman (1970b, p. 4-1, Eq. 4.28).

The conservative field: A field is said to be *conservative* when the work done by the motion is independent of the path of the motion. Thus the conservative field is related to the FTC, which states that the integral of the work depends on only the end points. Given that the force on a charge is proportional to the gradient of the potential, the above exercise shows that the integral of the gradient only depends on the end points. Thus the work done in moving a charge only depends on the limits of the integral, which defines the *conservative field*. This holds in the ideal case where $\boldsymbol{e}$ is determined by Eq. 6.1, namely the medium has no friction (the charge sees no other forces). The definitive answer must await the introduction of the *Fundamental theorem of vector calculus* (Eq. 6.70).

A few examples provide insight:

Example: The gradient of a scalar potential, e.g., a voltage (Eq. 6.1) causes a current in the load impedance. When the impedance is lossless, such as a combination inductors and capacitors, the system is reactive, leading to stored energy with no heat generated. When the impedance is infinite, the flow is zero and no energy is stored.

Example: At audible frequencies the viscosity of air is negligible, resulting in a lossless system. When the wavelength is small (e.g., at 100 [kHz] $\lambda = c_o/f = 345/10^5 = 3.45$ [mm]) the lossless air assumption breaks down, resulting in a significant propagation loss.[6] This is why airplanes can fly without burning up. When the viscosity is taken into account, the field is lossy and thus the field is no longer conservative. In narrow tubes, for example a flute, thermal loss plays a much larger role due to the walls (Appendix 3.10).

Example: If a temperature field is a time-varying constant [i.e., $T(\boldsymbol{x},t) = T_o(t)$], there is no "heat flux," since $\nabla T_o(t) = 0$. When there is no heat flux [i.e., flux, or flow], there is no heat power, since the power is the product of the force and the flow.

Example: The force of gravity is given by the gradient of Newton's gravitational potential (Eq. **??**):

$$F = -\nabla_r \phi_{\mathrm{N}}(r) = -\frac{\partial}{\partial r}\frac{1}{r} = \frac{1}{r^2}.$$

Historically speaking, $\phi_N(r)$ was the first conservative field, used by Galileo, Newton **and others**, to explain the elliptic orbits of the planets around the sun.[7] Galileo's law says that bodies fall with constant acceleration, giving rise to a parabolic path and a time of fall proportional to t^2. This behavior of falling objects directly follows from the Galilean potential:

$$\phi_{\mathrm{G}}(r) = \frac{1}{(r-r_o)} = \frac{-r_o}{1-r/r_o} \underset{r<r_o}{=} -r_o(1-r/r_o+(r/r_o)^2+\cdots) \underset{r\ll r_o}{\approx} r_o - r,$$

which, given the large radius r_o of the earth and the small distance of the object from the surface of the earth $r - r_o$, is equal to the distance above the ground. In actual fact, there is very small damping in our planetary system, due to the gravitational pull of the sun and moon on the oceans, leading to the damping of our daily tides. 6.6.1 Thus Galileo's law says that the force a falling body sees is constant:

$$F_G = -G_o\nabla_r\phi_G(r) = G_o.6.6.1$$

This can be scaled by G_o to account for the magnitude of the gravitational force.[8]

of the
many-
m.

Exercise #8

Galileo discovered that the height of a falling object is proportional to the square of 244 the time it falls. Based

[6] https://en.wikipedia.org/wiki/Laminar_flow#Examples

[7] Verify Galileo and Newton! Others?

[8] If three bodies are connected by springs, there are three coupled resonance that are not in sync, which can result in chaotic motion. Our solar system is just such a many-body system, yet we are blessed that it has remained stable over 13 billion years. There is damping in the planetary system due to the gravitational pull of the sun and moon on the oceans.

on Newton's follow-up analysis, today we would say this height $h(t)$ is

$$h(t) = \frac{1}{2} m G_o (t - t_o)^2 \quad [\text{m}],$$

where m is the object's mass and G_o is the gravitational constant for the earth at its surface r_o. Show that $h(t)$ directly follows from the potential $\phi_G = r_o - r$. This formula applies if you toss a ball into the air or if you drop it from a high place.

Solution: Given Galileo's potential $\phi_G(r) \underset{r \ll r_o}{\approx} m G_o (r_o - r)$, thus $\ddot{h}(t) = m G_o$. Given Galileo's formula for the height $h(t)$, the velocity is $v(t) = \dot{r}(t) = m G_o t$, and the acceleration is $\ddot{r}(t) = m G_o$.

Exercise #9

Find the time that it takes to fall from a distance $r = L$. That is solve $h(t) = L$ for the time the object takes to fall the distance L.

Solution: Setting $t_o = 0$ gives $t^2 = 2L/mG_o$. Thus the time to fall is $T(L) = \sqrt{2L/mG_o}$.

6.2 Partial differential equations and field evolution

The three main classes of partial differential equations (PDEs) are: elliptic, parabolic, and hyperbolic, distinguished by the order of the time derivative. These categories seem to have little mathematical utility (the categories are labels).

6.2.1 The Laplacian ∇^2:

In the most important case the space operator is the Laplacian ∇^2, the definition of which depends on the dimensionality of the waves—that is, the coordinate system being used. We first discussed the Laplacian as a 2D operator on p. 149 where we studied complex–analytic functions. An expression for ∇^2 for one, two, and three-dimensions is (Eq. 6.9). In three dimensional rectangular coordinates, it is defined as

$$\nabla^2 T(\boldsymbol{x}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T(\boldsymbol{x}). \tag{6.16}$$

The Laplacian operator is ubiquitous in mathematical physics, starting with simple complex–analytic functions (Laplace's equation) and progressing to Poisson's equation, the diffusion equation, and finally the wave equation. Only the wave equation results in a delay. The diffusion equation "wave" has an instantaneous spread (the effective "wavefront" velocity is infinite, yet the wavelength is long; it's not a traveling wave).

Examples of elliptic, parabolic, and hyperbolic equations are

1. *Laplace's equation:* The equation
$$\nabla^2 \Phi(\boldsymbol{x}) = 0 \tag{6.17}$$
describes, for example, the voltage inside a closed chamber that has a given voltage on the walls or the steady-state temperature within a closed container having a specified temperature distribution on the walls. There are no dynamics to the potential, even when it is changing, since the potential instantaneously follows the potential on the walls.

2. *Poisson's equation:* In the steady state, the diffusion equation degenerates to either Poisson's or Laplace's equation; both are classified as *elliptic* equations (second-order in space, zero-order in time). As in the diffusion equation, the evolution has a wave velocity that is functionally infinite. For example,
$$\nabla^2 \Phi(\boldsymbol{x}, t) = \rho(x, t)$$
holds for gravitational fields or the voltage around a charge. It does not describe gravity waves, which travel at the speed of light.

3. *Fourier diffusion equation:* Equation 6.18 describes the evolution of the scalar temperature $T(\boldsymbol{x}, t)$ (a scalar potential), gradients of solution concentrations (i.e., ink in water), and Brownian motion. Diffusion is first-order in time, which is categorized as *parabolic* (first-order in time, second-order in space). When these equations are Laplace transformed, diffusion has a single real root, resulting in a real solution (e.g., $T \in \mathbb{R}$). There is no wavefront in the case of the diffusion equation. As soon as the source is turned on, the field is nonzero at every point in the bounded container. As an example,

$$\nabla^2 T(\boldsymbol{x}, t) = \kappa_o \frac{\partial T(\boldsymbol{x}, t)}{\partial t} \leftrightarrow s\kappa_o T(\boldsymbol{x}, s) \tag{6.18}$$

describes the temperature $T(\boldsymbol{x}, t) \leftrightarrow T(\boldsymbol{x}, \omega)$, as proposed by Fourier in 1822, or the diffusion of two miscible liquids (Fick, 1855) or Brownian motion (Einstein, 1905). The diffusion equation is not a wave equation, since the temperature wavefront propagates instantaneously. The diffusion equation does a poor job of representing the velocity of molecules banging into each other, since such collisions have a mean free path, and thus the velocity cannot be infinite.

4. *Wave equations:* There are scalar and vector forms of wave equations.

   (a) Scalar wave equations: Equation **??** describes the evolution of a scalar potential field, such as pressure $\varrho(\boldsymbol{x}, t)$ (sound) or the displacement of a string or membrane under tension. The wave equation is second-order in time. When transformed into the frequency domain, the solution has pairs of complex conjugate roots, leading to two real solutions. The wave equation is classified as *hyperbolic* (second-order in time and space).

   (b) Vector wave equations: Maxwell's equations describe the propagation of the electric $\boldsymbol{e}(\boldsymbol{x}, t)$ and magnetic $\boldsymbol{h}(\boldsymbol{x}, t)$ field strength vectors, as well as the electric $\boldsymbol{d}(\boldsymbol{x}, t) = \epsilon_o \boldsymbol{e}(\boldsymbol{x}, t)$ and magnetic $\boldsymbol{b}(\boldsymbol{x}, t) = \mu_o \boldsymbol{h}(\boldsymbol{x}, t)$ flux vectors. ME are anti-reciprocal (P6).

Solution evolution: The partial differential equation defines the evolution of the scalar field [pressure $\varrho(\boldsymbol{x}, t)$ and temperature $T(\boldsymbol{x}, t)$], or vector field ($\boldsymbol{e}, \boldsymbol{d}, \boldsymbol{b}, \boldsymbol{h}$), as functions of space $\boldsymbol{x}$ and time t. There are two basic categories of field evolution: diffusion and propagation.

1. *Diffusion*: The simplest and easiest PDE example, easily visualized, is a static(time-invariant) scalar temperature field $T(\boldsymbol{x})$ [°C]. Just like an impedance or admittance, a field has regions where it is analytic, and for the same reasons, $T(\boldsymbol{x}, t)$ satisfies Laplace's equation

$$\nabla^2 T(\boldsymbol{x}, t) = 0.$$

   Since there is no current when the field is static, such systems are lossless and thus are conservative.

   When $T(\boldsymbol{x}, t)$ depends on time (is not static), it is described by the *diffusion equation* (Eq. 6.18), a rule for how $T(\boldsymbol{x}, t)$ evolves with time from its initial state $T(\boldsymbol{x}, 0)$. The constant κ_o is called the *thermal conductivity*, which depends on the properties of the fluid in the container, with $s\kappa_o$ being the thermal admittance per unit area. The conductivity is a measure of how the heat gradients induce heat currents $\boldsymbol{j} = -\kappa_o \nabla T$, analogous to Ohm's law for electricity.

   Note that when $T(\boldsymbol{x}, t \to \infty)$ the temperature reaches a steady state, $\boldsymbol{j} = 0$ and $\nabla^2 T = 0$. This all depends on what is happening at the boundaries. When the wall temperature of a container is a function of time, the internal temperature $T(\boldsymbol{x}, t)$ will continue to change, but with a frequency-dependent delay that depends on the thermal conductivity κ_o.

   Such a system is analogous to an electrical resistor–capacitor series circuit connected to a battery. For example, the wall temperature (voltage across the battery) represents the potential driving the system. The thermal conductivity κ_o (the electrical resistor) is likewise analogous. The fluid (the electrical capacitor) is being heated (charged) by the heat (charge) flux. In all cases Ohm's law defines the ratio of the potential (voltage) to the flux (current). How this happens can be understood only once the solution to the equations has been established. The fluid has a heat capacity analogous to that of an electrical capacitor (Kirchhoff, 1868, 1974). Sometimes the diffusion equation is called the telegraph equation.

2. *Propagation:* Pressure and electromagnetic waves are described by a scalar potential (pressure) (Eq. **??**) and a vector potential (Eq. 6.84, and vector wave equations.

The Taylor series of $f(\boldsymbol{x})$: Next we extend the concept of the Taylor series of one variable to $\boldsymbol{x} \in \mathbb{R}^3$. Just as we generalized the derivative with respect to a real frequency variable $\omega \in \mathbb{R}$ to complex frequency $s = \sigma + \omega \jmath \in \mathbb{C}$, here we generalize the derivative with respect to $x \in \mathbb{R}$ to the vector $\boldsymbol{x} \in \mathbb{R}^3$.

Since the scalar field is analytic in $\boldsymbol{x}$, it is a good place to start. Assuming we have carefully defined the Taylor series (Eq. 3.14), the Taylor series of $f(\boldsymbol{x})$ in $\boldsymbol{x} \in \mathbb{R}^3$ about $\boldsymbol{x} = 0$ may be defined as

$$f(\boldsymbol{x} + \delta\boldsymbol{x}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x}) \cdot \delta\boldsymbol{x} + \frac{1}{2!}\sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\partial^2 f(\boldsymbol{x})}{\partial x_k \partial x_l}\delta x_k \delta x_l + \text{HOT}, \tag{6.19}$$

where HOT stands for Higher Order Terms (Greenberg, 1988, p 639). From this definition, it is clear that the gradient is the generalization of the second term in the one-dimensional Taylor series expansion.

Summary: For every potential $\phi(\boldsymbol{x}, t)$ there exists a force density $\boldsymbol{f}(\boldsymbol{x}, t) = -\nabla\phi(\boldsymbol{x}, t)$, proportional to the potentials, that drives a generalized flow $\boldsymbol{u}(\boldsymbol{x}, t)$. If the normal components of the force and flow are averaged over a surface, the mean force and volume flow (i.e, volume velocity for the acoustic case) are defined. In such cases the impedance $Z(s)$ is the net force through the surface force over the net flow, and Gauss's law and quasi-statics (Postulate P10, p. 127) come into play (Feynman, 1970a). We call this the *generalized impedance*. An example is $Z(s) = \sqrt{s}$.

Assuming linearity (Postulate P2), the product of the force and flow is the power, and the ratio (force/flow) is an impedance (Table 3.3.2, p. 86). This impedance statement is called Ohm's law, Kirchhoff's laws, Laplace's law, or Newton's laws. In the simplest cases, they are all linearized (proportional) complex relationships between a force and a flow. Very few impedance relationships are inherently linear over a large range of force or current, but for physically useful studies, they are treated as linear. Nonlinear interactions require a more sophisticated approach, typically involving numerical methods, working in the time domain.

In electrical circuits it is traditional to define a zero potential *ground* point that all voltages use as the reference potential. The ground is a useful convention as a simplifying rule, but it obscures the physics and obscures the fact that the voltage is *not* the force. Rather, the force is the voltage difference, referenced to the ground, which is defined as zero volts. This results in abstracting away (i.e., hiding) the difference in voltage. It seems misleading (more precisely, it is wrong) to state Ohm's law as the voltage over the current, since Ohm's law actually says that the *voltage difference* (i.e., voltage gradient) over the current defines an impedance (Kennelly, 1893).

When we measure the voltage between two points, it is a crude approximation to the gradient based on the quasi-static approximation (Postulate P10). The pressure is also a potential, the gradient of which is a force density, which drives the volume velocity (flow).

In Sec. 6.8.1 we introduce the fundamental theorem of vector calculus (otherwise known as Helmholtz's decomposition theorem), which generalizes Ohm's law to include circulation (e.g., angular momentum, vorticity, and the related magnetic effects). To understand these generalizations in flow, we need to understand compressible and rotational fields (Table 6.3, p. 247), complex–analytic functions, and more mathematical physics history.

It is the *difference* in the potential (i.e., voltage, temperature, pressure) that is proportional to the flux. This can be viewed as a major simplification of the gradient relationship, justified by the quasi-static assumption (Postulate P10, p. 127).

The roots of the impedance are related to the eigenmodes of the system equations. The solutions to the equations are the eigen-functions, evaluated at the eigenvalues (p. 149)

6.2.2 Scalar wave equation (Acoustics)

In this section we discuss the general solution to the wave equation, which has two forms: scalar waves (acoustics) and vector waves (electromagnetics). These have an important mathematical distinction but a similar solution space, one scalar and the other vector. We start with the scalar wave equation. The vector case will be discussed §6.9.2 (p. 253).

A good starting point for understanding PDEs is to explore the scalar wave equation (Eq. **??**, p. **??**). Acoustic wave propagation was first analyzed mathematically by Isaac Newton in his famous book *Principia* (1687), in which he first calculated the speed of sound based on the conservation of mass and momentum.

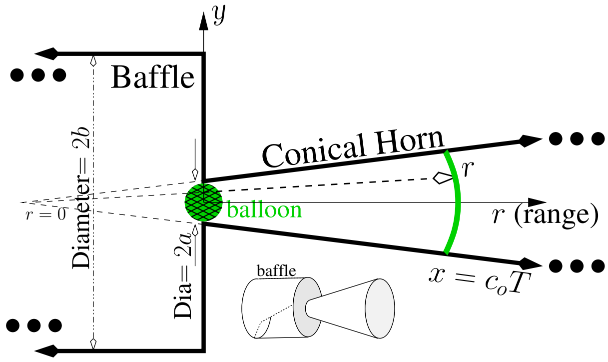


Figure 6.2: *Experimental setup showing a large pipe on the left terminating at the wall containing a small hole with a balloon, shown in green. At time $t = 0$ the balloon is pricked and a pressure pulse is released. The baffle on the left represents a semi-∞ long tube having a large radius compared to the horn input diameter $2a$. At time T the outbound pressure pulse $\varrho(r,T) = \delta(t - x/c_o)/r$ has reached a radius $x = r - r_o = c_oT$, where $r = x$ is the location of the source at the throat of the horn and r is measured from the vertex. At the throat of the horn $\mathcal{V}_+/A_+ = \mathcal{V}_-/A_-$. The term "horn" is used for the case of scalar waves, while the term "wave-guide" is used when speaking of EM waves. When the propagation is constrained by a horn, or wave-guide, the waves are "guided."*

Early history: The study of wave propagation begins at least as early as Huygens (ca. 1678) (Pierce, 1981, p 15). The acoustic variables are the *pressure*,

$$\varrho(\boldsymbol{x},t) \leftrightarrow \mathcal{P}(\boldsymbol{x},s),$$

and the particle velocity,

$$\nu(\boldsymbol{x},t) \leftrightarrow \mathcal{U}(\boldsymbol{x},s).$$

To obtain a wave, we must include two basic components: the stiffness of air and its mass. The two equations are called (1) Newton's second law ($F = ma$) and (2) Hooke's law ($F = kx$), respectively. In vector form these equations are (1) Euler's equation (i.e., conservation of momentum density),

$$-\nabla\varrho(\boldsymbol{x},t) = \rho_o \frac{\partial}{\partial t}\nu(\boldsymbol{x},t) \leftrightarrow \rho_o s\mathcal{U}(\boldsymbol{x},s), \tag{6.20}$$

which assumes the time-average density ρ_o is independent of time and position $\boldsymbol{x}$, and (2) the continuity equation (i.e., conservation of mass density),

$$-\nabla\cdot\nu(\boldsymbol{x},t) = \frac{1}{\eta_o P_o}\frac{\partial}{\partial t}\varrho(\boldsymbol{x},t) \leftrightarrow \frac{s}{\eta_o P_o}\mathcal{P}(\boldsymbol{x},s) \tag{6.21}$$

(Pierce, 1981; Morse, 1948, p 295). Here $P_o = 10^5$ [Pa] is the barometric pressure and $\eta_o P_o$ is the dynamic (adiabatic) stiffness, with $\eta_o = 1.4$ (See p. 130). Solving for $\nu(\boldsymbol{x},t)$) and removing it, results in the three-dimensional scalar pressure wave equation

$$\nabla^2\varrho(\boldsymbol{x},t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(\boldsymbol{x},t) \leftrightarrow \frac{s^2}{c_o^2}\mathcal{P}(\boldsymbol{x},s) \tag{6.22}$$

with $c_o = \sqrt{\eta_o P_o/\rho_o}$, the speed of sound and $\nu = \sqrt{\eta_o P_o \rho_o}$, the air resistance, assuming no visco-thermal loss.

Exercise #*10*

Show that Eqs. 6.20 and 6.31 can be reduced to Eq. 6.22.

Solution: Taking the divergence of Eq. 6.20 gives

$$-\nabla\cdot\nabla\varrho(\boldsymbol{x},t) = \rho_o\frac{\partial}{\partial t}\nabla\cdot\nu(\boldsymbol{x},t). \tag{6.23}$$

Note that $\nabla\cdot\nabla = \nabla^2$ (Table 6.1). Next, substituting Eq. 6.31 into the above equation results in the scalar wave equation, Eq. 6.22, since $c_o = \sqrt{\eta_o P_o/\rho_o}$.

6.2.3 The Webster horn equation (WHEN)

An important generalization of the problem of lossless plane-wave propagation in one-dimensional uniform tubes is known as *transmission line theory*. As depicted in Fig. 6.2, by allowing the area $A(r)$ [e.g., for the conical horn $A(r) = A_o(r/L)^2$ with $L = 1$ [m] and $A_0 \leq 4\pi$] of an acoustical wave-guide (horn) to vary along the range axis r (the direction of wave propagation), we can explore general solutions to the wave equation. Classic applications of horns include vocal tract acoustics, loudspeaker design, cochlear mechanics, quantum mechanics (e.g., the hydrogen atom), and wave propagation in periodic media (Brillouin, 1953).

We must be more precise when defining the area $A(x)$: The area is *not* the cross-sectional area of the horn; rather it is the wavefront (iso-pressure) area, which is related to Gauss' law, since the gradient of the pressure defines the force that drives the mass flow (also called volume velocity).

For the scalar wave equation,

$$\nabla_r^2\, \varrho(r,t) = \frac{1}{A(r)}\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]\varrho(r,t). \tag{6.24}$$

This Laplacian is based on the quasi-static approximation (Postulate P10) which requires that the frequency lie below the critical value $f_c = c_o/2d$—namely, that a half wavelength be greater than the horn diameter d (i.e., $d < \lambda/2$).[9] For the adult human ear canal, $d = 7.5$ [mm] and $f_c = (343/2\cdot 7.5)\times 10^{-3} \approx 22.87$ [kHz], which is above the upper range of human hearing.

The term on the right of Eq. 6.24 is also the Laplacian for thin tubes (e.g., rectangular, spherical, and cylindrical coordinates). Thus the Webster horn "wave" equation is

$$\frac{1}{A(r)}\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]\varrho(r,t) = \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(r,t) \leftrightarrow \frac{s^2}{c_o^2}\mathcal{P}(r,s), \tag{6.25}$$

where $\varrho(r,t) \leftrightarrow \mathcal{P}(r,s)$ is the acoustic pressure in Pascals [Pa] (Hanna and Slepian, 1924; Mawardi, 1949; Eisner, 1967; Morse, 1948); Olson (1947, p. 101); Pierce (1981, p. 360). Extensive experimental analyses for various types of horns (conical, exponential, parabolic) along with a review of horn theory may be found in Goldsmith and Minton (1924). Of special interest is Eisner (1967) due to his history section and long list of relevant articles.

