CHAPTER

The wave equation and solutions

2

CHAPTER OUTLINE

PART III: THE WAVE EQUATION

2.1 INTRODUCTION

We have already outlined in a qualitative way the nature of sound propagation in a gas. In this chapter we shall put the physical principles described earlier into the language of mathematics. The approach is in two steps. First, we shall establish equations expressing Newton's second law of motion, the gas law, and the laws of conservation of mass. Second, we shall combine these equations to produce a wave equation.

The mathematical derivations are given in two ways: with and without use of vector algebra. Those who are familiar with vector notation will appreciate the generality of the three-dimensional vector

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approach. The two derivations are carried on in parallel; on the left sides of the pages, the one-dimensional wave equation is derived with the use of simple differential notation; on the right sides, the three-dimensional wave equation is derived with the use of vector notation. The simplicity of the vector operations is revealed in the side-by-side presentation of the two derivations.

2.2 DERIVATION OF THE WAVE EQUATION

2.2.1 The equation of motion

If we write Newton's second law for a small volume of gas located in a homogeneous medium, we obtain the equation of motion, or the force equation as it is sometimes called. Imagine the small volume of gas to be enclosed in a box with weightless flexible sides (Fig.2.1).

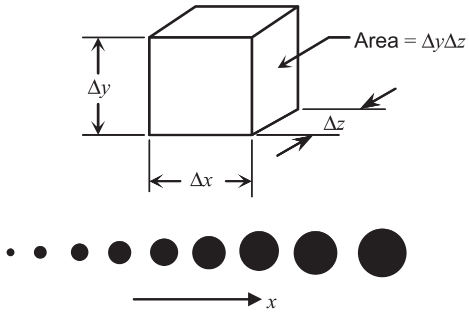

FIG. 2.1 The very small "box" of air shown here is part of a gaseous medium in which the sound pressure increases from left to right at a space rate of $\partial p/\partial x$ (or, in vector notation, grad p). The sizes of the dots indicate the magnitude of the sound pressure at each point.

One-dimensional derivation [1]

Let us suppose that the box is situated in a medium where the sound pressure p increases from left to right at a space rate of $\partial p/\partial x$ (see Fig. 2.1).

Three-dimensional derivation [2]

Let us suppose that the box is situated in a medium (see Fig. 2.1) where the sound pressure p changes in space at a space rate of

$$\textbf{grad}\, p = \nabla p = \mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z},$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the x, y, and z directions, respectively, and p is the pressure at a point.

Assume that the sides of the box are completely frictionless; i.e., any viscous drag between gas particles inside the box and those outside is negligible. Thus the only forces acting on the enclosed gas are due to the pressures at the faces of the box.

The difference between the forces acting on the two sides of our tiny box of gas is equal to the rate at which the force changes with distance times the incremental length of the box:

Force acting to accelerate the box in the positive x direction

$$= -\left(\frac{\partial p}{\partial x}\Delta x\right)\Delta y \Delta z. \qquad (2.1a)$$

Force acting to accelerate the box in the positive direction

$$= -\mathbf{i}\left[\left(\frac{\partial p}{\partial x}\Delta x\right)\Delta y\Delta z + \mathbf{j}\left(\frac{\partial p}{\partial y}\Delta y\right)\Delta x\Delta z + \mathbf{k}\left(\frac{\partial p}{\partial z}\Delta z\right)\Delta x\Delta y\right]. \qquad (2.1b)$$

Note that the positive gradient causes an acceleration of the box in the negative direction of x.

Division of both sides of the above equation by Δx Δy $\Delta z = V$ gives the force per unit volume acting to accelerate the box:

$$\frac{f}{V} = -\frac{\partial p}{\partial x}. \qquad (2.2a)$$

Division of both sides of the equation by Δx Δy $\Delta z = V$ gives the force per unit volume acting to accelerate the box:

$$\frac{f}{V} = -\nabla p. \qquad (2.2b)$$

By Newton's Law, the force per unit volume (f/V) of Eq. (2.2) must be equal to the time rate of change of the momentum per unit volume of the box. We have already assumed that our box is a deformable packet so that the mass of the gas within it is always constant. That is,

$$\frac{f}{V} = -\frac{\partial p}{\partial x} = \frac{M}{V}\frac{\partial u}{\partial t} = \rho'\frac{\partial u}{\partial t}, \qquad (2.3a)$$

where u is the average velocity of the gas in the "box" in the x direction, ρ' is the space average of the instantaneous density of the gas in the box, and $M = \rho' V$ is the total mass of the gas in the box.

$$\frac{f}{V} = -\nabla p = \frac{M}{V}\frac{D\mathbf{q}}{Dt} = \rho'\frac{D\mathbf{q}}{Dt}, \qquad (2.3b)$$

where $\mathbf{q}$ is the average vector velocity of the gas in the "box," ρ' is the average density of the gas in the box, and $M = \rho' V$ is the total mass of the gas. in the box. D/Dt is not a simple partial derivative but represents the total rate of the change of the velocity of the particular bit of gas in the box regardless of its position, i.e.,

$$\frac{D\mathbf{q}}{Dt} = \frac{\partial \mathbf{q}}{\partial t} + q_x\frac{\partial \mathbf{q}}{\partial x} + q_y\frac{\partial \mathbf{q}}{\partial y} + q_z\frac{\partial \mathbf{q}}{\partial z},$$

where q_x, q_y, and q_z are the components of the vector particle velocity $\mathbf{q}$.

If the change in density of the gas due to the sound wave is small enough, then the instantaneous density ρ' is approximately equal to the average density ρ_0.
Then,

$$-\frac{\partial p}{\partial x} = \rho_0\frac{\partial u}{\partial t}. \qquad (2.4a)$$

If the vector particle velocity $\mathbf{q}$ is small enough, the rate of change of momentum of the particles in the box can be approximated by the rate of change of momentum at a fixed point, $D\mathbf{q}/Dt \approx \partial\mathbf{q}/\partial t$, and the instantaneous density ρ' can be approximated by the average density ρ_0. Then,

$$-\nabla p = \rho_0\frac{\partial \mathbf{q}}{\partial t}. \qquad (2.4b)$$

The approximations just given are generally acceptable provided the sound pressure levels being considered are below about 110 dB re 20 μPa. Levels above 110 dB are so large as to create hearing discomfort in many individuals.

2.2.2 The gas law

If we assume an ideal gas, the Charles–Boyle gas law applies to the box. It is

$$PV = RT \tag{2.5}$$

where P is the total pressure in the box, V is the volume equal to $\Delta x\, \Delta y\, \Delta z$, T is the absolute temperature in °K, and R is a constant for the gas whose magnitude is dependent upon the mass of gas chosen. [3] Using this equation, we can find a relation between the sound pressure (excess pressure) and an incremental change in V for our box. Before we can establish this relation, however, we must know how the temperature T varies with changes in P and V and, in particular, whether the phenomenon is adiabatic or isothermal.

