ECE 403 **HWa – Version 1.05 February 2, 2017** **Spring 2016**

Univ. of Illinois **Due: Tu, Feb 7, 2017** **Prof. Allen**

Topic of this homework: Convolution, Fourier and Laplace Transforms, Impedance functions of Laplace frequency $s = \sigma + \jmath\omega$, Filter classes.

Deliverable: Show your work. If you hand it in late, you will get zero credit (Some credit is better than NONE)! I need a **stapled paper copy**, with your name on it. **No** files.doc

Note: Each person is to do their own final writeup, but you may, and should, discuss it as much as you wish between yourselves. You're crossing the line if you share computer files. The general rule is, "look, understand, but do not copy." In otherwords, you need to process all the words you write through your eyes and fingers. If you use material from elsewhere, you must cite the source.

Terminology and acoustic constants

Definitions of common acoustic variables,[1] the mathematical symbols and the units, as used in the text (see p. 180-181).

Variable name	Symbol	[Units]
pressure	$P = P_o + p(t)$	[N/m^2 = Pa]
density	$\rho = \rho_o + \delta(t)$	[kg/m^3]
temperature	$T_o + \tau(t)$	[K°] Kelvin
Constants		
Atm Pressure	$P_o = 10^5$	[Pa]
Abs temperature	$T_o = 273$	[K°]
sound speed	$c = 345$	[m/s]
density	$\rho_o = 1.18$	[kg/m^3]
specific impedance	$\rho c = 407$	[Rayls]
viscosity	$\mu = 1.86 \times 10^{-5}$	[Ns/m^2] Poise
Boyle's Law	$P_o = \rho_o T_o$ Const.	[Pa]
adiabatic law	$p = \delta^\gamma$ Const.	[Pa]
thermal conductivity	$\kappa = 25.4 \times 10^3$	[N/sK]
specific heat cap @V$_o$	c_v	[J/kg]
specific heat cap @P$_o$	c_p	[J/kg]
Boltzman's const.	$k = 1.38 \times 10^{-23}$	J/molecule
ratio of specific heats	$\gamma = c_p/c_v = 1.4$	
Lossy $\gamma\sqrt{\mu}$	$\gamma' = 6.6180 \times 10^{-3}$	
Propagation function	$\kappa(s) = \left(s + \beta_0\sqrt{s}\,\right)/c$	$\mathcal{P}^+(s) = e^{-\kappa(s)x}$

The lossy wave-number $\kappa(s)$, including thermal and viscous loss, is (p. 154, *anInvitation*) $\kappa(s) = (s + \beta_0\sqrt{s})/c$. If the tube diameter D [m] is the tube diameter, then $\beta_0 D = 2\gamma'/\sqrt{\rho} = 12.0828$, with

$$\gamma' = \sqrt{\mu}\left(1 + \sqrt{5/2}\left(\sqrt{\gamma} - \frac{1}{\sqrt{\gamma}}\right)\right) = 6.618045 \times 10^{-3}.$$

For example, with $D = 1$ [cm], $\beta_0 = 1.2083 \times 10^3$, and for $D = 1$ [mm], $\beta_0 = 12.083 \times 10^3$.

[1] C_p, C_v: `https://en.wikipedia.org/wiki/Heat_capacity#Specific_heat_capacity`

1 Basic Acoustics

1. What is the formula for the speed of sound? Solution: The formula (Eq. (2.19) is $c = \sqrt{\gamma P_0/\rho_0}$.

   (a) Identify the variables in the formula for the speed of sound: Names, units, and values of consonants. Solution: On the left is the speed of sound c, which is a derived quantity, and on the right are physical parameters $\gamma \equiv c_p/c_v = 1.4$ [dimensionless], the barometric pressure (e.g., at sea level) $P_0 = 10^5$ [Pa], and the density $\rho_0 = 1.18$ [kgm/m^3].

   (b) What is the meaning of γP_0? Solution: This combination of variables represents the adiabatic compressibility of air. The $\gamma = c_p/c_v$ results from holding the temperature constant during the cycle of the wave. Heat diffusion is slow compared to the cycle at acoustic frequencies, such that the thermal energy is trapped in the air.