6.3 Webster horn equation derivation

6.3.1 WHEN Overview

If we transform the acoustic equations, Eqs. 6.20 and 6.31 (p. 228) into their equivalent integral form, we obtain Eq. 6.25. This derivation is similar to that of Hanna and Slepian (1924) and Pierce (1981, p. 360).

6.3.2 Conservation of momentum

The first step is to integrate the normal component of Eq. 6.20 over the isopressure surface $\mathcal{S}$, defined by $\nabla p = 0$

$$-\int_{\mathcal{S}} \nabla p(\mathbf{x},t)\cdot d\mathbf{A} = \rho_o \frac{\partial}{\partial t}\int_{\mathcal{S}} \mathbf{u}(\mathbf{x},t)\cdot d\mathbf{A}.$$

The average pressure $\varrho(x,t)$ is defined by dividing by the total area

$$\varrho(x,t) \equiv \frac{1}{A(x)}\int_{\mathcal{S}} p(x,t)\,\hat{\boldsymbol{n}}\cdot d\boldsymbol{A}. \tag{6.26}$$

From the definition of the gradient operator, we have

$$\nabla p = \frac{\partial p}{\partial x}\hat{\boldsymbol{n}}, \tag{6.27}$$

where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to the isopressure surface $\mathcal{S}$. Thus the left side of Eq. 6.20 reduces to $\partial\varrho(x,t)/\partial x$.

[9] This condition may be written in several ways, the most common being $ka < 1$, where $k = 2\pi/\lambda$ and a is the horn radius. This may be expressed in terms of the diameter as $\frac{2\pi}{\lambda}\frac{d}{2} < 1$, or $d < \lambda/\pi < \lambda/2$. Thus $d < \lambda/2$ may be a more precise metric by the factor $\pi/2 \approx 1.6$. This is called the *half-wavelength assumption*, a synonym for the quasi-static approximation, and the Nyquist theorem (See DE-3, #2).

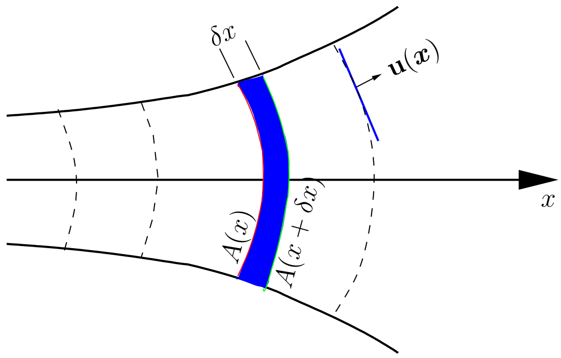


Figure 6.3: *Derivation of the horn equation using Gauss's law: The divergence of the velocity $\nabla\cdot\mathbf{u}$ within δx, shown as the shaded region, is integrated over the enclosed volume. Next the divergence theorem is applied, transforming the integral to a surface integral normal to the surface of propagation. This results in the difference of the two volume velocities $\delta\nu = \nu(x+\delta x)-\nu(x) = [\mathbf{u}(x+\delta x)\cdot\mathbf{A}(x+\delta x) - \mathbf{u}(x)\cdot\mathbf{A}(x)]$. The flow is always perpendicular to the isopressure contours.*

The integral on the right side defines the volume velocity,

$$\nu(x,t) \equiv \int_S \mathbf{u}(x,t)\cdot d\boldsymbol{A}. \tag{6.28}$$

Thus the integral form of Eq. 6.20 becomes

$$-\frac{\partial}{\partial x}\varrho(x,t) = \frac{\rho_o}{A(x)}\frac{\partial}{\partial t}\nu(x,t) \leftrightarrow \mathcal{Z}(x,s)\,\mathcal{V}(x,s), \tag{6.29}$$

where

$$\mathcal{Z}(s,x) = s\rho_o/A(x) = sM(x), \tag{6.30}$$

and $M(x) = \rho_o/A(x)$ [kgm/m^5] is the per-unit-length mass density of air.

6.3.3 Conservation of mass

Integrating Eq. 6.31 over the volume V gives

$$-\int_V \nabla\cdot\boldsymbol{u}\,dV = \frac{1}{\eta_o P_o}\frac{\partial}{\partial t}\int_V p(\boldsymbol{x},t)dV.$$

The volume V is defined by two isopressure surfaces between x and $x+\delta x$ (the shaded region of Fig. 6.3). On the right-hand side we use the definition of the average pressure (i.e., Eq. 6.26) integrated over the volume dV.

Applying Gauss's law to the left-hand side[10] and using the definition of ϱ (on the right) in the limit $\delta x \to 0$ give

$$-\frac{\partial}{\partial x}\nu(x,t) = \frac{A(x)}{\eta_o P_o}\frac{\partial}{\partial t}\varrho(x,t) \leftrightarrow \mathcal{Y}(x,s)\,\mathcal{P}(x,s), \tag{6.31}$$

where

$$\mathcal{Y}(x,s) = sA(x)/\eta_o P_o = sC(x). \tag{6.32}$$

$C(x) = A(x)/\eta_o P_o$ [m^4/N] is the per-unit-length compliance of the air. Equations 6.29 and 6.3.3 accurately characterize the Webster plane-wave mode up to the frequency where the higher-order eigenmodes begin to propagate (i.e., $f > f_c$).

6.3.4 Horn properties

Speed of sound c_o

In terms of $M(x)$ and $C(x)$, the speed of sound and the acoustic admittance are

$$c_o = \sqrt{\frac{\text{stiffness}}{\text{mass}}} = \frac{1}{\sqrt{C(x)M(x)}} = \sqrt{\frac{\eta_o P_o}{\rho_o}}. \tag{6.33}$$

This assumes the medium is lossless. For a discussion of lossy propagation, see Appendix 3.10 (p. 127).

[10] As shown in Fig. 6.3, taking the limit of the difference between the two volume velocities $\nu(x+\delta x)-\nu(x)$ divided by δx results in $\partial\nu/\partial x$.

6.3.5 Characteristic admittance $\mathcal{Y}_r(x)$:

Since the horn equation (Eq. 6.25) is second-order, it has two eigenfunction solutions $\mathcal{P}^{\pm}$. The ratios of Eq. 6.32 to Eq. 6.30 are determined by the local stiffness $1/C(x)$ and mass $M(x)$. The ratio C/M determines the area-dependent characteristic admittance $\mathcal{Y}_r(x)$ ($\in \mathbb{R}$):

$$\mathcal{Y}_r(x) = \frac{1}{\sqrt{\text{stiffness} \cdot \text{mass}}} = \sqrt{\frac{\mathcal{Y}(x,s)}{\mathcal{Z}(x,s)}} = \sqrt{\frac{C(x)}{M(x)}} = \sqrt{\frac{A(x)}{\cancel{s}\rho_o}\frac{\cancel{s}A(x)}{\eta_o \mathcal{P}_o}} = \frac{A(x)}{\rho_o c_o} > 0 \tag{6.34}$$

(Campbell, 1903, 1910, 1922). The characteristic impedance is $\mathcal{Z}_r(x) = 1/\mathcal{Y}_r(x)$. Based on a physical argument, $\mathcal{Y}_r(x)$ must be positive and real; thus only the positive square root is allowed. As long as $A(x)$ has no jumps (is continuous), $\mathcal{Y}_r(x)$ must be the same in both directions. It is locally determined by the isopressure surface and its volume velocity.

Radiation admittance

The radiation admittance is defined looking into a horn with no termination (infinitely long) from the input at $x = 0$:

$$Y_{rad}^{\pm}(s) = \frac{\mathcal{V}^{\pm}}{\mathcal{P}^{\pm}} \in \mathbb{C}. \tag{6.35}$$

The impedance depends on the direction, with $+$ looking to the right and $-$ to the left.

The input admittance $Y_{in}^{\pm}(x,s)$ is computed using the upper equation of Eq. 6.37 (p. 232) for $\mathcal{V}(x,s)$ and then dividing by the pressure eigenfunction $\mathcal{P}^{\pm}$. This results in the logarithmic derivative of $\mathcal{P}^{\pm}(x,s)$:

$$Y_{in}^{\pm}(x,s) \equiv \frac{\mathcal{V}^{\pm}}{\mathcal{P}^{\pm}} = \frac{-1}{sM(x)}\frac{\partial}{\partial r}\ln \mathcal{P}^{\pm}(x,s).$$

For example, for the conical horn (last column of Table 6.2, p. 237)

$$Y_{in}^{\pm} = \mathcal{Y}_r(1 \pm c_o/sr_o). \tag{6.36}$$

Note that $Y_{in}^{+}(x,s) + Y_{in}^{-}(x,s) = 2\mathcal{Y}_r = 2A_0 r^2/\rho_o c_o \in \mathbb{R}$, which shows that the frequency-dependent parts of the two admittances, being equal and opposite in sign, exactly cancel.

As the wavefront travels down the variable-area horn, there is a mismatch in the characteristic admittance due to the change in area. This mismatch creates a reflected wave, which in the case of the conical horn is $= -c_o/sr_o$. Due to conservation of volume, there is a corresponding identical forward component that travels forward, equal to $+c_o/sr_o$. The sum of these two responses to the change in area must be zero in order to conserve volume velocity.

The resulting equation for the velocity eigenfunctions is therefore

$$\mathcal{V}^{\pm}(x,s) = Y_{in}^{\pm}(x,s)\,\mathcal{P}^{\pm}(x,s).$$

Propagation function $\kappa(s)$ The eigenfunctions of the lossless wave equation propagate as

$$\mathcal{P}^{\pm}(x,s) = \frac{e^{\mp\kappa(s)x}}{\sqrt{A(x)}},$$

where $\kappa(s) = \sqrt{\mathcal{Z}(x,s)\mathcal{Y}(x,s)} = \pm s\sqrt{MC}$. The velocity eigenfunctions $\mathcal{V}^{\pm}(x,s)$ may be computed from Eq. 6.29.

From the above definitions,

$$\kappa(s) = \sqrt{\frac{s\rho_o}{\cancel{A(x)}}\frac{s\cancel{A(x)}}{\eta_o P_o}} = \frac{s}{c_o}.$$

Thus $\kappa(s)$ and s are the eigenvalues of the differential operations $\partial/\partial x$ and $\partial/\partial t$ on the pressure $\mathcal{P}(x,s)$. See Appendix 3.10 for the inclusion of visco-thermal losses.

The limits of the Webster horn equation: It is commonly stated that the Webster horn equation (WHEN) is fundamentally limited and thus is an approximation that applies only to frequencies much lower than f_c

(Morse, 1948; Shaw, 1970; Pierce, 1981). However, in all these discussions it is assumed that the area function $A(r)$ is the horn's cross-sectional area, not the area of the iso-pressure wavefront.

In the next section we show that this "limitation" may be avoided by making the Webster horn theory an "exact" solution for plane-wave" eigen-functions of Eq. 6.25.

The limitation of the quasi-static approximation is that it "ignores" higher-order "evanescent modes," which are naturally small since, being evanescent modes below their cutoff frequency, the wave number is real and thus they do not propagate (Hahn, 1941; Karal, 1953). This method is frequently called a *modal analysis* or *eigen-analysis*. This is the same approximation that is required to define an impedance, since every eigenmode has an impedance (Miles, 1948).

As derived in Appendix 6.3 the acoustic variables (eigen-functions) are redefined on the iso-pressure wavefront boundary for the pressure and the corresponding volume velocity (Hanna and Slepian, 1924; Morse, 1948; Pierce, 1981). The resulting acoustic impedance is then the ratio of the pressure to the volume velocity. This approximation is valid up to the frequency where the first cross-mode begins to propagate ($f > f_c$), which may be estimated given the roots of the Bessel eigen-functions (Morse, 1948). Perhaps it should be noted that these ideas, which may come from acoustics, apply equally well to electromagnetics and quantum mechanics, and other wave phenomena.

Visco-thermal losses: When losses are to be included, the wave number $\kappa(s) = s/c_o$ must be replaced with Eq. 3.25. This introduces dispersion in the wavefront due to the very small dispersive term $2\beta_0\sqrt{s}$, which contains a branch cut. When calculating the losses, we must be careful that they are always on the correct Riemann sheet. In cases where precise estimates of the wave properties and input impedance are required, this term is critical.

The best known examples of wave propagation are electrical and acoustic transmission lines. Such systems are loosely referred to as the *telegraph* or *telephone equations*, harking back to the early days of their discovery (Heaviside, 1892; Campbell, 1903; Brillouin, 1953; Feynman, 1970a). The telegraph equation characterizes the large resistance of the wire over long distances along with the stray capacitance of the wire to the ground (which at the time was taken as the second conductor, to save wire). Thus the telegraph equation is best modeled by a diffusion line. The telephone equation included loading coils, consisting of inductors placed periodically in the wire, to increase the circuits inductance. This converted the circuit into a true transmission line. The loading coils were introduced by the AT&T engineer and mathematician George Ashley Campbell (Campbell, 1903, 1937), however they were first proposed and promoted by Heaviside.

In acoustics, waveguides are known as horns, such as the horn connected to the first phonographs from around the turn of the century (Webster, 1919). Thus the names reflect the historical development, back to a time when mathematics and its applications were related.

6.3.6 Matrix formulation of the WHEN

Newton's laws of conservation of momentum (Eq. 6.20) and mass (Eq. 6.31) are modern versions of Newton's starting point for calculating the horn lowest-order plane-wave eigenmode wave speed.

The acoustic equations for the average pressure $\mathcal{P}(r,\omega)$ and the volume velocity are derived in Appendix 6.3, where the pressure and particle velocity equations (Eqs. 6.29 and 6.3.3) are transformed into a 2×2 matrix of acoustical variables, average pressure $\mathcal{P}(r,\omega)$ and volume velocity $\mathcal{V}(r,\omega)$:

$$-\frac{d}{dr}\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix} = \begin{bmatrix}0 & \frac{s\rho_0}{A(r)}\\ \frac{sA(r)}{\eta_o P_o} & 0\end{bmatrix}\begin{bmatrix}\mathcal{P}(r,\omega)\\ \mathcal{V}(r,\omega)\end{bmatrix}. \tag{6.37}$$

The equations

$$M(r) = \rho_o/A(r) \quad \text{and} \quad C(r) = A(r)/\eta_o P_o \tag{6.38}$$

define the per-unit-length mass and compliance of the horn (Ramo et al., 1965, p 213). The product of $\mathcal{P}(r,s)$ and $\mathcal{V}(r,s)$ defines the acoustic power [W/m^2], while their ratio defines the horn's admittance $Y_{in}^{\pm}(r,s) = \mathcal{V}^{\pm}/\mathcal{P}$, looking in the two directions (Pierce, 1981, p 37–41).

To obtain Eq. 6.25 from Eq. 6.37, we take the partial derivative of the top equation

$$-\frac{\partial^2\mathcal{P}}{\partial r^2} = s\frac{\partial M(r)}{\partial r}\mathcal{V} + sM(r)\frac{\partial\mathcal{V}}{\partial r}$$

and then use the lower equation to remove $\partial\mathcal{V}/\partial r$,

$$\frac{\partial^2 \mathcal{P}}{\partial r^2} - s\frac{\partial M(r)}{\partial r}\mathcal{V} = s^2 M(r)C(r)\,\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P}.$$

Note that $c_o^2 = MC = \left(\frac{\rho_o}{\cancel{A(r)}}\right)\cdot\left(\frac{\cancel{A(r)}}{\eta_o P_o}\right)$. In air $c_o = \sqrt{\eta_0 P_o/\rho_o}$.

We must then use the upper equation a second time to remove $\mathcal{V}$:

$$\frac{\partial^2}{\partial r^2}\mathcal{P} + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\frac{\partial}{\partial r}\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P}(r,s). \tag{6.39}$$

By use of the chain rule, equations of this form may be directly integrated, since

$$\begin{aligned}\nabla_r\mathcal{P} &= \frac{1}{A(r)}\frac{\partial}{\partial r}\left[A(r)\frac{\partial}{\partial r}\right]\mathcal{P}(r,s)\\ &= \frac{\partial^2}{\partial r^2}\mathcal{P}(r,s) + \frac{1}{A(r)}\frac{\partial A(r)}{\partial r}\mathcal{P}(r,s).\end{aligned} \tag{6.40}$$

This is equivalent to integration by parts, with integration factor $A(r)$. Finally we set $\kappa(s) \equiv s/c_o$, which later may be generalized to include visco-thermal losses (Eq. 3.25.

Merging Eqs. 6.39 and 6.40 results in the Webster horn equation (WHEN) (Eq. 6.25:

$$\frac{1}{A(r)}\frac{\partial}{\partial r}A(r)\frac{\partial}{\partial r}\mathcal{P}(r,s) = \kappa^2(s)\,\mathcal{P}(r,s) \leftrightarrow \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\varrho(r,t). \tag{6.41}$$

Equations having this form are known as *Sturm-Liouville equations*. This important class of ordinary differential equations follows from the use of separation of variables of the Laplacian in any (i.e., every) separable coordinate systems (Morse and Feshbach, 1953, p 494–523). The frequency domain eigen-solutions are denoted $\mathcal{P}^{\pm}(r,s)$.

We transform the three-dimensional acoustic wave equation into acoustic variables (Eq. 6.22) in Appendix 6.3 by the application of Gauss's law, resulting in the one-dimensional WHEN (Eq. 6.25), which is a non-singular Sturm-Liouville equation.[11] It seems significant that the integration factor corresponds to the horn's area function. Thus we have demonstrated that Eqs. 6.22 and 6.37 reduce to to Eq. 6.41 in a horn.

6.3.7 Scalar fields and the ∇ operator

Problem # *57: Let $T(x,y) = x^2 + y$ be an analytic scalar temperature field in two dimensions (single-valued $\in \mathbb{R}^2$).*

– 57.1: Find the gradient of $T(\boldsymbol{x})$ and make a sketch of T and the gradient.

Solution: $\nabla(x^2 + y) = 2x\hat{\mathbf{x}} + \hat{\mathbf{y}}$. The temperature is quadratic in x and linear in y, which has the shape of a trough in x, linearly increasing in y. In the y ($\hat{\mathbf{y}}$) direction the gradient is constant, and in the $\hat{\mathbf{x}}$ direction, it is linear, and goes through zero at $x = 0$, with $T(0) = 0$. Skiing in the y direction would be a constant ride of slope 1. If the snow had no friction, you would accelerate, but the terminal velocity would be due to the friction of the snow on the skis. Along the x direction, you would accelerate, at first, coming down, and at $x = 0$ you would stop accelerating, and begin slow down. This would be a more interesting problem if you treated it in terms of the forces on the skis and included friction as well as gravity.

– 57.2: Compute $\nabla^2 T(\boldsymbol{x})$ to determine whether $T(\boldsymbol{x})$ satisfies Laplace's equation.

Solution: Forming this operation we find that

$$\frac{\partial^2}{\partial x^2}x^2 + \cancel{\frac{\partial^2}{\partial y^2}y} = 2.$$

[11] The Webster horn equation is closely related to Schrödinger's equation (Salmon, 1946a).

So $T(\boldsymbol{x})$ does not satisfy laplace's equation, rather it satisfies the Poisson equation $\nabla^2 T(\boldsymbol{x}) = 2$.

– 57.3: Sketch the iso-temperature contours at $T = -10, 0, 10$ degrees.

Solution: The iso-potential contours are the concave parabolas $y = T_0 - x^2$.

– 57.4: The heat flux[12] *is defined as $\boldsymbol{J}(x, y) = -\kappa(x, y)\nabla T$, where $\kappa(x, y)$ is denotes the thermal conductivity at the point (x, y). Given that $\kappa = 1$ everywhere (the medium is homogeneous), plot the vector $\boldsymbol{J}(x, y) = -\nabla T$ at $x = 2$, $y = 1$. Be clear about the origin, direction, and length of your result.*

Solution: $\vec{J} = \nabla T = -2x\hat{\mathbf{x}} + -\hat{\mathbf{y}}$ thus $-\kappa\nabla T(2, 1) = \vec{J} = -(4\hat{\mathbf{x}} + \hat{\mathbf{y}})$, which has a length of $\sqrt{17}$ and is pointed $1\sqrt{1}$ unit down and $4/\sqrt{17}$ units to the left.

– 57.5: Find the vector $\perp$ to $\nabla T(x, y)$—that is, tangent to the iso-temperature contours. Hint: Sketch it for one (x, y) point (e.g., $2, 1$) and then generalize.

Solution: We may invoke the third dimension $\hat{\mathbf{z}}$ to generate this vector: $\pm\hat{\mathbf{z}} \times \nabla T = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \pm 1 \\ 2x & 1 & 0 \end{bmatrix} = \mp(1\hat{\mathbf{x}} - 2x\hat{\mathbf{y}} + 0\hat{\mathbf{z}})$. Alternatively, rotate ∇T by $\pm\pi/2$ in the (x, y) plane.

– 57.6: The thermal resistance R_T is defined as the potential drop ΔT over the magnitude of the heat flux $|\boldsymbol{J}|$. At a single point the thermal resistance is

$$R_T(x, y) = -\nabla T/|\boldsymbol{J}|.$$

How is $R_T(x, y)$ related to the thermal conductivity $\kappa(x, y)$?

Solution: $R_T(x, y) = 1/\kappa(x, y)$. In general, resistance is the reciprocal of conductivity (conductance). This is true for electrical and acoustic systems as well.

Problem # *58: Acoustic wave equation*

Note: In this problem, we will work in the frequency domain.

– 58.1: The basic equations of acoustics in one dimension are

$$-\frac{\partial}{\partial x}\mathcal{P} = \rho_o s\vec{\mathcal{V}} \quad \text{and} \quad -\frac{\partial}{\partial x}\vec{\mathcal{V}} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

Here $\mathcal{P}(x, \omega)$ is the pressure (in the frequency domain), $\mathcal{V}(x, \omega)$ is the volume velocity (the integral of the velocity over the wavefront with area A), $s = \sigma + \omega\jmath$, $\rho_o = 1.2$ is the specific density of air, $\eta_o = 1.4$, and P_o is the atmospheric pressure (i.e., 10^5 Pa). Note that the pressure field $\mathcal{P}$ is a scalar (pressure does not have direction), while the volume velocity field $\vec{\mathcal{V}}$ is a vector (velocity has direction).