At audible frequencies the wavelength of a sound is long compared with the spacing between air molecules. For example, at 1000 Hz, the wavelength λ equals 0.34 m, as compared with an intermolecular spacing of 10^{-9} m. Now, whenever a portion of any gas is compressed rapidly, its temperature rises, and, conversely, when it is expanded rapidly, its temperature drops. At any one point in an alternating sound field, therefore, the temperature rises and falls relative to the ambient temperature. This variation occurs at the same frequency as that of the sound wave and is in phase with the sound pressure.

Let us assume, for the moment, that the sound wave has only one frequency. At points separated by one-half wavelength, the pressure and the temperature fluctuations will be 180° out of phase with each other. Now the question arises, is there sufficient time during one-half an alternation in the temperature for an exchange of heat to take place between these two points of maximally different temperatures?

It has been established [4] that under normal atmospheric conditions the speed of travel of a thermal diffusion wave at 1000 Hz is about 0.5 m/s, and at 10,000 Hz it is about 1.5 m/s. The time for one-half an alternation of 1000 Hz is 0.0005 s. In this time, the thermal wave travels a distance of only 0.00025 m. This number is very small compared with one-half wavelength (0.17 m) at 1000 Hz. At 10,000 Hz the heat travels 7.5×10^{-5} m, which is a small distance compared with a half wavelength (1.7×10^{-2} m). It appears safe for us to conclude, therefore, that there is negligible heat exchange in the wave in the audible frequency range. Gaseous compressions and expansions of this type are said to be adiabatic.

For adiabatic expansions, the relation between the total pressure and the volume is known to be [5]

$$PV^{\gamma} = \text{constant}, \tag{2.6}$$

where γ is the ratio of the specific heat of the gas at constant pressure to the specific heat at constant volume for the gas. This equation is obtained from the gas law in the form of Eq. (2.5), assuming adiabatic conditions. For air, hydrogen, nitrogen, and oxygen, i.e., gases with diatomic molecules,

$$\gamma = 1.4.$$

Expressing Eq. (2.6) in differential form, we have

$$\frac{dP}{P} = -\frac{\gamma dV}{V}. \tag{2.7}$$

Let

$$P = P_0 + p, \qquad V = V_0 + \tau, \tag{2.8}$$

where P_0 and V_0 are the undisturbed pressure and volume, respectively, and p and τ are the incremental pressure and volume, respectively, owing to the presence of the sound wave. Then, to the same approximation as that made preceding Eq. (2.4) and because $p << P_0$ and $\tau << V_0$,

$$\frac{p}{P_0} = -\frac{\gamma\tau}{V_0}. \tag{2.9}$$

The time derivative of this equation gives

$$\frac{1}{P_0}\frac{dp}{dt} = -\frac{\gamma}{V_0}\frac{d\tau}{dt}. \tag{2.10}$$

2.2.3 The continuity equation

The continuity equation is a mathematical expression stating that the total mass of gas in a deformable "box" must remain constant. Because of this law of conservation of mass, we are able to write a unique relation between the time rate of change of the incremental velocities at the surfaces of the box.

One-dimensional derivation

Refer to Fig. 2.2. If the mass of gas within the box remains constant, the change in volume τ depends only on the difference of displacement of the air particles on the opposite sides of the box. Another way of saying this is that, unless the air particles adjacent to any given side of the box move at the same velocity as the box itself, some will cross into or out of the box and the mass inside will change.

In a given interval of time the air particles on the left-hand side of the box will have been displaced ξ_x. In this same time, the air particles on the right-hand side will have been displaced

$$\xi_x + \frac{\partial \xi_x}{\partial x}\Delta x.$$

The difference of the two quantities above multiplied by the area $\Delta y \Delta z$ gives the increment in volume τ

$$\tau = \frac{\partial \xi_x}{\partial x}\Delta x \Delta y \Delta z \tag{2.11a}$$

or

$$\tau = V_0 \frac{\partial \xi_x}{\partial x}. \tag{2.12}$$

Three-dimensional derivation

If the mass of gas within the box remains constant, the change in incremental volume τ depends only on the divergence of the vector displacement. Another way of saying this is that, unless the air particles adjacent to any given side of the box move at the same velocity as the side of the box itself, some will cross into or out of the box and the mass inside will change; so

$$\tau = V_0 \operatorname{div} \xi = V_0 \nabla \cdot \xi \tag{2.11b}$$

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Differentiating with respect to time yields,

$$\frac{\partial \tau}{\partial t} = V_0 \frac{\partial u}{\partial x}, \qquad (2.13a)$$

where u is the instantaneous particle velocity.

Differentiating with respect to time yields,

$$\frac{\partial \tau}{\partial t} = V_0 \, \nabla \cdot q, \qquad (2.13b)$$

where **q** is the instantaneous particle velocity.

Example 2.1. In the steady state, that is,

$$\partial u / \partial t = j\omega \tilde{u} = \sqrt{2} u_{rms},$$

determine mathematically how the sound pressure in a plane progressive sound wave (one-dimensional case) could he determined from measurement of particle velocity alone.

Solution. From Eq. (2.4a) we find in the steady state that

$$-\frac{\partial p_{rms}}{\partial x} = j\omega\rho_0 u_{rms}.$$

Written in differential form,

$$-\Delta p_{rms} = j\omega\rho_0 u_{rms} \Delta x.$$

If the particle velocity is 1 cm/s, ω is 1000 rad/s, and Δx is 0.5 cm, then

$$\begin{aligned}\Delta p_{rms} &= -j0.005 \times 1000 \times 1.18 \times 0.01 \\ &= -j0.059 \text{ Pa.}\end{aligned}$$

We shall have an opportunity in Chapter 5 of this text to see a practical application of these equations to the measurement of particle velocity by a velocity microphone.

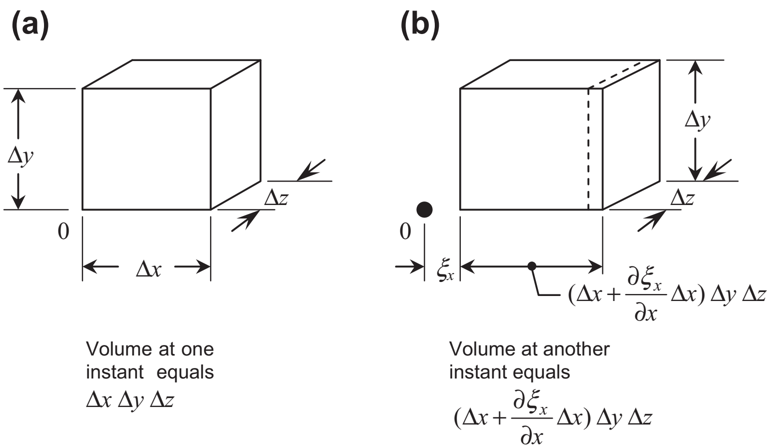

FIG. 2.2 Change in volume of the box with change in position.

From (a) and (b) it is seen that the incremental change in volume of the box is $\tau = (\partial \xi_x / \partial x)\, \Delta x\, \Delta y\, \Delta z$.