   (c) Does P_0 depend on temperature? Explain? Solution: No. The barometric pressure depends on the weight of the air above us. While its density dependents on temperature ($\rho \propto T$), the total weight (i.e., mass) is constant, and is therefore independent of temperature. That leaves the question as to why the barometric pressure varies over time, but that is more about the humidity, winds, and many other variables.

   (d) Does ρ_0 depend on temperature? Explain. Solution: Yes, as discussed in class

   $$\rho(T, P_0) = 1.275 \frac{273}{273 + C} \frac{P_0}{10^5},$$

   where C is the temperature in degrees Celsius and $K = 273 + C$ is the absolute temperature [K°].

2. State the Gas law, in terms of the universal gas constant R_0.

   (a) Specifically discuss the value and nature of the constant PV/T. Solution: The one everyone is taught in school is

   $$\frac{PV}{T} = constant.$$

   This constant is given by nR_0 where n is the number of moles (unit [mol]) of the gas in question, and $R_0 = 8.314$ [J/(mol K)] is the universal gas constant. There is also an 'individual' gas constant $R_{ind} = R_0/M$, which has units [J/(mol K) · mol/kg = m^2/ (s^{2}K)]. If we let the total mass of the gas $nM = m$,

   $$PV = nR_0T = n(R_{ind}M)T = mR_{ind}T$$

   which gives the formula for the density

   $$\rho = \frac{P}{R_{ind}T}.$$

   For air, which is a mixture of different molecules, $R_{ind} = 286.9$ [m^2/ (s^{2}K)], and the molar mass $M = 28.97$ [g/mol] = 0.02897 [kg/mol].

(b) What is 1 [mol] of air? Solution: One mole is 6.022×10^{23} molecules (where $N_A = 6.022 \times 10^{23}$ is Avogadro's number). There are many different types of molecules, in specific proportions (e.g. Wikipedia: 78% nitrogen, 21% oxygen...). The mass of 1 [mol] of air is 28.97 [g].

(c) What is the relationship between the density of air ρ and the volume V. Solution: Let m = the total mass of the gas contained in volume V, M = the molar mass of the gas, and n = the number of moles of the gas contained in volume V. This gives

$$\rho = \frac{m}{V} = \frac{nM}{V}$$

(d) Describe the relationship between the gas constant R, Boltzman's constant k and Avagadro's number N_A. Solution: Avogadro's number, $N_A \approx 6.022 \times 10^{23}$[1/mol], and Boltzman's constant $k_B = 1.3806 \times 10^{-23}$ [J/K] are related to the universal gas constant by $R_0 = k_B N_A = 8.314$ [J/(mol K)].

3. What is the form of the dependence of the speed of sound on temperature? Namely give the formula for $c(T)$, and explain the dependence. Solution: Since $c = \sqrt{\gamma P_0/\rho_0}$ and following the state equation for a gas, $\rho_0 \propto 1/T$, we may conclude that $c(T) \propto \sqrt{T}$, where T is in degrees Kelvin. Because the percentage change in degrees Kelvin is much smaller than in degrees C, this dependence is relatively small.

1.1 Basic equations of sound propagation:

1. Write out the 2x2 matrix equation that describes, in the frequency domain, the propagation of 1 dimensional sound waves in a tube having area $A(x)$, in terms of the pressure $P(x,s)$ and volume velocity $U(x,s)$: Solution: As shown in class the basic equations are:

$$\frac{d}{dx}\begin{bmatrix} P(x,\omega) \\ U(x,\omega) \end{bmatrix} = -\begin{bmatrix} 0 & \mathcal{Z}(x,s) \\ \mathcal{Y}(x,s) & 0 \end{bmatrix}\begin{bmatrix} P(x,\omega) \\ U(x,\omega) \end{bmatrix}. \tag{1}$$

where P is the pressure, U is the volume velocity, $\mathcal{Z} = s\rho_0/A(x)$ and $\mathcal{Y} = sA(x)/\eta P_0$, with $A(x)$ the area of the tube as a function of position along the length of the tube x.