We can generalize these equations to three dimensions using the ∇ operator

$$-\nabla\mathcal{P} = \rho_o s\vec{\mathcal{V}} \quad \text{and} \quad -\nabla\cdot\vec{\mathcal{V}} = \frac{s}{\eta_o P_o}\mathcal{P}.$$

– 58.2: Starting from these two basic equations, derive the scalar wave equation in terms of the pressure $\mathcal{P}$,

$$\nabla^2\mathcal{P} = \frac{s^2}{c_0^2}\mathcal{P},$$

where c_0 is a constant representing the speed of sound.

Solution: We wish to remove $\mathcal{V}$ from the two equations, to obtain a single equation in pressure. If we take the partial wrt x of the pressure equation, and then substitute the velocity equation, to remove the velocity:

$$\nabla^2\mathcal{P} = -\rho_o s\nabla\cdot\vec{\mathcal{V}} = \frac{s^2\rho_o}{\eta_o P_o}\mathcal{P} = \frac{s^2}{c_o^2}\mathcal{P}$$

[12]The heat flux is proportional to the change in temperature times the thermal conductivity κ of the medium.

– *58.3: What is c_0 in terms of η_0, ρ_0, and P_0?*

Solution: Comparing the last two terms from the previous solution we see that

$$c_o = \sqrt{\eta_o P_o / \rho_o}.$$

– *58.4: Rewrite the pressure wave equation in the time domain using the time derivative property of the Laplace transform [e.g., $dx/dt \leftrightarrow sX(s)$]. For your notation, define the time-domain signal using a lowercase letter, $p(x, y, z, t) \leftrightarrow \mathcal{P}$.*

Solution:

$$\nabla^2 p(x, y, z, t) = \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} p(x, y, z, t)$$

6.3.8 Vector fields and the ∇ operator

6.3.9 Vector algebra

Problem # *59: Let $\boldsymbol{R}(x, y, z) \equiv x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$.*

– *59.1: If a, b, and c are constants, what is $\boldsymbol{R}(x, y, z) \cdot \boldsymbol{R}(a, b, c)$?*
Solution: Using the formula for a scalar dot product:

$$\begin{aligned}\mathbf{R}(x, y, z) \cdot \mathbf{R}(a, b, c) &\equiv [x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}] \cdot [a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}}] \\ &= x(t)a + y(t)b + z(t)c.\end{aligned}$$

– *59.2: If a, b, and c are constants, what is $\frac{d}{dt}\left(\boldsymbol{R}(x, y, z) \cdot \boldsymbol{R}(a, b, c)\right)$?*
Solution: $\left(a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) + c\frac{d}{dt}z(t)\right)$.

Problem # *60: Find the divergence and curl of the following vector fields:*

– *60.1: $\boldsymbol{v} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\hat{\mathbf{z}}$*
Solution: $\nabla \cdot \vec{v} = 0$, $\nabla \times \vec{v} = 0$

– *60.2: $\boldsymbol{v}(x, y, z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + z^2\hat{\mathbf{z}}$*

Solution: $\nabla \cdot \vec{v} \equiv \partial_x x + \partial_y xy + \partial_z z^2 = 1 + x + 2z$ $\nabla \times \vec{v} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ x & xy & z^2 \end{vmatrix} = (0-0)\hat{\mathbf{x}} + (0-0)\hat{\mathbf{y}} + (y-0)\hat{\mathbf{z}} = y\hat{\mathbf{z}}$

– *60.3: $\boldsymbol{v}(x, y, z) = x\hat{\mathbf{x}} + xy\hat{\mathbf{y}} + \log(z)\hat{\mathbf{z}}$*
Solution: Divergence: $\partial_x x + \partial_y xy + \partial_z \log(z) = 1 + x + 1/z$, Curl: $\hat{\mathbf{x}}\left(\partial_y \log(z) - \partial_z xy\right) + \hat{\mathbf{y}}\left(\partial_z x - \partial_x \log(z)\right) + \hat{\mathbf{z}}\left(\partial_x xy - \partial_y x\right) = \hat{\mathbf{z}} y$

– *60.4: $\boldsymbol{v}(x, y, z) = \nabla(1/x + 1/y + 1/z)$*
Solution: First find $\mathbf{v} = -(\hat{\mathbf{x}}/x^2 + \hat{\mathbf{y}}/y^2 + \hat{\mathbf{z}}/z^2)$. Divergence of $\mathbf{v}$: $-(\partial_x 1/x^2 + \partial_y 1/y^2 + \partial_z 1/z^2) = 2(1/x^3 + 1/y^3 + 1/z^3)$, Curl of $\mathbf{v}$: 0, because the curl of the gradient is always zero.

6.3.10 Vector and scalar field identities

Problem # *61: Find the divergence and curl of the following vector fields:*

– *61.1: $\boldsymbol{v} = \nabla\phi$, where $\phi(x, y) = xe^y$*
Solution: $\nabla \times \nabla\phi = 0$, and $\nabla^2\phi = xe^y$

– *61.2: $\boldsymbol{v} = \nabla \times \boldsymbol{A}$, where $\boldsymbol{A} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$*
Solution: $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, and $\nabla \times (\nabla \times \mathbf{A}) = 0$

– *61.3:* $\boldsymbol{v} = \nabla \times \boldsymbol{A}$, *where* $\boldsymbol{A} = y\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Solution: $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, and $\nabla \times (\nabla \times \mathbf{A}) = -2\hat{\mathbf{y}}$

– *61.4: For any differentiable vector field* $\boldsymbol{V}$*, write two vector calculus identities that are equal to zero.*

Solution: *Curl of the gradient* $\nabla \times \nabla\Phi(x, y, z) = 0$ and the *divergence of the curl* $\nabla \cdot \nabla \times \vec{V}(x, y, z) = 0$ are both zero. (Page 780, Stillwell)

– *61.5: What is the most general form a vector field may be expressed in, in terms of scalar* Φ *and vector* $\boldsymbol{A}$ *potentials?*

Solution: $\vec{V} = \nabla\Phi(x, y, z) + \nabla \times \vec{A}(x, y, z)$, where Φ is the scalar potential and $\vec{A}$ is the vector potential.

Problem # *62: Perform the following calculations. If you can state the answer without doing the calculation, explain why.*

– *62.1: Let* $\boldsymbol{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$*. Find* $\nabla \cdot (\nabla \times \boldsymbol{v})$*.*

Solution: 0

– *62.2: Let* $\boldsymbol{v} = \sin(x)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$*. Find* $\nabla \times (\nabla\sqrt{\boldsymbol{v} \cdot \boldsymbol{v}})$

Solution: 0

– *62.3: Let* $\boldsymbol{v}(x, y, z) = \nabla(x + y^2 + \sin(\log(z))$*. Find* $\nabla \times \boldsymbol{v}(x, y, z)$*.*

Solution: It is zero because $\nabla \times \nabla f(x, y, z)$ is *always* zero.

6.3.11 Integral theorems

Problem # *63: For each of the following problems, in a few words, identify either Gauss's or Stokes's law, define what it means, and explain the formula that follows the question.*

– *63.1: What is the name of this formula?*

$$\int_S \hat{\mathbf{n}} \cdot \boldsymbol{v}\, dA = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{v}\, dV.$$

Solution: This is the integral form of *Gauss' law*. The unit normal vector is $\perp$ to the surface $\mathcal{S}$ having area $A \equiv \int_{\mathcal{S}} dA$ The integral represents the total flow normal to the surface. The surface integral is equal to the integral of the divergence of the vector field $\nabla \cdot \mathbf{v}$ over the volume contained by the surface, and defined as $\mathcal{V}$.

– *63.2: What is the name of this formula?*

$$\int_S (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{V} \cdot d\boldsymbol{R}$$

Give one important application. **Solution:** Stokes Theorem, which relates the differential to the integral form of Maxwell's equations.

– *63.3: Describe a key application of the vector identity*

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \boldsymbol{\nabla}^2\vec{V}.$$

Solution: When we wish to reduce Maxwell's two curl equations to the vector wave equation, we must use this identity.

6.4 Examples of finite-length horns

Figure 6.4 is taken from the classic book by Olson (1947, p. 101), showing the theoretical acoustical radiation impedance $Z_{rad}(r, \omega)$ for five horns having different area functions, resulting in frequency dependent impedance between 50 [Hz] to 10 [kHz]. The parabolic horn (#1) has a mild peak in the reactive input impedance just below 2 [kHz]. The conical (megaphone) horn (#2) has a maximum reactive peat at 5 [kHz], close to the frequency where the real and imaginary parts are equal. The most interesting horns are #3 and #4 which,

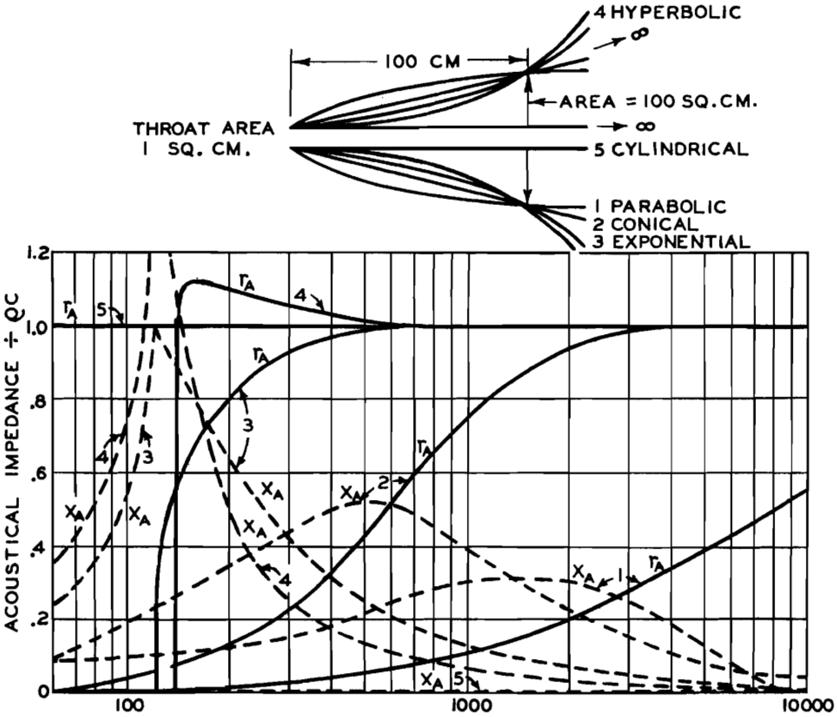


Figure 6.4: *:: Throat acoustical resistance r_A and acoustical reactance x_A, frequency characteristics of infinite eigen-functions of the parabolic, conical, exponential, hyperbolic, and cylindrical horns, having a throat area of 1 [cm^2]. Note how the "critical" frequency (defined here as the frequency where the reactive and real parts of the radiation impedance are equal) of the horn reduces dramatically with the type of horn. For the uniform horn, the reactive component is zero, so there is no cutoff frequency. For the parabolic horn (#1), the cutoff is around 3 kHz. For the conical horn (#2), the cutoff is at 0.6 [kHz]. For the exponential horn (#3), the critical frequency is around 0.18 [kHz], which is one-16th that of the parabolic horn. For each horn the cross-sectional area is taken as 100 [cm^2] at a distance of $L = 1$ [m] from the throat (Olson, 1947, p 101); (Morse, 1948, p 283).*

Table 6.2: *Horns and their properties for N = 1, 2, and 3 dimensions, along with the exponential horn (*EXP*). The range variable goes from $r_o \le r \le L$ [m] with area $1 \le A(r) \le 100$ [cm^2]. $F(r)$ is the coefficient on $\mathcal{P}_x$, $\kappa(s) \equiv s/c_o$, where c_o is the sound speed and $s = \sigma + \omega\jmath$ is the Laplace frequency. The horn's eigen-functions are $\mathcal{P}^{\pm}(\xi, s) \leftrightarrow \varrho^{\pm}(\xi, t)$. When $\pm$ is indicated, the outbound solution corresponds to the negative sign. Eigen-functions $H_0^{\pm}(\xi, s)$ are outbound and inbound Hankel functions. The rightmost column is the input radiation admittance normalized by the* characteristic admittance $\mathcal{Y}_r(r) = A(r)/\rho_o c_o$.

N	Name	radius	Area/A_o	$F(r)$	$\mathcal{P}^{\mp}(r, s)$	$\varrho^{\mp}(r_o, t)$	$Y^{\mp}_{rad}/\mathcal{Y}_r$
1D	uniform	1	1	0	$e^{\mp\kappa(s)r}$	$\delta(t)$	1
2D	parabolic	$\sqrt{r}$	r	$1/r$	$H_0^{\mp}\left(-j\kappa(s)r\right)$	—	$\frac{-\jmath r_o H_1^{\mp}}{H_0^{\mp}}$
3D	conical	r	r^2	$2/r$	$e^{\mp\kappa(s)r}/r$	$\delta(t) \pm \frac{c_o}{r_o}u(t)$	$1 \pm c_o/sr_o$
EXP	exponential	e^{mr}	e^{2mr}	$2m$	$e^{-\left(m\mp\sqrt{m^2+\kappa^2}\right)r}$	$e^{-mr}E(t)$	Eq. 6.52

have a razor sharp cutoff near 150 [Hz], where the impedance abruptly switches from resistive to reactive. The cylindrical horn (#5) has a constant impedance. All the impedance's have been normalized to have their impedance of 1 at high frequencies.

Table 6.2 summarizes the properties of four of these horns: uniform (cylindrical) ($A = A_o$), parabolic ($A(r) = A_o r$), conical (spherical) ($A(r) = A_o r^2$), and exponential ($A(r) = A_o e^{2mr}$), where m is called the flair parameter.

6.4.1 Uniform Horn #5

The one-dimensional wave equation [$A(r) = A_o = 1$ [cm^2]] is

$$\frac{d^2}{dr^2}\mathcal{P} = \kappa^2(s)\,\mathcal{P},$$

thus the roots (eigenvalues) are $\kappa^2_{\pm}(s) \equiv \pm s^2/c_o^2$.

Solution: The two eigen-functions of this equation are the two d'Alembert waves (Eq. 4.20,

$$\varrho(x,t) = \mathcal{P}_0^+\varrho^+(t - x/c) + \mathcal{P}_0^-\varrho^-(t + (x-L)/c) \;\leftrightarrow\; \mathcal{P}_0^+ e^{-\kappa(s)x} + \mathcal{P}_0^- e^{\kappa(s)(x-L)},$$

where $\mathcal{P}_0^{\pm} \in \mathbb{C}$ are wave amplitudes and $\kappa(s) = s/c_o = \jmath\omega/c$ is called the *propagation function* (also known as the wave-evolution function, propagation constant, and wave number).

Note that for the uniform lossless horn, $\omega/c_o = 2\pi/\lambda \ge 0$. It is convenient to normalize $\mathcal{P}_0^+ = 1$ and

$\mathcal{P}_L^- = 1$. The characteristic admittance $\mathcal{Y}_r(x)$ (Table 6.2) is independent of direction. The signs must be physically chosen, with the velocity $\mathcal{V}^\pm$ into the port, to ensure that the real part $\mathcal{Y}_r \geq 0$, to assure the system remains stable.

Applying the boundary conditions: The general solution in terms of the eigenvector matrix, evaluated at $x = L$, is

$$\begin{bmatrix} \mathcal{P}(x) \\ \mathcal{V}(x) \end{bmatrix}_L = \begin{bmatrix} e^{-\kappa x} & e^{\kappa(x-L)} \\ \mathcal{Y}_r e^{-\kappa x} & -\mathcal{Y}_r e^{\kappa(x-L)} \end{bmatrix}_L \begin{bmatrix} \mathcal{P}_0^+ \\ \mathcal{P}_0^- \end{bmatrix}_L = \begin{bmatrix} e^{-\kappa L} & 1 \\ \mathcal{Y}_r e^{-\kappa L} & -\mathcal{Y}_r \end{bmatrix} \begin{bmatrix} \mathcal{P}_0^+ \\ \mathcal{P}_0^- \end{bmatrix}_L, \tag{6.42}$$

where $\mathcal{P}_0^+$ and $\mathcal{P}_0^-$ are the relative amplitudes of the two unknown eigen-functions to be determined by the boundary conditions at $x = 0, L$, $\kappa = s/c$, and $\mathcal{Y}_r = 1/\mathcal{Z}_r = A_o/\rho_o c$.

Solving Eq. 6.42 for $\mathcal{P}_0^+$ and $\mathcal{P}_0^-$ with determinant $\Delta = -2\mathcal{Y}_r e^{-\kappa L}$,we get

$$\begin{bmatrix} \mathcal{P}_L^+ \\ \mathcal{P}_L^- \end{bmatrix} = \frac{-1}{2\mathcal{Y}_r e^{-\kappa L}} \begin{bmatrix} -\mathcal{Y}_r & -1 \\ -\mathcal{Y}_r e^{-\kappa L} & e^{-\kappa L} \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ \mathcal{V} \end{bmatrix}_L = \frac{1}{2} \begin{bmatrix} e^{\kappa L} & -\mathcal{Z} e^{\kappa L} \\ 1 & \mathcal{Z} \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ -\mathcal{V} \end{bmatrix}_L. \tag{6.43}$$

In the final step we swapped all the signs, including those on $\mathcal{V}$, and moved $\mathcal{Z}_r = 1/\mathcal{Y}_r$ inside the matrix.

We can uniquely determine these two weights $\mathcal{P}_L^+$, $\mathcal{P}_L^-$ given the pressure and velocity at the boundary $x = L$, which is typically determined by the load impedance ($Z_L(s) = \mathcal{P}_L/\mathcal{V}_L$).

The weights may now be substituted back into Eq. 6.42 to determine the pressure and velocity amplitudes at any point $0 \leq x \leq L$:

$$\begin{bmatrix} \mathcal{P} \\ \mathcal{V} \end{bmatrix}_x = \frac{1}{2} \begin{bmatrix} e^{-\kappa x} & e^{\kappa(x-L)} \\ \mathcal{Y}_r e^{-\kappa x} & -\mathcal{Y}_r e^{\kappa(x-L)} \end{bmatrix}_x \begin{bmatrix} e^{\kappa L} & -\mathcal{Z} e^{\kappa L} \\ 1 & \mathcal{Z} \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ -\mathcal{V} \end{bmatrix}_L. \tag{6.44}$$

Setting $x = 0$ and multiplying these out give the final transmission matrix:

$$\begin{bmatrix} \mathcal{P} \\ \mathcal{V} \end{bmatrix}_0 = \frac{1}{2} \begin{bmatrix} e^{\kappa L} + e^{-\kappa L} & \mathcal{Z}_r(e^{\kappa L} - e^{-\kappa L}) \\ \mathcal{Y}_r(e^{\kappa L} - e^{-\kappa L}) & e^{\kappa L} + e^{-\kappa L} \end{bmatrix}_x \begin{bmatrix} \mathcal{P} \\ -\mathcal{V} \end{bmatrix}_L. \tag{6.45}$$

Note that the diagonal terms are $\cosh \kappa L$ and the off-diagonal terms are $\sinh \kappa L$.

Applying the last boundary condition, we evaluate Eq. 6.43 to obtain the ABCD matrix at the input ($x = 0$) (Pipes, 1958),

$$\begin{bmatrix} \mathcal{P} \\ \mathcal{V} \end{bmatrix}_0 = \begin{bmatrix} \cosh \kappa L & \mathcal{Z}_r \sinh \kappa L \\ \mathcal{Y}_r \sinh \kappa L & \cosh \kappa L \end{bmatrix} \begin{bmatrix} \mathcal{P} \\ -\mathcal{V} \end{bmatrix}_L. \tag{6.46}$$

Note that the uniform horn is reversible (P7), reciprocal (P6) and causal(P1).

Exercise #*11*

Evaluate the expression in terms of the load impedance.

Solution: Since $Z_{\text{load}} = -\mathcal{P}_L/\mathcal{V}_L$, we have

$$\left.\frac{\mathcal{P}}{\mathcal{V}}\right|_0 = \frac{Z_{\text{load}} \cosh \kappa L - \mathcal{Z}_r \sinh \kappa L}{Z_{\text{load}} \mathcal{Y}_r \sinh \kappa L - \cosh \kappa L}$$

Impedance matrix: Equation 6.46 is an impedance matrix (algebra required)

$$\begin{bmatrix} \mathcal{P}_0 \\ \mathcal{P}_L \end{bmatrix} = \frac{\mathcal{Z}_r}{\sinh(\kappa L)} \begin{bmatrix} \cosh(\kappa L) & 1 \\ 1 & \cosh(\kappa L) \end{bmatrix} \begin{bmatrix} \mathcal{V}_0 \\ \mathcal{V}_L \end{bmatrix}.$$

Exercise #*12*

Write out the short-circuit ($\mathcal{V}_L = 0$) input impedance $Z_{in}(s)$ for the uniform horn.

Solution:

$$Z_{in}(s) = \left.\frac{\mathcal{P}}{\mathcal{V}}\right|_{\mathcal{V}_L=0} = \mathcal{Z}_r \frac{\cosh \kappa L}{\sinh \kappa L} = \mathcal{Z}_r \tanh \kappa L|_{\mathcal{V}_L=0}$$

where $\mathbb{Z}_r = \rho c_o/A_o$.

Input admittance Y_{in}: Given the input admittance of the horn, it is possible to determine whether it is uniform without further analysis. That is, if the horn is uniform and infinite in length, the input admittance at $x = 0$ is

$$Y_{in}(x=0,s) \equiv \frac{\mathcal{V}(0,\omega)}{\mathcal{P}(0,\omega)} = \mathcal{Y}_r,$$

since $\mathcal{P}_0^+ = 1$ and $\mathcal{P}_L^- = 0$. For an infinite uniform horn, there are no reflections.

When the uniform horn is terminated with a fixed impedance $\mathcal{Z}_r$ at $x = L$, we can substitute pressure and velocity measurements into Eq. 6.43 to find $\mathcal{P}_0$ and $\mathcal{P}_0$, and given these, we can calculate the pressure reflectance at $x = L$ (Eq. 3.32,

$$\Gamma_L(s) = \frac{\mathcal{P}_L}{\mathcal{P}_0} = \frac{\mathcal{P}(L,\omega) - \mathcal{Z}_r\,\mathcal{V}(L,\omega)}{\mathcal{P}(L,\omega) + \mathcal{Z}_r\,\mathcal{V}(L,\omega)} = \frac{Z_L - \mathcal{Z}_r}{Z_L + \mathcal{Z}_r}. \tag{6.47}$$

Given sufficiently accurate measurements of the throat pressure and assuming $\Gamma_L = 0$, the input impedance Z_{in} at the input $\boldsymbol{x} = 0$ is $\mathcal{Z}_r = \rho_o c/A_o$.