2.2.4 The wave equation in rectangular coordinates

One-dimensional derivation

The one-dimensional wave equation is obtained by combining the equation of motion (2.4a), the gas law (2.10), and the continuity equation (2.13a). Combination of (2.10) and (2.3a) gives

$$\frac{\partial p}{\partial t} = -\gamma P_0 \frac{\partial u}{\partial x}. \qquad (2.14a)$$

Differentiate (2.14a) with respect to t:

$$\frac{\partial^2 p}{\partial t^2} = -\gamma P_0 \frac{\partial^2 u}{\partial t \partial x}. \qquad (2.15a)$$

Differentiate (2.4a) with respect to x:

$$-\frac{\partial^2 p}{\partial x^2} = \rho_0 \frac{\partial^2 u}{\partial x \partial t}. \qquad (2.16a)$$

Assuming interchangeability of the x and t derivatives, and combining (2.15a) and (2.16a), we get

$$\frac{\partial^2 p}{\partial x^2} = \frac{\rho_0}{\gamma P_0} \frac{\partial^2 p}{\partial t^2}. \qquad (2.18a)$$

Three-dimensional derivation

The three-dimensional wave equation is obtained by combining the equation of motion (2.4b), the gas law (2.10), and the continuity equation (2.13b). Combination of (2.10) and (2.13b) gives

$$\frac{\partial p}{\partial t} = -\gamma P_0 \, \nabla \cdot \mathbf{q}. \qquad (2.14b)$$

Differentiate (2.14b) with respect to t:

$$\frac{\partial^2 p}{\partial t^2} = -\gamma P_0 \, \nabla \cdot \frac{\partial \mathbf{q}}{\partial t}. \qquad (2.15b)$$

Take the divergence of each side of Eq. (2.4b):

$$-\nabla \cdot (\nabla p) = \rho_0 \, \nabla \cdot \frac{\partial \mathbf{q}}{\partial t}. \qquad (2.16b)$$

Replacing the $\nabla \cdot (\nabla\ p)$ by $\nabla^2 p$, we get

$$-\nabla^2 p = \rho_0 \, \nabla \cdot \frac{\partial \mathbf{q}}{\partial t}, \qquad (2.17)$$

where ∇^2 is the operator called the Laplacian. Combining (2.15b) and (2.17), we get

$$\nabla^2 p = \frac{\rho_0}{\gamma P_0} \frac{\partial^2 p}{\partial t^2}. \qquad (2.18b)$$

Let us, by definition, set

$$c^2 = \frac{\gamma P_0}{\rho_0}. \qquad (2.19)$$

We shall see later that c is the speed of propagation of the sound wave in the medium. Also, the quantity γP_0 is the *bulk modulus* of the fluid medium.

We obtain the one-dimensional wave equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}. \qquad (2.20a)$$

We obtain the three-dimensional wave equation

$$\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}. \qquad (2.20b)$$

In rectangular coordinates

$$\nabla^2 p \equiv \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}. \qquad (2.21)$$

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We could also have eliminated p and retained u, in which case we would have

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{2.22a}$$

We could also have eliminated p and retained $\mathbf{q}$, in which case we would have

$$\nabla^2 \mathbf{q} = \frac{1}{c^2}\frac{\partial^2 \mathbf{q}}{\partial t^2}, \tag{2.22b}$$

where $\nabla^2 \mathbf{q} = \nabla(\nabla \cdot q)$when there is no rotation in the medium.

Equations (2.20) and (2.22) apply to sound waves of "small" magnitude propagating in a source-free, homogeneous, isotropic, frictionless gas at rest.

2.2.5 The wave equation in cylindrical coordinates

The one-dimensional wave equations derived above are for plane-wave propagation along one dimension of a rectangular coordinate system. In the case of a line source, such as a vertical stack of loudspeakers in an auditorium, the sound spreads out radially in all directions as a cylindrical wave. To apply the wave equation to cylindrical waves, we must replace the operators on the left side of Eqs. (2.20) and (2.22) by operators appropriate to cylindrical coordinates. Assuming equal radiation in all directions about the axis of symmetry, the wave equation in one-dimensional cylindrical coordinates is

$$\frac{\partial^2 p}{\partial w^2} + \frac{1}{w}\frac{\partial p}{\partial w} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{2.23}$$

where w is the radial distance from the axis of symmetry or source if it is a line source.

2.2.6 The wave equation in spherical coordinates

In an anechoic (echo-free) chamber or in free space, we frequently wish to express mathematically the radiation of sound from a spherical (nondirectional) source of sound. In this case, the sound wave will expand as it travels away from the source, and the wave front always will be a spherical surface. To apply the wave equation to spherical waves, we must replace the operators on the left side of Eqs. (2.20) and (2.22) by operators appropriate to spherical coordinates.

Assuming equal radiation in all directions, the wave equation in one-dimensional spherical coordinates is

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{2.24}$$

where r is the distance from the origin of the spherical coordinate system or source if it is a point source. Simple differentiation will show that (2.24) can also be written

$$\frac{\partial^2 (pr)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 (pr)}{\partial t^2}. \tag{2.25}$$

It is interesting to note that this equation has exactly the same form as Eq. (2.20a). Hence, the same formal solution will apply to either equation except that the dependent variable is $p(x,t)$ in one case and

$p(r,t)r$ in the other case. The latter suggests that the solution for the spherical wave equation is of the same form as that to the plane wave equation, but divided by r as will be shown further on in this text.

2.2.7 General one-dimensional wave equation (Webster's equation) [6]

A general one-dimensional equation which is often used to describe waves in flaring ducts or horns can be written

$$\frac{1}{S(x)}\frac{\partial}{\partial x}\left(S(x)\frac{\partial p}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{2.26}$$

which in expanded form becomes

$$\frac{\partial^2 p}{\partial x^2} + \frac{1}{S(x)}\left(\frac{\partial S(x)}{\partial x}\right)\frac{\partial p}{\partial x} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{2.27}$$

where $S(x)$ is a function that describes the variation of cross-sectional area with x. The first term is the Laplacian operator and is present in all plane wave equations. It describes the curvature of the pressure distribution along the x ordinate. The second term describes the pressure gradient due to the variation of cross-sectional area with x. Naturally, this term is absent in the case of a plane wave, where the cross-sectional area is constant (and can be infinite in theoretical models). In the case of a cylindrical wave, the area is given by $S(w) = 2\pi wl$, where l is the width of the wave along the axis of symmetry (again this can be infinite). Note that x is replaced by the radial ordinate w. Substituting $S(w) = 2\pi wl$ in Eq. (2.27) yields Eq. (2.23), the wave equation for a cylindrical wave. Likewise, substituting $S(r) = 4\pi r^2$ in Eq. (2.27) and replacing x with r yields Eq. (2.24), the wave equation for a spherical wave.