2. Assuming $A(x)$ is constant, rewrite these equations as a second order equation solely in terms of the pressure P (remove U), and thereby find the formula for the speed of sound in terms of Z and Y: Solution: If we let $P' \equiv \partial P/\partial x$ (i.e., the partial with respect to space) then

$$P' + ZU = 0 \tag{2}$$

and

$$U' + YP = 0. \tag{3}$$

Taking the partial wrt x of the first equation, and then using the second, gives

$$P'' + ZU' = P'' - \mathcal{Z}\mathcal{Y}P = 0. \tag{4}$$

Since the wave equation is

$$\frac{\partial^2 P}{\partial x^2} = \frac{s^2}{c^2} P \leftrightarrow \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \tag{5}$$

by inspection we see that $\omega^2/c^2 = \mathcal{Z}\mathcal{Y}$, which results in the final formula for the speed of sound.

2 deciBels [dB]

1. Express decibels in terms of the pressure ratio Solution: $20 \log(P/P_{\text{ref}})$ where P_{ref} is the reference pressure.

2. State the reference pressure for dB-SPL Solution: 20 micro Pascals, or -94 dB re 1 Pa.

3. What is the attenuator gain, expressed in dB, if the voltage is reduced by a factor of 2? Solution: -6 dB corresponds to dividing the voltage (or pressure) by 2.

4. How many millibels [mB] in 1 bel [B]? Solution: 1000 mB = 1 B since mB is a much smaller unit that the bel.

5. Give the formula for the intensity in mB units. Solution: Note that 1 [dB] = 0.1 [Bel]. The intensity in mB is $1000 \log_{10}(I/I_{ref})$ where $I_{ref} = 10^{-12}$ W/m^2.

6. Give the formula for the sound pressure level in cB (centibel) units. Solution: The sound pressure level in cB is $200 \log_{10}(P/P_{ref})$ where $P_{ref} = 20 \times 10^{-6}$ [Pa].

7. There are two different definitions of acoustic dB, one based on pressure

$$dB_p = 20 \log_{10}(P/P_{ref})$$

and a second based on acoustic intensity

$$dB_I = 10 \log_{10}(I/I_{ref})$$

where P is the pressure in Pascals and $I \equiv |P|^2/\rho c$ is the acoustic intensity. Here ρ is the density of air and c is the speed of sound. The product $\rho c = 407$ [Rayls] is called the *Specific acoustic impedance* of air.

Demonstrate that $P_{ref} \equiv 20$ μPa is the same as $I_{ref} \equiv 10^{-12}$ [W/m^2]. Solution: The first is given by the formula $|P_{ref}|^2/\rho c$ and the second is given by I_{ref}. Thus we need to show that numerically these two quantities are similar (almost identical). Since $(20 \times 10^{-6})^2/407 \approx 10^{-12}$, the two references are nearly, but not exactly, the same.

8. What is the acoustic impedance observed by a plane wave? What are its units? Solution: The *specific acoustic impedance* of a plane wave is $\rho c = 407$ [Rayls]. The *acoustic impedance* of a plane wave in a tube of area A_0 is $\rho c/A_0$.

3 The Helmholtz Resonator

A bottle has a neck diameter of 1 [cm] and is l =1 cm long. It is connected to the body of the bottle "barrel" which is 5 cm in diameter and L =10 cm long. Treat the barrel as a short piece of transmission line, closed at one end, which looks like a compliance $C = V_{barrel}/\gamma P_0$, and the neck which look like a mass $M = \rho_0 l/A_{neck}$. These two impedances are in series, since they both see the same volume velocity (flow).