6.4.2 Conical horn #2

Using the conical horn area $A(r) \propto r^2$ in Eq. 6.25 and 6.37 results in the spherical wave equation

$$\mathcal{P}_{rr}(r,\omega) + \frac{2}{r}\mathcal{P}_r(r,\omega) = \kappa^2\,\mathcal{P}(r,\omega), \tag{6.48}$$

where $\kappa(s) = \pm s/c_o$. The eigen-solutions of Eq. 6.48 are

$$\mathcal{P}^\pm(r,s) = \frac{e^{\mp\kappa r}}{r} \leftrightarrow \frac{1}{r}\delta(t \mp r/c_o).$$

Radiation admittance for the conical horn: The conical horn's acoustic input admittance $Y_{in}(r,s)$ at any location r is found by dividing $\mathcal{V}(r,s)$ by $\mathcal{P}(r,s)$:

$$Y_{in}^\pm(r,s) = \frac{\mathcal{V}^\pm}{\mathcal{P}^\pm} = -\frac{A(r)}{s\rho_o}\frac{d}{dr}\ln\mathcal{P}^\pm(r,s) \tag{6.49}$$

$$= \mathcal{Y}_r(r)\left[1 \pm \frac{c_o}{sr}\right] \leftrightarrow \frac{A(r)}{\rho_o c_o}\left(\delta(t - r/c_o) \pm \frac{c_o}{r}u(t - r/c_o)\right). \tag{6.50}$$

The pressure pulse is delayed by r/c_o due to $e^{-\kappa(s)r}$. As the area of the horn increases, the pressure decreases as $1/r = 1/\sqrt{A(r)}$. This results in the uniform back-flow $c_o u(t)/r \leftrightarrow c_o/sr$ due to conservation of mass, and the characteristic admittance $\mathcal{Y}_r(r)$ variation with r.

6.4.3 Exponential horn #3

If we define the area as $A(r) = A_o e^{2mr}$, the eigen-functions of the horn are

$$\mathcal{P}^\pm(r,\omega\jmath) = e^{-mr}e^{\mp\jmath\sqrt{\omega^2-\omega_c^2}\;r/c}, \tag{6.51}$$

which may be shown by the substitution of $\mathcal{P}_c^\pm(r,\omega\jmath)$ into Eq. 6.25

This case is of special interest because the radiation impedance is purely reactive below the horn's cutoff frequency ($\omega < \omega_c = mc_o$), as may be seen from curves 3 and 4 of Fig. 6.4. As a result, no energy can radiate from an open horn for $\omega < \omega_c$ because the eigenvalues

$$\kappa(s) = -m \pm \frac{\jmath}{c_o}\sqrt{\omega^2 - \omega_c^2} = -m \pm \sqrt{s^2 - s_c^2}$$

are purely real (this is the case of non-propagating evanescent waves).

Solution: Here we have replaced ω by $s = \jmath\omega$, by moving s under the square root.

If we use Eq. 4.26 the input admittance is

$$Y_{in}^{\pm}(x,s) = -\frac{A(x)}{s\rho_o}\left(m \pm \sqrt{m^2+\kappa^2}\right)x. \tag{6.52}$$

Kleiner (2013) gives an equivalent expression for $Y_{in}(x,\omega)$ given area $S(x) = e^{mx}$,

$$Y_{in}(x,\omega) = \frac{S(x)}{\jmath\omega\rho}\left[\frac{m}{2} + \jmath\frac{\sqrt{4\omega^2-(mc)^2}}{2c}\right],$$

and impedance

$$Z_{in}(r,\omega) = \frac{\rho c}{S_T}\left[\jmath\frac{\omega_c}{\omega} + \sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}\right],$$

where $\omega_c(r)$ is the cutoff frequency. Given this exact solution to the exponential horn, we could use a series expansion of the form

$$A(r) = \sum_k a_k e^{b_k x} \quad \in \mathbb{R} \tag{6.53}$$

to obtain the general solution, for an arbitrary analytic $A(r)$.

6.5 Solution methods

To model the wave equation, two distinct mathematical techniques are available. The first of these is called *separation of variables*. This method is limited to a small and restrictive number of separable coordinate systems (SCS). Once the SCS is chosen, the eigen-functions are known as solutions to that specific Sturm-Liouville equation, which is always a scalar (ordinary) differential equation (ODEs).

The second method uses the transmission matrix (a lumped-parameter method). This solution method assumes a limit on the upper frequency response. This is because quasi-statics requires that the wavelength is larger than the size of the object being modeled. Thus other than the upper frequency limit, there are no limitations in the analytical and numerical solutions using the transmission matrix method. The calculation may be done in either the frequency or time domains (Fettweis, 1986).

1a. **Separable coordinate systems:** Classically PDEs are solved by *separation of variables*. This method is limited to a few ortho-normal coordinate systems, such as rectangular, cylindrical, and spherical coordinates (Morse, 1948, p 296–97). Even a slight deviation from separable specific coordinate systems represents a major barrier toward further analysis and understanding, blocking insight into more general cases. Separable coordinate systems have a high degree of symmetry. Note that the solution of the wave equation is not tied to a specific coordinate system.

1b. **Sturm-Liouville methods and eigenvectors:** When the coordinate system is separable, the resulting PDEs are always reduced to a system of Sturm-Liouville equations. The solutions of this important class of Sturm-Liouville eigen-functions are all tabulated.

   The Webster horn equation (Webster, 1919; Morse, 1948; Pierce, 1981) is a generalized Sturm-Liouville equation that includes the physics in the form of the horn's area function. The Webster equation sidesteps the seriously limiting problem of separation of variables, by using the alternative quasi-static solution, which ignores the non propagating high-frequency evanescent modes. This is essentially a one-dimensional low-pass approximation to the wave equation. While mathematics provides rigor, physics provides understanding.

2. **Lumped-element method:** A system may be represented in terms of lumped elements, as either electrical inductors, capacitors, and resistors or their mechanical counterparts, masses, springs, and dash pots. Such systems are represented by 2×2 transmission matrices in the s (i.e., Laplace) domain (Ramo et al., 1965, Appendix IV).

   When a system of lumped-element networks contains only resistors and capacitors or resistors and inductors, the solution is a diffusion equation, which does not support propagated waves. Depending on the elements in the system of equations, there can be an overlap between a diffusion process and scalar waves, represented as transmission lines, both modeled as lumped-element networks of 2×2 matrices (Eq. 3.3) (Campbell, 1922; Brillouin, 1953; Ramo et al., 1965).

Nyquist sampling and quasi-statics: Quasi-static methods provide band-limited solutions below a critical frequency f_c for a much wider class of geometries, by avoiding high-frequency cross-modes. The model of a train as depicted in Fig. 5.8 (p. 202) is an example.

Example: Train-mission-line problem. The mechanical mass-spring system of Fig. 6.5 is the electrical equivalent circuit. The mass is modeled as an inductor and the springs as capacitors to ground. The velocity is analogous to a current and the force $f_n(t)$ to the voltage $\phi_n(t)$. The length of each cell is Δ [m].

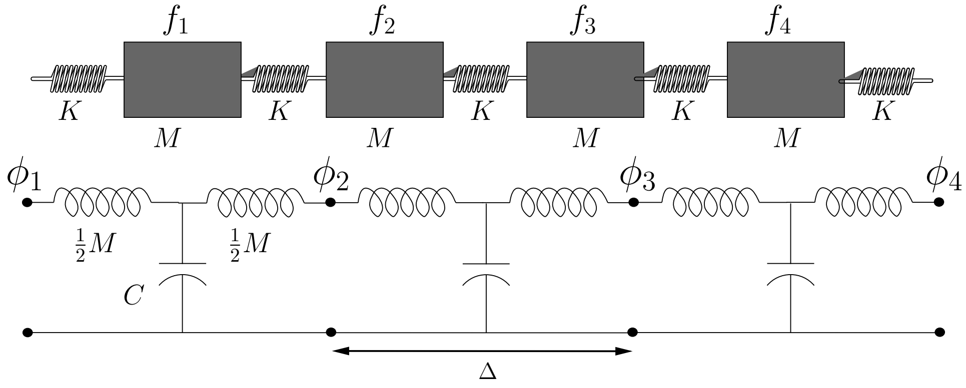


Figure 6.5: *Depiction of a train consisting of cars, treated as a mass M and linkages, treated as springs of stiffness K or compliance $C = 1/K$. The equivalent electrical circuit is shown below the mass-spring system, with the masses modeled by inductors (M) and springs modeled as capacitors (C). For this model to accurately represent a transmission line the frequency must be less than the equivalent Nyquist frequency f_c. The delay of one cell is $T_o = \Delta/c_o$. One can measure either T_o or Δ and $c_o = 1/\sqrt{MC}$. Since $\lambda_c = 2\Delta$, it follows that the cutoff frequency is $f_c = c_o/2\Delta = 1/2T_o$. For $f < f_c$, the frequency response is independent of frequency, thus acting as a pure delay. As the frequency is increased above f_c, the wavelength becomes shorter that the critical wavelength $\lambda_c = 2\Delta$, and the delay line becomes a low-pass filter, strongly departing from transmission line properties. This transition is known as the Nyquist cutoff frequency (Nyquist (1924)).*

When the wavelength $\lambda = c_o/f_c$ is greater than twice the physical distance Δ between the elements

$$\lambda > \lambda_c = 2\Delta \quad \text{[m]},$$

the approximation is mathematically equivalent to a transmission line. As described in DE-3, problem #2, the velocity is $c_o = 1/\sqrt{MC}$ [m/s]. As the frequency increases, the wavelength becomes shorter. When the frequency is equal to the critical frequency f_c the critical wavelength $\lambda_c = c_o/f_c = 2\Delta$. Above the critical frequency the quasi-static (lumped-element) model breaks down and transitions from a delay line to a low-pass filter, as discussed in DE-3, problem #2.

The frequency is under the control of the modeling process, since more elements may be added to allow for higher frequencies (shorter wavelengths). If the nature of the solution at high frequencies ($f > f_c$) is desired, we may add more sections (make Δ smaller). For many (perhaps most) problems, lumped elements are easy to use and accurate, as long as we don't violate the Nyquist condition (Nyquist, 1924; Brillouin, 1953; Ramo et al., 1965).

6.5.1 Eigen-functions $\varrho^{\pm}(r,t)$ of the WHEN

Because the wave equation (Eq. 6.22) is second-order in time, there are two causal independent eigenfunction solutions: an outbound (right–traveling) $\varrho^{+}(r,t)$ wave, and an inbound (left-traveling) $\varrho^{-}(r,t)$ wave.

Every eigen-function depends on an area function $A(r)$. In theory, given an eigen-function, it be possible to find the area $A(r)$. This is known as the *inverse problem*, which is generally believed to be a difficult problem.

Because the characteristic impedance $\mathcal{Y}_r(r)$ of the wave in the horn changes with location, there are local reflections due to these area variations. Thus there are fundamental relationships between the area change $dA(r)/dr$, the horn's eigen-functions $\mathcal{P}^{\pm}(r,s)$, the eigenmodes, and the input impedance.

Complex vs. real frequency: We shall continue to maintain the distinction that functions of ω are Fourier transforms and causal functions of Laplace frequency s correspond to Laplace transforms, which are necessarily complex–analytic in s in the right half-plane (RHP) region of convergence (RoC). This distinction is critical, since we typically describe impedance $Z(s)$ and admittance $Y(s)$ as complex–analytic functions in s in terms of their poles and zeros. The eigen-functions $\mathcal{P}^{\pm}(r,s)$ of Eq. 6.25 are also causal complex–analytic functions of s.

Plane-wave eigen-function solutions: In 1690, nine years before Newton's publication of *Principia*, Christiaan Huygens was the first to gain insight into wave propagation, today known as *Huygens's principle*. While his concept showed a deep insight, we now know it was flawed, as it ignored the backward-traveling wave (Miller, 1991). In 1747 d'Alembert published the first correct solution for the plane-wave scalar wave equation,

$$\varrho(x,t) = f(t - x/c_o) + g(t + x/c_o), \tag{6.54}$$

where $f(\cdot)$ and $g(\cdot)$ are general functions of their argument. That this is the solution may be shown by use of the chain rule, by taking partials with respect to x and t.

In terms of physics, d'Alembert's general solution describes two arbitrary wave forms $f(\cdot)$ and $g(\cdot)$ traveling at a speed c_o, one forward and one reversed. This solution is quite easily visualized.

Exercise #*13*

By the use of the chain rule, show that d'Alembert's formula satisfies the one-dimensional wave equation.

Solution: Taking a derivative with respect to t and r gives

$$\partial_t \varrho(r,t) = -c_o f'(r - c_o t) + c_o g'(r + c_o t)$$
$$\partial_r \varrho(r,t) = f'(r - c_o t) + g'(r + c_o t),$$

and a second derivative gives

$$\partial_{tt} \varrho(r,t) = c_o^2 f''(r - c_o t) + c_o^2 g''(r + c_o t)$$
$$\partial_{rr} \varrho(r,t) = f''(r - c_o t) + g''(r + c_o t).$$

From these last two equations we have the one-dimensional wave equation

$$\partial_{rr} \varrho(r,t) = \frac{1}{c_o^2} \partial_{tt} \varrho(r,t),$$

which has solutions in Eq. 6.54.

Exercise #*14*

Assuming $f(\cdot)$ and $g(\cdot)$ are $\delta(\cdot)$, find the Laplace transform of the solution corresponding to the uniform horn $A(x) = 1$.

Solution: Using the Table of Laplace transforms on Eq. 6.54 gives

$$\varrho(x,t) = \delta(t - x/c_o) + \delta(t + x/c_o) \leftrightarrow e^{-sx/c_o} + e^{sx/c_o}. \tag{6.55}$$

Note that the delay $T_o = \pm x/c_o$ depends on the range x.

Three-dimensional d'Alembert spherical eigen-functions: We can generalize the d'Alembert solution to spherical waves by changing the area function of Eq. 6.25 to $A(r) = A_o r^2$ (see Eq. 6.9, and Table 6.2. The radial wave equation then becomes

$$\nabla_r^2 \varrho(r,t) = \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \varrho(r,t). \tag{6.56}$$

where

$$\nabla_r^2\, p(r,t) \equiv \nabla \cdot \nabla\, p(r,t). \tag{6.57}$$

Note that

$$\nabla_r \phi(r) = \frac{1}{r} \frac{\partial r\phi(r)}{\partial r} = \frac{1}{r}\left(\phi + r\frac{\partial \phi(r)}{\partial r}\right)$$

Multiplying by r (see the discussion of the conical horn in §6.48, p. 243) provides the d'Alembert wave equation solution for arbitrary forward $f(\cdot)$ and retrograde $g(\cdot)$ waveform

$$r\varrho(r,t) = f(t - r/c_o) + g(t + r/c_o). \tag{6.58}$$

Finally dividing by r results in the well known spherical (3D) d'Alembert wave equation eigen-functions (Eq. 6.56)

$$\varrho(r,t) = \frac{f(t - r/c_o)}{r} + \frac{g(t + r/c_o)}{r}. \tag{6.59}$$

This dependence on r is required for the conservation of energy for spherical waves. The use of the image method allows to solve the problem of a sound source in a room (Allen and Berkley, 1979).

6.6 Integral definitions of $\nabla()$ $\nabla\cdot()$ $\nabla\times()$ and $\nabla\wedge()$

We present two forms of wave equations, scalar and vector. Up to here we have only discussed the scalar case. The *vector wave equation* describes the evolution of a vector field, such as Maxwell's electric field vector $\boldsymbol{e}(\boldsymbol{x},t)$. I have switched to lower case for the vector fields to indicate the time domain and to upper case for the frequency domain (Laplace's transforms).

There are two equivalent definitions for each of the four operators: differential and integral. The integral form provides a more intuitive view of the operator, which in the limit converges to the differential form. Following a discussion of the gradient, divergence, and curl integral operators, we discuss these two forms.

In addition there are three fundamental vector theorems: Gauss's law (divergence theorem), Stokes's law (curl theorem), and Helmholtz's decomposition theorem. Without the use of these fundamental vector calculus theorems, we could not understand Maxwell's equations.

6.6.1 Maxwell's equations

Maxwell's two curl electric equations, for the electrical and magnetic field strength $\boldsymbol{e}(\boldsymbol{x},t)$ and $\boldsymbol{h}(\boldsymbol{x},t)$, are

$$\nabla\times\boldsymbol{e} = -\partial_t\boldsymbol{b}. \tag{6.60}$$

and

$$\nabla\times\boldsymbol{h} = \boldsymbol{j} + \partial_t\boldsymbol{d}. \tag{6.61}$$

An alternate popular notation for $\partial_t\boldsymbol{b}$ is $\dot{\boldsymbol{b}}$, but the dot is easily missed.

The units for $\boldsymbol{e}(\boldsymbol{x},t)$ are [*Volts/m*] while those of $\boldsymbol{h}(\boldsymbol{x},t)$ are [Amps/m^2]. The *conduction current* $\boldsymbol{j}$ obeys Ohm's Law ($\boldsymbol{j} = \sigma\boldsymbol{e}$), where σ is the conductivity in $\mho/m^2$ and $\boldsymbol{d}$ is the *displacement current*, first identified and then named by the brilliant Maxwell. In my view, he was every bit as brilliant as Einstein, both being creative genius of comparable stature. Maxwell was the first to conceived these these most important and famous four equations around 1862, while working at King's college London. There is a Planck on the wall and a chair called the Maxwell chair, [13] [14]

The last two equations are know as *constitutive relations* are define as $\boldsymbol{d} = \epsilon_o\boldsymbol{e}$ and $\boldsymbol{b} = \mu_o\boldsymbol{h}$

$$\nabla\cdot\boldsymbol{d} = \rho \tag{6.62}$$

and

$$\nabla\times\boldsymbol{b} = 0. \tag{6.63}$$

Two additional important constants are the *characteristic impedance* of free space $z_o = \sqrt{\epsilon_o/\mu_o} = 377\ \Omega$, and the speed of light $c_o = 1/\sqrt{\epsilon_o\mu_o} \approx 3\times10^8$ [m/s]. Due to the extreme temperature of the hot plasma inside the sun, it is likely that $\epsilon(\boldsymbol{x},t)$ and $\mu(\boldsymbol{x},t)$ are functions of space and time, making Maxwell's equations both nonlinear and time varying.

These are know as the most significant equations in science, only rivaled by Newton's $F = ma$ and Einstein's famous theory of relativity energy-mass equivalence relation $E = mc_o^2$, which is generalize by the use of the Lorentz transformation γ.[15]

[13] https://en.wikipedia.org/wiki/James_Clerk_Maxwell
[14] https://www.viator.com/tours/San-Francisco/Exploratorium-General-Admission/
[15] https://en.wikipedia.org/wiki/Lorentz_factor

The wave-particle duality of electrons raises the question about the nature of their makeup in the hot plasma. These views were well summarized by Einstein:[16] l

The dual wave-particle character of light and electrons are an enigma.

The extreme temperatures fundamentally change the problem.

6.7 Electric fields $e, \boldsymbol{d}$

Gradient: $\boldsymbol{e} = -\nabla\phi(\boldsymbol{x})$ As shown in Fig. 6.1 the gradient maps $\mathbb{R}^1 \underset{\nabla}{\mapsto} \mathbb{R}^3$. The gradient is defined as the unit-normal $\hat{\mathbf{n}}$ weighted by the potential $\phi(\boldsymbol{x})$ averaged over a closed surface $\mathcal{S}$,

$$\nabla\phi(\boldsymbol{x}) \equiv \lim_{\mathcal{S},\mathcal{V}\to 0} \left\{ \frac{\iint_{\mathcal{S}} \phi(\boldsymbol{x})\, \hat{\mathbf{n}}\, d\mathcal{S}}{\mathcal{V}} \right\} \quad [\text{V/m}]\,, \tag{6.64}$$

having area $\mathcal{S}$ and volume $\mathcal{V}$ and centered at $\boldsymbol{x}$ (Greenberg, 1988, p 773).[17] Here $\hat{\mathbf{n}}$ is a dimensionless unit vector perpendicular to the surface $\mathcal{S}$:

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{\|\nabla\phi\|}. \tag{6.65}$$

The dimensions of Eq. 6.64 are in the units of the potential times the area, divided by the volume, as needed for a gradient (e.g., [V/m]). The units depend on the potential. If ϕ were temperature, the units would be [deg/m].

Exercise #*15*

Justify the units of Eq. 6.64.

Solution: The units depend on ϕ per unit length. If ϕ is voltage, then the gradient has units of [V/m]. Under the limit, $d|\mathcal{S}|/||\mathcal{S}||$ must have units of m^{-1}.

The natural way to define the surface and volume is to place the surface on the iso-potential surfaces, forming either a cube or a pill-box-shaped volume. As the volume $||\mathcal{S}||$ goes to zero, so must the area $|\mathcal{S}|$. One must avoid irregular volumes where the area is finite as the volume goes to zero (Greenberg, 1988, footnote p 762).

A well-known example is the potential

$$\phi(x, y, z) = \frac{Q}{\epsilon_o\sqrt{x^2+y^2+z^2}} = \frac{Q}{\epsilon_o R} \quad [\text{V}]$$

around a point charge Q [SI units of coulombs]. The constant ϵ_o is the permittivity [F/m^2]. A second well-known example is the acoustic pressure potential around an oscillating sphere, which has the same form (see Table 6.2).

How does this work? To better understand Eq. 6.64, consider a three-dimensional Taylor series expansion (Eq. 6.19) of the potential in $\boldsymbol{x}$ about the limit point $\boldsymbol{x}_o$:

$$\phi(\boldsymbol{x}) \approx \phi(\boldsymbol{x}_o) + \nabla\phi(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{x}_o) + \cdots$$

The gradient is

$$\nabla\phi(\boldsymbol{x}_o) = \lim_{\boldsymbol{x}\to\boldsymbol{x}_o} \frac{\phi(\boldsymbol{x}) - \phi(\boldsymbol{x}_o)}{\boldsymbol{x} - \boldsymbol{x}_o}.$$

For this definition to apply, $\boldsymbol{x}$ must approach $\boldsymbol{x}_o$ along $\hat{\mathbf{n}}$. To compute the higher-order terms (HOT), we would need the Hessian matrix.[18]

[16]\small\url{https://www.fossilhunters.xyz/hydrogen-atom/werner-heisenberg-and-wolfgang-paul html}

[17]For further discussions, see Greenberg (1988, pp. 778, 791, 809).

[18]$H_{i,j} = \partial^2(\boldsymbol{x})\phi/\partial x_i \partial x_j$, which is well defined, as long as the potential is analytic in $\boldsymbol{x}$ at $\boldsymbol{x}_o$.