PART IV: SOLUTIONS OF THE WAVE EQUATION IN ONE DIMENSION

2.3 GENERAL SOLUTIONS OF THE ONE-DIMENSIONAL WAVE EQUATION

The one-dimensional wave equation was derived with either sound pressure or particle velocity as the dependent variable. Particle displacement, or the variational density, may also be used as the dependent variable. This can be seen from Eqs. (2.4a) and (2.13a) and the conservation of mass, which requires that the product of the density and the volume of a small box of gas remain constant. That is,

$$\rho' V = \rho_0 V_0 = \text{constant} \tag{2.28}$$

and so

$$\rho' dV = -V d\rho'. \tag{2.29}$$

Let

$$\rho' = \rho_0 + \rho, \tag{2.30}$$

where ρ is the incremental change in density. Then, approximately, from Eqs. (2.8) and (2.29),

$$\rho_0 \tau = -V_0 \rho. \tag{2.31}$$

Differentiating,

$$\frac{\partial \tau}{\partial t} = -\frac{V_0}{\rho_0}\frac{\partial \rho}{\partial t}$$

so that, from Eq. (2.13a),

$$\frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial u}{\partial x}. \tag{2.32}$$

Also, we know that the particle velocity is the time rate of change of the particle displacement.

$$u = \frac{\partial \xi_x}{\partial x}. \tag{2.33}$$

Inspection of Eqs. (2.4a), (2.13a), (2.32), and (2.33) shows that the pressure, particle velocity, particle displacement, and variational density are related to each other by derivatives and integrals in space and time. These operations performed on the wave equation do not change the form of the solution, as we shall see shortly. Since the form of the solution is not changed, the same wave equation may be used for determining density, displacement, or particle velocity as well as sound pressure by substituting p, or ξ_x, or u for p in Eq. (2.20a) or ρ, $\boldsymbol{\xi}$, or $\mathbf{q}$ for p in Eq. (2.20b), assuming, of course, that there is no rotation in the medium.

2.3.1 General solution

With pressure as the dependent variable, the wave equation is

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}. \tag{2.34}$$

The general solution to this equation is a sum of two terms,

$$p = f_1\left(t - \frac{x}{c}\right) + f_2\left(t + \frac{x}{c}\right), \tag{2.35}$$

where f_1 and f_2 are arbitrary functions. We assume only that they have continuous derivatives of the first and second order. Note that because t and x occur together, the first derivatives with respect to x and t are exactly the same except for a factor of $\pm c$.

The ratio x/c must have the dimensions of time, so that c is a speed. From

$$c^2 = \gamma P_0/\rho_0 \text{ [Eq.(2.19)]}$$

we find that

$$c = \left(1.4 \times \frac{10^5}{1.18}\right)^{1/2} = 344.4 \text{ m/s}$$

in air at an ambient pressure of 10^5 Pa and at 22°C. This quantity is nearly the same as the experimentally determined value of the speed of sound, 344.8 [see Eq. (1.8)], so that we recognize c as the speed at which a sound wave is propagated through the air.

From the general solution to the wave equation given in Eq. (2.35) we observe two very important facts:

The sound pressure at any point x in space can be separated into two components: an outgoing wave, $f_1(t - x/c)$, and a backward-traveling wave, $f_2(t + x/c)$.

Regardless of the shape of the outward-going wave (or of the backward-traveling wave), it is propagated without change of shape. To show this, let us assume that, at $t = t_1$, the sound pressure at $x = 0$ is $f_1(t_1)$. At a time $t + t_1 + t_2$ the sound wave will have traveled a distance x equal to $t_2 c$ m. At this new time the sound pressure is equal to

$$p = f_1(t_1 + t_2 - t_2 c) = f_1(t_1).$$

In other words the sound pressure has propagated without change. The same argument can be made for the backward-traveling wave which goes in the $-x$ direction.

It must be understood that inherent in Eqs. (2.34) and (2.35) are two assumptions. First, the wave is a plane wave, i.e., it does not expand laterally. Thus the sound pressure is not a function of the y and z ordinates but is a function of distance only along the x ordinate. Second, it is assumed that there are no losses or dispersion (scattering of the wave by turbulence or temperature gradients, etc.) in the air, so that the wave does not lose energy as it is propagated.

2.3.2 Steady-state solution

In nearly all the studies that we make in this text we are concerned with the steady state. Let us first consider the time-dependent part of the solution at a fixed point in space so that the pressure is only dependent upon time. As is well known from the theory of Fourier series, a steady-state periodic wave of arbitrary shape can be represented by a linear summation of sine-wave functions, each of which is of the form

$$p(t) = \sum_{n=-\infty}^{\infty} p_n(t), \tag{2.36}$$

where

$$p_n(t) = c_n e^{j\omega_n t} = c_n(\cos \omega_n t + j \sin \omega_n t), \tag{2.37}$$

where $\omega_n = n\omega = 2\pi\, nf$ is the angular frequency and c_n is the peak amplitude of the n^{th} component of the wave given by

$$c_n = \frac{1}{T}\int_0^T p(t) e^{-j\omega_n t} dt, \tag{2.38}$$

where $T = 1/f$ is the period of the wave. Taking the second time derivative of p_n yields

$$\frac{\partial^2}{\partial t^2}\, p_n(t) \;=\; \frac{\partial^2}{\partial t^2}\, c_n e^{j\omega_n t} \;=\; -\omega_n^2 c_n e^{j\omega_n t} \;=\; -\omega_n^2 p_n(t), \tag{2.39}$$

which gives the identities

$$\frac{\partial}{\partial t} \;=\; j\omega_n, \tag{2.40}$$

$$\frac{\partial^2}{\partial t^2} \;=\; -\omega_n^2. \tag{2.41}$$

Hence the steady-state plane-wave equation for any point in space can be written in the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\omega_n^2}{c^2}\right) p_n(x,t) \;=\; 0. \tag{2.42}$$

which is generally known as the *Helmholtz wave equation*. Because the wave is propagated without change of shape, we need consider, in the steady state, only those solutions to the wave equation for which the time dependence at each point in space is sinusoidal and which have the same angular frequencies $n\omega$ as the source. A general solution that satisfies this equation is given by

$$p_n(x,t) \;=\; \left(p_{n+} e^{-j\omega_n x/c} + p_{n-} e^{j\omega_n x/c}\right) e^{j\omega_n t}, \tag{2.43}$$

where the $+$ and $-$ subscripts indicate the forward and backward traveling waves respectively. In the steady state, therefore, we may replace f_1 and f_2 of Eq. (2.35) by a sum of functions each having a particular angular driving frequency ω_n so that

$$p(x,t) \;=\; \sum_{n=-\infty}^{\infty} p_n(x,t) \;=\; \sum_{n=-\infty}^{\infty} \Re\left(\left(p_{n+} e^{-j\omega_n x/c} + p_{n-} e^{j\omega_n x/c}\right) e^{j\omega_n t}\right). \tag{2.44}$$

Generally we omit writing $\Re$ although it always must be remembered that the real part must be taken when using the final expression for the sound pressure that would actually be observed, for example, when making an animated plot of a sound field.