1. Set the impedance to zero and solve for the bottle's resonant frequency, in terms of M and C. Solution: Solving for the resonant frequency, $0 = s_0 M + 1/(s_0 C)$, gives

$$s_0 = j\omega_0 = \sqrt{\frac{-1}{MC}} \quad \text{which gives} \quad \omega_0 = \sqrt{\frac{1}{MC}}$$

2. Write out the formula for the resonant frequency in terms of the physical dimensions of the bottle. Solution: The formula for the Helmholtz resonator was derived in class, where it was shown to be

$$f_0 = \frac{c}{2\pi}\sqrt{A/(Vl)}, \tag{6}$$

where A, l are the area and length of the neck and V is the volume of the bottle.

3. Calculate the resonant frequency in Hz for the dimensions given. Solution: From the numbers given $A = \pi \times 0.005^2$ [m^2], $l = 0.01$ while $V = \pi \times 0.025^2 \times 0.1$ [m^3]. Thus $f_0 = \frac{345}{2\pi}\sqrt{\pi \times 0.005^2/(0.1 \times 0.0025^2 \times \pi \times 0.01)} \approx 347.3$ [Hz].

4. **Extra credit:** Blow into a blottle and measure the resonant frequency by recording the tone, and taking the FFT of the resulting waveform, and finding the frequency.

4 Fourier and Laplace transforms

1. Derive the Fourier transform for the step function $u(t-1)$.

Solution: Since the integral does not converge, one must fake it by using the time-symmetric relationship $2u(t) = 1 - \text{sgn}(t)$, delayed:

$$\widetilde{U}(\omega) \equiv \int_{-\infty}^{\infty} \tilde{u}(t-1)e^{-j\omega t}dt = \mathcal{F}\left\{\frac{1-\text{sgn}(t-1)}{2}\right\} = \pi\tilde{\delta}(\omega) + \frac{e^{-j\omega}}{j\omega}$$

$$\neq \int_{1}^{\infty} e^{-j\omega t}dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_1^\infty = \frac{e^{-j\omega} - e^{-j\omega\infty}}{j\omega} = \frac{e^{-j\omega}}{j\omega} - \frac{e^{-j\omega\infty}}{j\omega}$$

2. Derive the Laplace transform for the step function $u(t-1)$. Solution: e^{-s}/s

3. Find the Fourier transform $X(\omega)$ of

$$x(t) = u(t) - u(t-.001) \leftrightarrow X(\omega)$$

Solution: Note the $\pi\delta(\omega)$ term cancels out in this case, making the rect(t) function much nicer than the step allowing one to compute rect$(t) \star$ rect(t).

4. Find the Laplace transform $\mathcal{X}(s)$ of $x(t) = u(t) - u(t - .001) \leftrightarrow \mathcal{X}(s)$

   Solution: $(1 - e^{-s/1000})/s$.

5. If $\tilde{u}(t) \leftrightarrow \widetilde{U}(\omega)$ is the $\mathcal{F}$ step function, what is $\tilde{u}(t) \star \tilde{u}(t)$? Solution: This is a really dirty question, because its not in any book I know of, and the solution is a surprise: The problem here is that $2\tilde{u}(t) \equiv 1 + \mathrm{sgn}(t)$, and $1 \star 1$ badly blows up, and does not exist. Another way to say this is to work in the frequency domain and use the fact that $1 \leftrightarrow 2\pi\delta(\omega)$, and that convolution in time is the product in frequency. In this case $1 \star 1 \leftrightarrow (2\pi)^2 \left(\delta(\omega)\right)^2$. Probably nobody told you, that you cannot make a function out of $\delta()$. (e.g., It is not legal to square a delta function. (The Taylor series of a delta functions cannot be defined.) Please show me if you find this written down anywhere.)

6. Hand-plot (or describe the plot of) $|X(\omega)|$ and $|\mathcal{X}(s)|$. Solution: Since the numerator has $1 - e^{-j\omega T}$ the magnitude is sinusodial varying. The phase factor $e^{-j\omega T/2}$ is removed by taking the magnitude.

7. Where are the poles and zeros in each case above. Solution: Fourier transforms dont have poles and zeros. The Laplace transforms for these examples have poles at $s = 0$, and zeros at $e^{-sT} = 1$ and $e^{s} = 0$, depending on the case.