To define a surface $|\mathcal{S}|$ one must identify iso-potential contours. The gradient is in the direction of maximum change in the potential, thus perpendicular to the iso-potential surface. The key to the integral definition is in taking the limit. As the volume $||\mathcal{S}||$ shrinks to zero, the HOT shrinks to zero and the integral reduces to the first-order term in the Taylor expansion, since the constant term integrates to zero. Such a construction is used in the proof of the Webster horn equation (Appendix 6.3, Fig. 6.3).

One serious problem with Eq. 6.64 is that this definition is self-referencing, since $\hat{\mathbf{n}}$ is based on the gradient. In other-words, the integral definition of the gradient depends on the gradient. Equation 6.64 is similar to the mean value theorem for the gradient.

6.7.1 Divergence: $\nabla\cdot\boldsymbol{d} = \rho$ [C/m^3]

The definition of the divergence at $\boldsymbol{x} = [x, y, z]^T$ (p. 217) is

$$\nabla\cdot\boldsymbol{d}(\boldsymbol{x}, t) \equiv [\partial_x, \partial_y, \partial_z] \cdot \boldsymbol{d}(\boldsymbol{x}, t) = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right](\boldsymbol{x}, t) = \rho(\boldsymbol{x}, t),$$

which maps $\mathbb{R}^3 \underset{\nabla\cdot}{\mapsto} \mathbb{R}^1$. Here $\boldsymbol{d}(\boldsymbol{x}, t) = \epsilon_o \boldsymbol{e}(\boldsymbol{x}, t)$ is the displacement vector of Maxwell's equations and $\rho(\boldsymbol{x}, t)$ is the charge density.

6.7.2 Divergence and Gauss's law

Like the gradient, the divergence of a vector field may be defined as the surface integral of a compressible vector field in the limit as the volume enclosed by the surface goes to zero. As in the case of the gradient, if this definition is to make sense, the surface $\mathcal{S}$ must be closed, defining the volume $\mathcal{V}$. The difference here is that the surface integral is over the normal component of the vector field being operated on (Greenberg, 1988, p 762–763)

$$\nabla\cdot\boldsymbol{d} = \lim_{\mathcal{V},\mathcal{S}\to 0} \left\{ \frac{\iint_{\mathcal{S}} \boldsymbol{d}\cdot\hat{\mathbf{n}}\, d\mathcal{S}}{\mathcal{V}} \right\} = \rho(x, y, z) \quad \text{[C/m}^3\text{]}. \tag{6.66}$$

We have defined the surface with area $\mathcal{S}$, and volume $\mathcal{V}$.

As defined in (Eq. 6.65) and shown here in Fig. 6.7.2, $\hat{\mathbf{n}}$ is a unit vector normal to the closed iso-potential surface $\mathcal{S}$. The limit defines the total flux across the surface, as the volume and surface simultaneously go to zero, Thus the surface integral is a measure of the total flux perpendicular to the surface. It is helpful to compare this formula with that for the gradient, Eq. 6.64.

Gauss's law: The definitions in Fig. 6.7.2 resulted in Gauss's law, a major breakthrough in vector calculus. For the electrical case, this is equivalent to the observation that the total flux across the surface is equal to the net charge enclosed by the surface. Since the volume integral over the charge density $\rho(x, y, z)$ is the total charge enclosed Q_{enc},

$$Q_{\text{enc}} = \iiint_{\mathcal{V}} \nabla\cdot\boldsymbol{D}\, d\mathcal{V} = \iint_{\mathcal{S}} \boldsymbol{d}\cdot\hat{\boldsymbol{n}}\, d\mathcal{S} \quad \text{[C]}. \tag{6.67}$$

When the surface integral over the normal component of $\boldsymbol{d}(\boldsymbol{x})$ is zero, the total charge enclosed is zero. If there is only positive (or negative) charge inside the surface, $\nabla\cdot\boldsymbol{d} = \rho(\boldsymbol{x}) = 0$. It is clear that this result only holds in the quasi-static limit, which is always satisfied because $\mathcal{S} \to 0$.

Taking the derivative with respect to time gives the total current normal to the surface:

$$\mathcal{I}_{\text{enc}} = \iint_{\mathcal{S}} \boldsymbol{d}\cdot\hat{\boldsymbol{n}}\, d\mathcal{S} = \dot{Q}_{\text{enc}} = \iiint_{\mathcal{V}} \dot{\rho}_{\text{enc}}\, d\mathcal{V} \quad \text{[A]}. \tag{6.68}$$

As summarized by Feynman (1970b, p. 13-2):

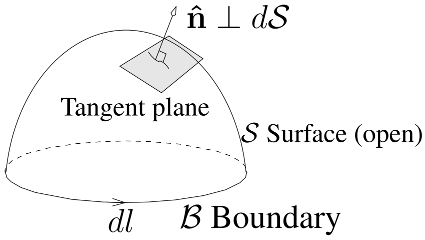
> The current leaving the closed surface $\mathcal{S}$ equals the rate of the charge leaving that volume $\mathcal{V}$, defined by that surface.

The integral definition reduces to a common-sense summary that can be grasped intuitively.

6.8 Magnetic fields $\boldsymbol{h}, \boldsymbol{b}$

Integral definition of the curl: $\nabla\times\boldsymbol{h} = \boldsymbol{c}$ The second of Maxwell's equations, the differential definition of the curl maps $\mathbb{R}^3 \underset{\nabla\times}{\mapsto} \mathbb{R}^3$, (page 218). The curl of the magnetic field strength $\boldsymbol{h}(\boldsymbol{x})$ is the current density $\boldsymbol{c} = \sigma\boldsymbol{e} + \dot{\boldsymbol{d}}$:

$$\nabla\times\boldsymbol{h} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{vmatrix} = \boldsymbol{c} \quad \text{[A/m}^2\text{]}.$$



$$\nabla\times\boldsymbol{h} \equiv \lim_{\mathcal{B},\mathcal{S}\to 0} \left\{ \frac{\iint_{\mathcal{S}} \hat{\mathbf{n}} \times \boldsymbol{h}\, d\mathcal{S}}{\mathcal{S}} \right\} \quad \text{[A/m}^2\text{]}$$

$$\mathcal{I}_{enc} = \iint_{\mathcal{S}} (\nabla\times\boldsymbol{h}) \cdot \hat{\mathbf{n}}\, d\mathcal{S} = \oint_{\mathcal{B}} \boldsymbol{h} \cdot d\boldsymbol{l} \quad \text{[A]}$$

Figure 6.6: *Left: The integral definition of the curl is related to that of the divergence (Eq. 6.66), as an integration over the tangent to the surface, except: (1) the curl is defined as the cross-product $\hat{\mathbf{n}} \times \boldsymbol{h}$ [A/m^2] of unit vector $\hat{\mathbf{n}}$ with the current density $\boldsymbol{h}$ (Greenberg, 1988, p 823), and (2) the surface is open, leaving a boundary $\mathcal{B}$ along the open edge. As with the divergence, which leads to Gauss's law, this definition leads to a second fundamental theorem of vector calculus:* Stokes's law *(also called the* curl theorem*). Right: Equations that summarize Stokes theorem (law). Compare this with Fig. 6.1.*

Curl and Stokes's law: Like the gradient and divergence, the curl may be written in integral form, allowing for a physical interpretation of its meaning. Stokes's law states that the open surface integral over the normal component of the curl of the magnetic field strength ($\hat{\mathbf{n}} \cdot \nabla\times\boldsymbol{h}$ [A/m^2]) is equal to the line integral $\oint_{\mathcal{B}} \boldsymbol{h} \cdot dl$ along the boundary $\mathcal{B}$, as summarized in Fig. 6.6. That is: *The line integral of $\boldsymbol{H}$ along the open surface's boundary $\mathcal{B}$ is equal to the total current enclosed* $\mathcal{I}_{\text{enc}}$. In many texts the normalization (denominator under the integral) is a volume $\mathcal{V}$ (Greenberg, 1988, p 778,823–4). However, because the surface is open, this volume does not exist.

Summary: Since integration is a linear process (sums of smaller elements), we can tile (tessellate) the surface, breaking it up into smaller surfaces and their boundaries, the sum of which is equal to the integral over the original boundary. This is an important concept that leads to the proof of Stokes's law.

Table 6.1 (p. 216) provides a description of the three basic integration theorems along with their mapping domains. The integral formulations of Gauss's and Stokes's laws use $\hat{\mathbf{n}} \cdot \boldsymbol{d}$ and $\boldsymbol{h} \times \hat{\mathbf{n}}$ in the integrands. The key distinction between the two laws naturally follows from the properties of the scalar ($\boldsymbol{A} \cdot \boldsymbol{B}$) and vector ($\boldsymbol{A} \times \boldsymbol{B}$) products, as discussed in Fig. 9 (p. 108). To fully appreciate the differences between Gauss's and Stokes's laws, we must master these two types of vector products.

Paraphrasing Feynman (1970b, 3-12), we have

1. $\Phi_2 = \Phi_1 + \int_1^2 \nabla\Phi \cdot d\mathcal{S}$

   The line integral of an analytic function depends only on the end points.

2. $\oint \boldsymbol{d} \cdot \hat{\mathbf{n}}\, d\mathcal{S} = \oint \nabla\cdot\boldsymbol{d}\, d\mathcal{V} = \oint \rho\, d\mathcal{V}$

   The normal component over a surface integral equal the divergence over the volume integral.

3. $\oint_{\mathcal{B}} \boldsymbol{e} \cdot d\boldsymbol{l} = \oint_{\mathcal{S}} (\nabla\times\boldsymbol{e}) \cdot \hat{\mathbf{n}}\, d\mathcal{S} = -\oint \dot{\mathcal{B}} \cdot \hat{\mathbf{n}}\, d\mathcal{S}$

   The line integral of the electric field (the induced EMF in any loop) equals the integral of the normal component of time-rate-of-change of the magnetic flux. The induced EMF is the Thévenin voltage in series with the loop. This is similar to an electrical transformer.

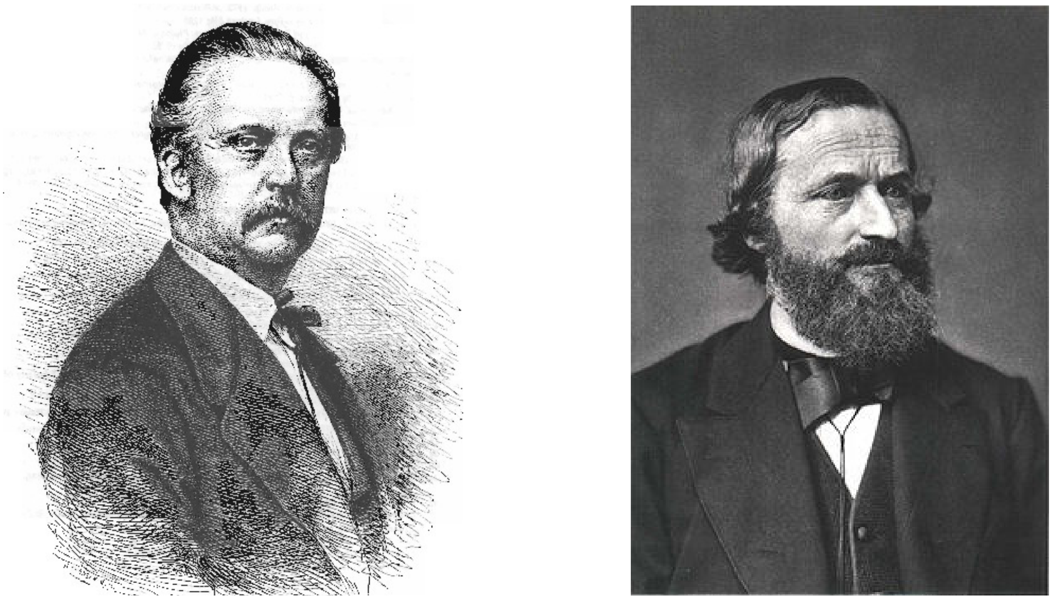


Figure 6.7: *Left: Hermann von Helmholtz (taken from Helmholtz (1978)), Right: Gustav Kirchhoff. Together they were the first to account for viscous (Helmholtz, 1858, 1978, 1863b) and thermal (Kirchhoff, 1868, 1974, English translation) losses in the acoustic propagation of airborne sound.*

6.8.1 Helmholtz's decomposition theorem

This is a good time to review everything we have defined, in terms of the two types of vector fields that decompose every analytic vector field, as shown in Table 6.3. The *irrotational field* is defined as one that is curl free. The *in-compressible field* is one that is divergence free.

Today the name for Helmholtz's decomposition theorem, because every analytic vector field may be decomposed into independent rotational and compressible components (Helmholtz, 1978). The theorem is also known as the *fundamental theorem of vector calculus* (FTVC). Gauss's and Stokes's theorems,[19] along with Helmholtz's decomposition theorem, are the three fundamental theorems of vector calculus. (p. 247).

Table 6.3: *The four possible classifications of scalar and vector potential fields: rotational/irrotational and compressible/in-compressible. Rotational fields are generated by the vector potential (e.g., $\boldsymbol{A}(\boldsymbol{x},t)$), while compressible fields are generated by the scalar potentials (e.g., voltage $\phi(\boldsymbol{x},t)$, velocity ψ, pressure $\varrho(\boldsymbol{x},t)$, or temperature $T(\boldsymbol{x},t)$).*

Field: $\nu(\boldsymbol{x},t)$	Compressible $\nabla\cdot\nu \neq 0$	In-compressible $\nabla\cdot\nu = 0$
Rotational $\nabla\times\nu \neq 0$	$\nu = \nabla\phi + \nabla\times\boldsymbol{\omega}$ Vector wave Eq. $\boldsymbol{\nabla}^2\nu = \frac{1}{c^2}\ddot{\nu}$	$\nu = \nabla\times\boldsymbol{w}$ Lubrication theory Boundary layers
Irrotational conservative $\nabla\times\nu = 0$	Acoustics $\nu = \nabla\psi$ $\nabla^2\varrho(\boldsymbol{x},t) = \frac{1}{c^2}\ddot{\varrho}(\boldsymbol{x},t)$	Statics $\nabla^2\phi = 0$ Laplace's Eq. ($c \to \infty$)

The four categories of linear fluid flow: The following is a summary of the four cases for fluid flow, as shown in Table. 6.3:

1,1 Compressible, rotational fluid (general case): $\nabla\psi \neq 0$, $\nabla\times\boldsymbol{w} \neq 0$. This is wave propagation in a medium where viscosity cannot be ignored, as in acoustics close to the boundaries, where viscosity contributes to losses (Batchelor, 1967).

[19]These theorems are mathematical relationships that follow from physical principles.

1,2 In-compressible, rotational fluid (lubrication theory): $\nu = \nabla\times\boldsymbol{w} \neq 0, \nabla\cdot\nu = 0, \nabla^2\psi = 0$. In this case, the flow is dominated by the walls, while the viscosity and heat transfer introduce shear. This is typical of lubrication theory (solenoidal fields).

2,1 Compressible, irrotational fluid (acoustics): $\nu = \nabla\psi$, $\nabla\times\boldsymbol{w} = 0$. Here losses (viscosity and thermal diffusion) are small (assumed to be zero). We can define a velocity potential ψ, the gradient of which gives the air particle velocity, thus $\nu = -\nabla\psi$. For an irrotational fluid, $\nabla\times\nu = 0$ (Greenberg, 1988, p 826). This is the case for the conservative field, where $\int \nu \cdot \hat{\mathbf{n}} dR$ depends on only the end points and $\oint_{\mathcal{B}} \nu \cdot \hat{\mathbf{n}} dR = 0$. When a fluid may be treated as having no viscosity, it is typically assumed to be irrotational, since it is viscosity and shear that lead to rotation (Greenberg, 1988, p 814). A fluid's angular velocity is $\Omega = \frac{1}{2}\nabla\times\nu = 0$; thus irrotational fluids have zero angular velocity ($\Omega = 0$).

2,2 Incompressible, irrotational fluid (statics): $\nabla\cdot\nu = 0$ and $\nabla\times\nu = 0$; thus $\nu = \nabla\psi$ and $\nabla^2\psi = 0$. An example is water in a small space at low frequencies, where the wavelength is long compared to the size of the container; the fluid may be treated as incompressible. When $\nabla\times\nu = 0$, the effects of viscosity may be ignored, as it is the viscosity that creates the shear that leads to rotation. This is the case in models of the cochlea, where fluid losses are ignored and the quasistatic limit is justified.

In summary, each of these cases is an approximation that best applies in the low-frequency limit. This is why it is called *quasi-static*, meaning small (but not zero) frequency, such that the wavelength is more than twice the diameter. A magnetic solenoidal field is a uniform-flux field $\boldsymbol{b}_z(\boldsymbol{x})$ that is generated by a solenoidal coil iis uniform inside the coil, thus similar to that of a permanent magnet. As a result, the divergence of a solenoidal field is approximately zero, which makes it incompressible ($\nabla\cdot = 0$) and rotational ($\nabla\times \neq 0$).

You need to know the term *solenoidal* since it has been widely used. The preferred terms are *incompressible* and *rotational*. Strictly speaking, the term *solenoidal field* applies to only a magnetic field produced by a solenoid.

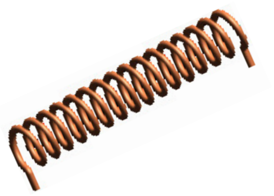


Figure 6.8: *A solenoid is a uniform coil of wire. The term comes from the Greek meaning "shaped like a pipe or channel." When a steady current is passed through the wire, a uniform magnetic field intensity $\boldsymbol{h}$ is created. Such a coil is indistinguishable from a permanent bar magnet with its north and south poles. Depending on the direction of the current, one end of a finite solenoidal coil is the north pole of the magnet, and the other end is the south pole. The uniform field inside the coil is called* solenoidal, *a synonym for* rotational *(Figure from Wikipedia).*

Application and derivation

Helmholtz's decomposition theorem is expressed as the linear sum of a scalar potential $\phi(x, y, z)$ (voltage) and a magnetic vector potential. Specifically

$$\boldsymbol{e}(\boldsymbol{x}, s) = \nabla\times\boldsymbol{a}(x, s) - \nabla\phi(\boldsymbol{x}, s) \tag{6.69}$$

$$\boldsymbol{b}(\boldsymbol{x}, s) = \nabla\times\boldsymbol{a}(\boldsymbol{x}, s) + s\boldsymbol{d} \tag{6.70}$$

where $\boldsymbol{a}$ is the vector potential, as a function of the Laplace frequency s. A similar expression in terms of potentials applies to linear fluid vector fields such as those in water and air.

In cases where rotation and dilation become coupled, this relationship may break down. In the sun the temperature could be as high as 10^6 (K°).

This theorem is easily stated and proved, but less easily appreciated (Heras, 2016). A physical description is helpful: Every vector field may be split into two independent parts: translation and rotation. We have seen this same idea in vector algebra, where the scalar and wedge products of two vectors are orthogonal.

For example, think of linear versus angular momentum, which are independent in that they represent different ways of delivering kinetic energy via different modalities. Linear and rotational motions are a common theme in physics, rooted in geometry. Thus it seems a natural extension to split a vector field into independent dilation and rotation parts.

Example: A fluid with mass and momentum can be moving along a path and independently be rotating. These independent modes of motion correspond to different types of kinetic energy (modes), such as translational, compressional, and rotational. Each eigenmode of vibration can be viewed as an independent degree of freedom.

Helmholtz's decomposition theorem (FTVC) quantifies these degrees of freedom. The second-order vector identities **DoC**: $\nabla\cdot\nabla\times()$ and **CoG**: $\nabla\times\nabla()$ may be used to verify the FTVC.

Helmholtz's decomposition theorem proof: To prove Eq. 6.70 we must understand how it splits the vector field into parts. The identities have a physical meaning: Every vector field may be split into its translational and rotational factors. If $\boldsymbol{e}$ is the electric field [V/m], ϕ is the voltage, and $\boldsymbol{a}$ is the current induced rotational part.

To do this we need the two key vector identities that are always zero for analytic fields: the curl of the gradient

$$\nabla\times\nabla\phi(\boldsymbol{x},t) = 0 \tag{6.71}$$

and the divergence of the curl[20]

$$\nabla\cdot(\nabla\times\boldsymbol{a}) = 0. \tag{6.72}$$

By applying these two identities to Helmholtz's decomposition, we can appreciate the theorem's significance. We can work backward via a physical argument that rotational momentum is independent of translational momentum. Once these forces are made clear, the vector operations take on meaning. The physics is simply related to geometry via the scalar and vector products.

For example, taking the divergence and curl of $\boldsymbol{e}$ gives

$$-\frac{1}{\epsilon_o}\rho = \nabla\cdot\boldsymbol{e} = \nabla\cdot\{-\nabla\phi + \cancelto{0}{\nabla\times\boldsymbol{a}}\} = -\nabla\cdot\nabla\phi = -\nabla^2\phi,$$

$$-\dot{\boldsymbol{b}} = \nabla\times\boldsymbol{e} = \nabla\times\{\cancelto{0}{-\nabla\phi} + \nabla\times\boldsymbol{a}\} = \nabla\times\nabla\times\boldsymbol{a} = \nabla^2\boldsymbol{a} - \boldsymbol{\nabla}^2\boldsymbol{a}.$$

6.8.2 Second-order operators

In addition to the first-order vector derivatives are second-order operators, the most important being the scalar Laplacian $\nabla^2() = \nabla\cdot\nabla()$ and vector Laplacian $\boldsymbol{\nabla}^2() = \nabla\cdot\nabla()$, which operates on vectors.[21]

Terminology

There are six second-order combinations of the DEL operator ∇, requiring six new mnemonics (Table 6.1, p. 216):

1. DoG: $(\nabla\cdot\nabla)$ Divergence of the gradient (*scalar Laplacian* operates on scalar potentials (Greenberg, 1988, p. 779)):

$$\begin{aligned}\nabla^2\phi &= (\nabla\cdot\nabla)\phi \\ &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\end{aligned} \tag{6.73}$$

2. **DoG**: (∇^2) Divergence of the Gradient (Bull-Dog, the *Laplacian* ∇^2 (Sommerfeld, 1952, p 33)):

$$\begin{aligned}\nabla^2\boldsymbol{a} &= (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{a} \\ &= \frac{\partial^2\boldsymbol{a}}{\partial x^2} + \frac{\partial^2\boldsymbol{a}}{\partial y^2} + \frac{\partial^2\boldsymbol{a}}{\partial z^2} \\ &= \nabla^2\boldsymbol{a} - \nabla\times\nabla\times\boldsymbol{a}\end{aligned} \tag{6.74}$$

3. **gOd**: (∇^2) Gradient of the Divergence ($\nabla^2\boldsymbol{a} = \nabla(\nabla\cdot\boldsymbol{a})$)

[20] Helmholtz was the first person to apply mathematics in modeling the eye and the ear (Helmholtz, 1863a).