It is customary in texts on acoustics to define a wave-number k where

$$k \;=\; \frac{\omega}{c} \;=\; \frac{2\pi f}{c} \;=\; \frac{2\pi}{\lambda}, \tag{2.45}$$

which can be considered as the spatial angular frequency in rad/m. When k is multiplied by a characteristic dimension such as the length of a tube or the radius of a circular radiator, it forms a useful dimensionless parameter that is proportional to the frequency. Let us now drop $\Re$ and the subscript n for convenience. Also, we will replace the factor $e^{j\omega t}$ with a tilde. Any one term of Eq. (2.44), with these changes, becomes

$$\tilde{p}(x) \;=\; \tilde{p}_+ e^{-jkx} + \tilde{p}_- e^{jkx}. \tag{2.46}$$

Equation (2.46) represents two traveling waves: one with amplitude $\tilde{p}_+$ traveling in the positive x direction and the other with amplitude $\tilde{p}_-$ traveling in the negative x direction, where the amplitudes are independent of position x. The appearance of these two solutions occurs because in solving the wave equation we have not specified the direction of travel or any boundary conditions and so the result simply tells us that these solutions can occur. The complex values of $\tilde{p}_+$ and $\tilde{p}_-$ are determined from the boundary conditions. The real parts of the forward and reverse traveling solutions are represented in Fig. 2.3 (a) and (b) respectively, which shows the waveforms in space at a snapshot in time, whereas if the plots were animated, they would be moving in the directions of the arrows. At any fixed point, the pressure or velocity would oscillate as the wave passed through it, with the oscillations having the same shape versus time as versus distance. This is a property of plane waves where the waves propagate without changing shape. Similarly, the solution to Eq. (2.22a) for velocity, assuming steady-state conditions is

$$\tilde{u}(x) = \tilde{u}_+ e^{-jkx} + \tilde{u}_- e^{jkx}. \tag{2.47}$$

A similar expression for the velocity can also be obtained from the expression for the pressure by applying Eq. (2.4a) to Eq. (2.46):

$$\begin{aligned} \tilde{u}(x) &= \frac{1}{-j\omega\rho_0}\frac{\partial}{\partial x}\tilde{p}(x) \\ &= \frac{1}{\rho_0 c}\left(\tilde{p}_+ e^{-jkx} - \tilde{p}_- e^{jkx}\right), \end{aligned} \tag{2.48}$$

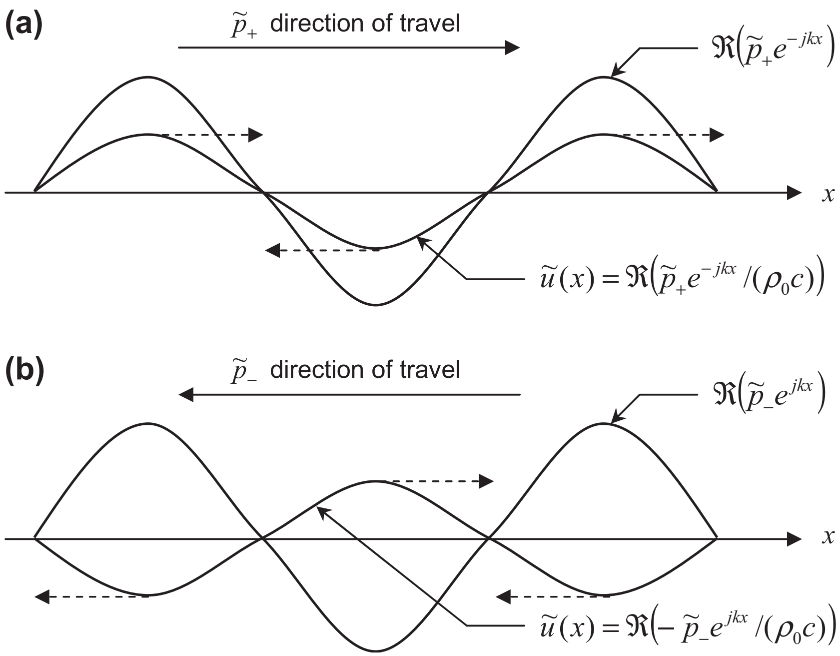

FIG. 2.3 Solutions to the steady-state one-dimensional wave equation.

Forward and reverse traveling waves.

the real part of which is also shown in Fig. 2.3. The wave equation (2.48) for velocity is similar to equation (2.46) for pressure except for one important difference, which is the minus sign preceding $\tilde{p}_-$. The reason for this is fairly simple. During a positive pressure half-cycle, the resulting velocity is always in the direction of travel. Therefore, in the case of the wave with amplitude $\tilde{p}_+$ traveling in the positive x direction, positive pressure produces positive velocity because it is in the positive x direction, as shown by the dashed arrows in Fig. 2.3a. However, in the case of the wave with amplitude $\tilde{p}_-$ traveling in the negative x direction, positive pressure produces negative velocity because it is in the negative x direction, as shown by the dashed arrows in Fig. 2.3b. Of course, the converse applies during a negative pressure half cycle. The ratio of pressure to particle velocity is the specific acoustic impedance Z_s of the medium, which is obtained by dividing the pressure from Eq. (2.46) by the velocity from Eq. (2.48) to give

$$Z_\mathrm{s} = \frac{\tilde{p}(x)}{\tilde{u}(x)} = \rho_0 c. \tag{2.49}$$

It is worth noting that in the case of freely traveling waves, which are also known as progressive waves, the pressure and particle velocity are in phase and hence the impedance has a real value. This is very much a characteristic of traveling longitudinal waves, a class that includes sound pressure waves because the particles oscillate in the direction of propagation as opposed to transverse waves whereby the medium oscillates in a direction at right angles to the direction of propagation. An example of the latter is the wave motion of a plucked string.

Example 2.2. Determine the power flow in a freely traveling wave at a fixed point as a function of time.

Answer:

$$\begin{aligned} p(t) &= K\cos\omega t \\ u(t) &= p(t)/\rho c \\ \textit{Power flow} &= p^* u = (K^2/\rho c)\cos^2\omega t = (K^2/\rho c)(1 - \sin^2\omega t) \end{aligned}$$

Thus the power flows by a point in a freely traveling wave like a series of "sausages". This is explained by referring back to Fig. 1.1. The vibrating surface sends power into the wave when it is moving either to the right or the left. At the instant whenever the surface changes direction, the power drops to zero.

Example 2.3. Assume that for the steady state, at a point $x = 0$, the sound pressure in a one-dimensional outward-traveling wave has the recurrent form shown by the dotted curve in Fig. Ex. 2.3a. This wave form is given by the real part of the equation

$$p(0,t) = 4e^{j628t} + 2e^{j1884t}.$$

(*a*) What are the particle velocity and the particle displacement as a function of time at $x = 5$ m? (*b*) What are the rms values of these two quantities? (c) Are the rms values dependent upon x?