5 Laplace Transforms

Laplace transforms: given that $f(t) \leftrightarrow F(s)$

1. Find the Laplace transform of $\delta(t)$, $df(t)/dt$, $\int_{-\infty}^{t} \delta(t)dt$, and $\int_{-\infty}^{t} u(t)dt$.

   Solution:

   (a) $\delta(t) \leftrightarrow 1$ is too trivial to repeat here.

   (b) $df/dt \leftrightarrow sF(s)$. This is shown using integration-by-parts, as follows:

   $$d[f(t)e^{-st}] = e^{-st}\frac{df}{dt}dt - se^{-st}f(t)dt$$

   Next integrate this from 0^- to ∞, giving

   $$f(t)e^{-st}|_{0^-}^{\infty} = \int_{0^-}^{\infty} e^{-st}\frac{df}{dt}dt - s\int_{0^-}^{\infty} e^{-st}f(t)dt$$

   Rearranging these and evaluating the limits gives the desired result

   $$\int_{0^-}^{\infty} \frac{df}{dt} e^{-st}dt = f(0^-) + sF(s),$$

   where $f(0^-) = 0$.

   (c) The integral of a delta function is a step function $u(t) \leftrightarrow 1/s$, while

   (d) the integral of a step is $tu(t) \leftrightarrow 1/s^2$.

   In each case it is important to carry along the $u(t)$ but it can be implied if you know its a causal function (i.e., if you are told the transform is a function of s (e.g., $F(s)$).

2. If $f(t) = 1/\sqrt{\pi t}$ has a Laplace transform $F(s) = 1/\sqrt{s}$:

   (a) What is the inverse Laplace transform of $\sqrt{s}$?
   Solution: Since $d/dt \leftrightarrow s$ then $\sqrt{s} = s/\sqrt{s}$, thus $\frac{d}{dt}\frac{u(t)}{\sqrt{\pi t}} \leftrightarrow \sqrt{s}$. In fact there is a major difficulty here, since $\int \frac{\delta(t)}{\sqrt{t}} dt = \infty$. This problem may be resolved by a small delay in either the numerator of denominator terms so that the delta function does not resolve at $t = 0$ in the $\sqrt{t}$ term.

   (b) What is $f(-1)$?
   Solution: Since $f(t)$ is causal, at $t = -1$ it is zero.

   (c) Integrate $I = \int_C \frac{1}{s} ds$ around the unit circle centered on $s = 0$.
   Solution: Let C be the unit circle, then $s = e^{j\theta}$, so

   $$I = \int_0^{2\pi} \frac{de^{j\theta}}{e^{j\theta}} = \int_{\theta=0}^{2\pi} \frac{je^{j\theta}}{e^{j\theta}} d\theta = j\int_{\theta=0}^{2\pi} d\theta = j\,\theta\big|_{\theta=0}^{2\pi} = 2\pi j$$

   (d) Integrate $\int_C \frac{1}{s} ds$ around the unit circle centered on $s = .5$ (i.e., $\sigma = .5, \omega = 0$), and $s = -2$. Solution: The first is $2\pi j$ and the second is 0. This is explained by the *Cauchy Residue Theorem.*

6 Convolution

Given two "causal" sequences $a_n = [\cdots, ∴, 0, 1, 0, -1, 0, \cdots]$ and $b_n = [\cdots, ∴, 1, -1, 0, 0, \cdots]$. Here the rising dots ∴ define $t = 0$, before and at which time the signal is zero.

1. Find causal sequence $c \equiv a \star b$ by direction convolution Solution: Time reverse either a or b and slide it against the other, forming the output sequence $c_n = [0, 1, -1, -1, 1, 0, 0, \cdots]$

2. Form the polynomials $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$, and find $C(z) = A(z) \cdot B(z)$ Solution: $C(z) = (z - z^3)(1 - z) = z - z^2 - z^3 + z^4$ which has a coef vector $[0, 1, -1, -1, 1, 0, \cdots]$

3. What can you say about the sequence c_n and the coefficients of $C(z)$? Solution: They are the same.