[21] `https://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates#Non-trivial_calculation_rules`

$$
\begin{aligned}
\nabla^2 \boldsymbol{a} &= \nabla(\nabla\cdot\boldsymbol{a}) \\
&= \nabla\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right) \\
&= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\left(\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\right) \\
&= \hat{\mathbf{x}}\frac{\partial}{\partial x}\nabla\cdot A+\hat{\mathbf{y}}\frac{\partial}{\partial y}\nabla\cdot A+\hat{\mathbf{z}}\frac{\partial}{\partial z}\nabla\cdot A
\end{aligned}
\tag{6.75}
$$

4. **CoC:** ($\nabla\times\nabla\times$) Curl of the curl ($\nabla\times\nabla\times$= **gOd - DoG**) is equivalent to

$$\nabla\times\nabla\times\boldsymbol{a} = \nabla(\nabla\cdot\boldsymbol{a}) - (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{a},$$

which is a reorganized version of Eq. 6.74 (Sommerfeld, 1952, p 33, Eq. 2b).

5. DoC: Divergence of the curl ($\nabla\cdot\nabla\times() = 0$)

6. CoG: Curl of the gradient $\nabla\times\nabla() = 0$

DoC($\cdot$) and CoG($\cdot$) are special because they are always zero:

$$\nabla\cdot\nabla\times\boldsymbol{a} = 0, \qquad \nabla\times\nabla\phi = 0, \tag{6.76}$$

making them useful in proving the FTVC (Eq. 6.70, p. 248).

The third special vector identity

$$\nabla\times\nabla\times\boldsymbol{a} = \nabla^2\boldsymbol{a} - \boldsymbol{\nabla}^2\boldsymbol{a}, \tag{6.77}$$

is the difference between **gOd** and **CoC** (i.e., **DoG** = **gOd** − **CoC**):

$$\boldsymbol{\nabla}^2() = \nabla^2() - \nabla\times\nabla\times(). \tag{6.78}$$

The role of **gOd** (∇^2) is commonly ignored because it is zero for the magnetic wave equation, due to there being no magnetic charge [$\nabla\cdot\boldsymbol{b}(x,t) = 0$; thus $\nabla^2\boldsymbol{b}(x,t) \equiv 0$]. However for the electric vector wave equation it plays a role, as an important source term:

$$\nabla^2\phi(\boldsymbol{x},t) = -\nabla\boldsymbol{e}(\boldsymbol{x},t) = -\frac{1}{\epsilon_o}\nabla^2\boldsymbol{d}(x,t) = -\frac{1}{\epsilon_o}\nabla\rho(\boldsymbol{x},t),$$

or since $\nabla\cdot\boldsymbol{d} = \rho$,

$$\nabla^2\boldsymbol{d}(\boldsymbol{x},t) = \nabla\nabla\cdot\boldsymbol{d} = -\nabla\rho(\boldsymbol{x},t).$$

When the charge density is inomogeneous, such as the case of a plasma (e.g., the sun), this term plays an important role as a source term in the electric wave equation. This special case needs to be further explored through physical examples.

Exercise #*16*

Show that **DoG**: $\boldsymbol{\nabla}^2$ and **gOd**: ∇^2 differ.

Solution: Use $\nabla\times\nabla\times(\ )$ to explore this relationship. If **DoG** and **gOd** were the same, $\nabla\times\nabla\times(\ )$ would be null. An extremely interesting case, rarely discussed in the open literature, is the case of the two parameters μ, ϵ depending on time and $\boldsymbol{x}$, as in the Sun.

Exercise #*17*

What is the difference between **DoG**: ∇^2 and Bull-DoG: $\boldsymbol{\nabla}^2$?

Solution: **DoG** operates on scalar functions while Bull-DoG operates on vector functions.

Discussion: It is helpful to view these operators as playing fundamentally different roles:

utility operators $\nabla\times\nabla\times(\ )$: **DoG**: $\nabla^2(\ )$, and gOd: $\boldsymbol{\nabla}^2(\ )$,

and

identity operators DoC: $DoC(\ ) = 0$ and CoG: $\nabla\times\nabla(\ )$=0 (Eq. 6.76).

When using second-order differential operators, we must be careful with the order of operations, which can be subtle. Most of this is common sense. For example, don't operate on a scalar field with $\nabla\times$, and don't operate on a vector field with ∇.[22] **DoG** acts on on each vector component $\boldsymbol{\nabla}^2\boldsymbol{a} = \nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$ (Eq. 6.74), which is very different from the action of the Laplacian (DoG).

6.9 The unification of electricity and magnetism

Once you have mastered the three basic vector operations–gradient, divergence, and curl, we are ready to appreciate Maxwell's equations. From 1860-1865 Maxwell worked at Kings College London, on the north bank of the Thames river.[23]

While in London, James Maxwell "invented" what today are know as *Maxwell's equations*, which are without question the most important scientific invention in the last 1000 years. For example, they were the starting point for the work of Albert Einstein, known today as *AnnusMirabilis*.

Like the vector operations, these equations may be written in integral or differential form. An important difference is that with Maxwell's equations, we are dealing with well-defined physical quantities. The scalar and vector fields take on meaning and units. Thus, to understand these important equations, we must master both the names and units of the four fields $\boldsymbol{e}, \boldsymbol{h}, \boldsymbol{b}, \boldsymbol{d}$, as described in Table 6.4. Note that $\boldsymbol{j}_c$ is not $\sqrt{-1}$.

Table 6.4: *This table defines the four electro-magnetic variables that define Maxwell's equations, $\boldsymbol{e}, \boldsymbol{h}, \boldsymbol{d}, \boldsymbol{b}$. They are easily understood by noting their units. A fifth is called the conduction current (CC), define by $\boldsymbol{j}_c = \sigma E$, which is the current define by Ohm's Law. The variables of Maxwell's equations have names names shown in the table below (e.g., EF, ED, MF, MI) and units (in square brackets [SI units]). The units are necessary to obtain a full understanding of each of the four variables and their corresponding equations. For example, Eq.* EF *has units [V/m]. By integrating* $\mathbf{e}$ *from $x = a, b$, one obtains the voltage difference between the two points. The* speed of light in-vacuo *is $c = 3\times10^8 = 1/\sqrt{\mu_o\epsilon_o}$ [m/s], and the* characteristic resistance *of light is $r_o = 377 = \sqrt{\mu_o/\epsilon_o}$ [Ω] (i.e., ohms).*

Symbol	Name	Units	Maxwell's Eq.
$\boldsymbol{e}$	**EF**: Electric field strength	[V/m]	$\nabla\times\boldsymbol{e} = -\partial_t\boldsymbol{b}$
$\boldsymbol{d} = \epsilon_o\boldsymbol{e}$	**ED**: Electric displacement (flux density)	[C/m^2]	$\nabla\cdot\boldsymbol{d} = \rho$
$\boldsymbol{h}$	**MF**: Magnetic field strength	[A/m]	$\nabla\times\boldsymbol{h} = \boldsymbol{j_m} + \partial_t\boldsymbol{d}$
$\boldsymbol{b} = \mu_o\boldsymbol{h}$	**MI**: Magnetic induction (flux density)	[Wb/m^2]	$\nabla\cdot\boldsymbol{b} = 0$
$\boldsymbol{j}_c$	**CC**: conduction current	[A/m^2]	$\sigma\boldsymbol{e}$

If you wish to become an Electrical Engineer (there is an unlimited demand for well trained EEs), then the number one topic to master is ME.

Field strength $\boldsymbol{e}, \boldsymbol{h}$: As summarized by Eqs. 5.11.1, there are two field strengths: the electric $\boldsymbol{e}$ with units of [V/m] and the magnetic $\boldsymbol{h}$ with units of [A/m]. The ratio $\boldsymbol{j}_c = \boldsymbol{e}/\boldsymbol{h} = \sqrt{\mu_o/\epsilon_o} = 377$ [ohms] for in-vacuo plane-waves (μ_o, ϵ_o). It's a vector because of the direction of flow, as determined by a voltage drop. Once you grasp the concept of Ohm's Law, your there.

If two conducting plates are placed 1 [m] apart, with 1 [V] across them, the electric field is $\boldsymbol{e} = 1$ [V/m]. If a charge (i.e., an electron) is placed in an electric field, it feels a force $\boldsymbol{f} = q\boldsymbol{e}$ [N], where q is the magnitude of the charge [C].

To help us understand the meaning of $\boldsymbol{h}$, consider the solenoid made of wire, as shown in Fig. 6.8, that carries a current of I_θ [A]. The axial (along the long axis) magnetic field H_z inside such a solenoid is uniform, with a direction that depends on the polarity of I_θ, is

$$H_z = \frac{N}{L} I_\theta \quad \text{[Wb]},$$

where L is the length of the coil, N is the number of turns.

[22]This operation defines a *dyadic tensor*, which I view as a obtuse generalization of a vector.

[23]`www.google.com/maps/search/King's+College+London/`

Flux $\boldsymbol{d}, \boldsymbol{b}$: Flux is a flow, such as the mass flux of water flowing in a pipe [kg/s] driven by a force (pressure drop) across the ends of the pipe, or the heat flux in a thermal conductor, that has a temperature drop across it (i.e., a window or a wall). The flux is the same as the flow, be it charge, mass, or heat (Table 3.3.2, p. 86). In Maxwell's equations there are also two fluxes: the electric flux $\boldsymbol{d}$ and the magnetic flux $\boldsymbol{b}$. The flux density units for $\boldsymbol{d}$ is [C/m^2], and the magnetic flux density $\boldsymbol{b}$ is measured in either Webers [Wb/m^2] or its SI equivalent [tesla] [T].

6.9.1 Maxwell's equations

Maxwell's equations (ME) consist of two curl equations

$$\nabla\times\begin{bmatrix}\boldsymbol{e}(\boldsymbol{x},t)\\ \boldsymbol{h}(\boldsymbol{x},t)\end{bmatrix} = \partial_t\begin{bmatrix}-\boldsymbol{b}(\boldsymbol{x},t)\\ \boldsymbol{d}(\boldsymbol{x},t)\end{bmatrix} \tag{6.79}$$

$$= \begin{bmatrix}0 & -\mu_o\\ \epsilon_o & 0\end{bmatrix}\partial_t\begin{bmatrix}\boldsymbol{e}(\boldsymbol{x},t)\\ \boldsymbol{h}(\boldsymbol{x},t)\end{bmatrix}, \tag{6.80}$$

$$\tag{6.81}$$

and two divergence equations:

$$\nabla\cdot\begin{bmatrix}\boldsymbol{d}\\ \boldsymbol{b}\end{bmatrix} = \partial_t\begin{bmatrix}\rho(\boldsymbol{x})\\ 0\end{bmatrix}. \tag{6.82}$$

Exercise #*18*

When a static current is flowing in a wire in the $\hat{\mathbf{z}}$ direction, the magnetic flux is determined by Stokes's theorem (Fig. 6.6). Thus, just outside the wire we have

$$\mathcal{I}_{\text{enc}} = \iint_{\mathcal{S}}(\nabla\times\boldsymbol{h})\cdot\hat{\mathbf{n}}\, d|\mathcal{S}| = \oint_{\mathcal{B}}\boldsymbol{h}\cdot d\boldsymbol{l} \quad \text{[A]}.$$

For this simple geometry, the current in a wire is related to $\boldsymbol{h}(\boldsymbol{x},t)$ by

$$\mathcal{I}_{\text{enc}} = \oint_{\mathcal{B}}\boldsymbol{h}\cdot d\boldsymbol{l} = H_\phi 2\pi r,$$

where H_ϕ is perpendicular to both the radius r and the direction of the current $\hat{\mathbf{z}}$. Thus

$$\boldsymbol{h}_\phi = \frac{\mathcal{I}_{\text{enc}}}{2\pi r},$$

where $\boldsymbol{h}_\phi$ is attenuated by $1/r$ (Ramo et al., 1965, Eq. 9, page 244).

For the case of water, the material properties, depending on the state of the water (liquid, ice or vapor).

Exercise #*19*

Explain how Stokes's theorem may be applied to $\nabla\times\boldsymbol{e} = -\dot{\boldsymbol{b}}$, and explain what it means. Hint: This is the same argument given above for the current in a wire, but for the electric case.

Solution: Integrating the left side of equation Eq. 6.83 over an open surface results in a voltage (emf) induced in the loop closing the boundary $\mathcal{B}$ of the surface

$$\phi_{\text{induced}} = \iint_{\mathcal{S}}(\nabla\times\boldsymbol{e})\cdot\hat{\mathbf{n}}\, d|\mathcal{S}| = \oint_{\mathcal{B}}\boldsymbol{e}\cdot d\boldsymbol{l} \quad \text{[V]}. \tag{6.83}$$

The emf (electromagnetic force) is the same as the Thévenin source voltage induced by the rate of change of the flux. Integrating Eq. 6.83 over the same open surface $\mathcal{S}$ results in the source of the induced voltage ϕ_{induced}, which is proportional to the rate of change of the flux [weber]:

$$\phi_{\text{induced}} = -\frac{\partial}{\partial t}\iint_{\mathcal{S}}\boldsymbol{b}\cdot\hat{\mathbf{n}}\, dA = L\dot{\psi} \quad \text{[Wb/s] or [V]},$$

where L [H] is the inductance of the wire, with impedance $Z_L(s) = sL$. The area integral on the right is the total flux crossing normal to the surface ψ [Wb]. The rate of change of the total flux [Wb/s] is the induced (Thévenin) voltage [V].

As demonstrated next, applying Gauss's theorem to the divergence equations, we obtain the total flux that crosses the closed surface. Verify

Exercise #20
Apply Gauss's theorem to equation $\boldsymbol{d}$ (ED and MI are defined in Eq. 5.11.1), and explain what it means in physical terms.

5.11.1

Solution: The area of the normal component of $\boldsymbol{d}$ is equal to the volume integral over the charge density. Thus Gauss's theorem says that the total charge within the volume Q_{enc}, found by integrating the charge density $\rho(\boldsymbol{x})$ over the volume $\mathcal{V}$, is equal to the normal component of the flux $\boldsymbol{d}$ that crosses the surface $\mathcal{S}$:

$$Q_{\text{enc}} = \iiint_{\mathcal{V}} \nabla\cdot\boldsymbol{D}\, dV = \iint_{\mathcal{S}} \boldsymbol{d}\cdot\hat{\boldsymbol{n}}\, dA.$$

When equal amounts of positive and negative charge exist within the volume, regardless of its distribution, the integral is zero.

Summary: Maxwell's four equations relate the field strengths to the flux densities. There are two types of variables: field strengths $(\boldsymbol{e}, \boldsymbol{h})$ and flux densities $(\boldsymbol{d}, \boldsymbol{b})$. There are two classes: electric $(\boldsymbol{e}, \boldsymbol{d})$ and magnetic $(\boldsymbol{h}, \boldsymbol{b})$. This is a 2×2 matrix, with column being field strength and flux densities and rows being electric and magnetic variables.

	Strength	Flux density
Electric	$\boldsymbol{e}$ [V/m]	$\boldsymbol{d}$ [C/m^2]
Magnetic	$\boldsymbol{h}$ [A/m]	$\boldsymbol{b}$ [Wb/m^2]

Applying Stokes's curl theorem to the forces induces a Thévenin voltage (emf) or Norton current source. Applying Gauss's divergence theorem to the flows gives the total charge enclosed. The magnetic charge is zero ($\nabla\cdot\boldsymbol{b} = 0$) because magnetic mono-poles do not exist. However, magnetic dipoles do exist, as in the example of the electron, which contains a magnetic dipole.

6.9.2 Derivation of the vector wave equation

Next we provide the derivation of the vector wave equation starting from Maxwell's equations (Eq. 5.11.1), which is reminiscent of the derivation of the Webster horn equation (Eq. 6.9.2, p. 253). T

Here we use the same 2x2 matrix method we used in Eq 5.11.1, to both swap $\boldsymbol{b} = \mu_o\boldsymbol{h}$ and $\boldsymbol{d} = \epsilon_o\boldsymbol{e}$ and transform $\partial_t \leftrightarrow s$, due to the Laplace transform relation $\partial_t \leftrightarrow s = \sigma + \jmath\omega$. Working in the frequency domain and taking the curl of both sides gives

$$\begin{aligned}
\nabla\times\nabla\times \begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{h}\end{bmatrix}(\boldsymbol{x},t) &= \begin{bmatrix}0 & -s\mu_o\\ s\epsilon_o & 0\end{bmatrix}\nabla\times\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{h}\end{bmatrix}(\boldsymbol{x},t)\\
&= \begin{bmatrix}0 & -s\mu_o\\ s\epsilon_o & 0\end{bmatrix}\begin{bmatrix}0 & -s\mu_o\\ s\epsilon_o & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{h}\end{bmatrix}\\
&= -\frac{s^2}{c_o^2}\begin{bmatrix}\boldsymbol{e}\\ \boldsymbol{h}\end{bmatrix}.
\end{aligned}$$

Note that $s\mu_o s\epsilon_o = s^2/c_o^2$. If μ and ϵ were either functions of space or time, this transformation would not apply.

We must deal with the left hand side, which is $\nabla\times\nabla\times(\,) = \nabla^2(\,) - \boldsymbol{\nabla}^2(\,)$, which results in

$$\boldsymbol{\nabla}^2 \begin{bmatrix}\mathbf{e}\\ \mathbf{h}\end{bmatrix} - \nabla^2 \begin{bmatrix}\mathbf{e}\\ \mathbf{h}\end{bmatrix} = \frac{s^2}{c_o^2}\begin{bmatrix}\mathbf{e}\\ \mathbf{h}\end{bmatrix}$$

Rearranging terms results in driven vector wave equations, for $\boldsymbol{e}(\boldsymbol{x},t)$ and $\boldsymbol{h}(\boldsymbol{x},t)$

$$\begin{aligned}\boldsymbol{\nabla}^2 \begin{bmatrix}\mathbf{e}\\ \mathbf{h}\end{bmatrix} - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\begin{bmatrix}\mathbf{e}\\ \mathbf{h}\end{bmatrix} &= \nabla \begin{bmatrix}\frac{1}{\epsilon_o}\nabla\cdot\mathbf{d}\\ \frac{1}{\mu_o}\cancel{\nabla\cdot\mathbf{b}}^{0}\end{bmatrix}\\ &= \frac{1}{\epsilon_o}\begin{bmatrix}\nabla\rho(\boldsymbol{x},s)\\ 0\end{bmatrix},\end{aligned} \tag{6.84}$$

with the charge term $\nabla\rho(\boldsymbol{x},s)$ being the sole source of excitation. Note that if μ and ϵ depended on $\boldsymbol{x}$, the terms on the right would not be zero, thus $c(x,t) < c_o$ must be time dependent.

In remote outer space, where black holes and hot plasmas frequently exist (e.g., inside the sun), this seems likely.

Recall the d'Alembert solutions of the scalar wave equation (Eq. 4.20, p. 155)

$$\boldsymbol{e}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x}-ct) + \boldsymbol{g}(\boldsymbol{x}+ct),$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are arbitrary vector fields and $c < c_o$ is the wave front velocity.

Poynting vector: The EM power flux density $\boldsymbol{\mathcal{P}}$ [W/m^2] is perpendicular to $\boldsymbol{e}$ and $\boldsymbol{h}$,

$$\boldsymbol{\mathcal{P}} = \boldsymbol{e}\times\boldsymbol{h} \quad [\text{W/m}^2].$$

The corresponding EM momentum flux density $\boldsymbol{\mathcal{M}}$ (hence ME are related to mass, thus gravity) is

$$\boldsymbol{\mathcal{M}} = \boldsymbol{d}\times\boldsymbol{b} = \epsilon_o\mu_o\boldsymbol{e}\times\boldsymbol{h}.$$

Since the speed of light is $c_o = 1/\sqrt{\mu_o\epsilon_o}$, we see that

$$\boldsymbol{\mathcal{P}} = c_o^2\boldsymbol{\mathcal{M}} \quad [\text{W/m}^2],$$

which is related to Einstein's *theory of relativity* and his *energy–mass equivalence* $\mathcal{E} = mc^2$, as derived on p. 90, in a complex–analytic (i.e, causal) form (Sommerfeld, 1952).

For example, the power emitted by the sun is about 1360 [W/m^2], with a radiation pressure of 4 [µN/m^2] (i.e., 4 [µPa]) (Fitzpatrick, 2008). By way of comparison, the threshold audible acoustic pressure at the human eardrum at 1 [kHz] is 20 [µPa], which is 14 [dB] (a factor of 5) greater than the solar radiation pressure.

A relevant quote is:

> *The lasers used in Inertial Confinement Fusion (e.g., the NOVA experiment in Lawrence Livermore National Laboratory) typically have energy fluxes of 10^{18} [W/m^2]. This translates to a radiation pressure of about 10^4 atmospheres!*
>
> –*Fitzpatrick (2008, p. 291)* One [atm] is 10^5 [Pa].

Electrical impedance seen by an electron: Up to now we have considered only the Brune impedance, which is a special case, having no branch points or branch cuts. We can define an *impedance* for diffusion, as in the diffusion of heat. There is also the diffusion of electrical and magnetic fields at the surface of a conductor, where the resistance of the conductor dominates the dielectric properties. This is called the electrical *skin effect*, where the conduction currents are dominated by the conductivity of the metal rather than the displacement currents. In such cases, the impedance is proportional to $\sqrt{s}$, which requires that it has a branch cut. Still, the real part of the impedance must be positive in the right s half-plane, the required condition of all impedances, such that Postulate P3 (p. 126) is satisfied. The same effect is observed in acoustics (see Appendix 3.10).

The Lorentz force

$$\boldsymbol{f} = q\boldsymbol{e} + q\nu\times\boldsymbol{b} = q\boldsymbol{e} + \boldsymbol{c}\times\boldsymbol{b},$$

which is the force on a charge (e.g., electron) due to the electric $\boldsymbol{e}$ and magnetic $\boldsymbol{b}$ fields. This is related to the Lorentz transformation γ. The magnetic field plays a role when the charge has a velocity ν. When a charge is moving with velocity ν, it may be viewed as a current $\boldsymbol{c} = q\nu$ (see the discussion on p. 125). These results ignore gravity and relativistic effects.