Solution. a. We have for the solution of the wave equation giving both x and [see Eq. (2.46)]

$$p(x,t) = 4e^{j628(t-x/c)} + 2e^{j1884(t-x/c)}.$$

From Eq. (2.4a) we see that

$$u(x,t) = -\frac{1}{j\omega\rho_0}\frac{\partial p(x,t)}{\partial x}$$

or

$$u(x,t) = \frac{1}{\rho_0 c}\, p(x,t).$$

And from Eq. (2.33) we have

$$\xi(x,t) = \frac{1}{j\rho_0 c}\left(\frac{4}{628}\, e^{j628(t-x/c)} + \frac{2}{1884}\, e^{j1884(t-x/c)}\right).$$

At $x = 5$ m, $x/c = 5/344.8 = 0.0145$ s,

$$u(5,t) = \frac{1}{407}\left(4e^{j628(t-0.0145)} + 2e^{j1884(t-0.0145)}\right)$$

and

$$\xi(5,t) = \frac{1}{407}\left(\frac{4}{628}e^{j[628(t-0.0145)-(\pi/2)]} + \frac{2}{1884}e^{j[1884(t-0.0145)-(\pi/2)]}\right).$$

Taking the real parts of the two preceding equations,

$$u(5,t) = \frac{1}{407}\,(4\cos(628t - 9.1) + 2\cos(1884t - 27.3))$$

$$\xi(5,t) = \frac{1}{407}\left(\frac{4}{628}\,\sin(628t - 9.1) + \frac{2}{1884}\,\sin(1884t - 27.3)\right).$$

Note that each term in the particle displacement is 90° out of time phase with the velocity and that the wave shape is different. As might be expected, integration diminishes the higher frequencies. These equations are plotted in Fig. Ex. 2.3b.

b. The rms magnitude of a sine wave is equal to its peak amplitude divided by $\sqrt{2}$. This may be verified by squaring the sine wave and finding the average value over one cycle and then taking the square root of the result. If two sine waves of different frequencies are present at one time, the rms value of the combination is equal to the square root of the sums of the squares of the individual peak amplitudes divided by $\sqrt{2}$, so that

$$p = \frac{1}{\sqrt{2}}\sqrt{4^2 + 2^2} = 3.16 \text{ Pa},$$

$$u = \frac{1}{407\sqrt{2}}\sqrt{4^2 + 2^2} = 7.77 \times 10^{-3} \text{ m/s},$$

$$\xi_x = \frac{1}{407\sqrt{2}}\sqrt{\left(\frac{4}{628}\right)^2 + \left(\frac{2}{1884}\right)^2} = 1.12 \times 10^{-5} \text{ m}.$$

c. The rms values of u and ξ_x are independent of x for a plane progressive sound wave.

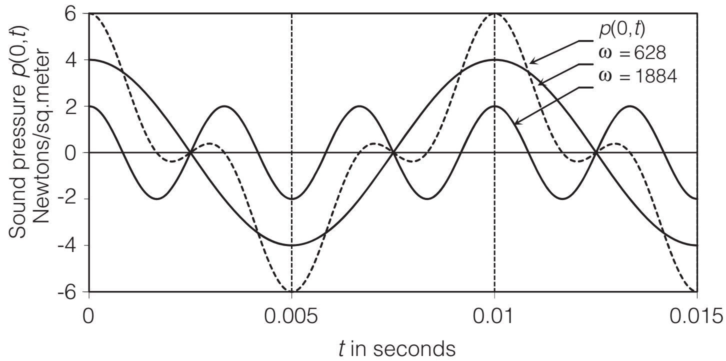

FIG. EX. 2.3A

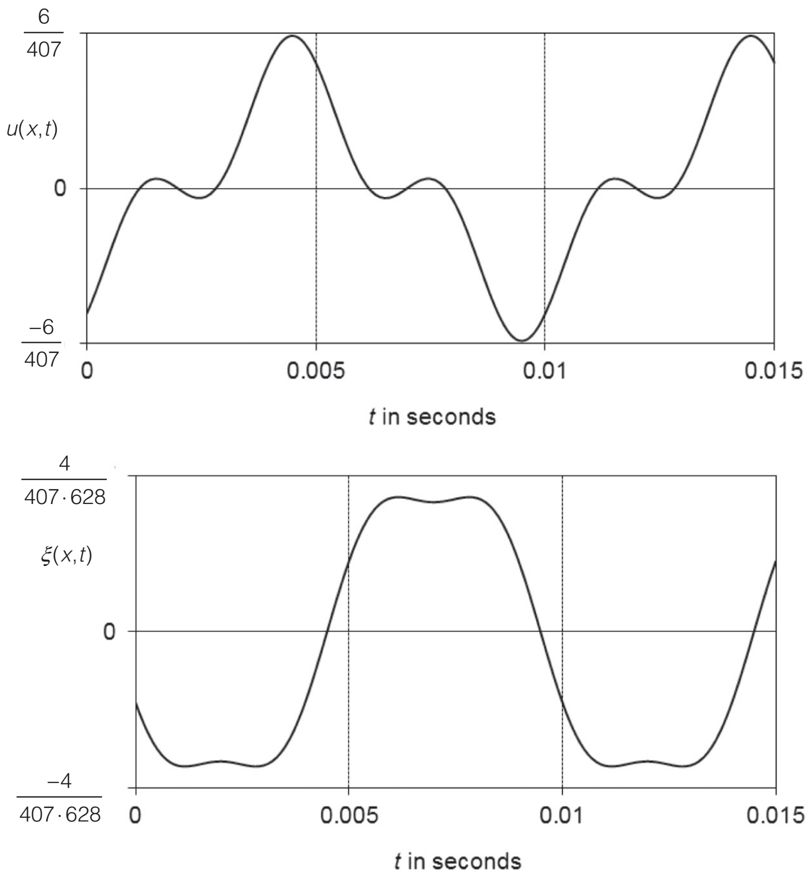

FIG. EX. 2.3B

2.4 SOLUTION OF WAVE EQUATION FOR AIR IN A TUBE TERMINATED BY AN IMPEDANCE

For this example of wave propagation, we shall consider a hollow cylindrical tube, terminated at one end ($x = 0$) by an impedance Z_T and at the other ($x = l$) end by a flat vibrating piston (see Fig. 2.4). Alternatively, we could have interchanged the positions of the piston and termination impedance, but the arrangement shown has been chosen because it simplifies the equations. For example, in the case of a rigid termination the particle velocity is shown to be proportional to $\sin x$ as opposed to $\sin(l - x)$. However, care needs to be taken when calculating the impedance where the velocity has to be taken as that in the negative x direction. The angular frequency of vibration of the piston is ω, and its rms velocity is $\tilde{u}_0$ at $x = l$. We shall assume that the diameter of the tube is sufficiently small so that the waves travel down the tube with plane wave fronts. In order for this be true, the ratio of the wavelength of the sound wave to the diameter of the tube must be greater than about 6.