The complex admittance density is (Feynman, 1970b, p 13–1)

$$Y(s) = \sigma_o + s\epsilon_o \quad [\mho/\mathrm{m}^2].$$

Here σ_o is the electrical conductivity, and ϵ_o is the electrical permittivity. Since $\omega\epsilon_o \ll \sigma_o$, the conductivity reduces to the resistance of the wire per unit length.[24]

6.10 Potential solutions of Maxwell's equations

One primary reason for using potentials is to generate solutions to Maxwell's equations. For example, if we extend Eq. 6.1 (p. 215), we can express Maxwell's equations in terms of scalar and vector potentials. These relationships are (Sommerfeld (1952, p. 146), Feynman (1970d, p. 18-10)):

$$\boldsymbol{e}(\boldsymbol{x},t) = -\nabla\phi(\boldsymbol{x},t) - \frac{\partial \boldsymbol{a}(\boldsymbol{x},t)}{\partial t} \quad [\mathrm{V/m}] \tag{6.85}$$

and

$$\boldsymbol{h}(\boldsymbol{x},t) = \frac{1}{\mu_o}\left[\nabla\times\boldsymbol{a}(\boldsymbol{x},t) + \frac{\partial \boldsymbol{d}(\boldsymbol{x},t)}{\partial t}\right]. \quad [\mathrm{A/m}] \tag{6.86}$$

Here we have extended $\boldsymbol{h}(\boldsymbol{x},t)$ to include the electric potential term

$$\boldsymbol{d}(\boldsymbol{x},t) = \epsilon(\boldsymbol{x},t)\boldsymbol{e}(\boldsymbol{x},t) = -\epsilon(\boldsymbol{x},t)\nabla\phi(\boldsymbol{x},t),$$

normally taken to be zero because taking the curl of $\boldsymbol{h}(t)$ naturally removes any electrical potential term due to CoG.[25] The extension makes the potential solutions symmetric so that $\boldsymbol{e}$ and $\boldsymbol{h}$ each have electrical and magnetic excitation.

Exercise #*21*

Explain why some dependence on $\phi(\boldsymbol{x},t)$ do not appear in Eq. 6.86 but do in 6.85.

Solution: For $\boldsymbol{h}(\boldsymbol{x},t)$ to depend on $\phi(x,t)$ it must appear through the electric strength, as $\boldsymbol{e}(x,t) = -\nabla\Phi(x,t)$. But then $\nabla\times\boldsymbol{h}(x,t)$ would mean applying CoG (i.e., $\nabla\times\nabla\phi = 0$) on the right side of the equation. Since this term would be zero; it is assumed to be zero, thus $\boldsymbol{h}(x,t)$ dependents on only $\boldsymbol{a}(x,t)$. To fill out the symmetry, we have added $\partial_t\boldsymbol{d}(\boldsymbol{x},t)$ to Eq. 6.86, to see what might happen in a more general case.

Use of Helmholtz's theorem on potential solutions: The generalized solutions to Maxwell's equations (Eqs. 6.85 and 6.86) have been expressed in terms of EM potentials $\phi(\boldsymbol{x})$ and $\boldsymbol{a}(\boldsymbol{x})$ and Helmholtz's theorem. These are solutions to Maxwell's equations, expressed in terms of the potentials $\phi(\boldsymbol{x},s)$ and $\boldsymbol{a}(\boldsymbol{x},s)$, as determined at the boundaries (Sommerfeld, 1952, p 146). These relationships are invariant to certain functions added to each potential, as shown below. They are equivalent to Maxwell's equations following the application of $\nabla\cdot$ and $\nabla\times$.

Finally we show that the potential equations are consistent with Maxwell's equations.

---

[24] For copper $\omega \ll \omega_c = \sigma_o/\epsilon_o \approx 6\times10^7/9\times10^{-12} \approx 6.66\times10^{18}$ [rad/s], or $f_c = 10^{18}$ [Hz]. This corresponds to a wavelength of $\lambda_o \approx c_o/f_c = 0.30\times10^{-9}$ [m]. For comparison, the Bohr radius (hydrogen) is $\approx 0.053\times10^{-9}$ [m] (5.66 times smaller) and the Lorentz radius (of the electron) is estimated to be $\mu_o = 2.8\times10^{-15}$ [m] (2.8 [femtometers]).

[25] In-vacuo $\epsilon_o = 8.85\times10^{-12}$ [F/m^2] is the capacitance, and $s\epsilon_o$ is the electric compliance-density of light. The related magnetic mass-density is the permeability $\mu_o = 4\pi\times10^{-7}$ [H/m^2] having an inductive impedance of $s\mu_o$ [Ω/m]. It is helpful to think of ϵ_o as a capacitance per unit area and μ_o as an inductance per unit area (consistent with their units). The in-vacuo speed of light is $c_o = 1/\sqrt{\epsilon_o\mu_o} = 3\times10^8$ [m/s], but is slower when traveling in matter (Brillouin, 1960).

ME for $\boldsymbol{e}(\boldsymbol{x},t)$: Taking the curl of Eq. 6.85, applying CoG $= 0$, and using Eq. 6.86,

$$\begin{aligned} \nabla\times\boldsymbol{e} &= -\cancelto{0}{\nabla\times\nabla\Phi} - \nabla\times\frac{\partial\boldsymbol{a}}{\partial t} \\ &= -\frac{\partial\boldsymbol{b}}{\partial t} \end{aligned} \tag{6.87}$$

which recovers Maxwell's equation for $\boldsymbol{e}(\boldsymbol{x})$ (Eq. 5.11.1).

Taking the divergence of Eq. 6.86 and applying DoC $= 0$ give Eq. 6.82 for $\boldsymbol{b}(\boldsymbol{x})$:

$$\nabla\cdot\boldsymbol{b}(\boldsymbol{x}) = \cancelto{0}{\nabla\cdot\nabla\times\boldsymbol{a}(\boldsymbol{x})} = 0.$$

ME for $\boldsymbol{h}(\boldsymbol{x},t)$: To recover Maxwell's equation for $\boldsymbol{h}(\boldsymbol{x})$ (Eq. 5.11.1, $\nabla\times\boldsymbol{h} = \boldsymbol{c}$) from the potential equation (Eq. 6.86), we take the curl and use $\boldsymbol{b} = \epsilon_o\boldsymbol{h}$ (Table 6.4, p. 251):

$$\begin{aligned} \nabla\times\boldsymbol{b}(\boldsymbol{x}) &= \mu_o\nabla\times\boldsymbol{h}(\boldsymbol{x}) \\ &= \nabla\times\nabla\times\boldsymbol{a}(\boldsymbol{x}) \\ &= \boldsymbol{\nabla}^2\,\boldsymbol{a}(\boldsymbol{x},t) - \boldsymbol{\nabla}^2\boldsymbol{a}(\boldsymbol{x},t) \\ &= \nabla\nabla\cdot\boldsymbol{a}(\boldsymbol{x},t) - \frac{1}{c_o^2}\frac{\partial^2}{\partial_t^2}\boldsymbol{a}(\boldsymbol{x},t) \\ &= -\frac{1}{c^2}\left(\ddot{\boldsymbol{a}} + \nabla\dot{\Phi}\right) + \mu_o\mathbf{J}. \end{aligned}$$

This last equation may be split into two independent equations by the use of Helmholtz's theorem:

$$\boldsymbol{\nabla}^2\boldsymbol{a} - \frac{1}{c_o^2}\ddot{\boldsymbol{a}} = -\mu_o\mathbf{J} \quad \text{and} \quad \nabla\cdot\boldsymbol{a} + \frac{1}{c_o^2}\ddot{\Phi} = 0.$$

Taking the divergence of Eq. 6.86 and applying DoC = 0 gives Eq. 6.82 ($\nabla\cdot\boldsymbol{d} = -\dot{\rho}$). Alternatively,

$$\nabla^2\Phi - \frac{1}{c_o^2}\Phi = -\frac{\rho}{\epsilon_o},$$

which is the scalar potential wave equation driven by the charge (Sommerfeld, 1952, p 146).

Summary: In conclusion, Eq. 6.85, along with DoC $= 0$ and CoG $= 0$, gives Maxwell's Eqs. 5.11.1 and 6.82 for $\boldsymbol{e}$. Likewise, Eq. 6.86, along with DoC $= 0$ and CoG $= 0$, gives Maxwell's Eqs. 5.11.1 and 6.82 for $\boldsymbol{h}$. The above derivation for $H(x,t)$ from A and Φ derives the magnetic component of the field, expressed in terms of its vector potential, in the same way as Eq. 5.11.1 describes $\boldsymbol{e}(x,t)$ in terms of the potentials.

Exercise #*22*

Starting with the values of the speed of light $c_o = 3\times10^8$ [m/s] and the characteristic resistance of light waves $r_o = 377$ [ohms], use the formulas $c_0 = 1/\sqrt{\mu_o\epsilon_o}$ and $r_o = \sqrt{\mu_o/\epsilon_o}$ to find values for ϵ_o and μ_o.

Solution: Squaring $c_o^2 = 1/\mu_o\epsilon_o$ and $r_o^2 = \mu_o/\epsilon_o$, we may solve for the two unknowns: $c_o^2r_o^2 = \frac{1}{\cancel{\mu_o}\epsilon_o}\frac{\cancel{\mu_o}}{\epsilon_o} = 1/\epsilon_o^2$; thus $\epsilon_o = 1/c_or_o = 10^{-8}/2.998\cdot377 \approx 8.85\times10^{-12}$ [F/m]. Likewise, $\mu_o = r_o/c_o = (377/2.998)\times10^{-8} \approx 125.75\times10^{-8}$ [H/m]. The value of μ_o is defined in international SI units as the constant $4\pi.\times10^{-7} \approx 12.566\times10^{-7}$ [H/m].

In conclusion, it is more productive to memorize c_0 and r_0, from which ϵ_o and μ_o may be easily derived.

Exercise #*23*

Starting from Eq. 5.11.1, with $\boldsymbol{j_m} = \sigma_o\boldsymbol{e}$, Maxwell's equation, including the magnetic intensity, is

$$\nabla\times\boldsymbol{h}(\boldsymbol{x},t) = \boldsymbol{j_m}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\boldsymbol{d}(\boldsymbol{x},t).$$

Find the equation for the magnetically induced current $\boldsymbol{j_m}(\boldsymbol{x},t)$.

Solution: The divergence of the curl is zero (DoC = 0),

$$\cancelto{0}{\nabla\cdot\nabla\times\boldsymbol{h}}(\boldsymbol{x},t) = \nabla\cdot\boldsymbol{j_m}(\boldsymbol{x},t) + \frac{\partial}{\partial t}\rho(\boldsymbol{x},t) = 0, \tag{6.88}$$

which is conservation of charge (i.e., Gauss's theorem).

6.10.1 Partial differential equations (PDEs): Wave equation

Problem # *64: Solve the wave equation in one dimension by defining $\xi = t \mp x/c$.*

– 64.1: Show that d'Alembert's solution, $\varrho(x,t) = f(t-x/c) + g(t+x/c)$, is a solution to the acoustic pressure wave equation in one dimension:

$$\frac{\partial^2\varrho(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\varrho(x,t)}{\partial t^2},$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions. **Solution:**

$$\frac{\partial}{\partial x}\varrho(x,t) = \frac{\partial}{\partial x}f(t-x/c) + \frac{\partial}{\partial x}g(t+x/c) = \frac{-1}{c}f'(t-x/c) + \frac{1}{c}g'(t+x/c) \tag{6.89}$$

$$\frac{\partial^2}{\partial x^2}\varrho(x,t) = \frac{\partial^2}{\partial x^2}f(t-x/c) + \frac{\partial^2}{\partial x^2}g(t+x/c) = \frac{1}{c^2}f''(t-x/c) + \frac{1}{c^2}g''(t+x/c) \tag{6.90}$$

$$\frac{\partial^2}{\partial t^2}\varrho(x,t) = \frac{\partial^2}{\partial t^2}f(t-x/c) + \frac{\partial^2}{\partial t^2}g(t+x/c) = f''(t-x/c) + g''(t+x/c) \tag{6.91}$$

Problem # *65: Solving the wave equation in spherical coordinates (i.e., three dimensions)*

– 65.1: Write the wave equation in spherical coordinates $\varrho(r,\theta,\phi,t)$. Consider only the radial term r (i.e., dependence on angles θ and ϕ is assumed to be zero). Hint: The form of the Laplacian as a function of the number of dimensions is given in Eq. 6.9 (page 219). **Solution:** Given the formula for the Laplacian in spherical coordinates, the wave equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\varrho(r,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varrho(r,t)$$

– 65.2: Show that

$$\nabla_r^2\varrho(r) \equiv \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\varrho(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\varrho(r). \tag{6.92}$$

Hint: Expand both sides of the equation. **Solution:** Both sides of the equation expand to

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}$$

– 65.3: Use the results from Eq. 6.92 to show that the solution to the spherical wave equation is

$$\nabla_r^2\,\varrho(r,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,\varrho(r,t) \tag{6.93}$$

$$\varrho(r,t) = \frac{f(t-r/c)}{r} + \frac{g(t+r/c)}{r}. \tag{6.94}$$

Solution: This proceed exactly as in the rectangular case (see above) except one must first recognize that the Laplacian in spherical coordinates may be written as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} r\varrho(r). \tag{6.95}$$

One then may proceed to use the solution for the rectangular case, but for $r\varrho(r)$, and then divide that solution by r.

– 65.4: Using $f(\xi) = \sin(\xi)u(\xi)$ and $g(\xi) = e^{\xi}u(\xi)$, write the solutions to the spherical wave equation, where $u(\xi)$ is the Heaviside step function.

Solution: In each case replace $\xi = t - x/c$ to obtain the solution to the wave equation for 1 dimensional waves. Thus

$$\begin{aligned}\varrho(r,t) &= \frac{f(t-r/c)}{r} + \frac{g(t+r/c)}{r} \\ &= \frac{\sin(t-r/c)}{t-r/c}u(t-r/c) + \frac{e^{(t+r/c)}u(t+r/c)}{t+r/c}\end{aligned}$$

– 65.5: Sketch this $f(\xi)$ and $g(\xi)$ for several times (e.g., 0, 1, and 2 seconds), and describe the behavior of the pressure $\varrho(r,t)$ as a function of time t and radius r.

Solution: Plot the functions at several times (e.g., 0, 1 2 seconds), as a function of x. The first function becomes smaller as the radius grows. The second function becomes larger as the inbound waves approaches $r = 0$.

– 65.6: What happens when the inbound wave reaches the center at $r = 0$?

Solution: Stand back. It blows up. The equations fail when the solution becomes so large that the linearity assumption fails. I'm not sure what actually happens, in practice. This seems to be how they detonate nuclear weapons.

6.10.2 Helmholtz's formula

Every differentiable vector field may be written as the sum of a scalar potential ϕ and a vector potential $\boldsymbol{w}$. This relationship is best known as the fundamental theorem of vector calculus (also called Helmholtz's formula):

$$\boldsymbol{v} = -\nabla\phi + \nabla\times\boldsymbol{w}. \tag{6.96}$$

This formula seems to be a natural extension of the algebraic products $\boldsymbol{a}\cdot\boldsymbol{b} \perp \boldsymbol{a}\times\boldsymbol{b}$, since $\boldsymbol{a}\cdot\boldsymbol{b} \propto \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos(\theta)$ and $\boldsymbol{a}\times\boldsymbol{b} \propto \|\boldsymbol{a}\|\|\boldsymbol{b}\|\sin(\theta)$, as developed in Appendix 3.4.1, page 97. Thus these orthogonal components have magnitude 1 when we take the norm, due to Euler's identity ($\cos^2(\theta) + \sin^2(\theta) = 1$).

As shown in Table 6.1 (p. 216), Helmholtz's formula separates a vector field (i.e., $\mathbf{v}(\mathbf{x})$) into compressible and rotational parts:

1. The rotational (e.g., angular) part is defined by the vector potential $\boldsymbol{w}$, which requires that $\nabla\times\nabla\times\boldsymbol{w} \neq 0$. A field is irrotational (conservative) when $\nabla\times\boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only a scalar potential, $\boldsymbol{v} = \nabla\phi$ (note that this is how a conservative field is usually defined, by saying there exists some ϕ such that $\boldsymbol{v} = \nabla\phi$).[26]

2. The compressible (e.g., radial) part of a field is defined by the scalar potential ϕ, which requires that $\nabla\cdot\nabla\phi = \nabla^2\phi \neq 0$. A field is incompressible (solenoidal) when $\nabla\cdot\boldsymbol{v} = 0$, meaning that the field $\boldsymbol{v}$ can be generated using only a vector potential, $\boldsymbol{v} = \nabla\times\boldsymbol{w}$.

The definitions and generating potential functions of irrotational (conservative) and incompressible (solenoidal) fields naturally follow from two key vector identities: (1) $\nabla\cdot(\nabla\times\boldsymbol{w}) = 0$ and (2) $\nabla\times(\nabla\phi) = 0$.

[26] A note about the relationship between the generating function and the test: You might imagine special cases where $\nabla\times\boldsymbol{w} \neq 0$ but $\nabla\times\nabla\times\boldsymbol{w} = 0$ (or $\nabla\phi \neq 0$ but $\nabla^2\phi = 0$). In these cases, the vector (or scalar) potential can be recast as a scalar (or vector) potential. For example, consider a field $\boldsymbol{v} = \nabla\phi_0 + \boldsymbol{b}$, where $\boldsymbol{b} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Note that $\boldsymbol{b}$ can actually be generated by either a scalar potential ($\phi_1 = \frac{1}{2}[x^2+y^2+z^2]$, such that $\nabla\phi_1 = \boldsymbol{b}$) or a vector potential ($\boldsymbol{w}_0 = \frac{1}{2}[z^2\hat{\mathbf{x}}+x^2\hat{\mathbf{y}}+y^2\hat{\mathbf{z}}]$, such that $\nabla\times\boldsymbol{w}_0 = \boldsymbol{b}$). We find that $\nabla\times\boldsymbol{v} = 0$; therefore $\boldsymbol{v}$ must be irrotational. We say this irrotational field is generated by $\nabla\phi = \nabla(\phi_0 + \phi_1)$.

Problem # *66: Define the following:*

– *66.1: A conservative vector field*
Solution: A conservative vector field is defined as the gradient of a scalar potential $\vec{v} = \nabla\phi(x, y, z)$. Every conservative field is necessarily *irrotational* (the test for an irrotational field is $\nabla \times \vec{v} = 0$).

– *66.2: An irrotational vector field*
Solution: The vector field $\vec{v}$ is rotational if there exists a vector potential $\vec{w}$ such that $\vec{v} = \nabla \times \vec{w}(x, y, z)$. The for *irrotational* is $\nabla \times \vec{v} = 0$. A purely rotational field is not conservative.

– *66.3: An incompressible vector field*
Solution: A field $\vec{v}$ is incompressible if $\nabla \cdot \vec{v} = 0$.

– *66.4: A solenoidal vector field*
Solution: A rotational field is one having a divergence of zero, i.e., $\nabla \cdot \vec{v} = 0$, or alternatively, $\vec{v} \equiv \nabla \times \vec{w}(x, y, z)$, since any field defined by a curl is rotational, since the divergence of the curl is always zero.

– *66.5: When is a conservative field irrotational?*
Solution: Always!

– *66.6: When is an incompressible field irrotational?*
Solution: A field is incompressible if $\nabla \cdot v = 0$ and irrotational if $\nabla \times v = 0$. So, almost never. The only case is the trivial solution $\vec{v} = 0$, or a constant field $\vec{v} = x_0\hat{\mathbf{x}} + y_0\hat{\mathbf{y}} + z_0\hat{\mathbf{z}}$.

Problem # *67: For each of the following, (i) compute $\nabla \cdot \boldsymbol{v}$, (ii) compute $\nabla \times \boldsymbol{v}$, and (iii) classify the vector field (e.g., conservative, irrotational, incompressible, etc.).*

– *67.1:* $\boldsymbol{v}(x, y, z) = -\nabla(3yx^3 + y\log(xy))$
Solution: The field is <u>conservative</u> (or <u>irrotational</u>) because it is defined by a gradient. To test for irrotational, show that the curl is zero. But $\nabla \times \nabla\phi(x, y, z) = 0$ for any $\phi(x, y, z)$. Thus you do not need to do any computation, just state the answer.

– *67.2:* $\boldsymbol{v}(x, y, z) = xy\hat{\mathbf{x}} - z\hat{\mathbf{y}} + f(z)\hat{\mathbf{z}}$
Solution: To test for a irrotational field, take the curl, to see if it is zero:

$$\nabla \times \vec{v} \equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ xy & -z & f(z) \end{vmatrix} = \hat{\mathbf{x}} - x\hat{\mathbf{z}}, \tag{6.97}$$

which is not zero. We can also see by inspection that $\nabla \cdot \vec{v} \neq 0$. Thus the vector field is <u>rotational</u> and <u>compressible</u>.

– *67.3:* $\boldsymbol{v}(x, y, z) = \nabla \times (x\hat{\mathbf{x}} - z\hat{\mathbf{y}})$
Solution: $\vec{v} = \hat{\mathbf{x}}$. Therefore, $\nabla \times \vec{v} = 0$, and $\nabla \cdot \vec{v} = 0$. This field is technically <u>incompressible</u> and <u>irrotational</u>, but it is also very boring, since it is a constant.

6.11 Suggested readings

The above concepts come straight from mathematical physics, as developed in the 17th–19th centuries. Much of this was first developed in acoustics by Helmholtz, Stokes, and Rayleigh, following through Green's footsteps, as described by Rayleigh (1896). When it comes to fully appreciating Green's theorem and reciprocity, I suggest the following readings, in no particular order, with Rayleigh (1896) as the primary key reference. The specific order depends on your present background in either math or physics. To repeat my reading experience, start with Brillouin (1953, 1960), followed by Sommerfeld (1952) and Pipes (1958).

Second-tier reading contains many items: Morse (1948); Sommerfeld (1949); Morse and Feshbach (1953); Ramo et al. (1965); Feynman (1970a); Boas (1987).

A third tier might include Helmholtz (1863a); Fry (1928); Lamb (1932); Bode (1945); Montgomery et al. (1948); Beranek (1954); Fagen (1975); Lighthill (1978); Hunt (1952); Olson (1947). Other physics writings include the series of mathematical-physics text books by authors J.C. Slater, and Landau and Lifshitz.[27] Hunt's book outlines a detailed history of loudspeakers, with deep insight into acoustics.

Successful reading of these books critically depends on what you already know. It assumes a rudimentary (high school) level math has been mastered. You must enter the reading list at a level that assumes you to understand the previous material. Read in the order that helps you best understand the material. Pick topics you are interested in, you will make progress faster. An example that almost everyone likes is acoustics, such as loudspeakers and microphones. Electronics is a good match for these acoustic electro-mechanical gadgets.