Particle velocity. The form of solution we shall select is Eq. (2.48). If l is the length of the tube, then at $x = l$ the particle velocity must be equal to the velocity $\tilde{u}_0$ of the piston. The boundary conditions are:

$$\text{At } x = l,\ \tilde{u}(l) = \tilde{u}_0, \text{ so that}$$

$$\tilde{u}(l) = \frac{\tilde{p}_+ e^{-jkl} - \tilde{p}_- e^{jkl}}{\rho_0 c} = \tilde{u}_0. \tag{2.50}$$

At $x = 0$,

$$\tilde{p}(0)/(-\tilde{u}(0)) = Z_s(0) = Z_T,$$

where the pressure is taken from Eq. (2.46). Note that velocity is negative here because it is in the reverse x direction. Hence

$$\frac{\tilde{p}(0)}{-\tilde{u}(0)} = \frac{\tilde{p}_+ + \tilde{p}_-}{\tilde{p}_- - \tilde{p}_+}\rho_0 c = Z_T. \tag{2.51}$$

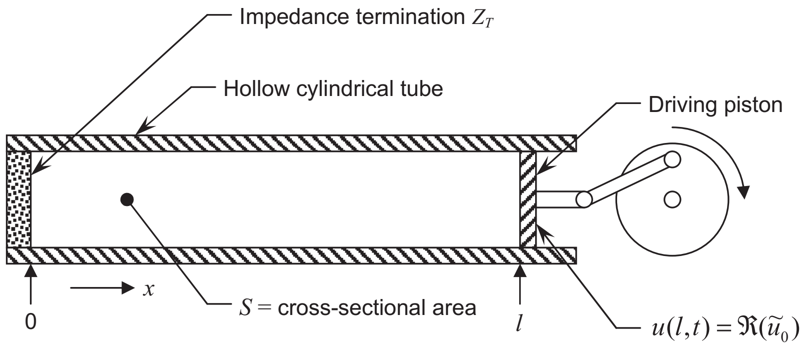

FIG. 2.4 Tube with rigid side walls and termination impedance Z_T.

The velocity at $x = 0$ has a value of $u_0 \cos \omega t$ m/s.

Transmitted and reflected pressures. Eliminating $\tilde{p}_-$ between Eqs. (2.50) and (2.51) yields

$$\tilde{p}_+ = \frac{\left(\rho_0 c - Z_T\right)\rho_0 c\tilde{u}_0}{\rho_0 c(e^{jkl} + e^{-jkl}) + Z_T(e^{jkl} - e^{-jkl})}. \tag{2.52}$$

Similarly, eliminating $\tilde{p}_+$ between Eqs. (2.50) and (2.51) yields

$$\tilde{p}_- = \frac{-(\rho_0 c + Z_T)\rho_0 c\tilde{u}_0}{\rho_0 c(e^{jkl} + e^{-jkl}) + Z_T(e^{jkl} - e^{-jkl})}. \tag{2.53}$$

Remember that

$$\sin y = (e^{jy} - e^{-jy})/(2j) \quad \text{and} \quad \cos y = (e^{jy} + e^{-jy})/2.$$

Hence

$$\tilde{p}_+ = \frac{(\rho_0 c - Z_T)\rho_0 c\tilde{u}_0}{2(\rho_0 c \cos kl + jZ_s \sin kl)} \tag{2.54}$$

and

$$\tilde{p}_- = \frac{-(\rho_0 c + Z_T)\rho_0 c\tilde{u}_0}{2(\rho_0 c \cos kl + jZ_s \sin kl)}, \tag{2.55}$$

where $\tilde{p}_-$ is the *transmitted* pressure magnitude and $\tilde{p}_+$ is *reflected* pressure magnitude. The amount of sound reflected depends on how the tube is terminated. The reflection coefficient Γ is given by

$$\Gamma = \frac{\tilde{p}_-}{\tilde{p}_+} = \frac{Z_T - \rho_0 c}{Z_T + \rho_0 c}. \tag{2.56}$$

In some places along the tube, the reflected wave will interfere *constructively* with the transmitted wave, thus producing a pressure *maximum*, and at others it will interfere *destructively* causing a pressure *minimum*. If the reflection is 100%, these maxima and minima become *anti-nodes* and *nodes* respectively. We shall examine these in greater detail in the next paragraph which describes the case of a *rigid termination*. The ratio of maximum to minimum pressure along the tube is given by the *Standing Wave Ratio* or *SWR* where

$$SWR = \frac{Z_T}{\rho_0 c} = \frac{1 + \Gamma}{1 - \Gamma}. \tag{2.57}$$

Of particular interest are the cases where (1) the pressure is zero at the termination (resilient termination), (2) the termination impedance is equal to the characteristic impedance of the tube (anechoic termination), and (3) the velocity is zero at the termination (rigid termination). All three cases are summarized in Table 2.1. The first case produces maximum negative reflection (that is, with reversed phase), the second zero reflection, and the third maximum positive reflection.
Sound proofing materials are often defined by the *absorption coefficient* α, which is given by

$$\alpha = 1 - \Gamma = 2\rho_0 c/(\rho_0 c + Z_T).$$

Impedance. Inserting Eqs. (2.54) and (2.55) into Eqs. (2.46) and (2.48) gives us

$$\tilde{p}(x) = -\frac{Z_T \cos kx + j\rho_0 c \sin kx}{\rho_0 c \cos kl + jZ_T \sin kl}\rho_0 c\tilde{u}_0, \tag{2.58}$$

Table 2.1 Termination Impedances, Standing Wave Ratios, and Reflection Coefficients for Three Types of Tube Termination

Quantity	Resilient termination	Anechoic termination	Rigid termination
Termination impedance (Z_T)	0	$\rho_0 c$	∞
Standing wave ratio (*SWR*)	0	1	∞
Reflection coefficient (Γ)	−1	0	1
Absorption coefficient (α)	2	1	0

$$\tilde{u}(x) = \frac{\rho_0 c \cos kx + jZ_T \sin kx}{\rho_0 c \cos kl + jZ_T \sin kl}\tilde{u}_0. \tag{2.59}$$

The specific acoustic impedance Z_s along the tube is then given by the ratio of pressure to velocity:

$$Z_s(x) = \frac{\tilde{p}(x)}{-\tilde{u}(x)} = \frac{\dfrac{Z_T}{\rho_0 c} + j\tan kx}{1 + j\dfrac{Z_T}{\rho_0 c}\tan kx}\rho_0 c. \tag{2.60}$$

Let us now recast this equation into two series impedances, as seen at the piston:

$$Z_s(l) = \left(\frac{1}{Z_T} + j\frac{1}{\rho_0 c}\tan kl\right)^{-1} + \left(\frac{Z_T}{\rho_0^2 c^2} - j\frac{\cot kl}{\rho_0 c}\right)^{-1}, \tag{2.61}$$

the equivalent circuit for which is shown in Fig. 2.5.