[27] https://www.amazon.com/Mechanics-Third-Course-Theoretical-Physics/dp/0750628960

Appendix A

Eigenanalysis

Eigenanalysis is ubiquitous in engineering applications. It is useful in solving differential and difference equations, data-science applications, numerical approximation and computing, and linear algebra applications. Typically one must take a course in linear algebra to become knowledgeable in the inner workings of this method. In this appendix we intend to provide sufficient basics to allow one to read the text.

A.1 The eigenvalue matrix (Λ)

Given 2×2 matrix $\boldsymbol{a}$, the related matrix eigen-equation is

$$\boldsymbol{a}\boldsymbol{e} = \boldsymbol{e}\Lambda. \tag{A.1}$$

Pre-multiplying by $\boldsymbol{e}^{-1}$ diagonalizes $\boldsymbol{a}$, resulting in the *eigenvalue matrix*

$$\Lambda = \boldsymbol{E}^{-1}\boldsymbol{a}\boldsymbol{e} \tag{A.2}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{A.3}$$

Post-multiplying by $\boldsymbol{e}^{-1}$ recovers $\boldsymbol{a}$

$$\boldsymbol{a} = \boldsymbol{e}\Lambda\boldsymbol{E}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{A.4}$$

Matrix product formula:

This last relation is the entire point of the eigenvector analysis, since it shows that any power of $\boldsymbol{a}$ may be computed from powers of the eigenvalues. Specifically,

$$\boldsymbol{a}^n = \boldsymbol{e}\Lambda^n\boldsymbol{E}^{-1}. \tag{A.5}$$

For example, $\boldsymbol{a}^2 = \boldsymbol{a}\boldsymbol{a} = \boldsymbol{e}\Lambda\, \cancelto{1}{(\boldsymbol{E}^{-1}\boldsymbol{e})}\, \Lambda\boldsymbol{E}^{-1} = \boldsymbol{e}\Lambda^2\boldsymbol{E}^{-1}$.

Equations A.1, A.3 and A.4 are the key to eigenvector analysis, and you need to memorize them. You will use them repeatedly throughout this text.

$\boldsymbol{a} - \lambda_\pm \boldsymbol{I}_2$ is singular:

If we restrict Eq. A.1 to a single eigenvector (one of $\boldsymbol{e}_\pm$), along with the corresponding eigenvalue $\lambda_\pm$, we obtain the two matrix equations

$$\boldsymbol{a}\boldsymbol{e}_\pm = \boldsymbol{e}_\pm\lambda_\pm = \lambda_\pm\boldsymbol{e}_\pm.$$

Note the swap in the order of $\boldsymbol{e}_\pm$ and $\lambda_\pm$. Since $\lambda_\pm$ is a scalar, this is legal (and critically important), since this allows us to factor out $\boldsymbol{e}_\pm$

$$(\boldsymbol{a} - \lambda_\pm\boldsymbol{I}_2)\boldsymbol{e}_\pm = 0. \tag{A.6}$$

The matrix $\boldsymbol{a} - \lambda_\pm\boldsymbol{I}_2$ must be singular because when it operates on $\boldsymbol{e}_\pm$, having nonzero norm, it must be zero.

It follows that its determinant (i.e., $|(\boldsymbol{a} - \lambda_\pm\boldsymbol{I}_2)| = 0$) must be zero. This equation uniquely determines the eigenvalues $\lambda_\pm$.

A.1.1 Calculating the eigenvalues $\lambda_\pm$

The eigenvalues $\lambda_\pm$ of $\boldsymbol{a}$ may be determined from $|(\boldsymbol{a}-\lambda_\pm \boldsymbol{I}_2)| = 0$. As an example we let $\boldsymbol{a}$ be Pell's equation (Eq. 2.23, p. 52). In this case the eigenvalues may be found from

$$\begin{vmatrix} 1-\lambda_\pm & N \\ 1 & 1-\lambda_\pm \end{vmatrix} = (1-\lambda_\pm)^2 - N = 0,$$

thus $\lambda_\pm = (1 \mp \sqrt{N})$.[1]

A.1.2 Calculating the eigenvectors $\boldsymbol{e}_\pm$

Once the eigenvalues have been determined, they are substituted into Eq. A.6, which determines the eigenvectors $\boldsymbol{e} = [\boldsymbol{e}_+, \boldsymbol{e}_-]$, by solving

$$(\boldsymbol{a} - \lambda_\pm)\mathbf{e}_\pm = \begin{bmatrix} 1-\lambda_\pm & 2 \\ 1 & 1-\lambda_\pm \end{bmatrix} \boldsymbol{e}_\pm = 0, \tag{A.7}$$

where $1-\lambda_\pm = 1-(1 \mp \sqrt{N}) = \pm\sqrt{N}$, thus the Pell equation eigenvalues are

$$\lambda_\pm = 1 \mp \sqrt{N}.$$

Recall that Eq. A.6 is singular because we are using an eigenvalue, and each eigenvector is pointing in a unique direction (this is why it is singular). You might expect that this equation has no solution. In some sense you would be correct. When we solve for $\boldsymbol{e}_\pm$, the two equations defined by Eq. A.6 are *co-linear* (the two equations describe parallel lines so their scalar product is one). This follows from the fact that there is only one eigenvector for each eigenvalue.

Since there is only one eigenvalue we are expecting trouble, yet we may proceeding to solve for $\boldsymbol{e}_+ = [e_1^+, e_2^+]^T$ with eigenvalue $+\sqrt{N}$

$$\begin{bmatrix} \sqrt{N} & N \\ 1 & \sqrt{N} \end{bmatrix} \begin{bmatrix} e_1^+ \\ e_2^+ \end{bmatrix} = 0.$$

If we divide the top row by $\sqrt{N}$ the two rows are identical, since the matrix must be singular. Thus this matrix equation gives two identical equations. This is the price of an over-specified equation (the singular matrix is degenerate).

We can determine each eigenvectors direction, but not their magnitudes.

Following the same procedure for $\lambda_- = -\sqrt{N}$, the equation for $\boldsymbol{e}_-$ is

$$\begin{bmatrix} -\sqrt{N} & N \\ 1 & -\sqrt{N} \end{bmatrix} \begin{bmatrix} e_1^- \\ e_2^- \end{bmatrix} = 0.$$

As before, this matrix is singular. Here $e_1^- - \sqrt{N}e_2^- = 0$, thus the eigenvector is $e^- = c\,[\sqrt{N}, 1]^T$ where c is a normalization constant.

Thus the *unnormalized* eigenmatrix is

$$\boldsymbol{e} = \begin{bmatrix} e_1^+ & e_2^- \\ e_2^+ & e_2^- \end{bmatrix} = \begin{bmatrix} \sqrt{N} & -\sqrt{N} \\ 1 & 1 \end{bmatrix}.$$

Normalization of the eigenvectors:

The constant c may be determined by normalizing the eigenvectors to have unit length. Since we cannot determine the length, we set it to 1. Thus the degeneracy may be resolved by the one degree of freedom normalization

$$\left(\pm\sqrt{N}\right)^2 + 1^2 = N + 1 = 1/c^2.$$

Thus the normalization factor to force each eigen vector to have length 1 is $c = 1/\sqrt{N+1}$.

[1] It is a convention to order the eigenvalues from largest to smallest.

A.2 Pell equation solution example

§A.2 (p. 263) showed that the solution $[x_n, y_n]^T$ to Pell's equation is given by powers of the Pell matrix $\boldsymbol{a}$. For $N = 2$, in §A.2 we found the explicit formula for $[x_n, y_n]^T$, based on powers of the Pell matrix

$$\boldsymbol{a} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{A.8}$$

This recursive solution to Pell's equation (Eq. 2.21) is Eq. 2.23 (p. 52). Thus we need powers of A, that is A^n, which gives an explicit expression for $[x_n, y_n]^T$. By the diagonalization of A, its powers are simply the powers of its eigenvalues.

From Matlab/Octave with $N = 2$ the eigenvalues of Eq. A.8 are $\lambda_\pm \approx [2.4142\jmath, -0.4142\jmath]$ (i.e., $\lambda_\pm = 1\jmath(1 \pm \sqrt{2})$). The solution for $N = 3$ is shown on page 263.

Once the matrix has been diagonalized, one may compute powers of that matrix as powers of the eigenvalues. This results in the general solution given by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = 1\jmath^n \boldsymbol{a}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\jmath^n \boldsymbol{e}\Lambda^n \boldsymbol{E}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The eigenvalue matrix D is diagonal with the eigenvalues sorted, largest first. The Matlab/Octave command `[E,D]=eig(A)` is helpful to find D and E given any A. As we saw above,

$$\Lambda = 1\jmath \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 2.414\jmath & 0 \\ 0 & -0.414\jmath \end{bmatrix}.$$

A.2.1 Pell equation eigenvalue-eigenvector analysis

Here we show how to compute the eigenvalues and eigenvectors for the 2×2 Pell matrix for $N = 2$

$$\boldsymbol{a} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

The Matlab/Octave command `[E,D]=eig(A)` returns the eigenvector matrix $\boldsymbol{e}$

$$\boldsymbol{e} = [\boldsymbol{e}_+, \boldsymbol{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.8165 & -0.8165 \\ 0.5774 & 0.5774. \end{bmatrix}$$

and the eigenvalue matrix Λ (Matlab/Octave's D)

$$\Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2.4142 & 0 \\ 0 & -0.4142 \end{bmatrix}.$$

Pell's equation for $N = 3$

In Table A.1, Pell's equation for $N = 3$ is given, with $\beta_0 = \jmath/\sqrt{2}$. Perhaps try other trivial solutions such as $[-1, 0]^T$ and $[\pm\jmath, 0]^T$, to provide clues to the proper value of β_0 for cases where $N > 3$.[2]

Example: I suggest that you verify $\boldsymbol{e}\Lambda \neq \Lambda\boldsymbol{e}$ and $\boldsymbol{a}\boldsymbol{e} = \boldsymbol{e}\Lambda$ with Matlab/Octave. Here is the Matlab/Octave program which does this:

```
A = [1 2; 1 1]; %define the matrix
[E,D] = eig(A); %compute the eigenvector and eigenvalue matrices
A*E-E*D %This should be $\approx 0$, within numerical error.
E*D-D*E %This is not zero
```

[2]My student Kehan found the general formula for β_o.

Table A.1: *Summary of the solution of Pell's equation due to the Pythagoreans using matrix recursion, for the case of N=3. The integer solutions are shown on the right. Note that $x_n/y_n \to \sqrt{3}$, in agreement with the Euclidean algorithm. The Matlab/Octave program for generating this data is* `PellSol3.m`. *It seems likely that the powers of β_0 could be absorbed in the starting solution, and then be removed from the recursion.*

Pell's Equation for $N = 3$
Case of $N = 3$ & $[x_0, y_0]^T = [1, 0]^T$, $\beta_0 = \jmath/\sqrt{2}$
Note: $x_n^2 - 3y_n^2 = 1, \qquad x_n/y_n \underset{\infty}{\longrightarrow} \sqrt{3}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \beta_0 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (1\beta_0)^2 - 3(1\beta_0)^2 = 1$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \beta_0^2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \beta_0^2 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \left(4\beta_0^2\right)^2 - 3\left(2\beta_0^2\right)^2 = 1$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \beta_0^3 \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \beta_0^3 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad \left(10\beta_0^3\right)^2 - 3\left(6\beta_0^3\right)^2 = 1$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \beta_0^4 \begin{bmatrix} 28 \\ 16 \end{bmatrix} = \beta_0^4 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} \qquad \left(28\beta_0^4\right)^2 - 3\left(16\beta_0^4\right)^2 = 1$$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \beta_0^5 \begin{bmatrix} 76 \\ 44 \end{bmatrix} = \beta_0^5 \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 28 \\ 16 \end{bmatrix} \qquad \left(76\beta_0^5\right)^2 - 3\left(44\beta_0^5\right)^2 = 1$$

Summary:

Thus far we have shown that for the case of Pell matrix with $N = 2$, the normalized eigenmatrix and its inverse is

$$\boldsymbol{e} = [\boldsymbol{e}_+, \boldsymbol{e}_-] = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \qquad \boldsymbol{e}^{-1} = \frac{\sqrt{6}}{4} \begin{bmatrix} 1 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix}$$

and the eigenmatrix is

$$\Lambda = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}.$$

A.3 Symbolic analysis of $\mathcal{T}\boldsymbol{e} = \boldsymbol{e}\Lambda$

A.3.1 The 2×2 transmission matrix

Here we assume

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$$

with $\Delta_\mathcal{T} = 1$.

The eigenvectors $\boldsymbol{e}_\pm$ of $\mathcal{T}$ are

$$\boldsymbol{e}_\pm = \begin{bmatrix} \frac{1}{2\mathcal{C}} \left[(\mathcal{A} - \mathcal{D}) \mp \sqrt{(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}\mathcal{C}} \right] \\ 1 \end{bmatrix} \tag{A.9}$$

and eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left[(\mathcal{A} + \mathcal{D}) \mp \sqrt{(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}\mathcal{C}} \right]. \tag{A.10}$$

Thus the expression under the radical may be rewritten in terms of the determinant of $\mathcal{T}$ (i.e., $\Delta_\mathcal{T} = \mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C}$) since

$$(\mathcal{A} - \mathcal{D})^2 - (\mathcal{A} + \mathcal{D})^2 = -4\mathcal{A}\mathcal{D}.$$

The for the ABCD matrix the expression under the radical becomes

$$\begin{aligned}(\mathcal{A}-\mathcal{D})^2 + 4\mathcal{BC} &= \mathcal{A}^2 + \mathcal{D}^2 - 4\mathcal{AD} + 4\mathcal{BC}\\ &= \mathcal{A}^2 + \mathcal{D}^2 - 4\Delta_\mathcal{T}.\end{aligned}$$

Rewriting the eigenvectors and eigenvalues in terms of $\Delta_\mathcal{T} = \pm 1$, we find

$$\boldsymbol{e}_\pm = \begin{bmatrix} \frac{1}{\mathcal{C}}\left[\frac{\mathcal{A}-\mathcal{D}}{2} \mp \sqrt{\left(\frac{\mathcal{A}+\mathcal{D}}{2}\right)^2 \mp \Delta_\mathcal{T}}\right] \\ 1 \end{bmatrix} \tag{A.11}$$

and

$$\lambda_\pm = \left[\frac{\mathcal{A}+\mathcal{D}}{2} \mp \sqrt{\left(\frac{\mathcal{A}+\mathcal{D}}{2}\right)^2 \mp \Delta_\mathcal{T}}\right]. \tag{A.12}$$

Note this may be further simplified since the radical is the same.

A.3.2 Matrices with symmetry

Reversible systems:

When $\mathcal{A} = \mathcal{D}$

$$\boldsymbol{e} = \begin{bmatrix} -\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} & +\sqrt{\frac{\mathcal{B}}{\mathcal{C}}} \\ 1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \mathcal{A}-\sqrt{\mathcal{BC}} & 0 \\ 0 & \mathcal{A}+\sqrt{\mathcal{BC}} \end{bmatrix} \tag{A.13}$$

the transmission matrix is said to be *reversible,*, and the properties greatly simplify.

In this case note that $r_o = \sqrt{\mathcal{B}/\mathcal{C}}$ is the *characteristic impedance* and $\kappa(s) = 1/\sqrt{\mathcal{BC}}$ is the *propagation function*.

Reciprocal systems

For the case of the ABCD matrix the eigenvalues depend critically on reciprocity. A reciprocal matrix is when $\Delta_\mathcal{T} = 1$ and is anti-reciprocal for $\Delta_\mathcal{T} = -1$. It is helpful to display the eigenfunctions and values in terms of $\Delta_\mathcal{T}$ so this distinction is clear.

When the matrix is symmetric ($\mathcal{B} = \mathcal{C}$), thus $r_o = 1$ and the corresponding system is said to be *reciprocal*. Most physical systems are reciprocal. The determinant of the transmission matrix of a reciprocal network $\Delta_\mathcal{T} = \mathcal{AD} - \mathcal{BC} = 1$. For example, electrical networks composed of inductors, capacitors and resistors are always reciprocal. It follows that the complex impedance matrix is symmetric (Van Valkenburg, 1964a).

Magnetic systems such as dynamic loudspeakers are anti-reciprocal, and correspondingly $\Delta_\mathcal{T} = -1$. The impedance matrix of these loudspeakers is *skew symmetric* (Kim and Allen, 2013). All impedance matrices are either symmetric or anti-symmetric, depending on whether they are reciprocal (LRC networks) or anti-reciprocal (magnetic networks). These systems have complex eigenvalues with negative real parts, corresponding to damped (lossy) causal systems. This follows from conservation of energy.

The impedance matrix cannot be Hermitian, or the losses would be zero, because Hermitian matrices have real eigenvalues. A physical system having power losses cannot be Hermitian because the eigenvalues must have a negative real parts.

In summary, given a reciprocal system, the $\mathcal{T}$ matrix has $\Delta_\mathcal{T} = 1$, and the corresponding impedance matrix is symmetric (*not* Hermitian).

A.3.3 The impedance matrix

As previously discussed in §3.7 (p. 113), the impedance matrix $\mathcal{Z}$ is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \boldsymbol{Z}(s)\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{\mathcal{C}}\begin{bmatrix} \mathcal{A} & \Delta_\mathcal{T} \\ 1 & \mathcal{D} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}.$$

Reciprocal systems have skew-symmetric impedance matrices, namely $z_{12} = z_{21}$ (i.e., $\Delta_\mathcal{T} = 1$). This condition is best understood using the T form of the impedance matrix, as shown in Fig. 3.6 (p. 115). When the system is both reversible $\mathcal{A} = \mathcal{D}$ and reciprocal ($\mathcal{B} = \mathcal{C}$), the impedance matrix simplifies to

$$\boldsymbol{Z}(s) = \frac{1}{\mathcal{C}} \begin{bmatrix} \mathcal{A} & 1 \\ 1 & \mathcal{A} \end{bmatrix}.$$

For such systems there are only two degrees of freedom, $\mathcal{A}$ and $\mathcal{C}$. As discussed previously in §3.7 (p. 113), each of these has a physical meaning: $1/\mathcal{A}$ is the *Thévenin source voltage* assuming a voltage drive, and $\mathcal{B}/\mathcal{A}$ is the *Thévenin impedance* (§3.5, p. 114).

Impedance is not Hermitian: By definition, when a system is Hermitian its matrix is conjugate symmetric

$$\mathbf{Z}(\mathbf{s}) = \mathbf{Z}^\dagger(\mathbf{s}),$$

a stronger condition than reciprocal, but not the symmetry of the Brune impedance matrix. A reciprocal Brune impedance is symmetric (not Hermitian).

In the case of a Hermitian matrix, the eigenvalues are always real. To show this start from the definition of an impedance eigen-equation ($\mathbf{V}$ is a vector of voltages, $\mathbf{I}$ a current vector and $\mathbf{Z}$ an impedance matrix)

$$\mathbf{V} = \mathbf{Z}\mathbf{I} = \mathbf{I}\boldsymbol{\Lambda}; \qquad \mathcal{P} = \mathbf{V}'\mathbf{I} = \mathbf{I}'\boldsymbol{\Lambda}\mathbf{I},$$

where $\mathbf{Z}, \mathbf{I}, \mathbf{V}, \boldsymbol{\Lambda} \in \mathbb{C}$, $A = A^\dagger$ is a square conjugate-symmetric matrix, and $\mathbf{I}, \mathbf{V}$ are vectors of the size of $\mathbf{Z}$ and $\mathcal{P}$ is the power. Here $\mathbf{Z}^\dagger$ is the complex transpose (see Appendix 2.1.1, p. 98). Cite result: $|\Gamma| \le 1 \equiv \Re Z_{in} > 0$ The power $\mathcal{P}$ is the real part of the voltage times the current

$$2\,\mathcal{P} = \mathbf{V}^\dagger\mathbf{I} + \mathbf{V}\mathbf{I}^\dagger = (\mathbf{Z}\mathbf{I})^\dagger\mathbf{I} + \mathbf{Z}\mathbf{I}\,\mathbf{I}^\dagger = \mathbf{I}^\dagger\mathbf{Z}^\dagger\mathbf{I} + \mathbf{Z}\mathbf{I}\,\mathbf{I}^\dagger.$$

ion to a HW Subtracting the two equations gives ....

Double roots not allowed

For the 2×2 case of double roots the matrix has Jordan form

$$\mathcal{T} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Then

$$\mathcal{T}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}.$$

This generalizes to nxn matrices having arbitrary combinations of degeneracies (multiple roots), as in symmetric (square) drums, for example ....

Octave vs. Matlab discrepency

Using Octave (version 4.2.2, June 21, 2018):

$$e_{1,1} = \frac{-b}{\frac{a}{2} - \frac{d}{2} + \frac{1}{2}\sqrt{a^2 - 2ad + 4bc + d^2}} = \frac{-2b}{(a-d) + \sqrt{(a-d)^2 + 4bc}}.$$

This looks very different from the Matlab output (above).

Here is what I get with the same command, from Matlab (June 21, 2018):

$$e_{1,1} = \frac{\frac{a}{2} + \frac{d}{2} - \frac{\sqrt{a^2 - 2\,a\,d + d^2 + 4\,b\,c}}{2}}{c} - \frac{d}{c} = \frac{1}{2c}\left[(a-d) - \sqrt{(a-d)^2 + 4\,b\,c}\right].$$

In the following we derive the eigenmatrix $\boldsymbol{e}$, and eigenvalue matrix Λ given a 2×2 transmission matrix

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix},$$

such that $\mathcal{T}\boldsymbol{e} = \boldsymbol{e}\Lambda$, using symbolic algebra methods, given by the Matlab/Octave's script

```
syms A B C D T E L %Use symbolic Matlab/Octave
   T=[A B;C D]      %Given matrix T
  [E,L]=eig(T)      %Find eigenvector matrix E and
                    %eigenvalue matrix L
```

These results have been numerically verified to be the same using `CkEig.m`. Conclusions from the numerical experiment with ($A = [1, 2; 3, 5]$) are:

1. The Octave and Matlab symbolic formulas give the same numeric results for eigenmatrix and eigenvalues.

2. The symbolic and numeric results correctly diagonalize the matrix, that is $\Gamma = E^{-1}AE$.

3. The numeric equations are normalized to 1: $norm(e_{\pm}) = 1$.

4. The symbolic Eigenvectors are not normalized to 1, rather $E(2, :) = 1$.

5. When the numeric result is normalized by the lower element (making it 1) all results agree.

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