Amazingly, a tube with any termination impedance Z_T can be represented by the impedance of a blocked tube (with $Z_T = \infty$) in series with an open tube (with $Z_T = 0$) and two external impedances connected across them, which are related to the termination impedance Z_T and characteristic impedance $\rho_0 c$. However, this makes more sense when we consider that when the impedance of the open tube is zero, the impedance of the blocked tube is infinite and vice versa. Hence the impedance seen between the input terminals simply alternates between Z_T and $(\rho_0 c)^2/Z_T$ as we sweep the piston generator frequency, which is entirely consistent with the standing wave ratio. When $Z_T = \rho_0 c$, the two

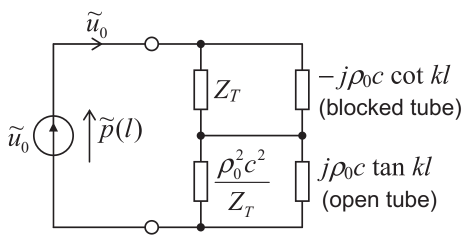

FIG. 2.5 Equivalent electrical circuit for a tube with a termination impedance Z_T, in which the single tube is represented by two tube impedances, each in parallel with an external impedance.

The piston is represented by a current generator. The reason for this will become clearer in the next chapter.

series impedances are the complex conjugates of each other and we just see the characteristic impedance $\rho_0 c$ at the input terminals. We shall use equivalent circuits extensively in this text. Also, it will be shown in Figs. 10.6 and 10.7 how the impedances of a blocked tube and open tube respectively may be represented by arrays of electrical circuit elements.

Impedance measurement. If we place two probe microphones in the tube, with one at $x = l_1$ and the other at $x = l_2$, then the ratio of the pressures $\tilde{p}(l_1)$ and $\tilde{p}(l_2)$ is given by

$$\frac{\tilde{p}(l_1)}{\tilde{p}(l_2)} = \frac{Z_T \cos kl_1 + j\rho_0 c \sin kl_1}{Z_T \cos kl_2 + j\rho_0 c \sin kl_2}, \tag{2.62}$$

which is independent of $\tilde{u}_0$. The termination impedance is then given by

$$\frac{Z_T}{\rho_0 c} = -j\frac{\sin kl_1 - (\sin kl_2)\tilde{p}(l_1)/\tilde{p}(l_2)}{\cos kl_1 - (\cos kl_2)\tilde{p}(l_1)/\tilde{p}(l_2)}, \tag{2.63}$$

which is the principle of an *impedance tube* which is used for measuring samples of material for which the impedance is unknown. An elegant feature of the method is that the measurement is independent of the piston velocity or actual magnitudes of the pressures. Only the relative pressure ratio is needed to calculate the impedance. However, when the impedance is a large multiple (or small fraction) of $\rho_0 c$, the calibration of the microphones becomes very critical, as does the accuracy of the distances l_1 and l_2 between them and the sample.

Rigid termination (infinite impedance). If we let $Z_T = \infty$ in Eq. (2.59), the tube is terminated with a rigid wall, which gives us

$$\tilde{u}(x) = \tilde{u}_0 \frac{\sin kx}{\sin kl} \tag{2.64}$$

or

$$u(x,t) = u_0 e^{j\omega t} \frac{\sin kx}{\sin kl}. \tag{2.65}$$

Refer to Fig. 2.6. If the length l and the frequency are held constant, the particle velocity will vary from a value of zero at $x = 0$ to a maximum at $x = \lambda/4$, that is, at x equal to one-fourth wavelength. In the entire length of the tube the particle velocity varies according to a sine function.

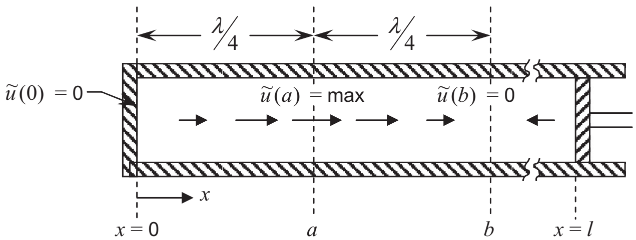

FIG. 2.6 Portion of the tube with a rigid termination showing the direction and magnitude of movement of the air particles as a function of *x*.

At position *a*, the particle velocity and displacement are a maximum. At position *b*, they are zero.

Between the end of the tube and the $\lambda/4$ point, the oscillatory motions are *in phase*. In other words, there is no progressive phase shift with x. This type of wave is called a standing wave [5] because, in the equation, x and ct do not occur as a difference or a sum in the argument of the exponential function. Hence the wave is not propagated. In cases where there are absolutely no losses, the term stationary wave [5] is also used, although this can only be approximated in practice.

In the region between $x = \lambda/4$ and $x = \lambda/2$, the particle velocity still has the same phase except that its amplitude decreases sinusoidally. At $x = \lambda/2$, the particle velocity is zero. In the region between $x = \lambda/2$ and $x = \lambda$ the particle velocity varies with x according to a sine function, but the particles move 180° out of phase with those between 0 and $\lambda/2$. This is seen from Eq. (2.64), wherein the sines of arguments greater than π are negative.

If we fix our position at some particular value of x and assume constant l, then, as we vary frequency, both the numerator and denominator of Eq. (2.64) will vary. When kl is some multiple of π, the particle velocity will become very large, except at $x = l$ or at points where kx is a multiple of π, that is, at points where x equals multiples of $\lambda/2$. Then for $kl = n\pi$

$$l\big|_{\tilde{u}=\infty} = \frac{n\lambda}{2} \quad n = 1, 2, 3, \ldots \tag{2.66}$$

Equation (2.64) would indicate an infinite velocity under this condition. In reality, the presence of some dissipation in the tube, which was neglected in the derivation of the wave equation, will keep the particle velocity finite, though large.

The particle velocity $\tilde{u}(x)$ will be zero at those parts of the tube where $kx = n\pi$ and n is an integer or zero. [7] That is,

$$x\big|_{\tilde{u}=0} = \frac{\lambda}{2}n \quad n = 1, 2, 3, \ldots \tag{2.67}$$

In other words, there will be planes of zero particle velocity at points along the length of the tube whenever l is greater than $\lambda/2$.

Some examples of the particle velocity for l slightly greater than various multiples of $\lambda/2$ are shown in Fig. 2.7. Two things in particular are apparent from inspection of these graphs. First, the quantity n determines the approximate number of half wavelengths that exist between the two ends of the tube. Secondly, for a fixed $\tilde{u}_0$, the maximum velocity of the wave in the tube will depend on which part of the sine wave falls at $x = l$. For example, if $l - n\lambda/2 = \lambda/4$, the maximum amplitude in the tube will be the same as that at the piston. If $l - n\lambda/2$ is very near zero, the maximum velocity in the tube will become very large.

Let us choose a frequency such that $n = 2$ as shown. Two factors determine the amplitude of the sine function in the tube. First, at $x = l$ the sine curve must pass through the point u_0. Second, at $x = 0$ the sine curve must pass through zero. It is obvious that one and only one sine wave meeting these conditions can be drawn so that the amplitude is determined. Similarly, we could have chosen a frequency such that $n = 2$, but where the length of the tube is slightly less than two half wavelengths. If this case had been asked for, the sine wave would have ended with a negative instead of a positive slope at $x = l$.

Sound pressure. The sound pressure in the tube may be found from the velocity with the aid of the equation of motion [Eq. (2.4a)], which, in the steady state, becomes

$$\tilde{p}(x) = -j\omega\rho_0 \int \tilde{u}(x)\, dx. \tag{2.68}$$

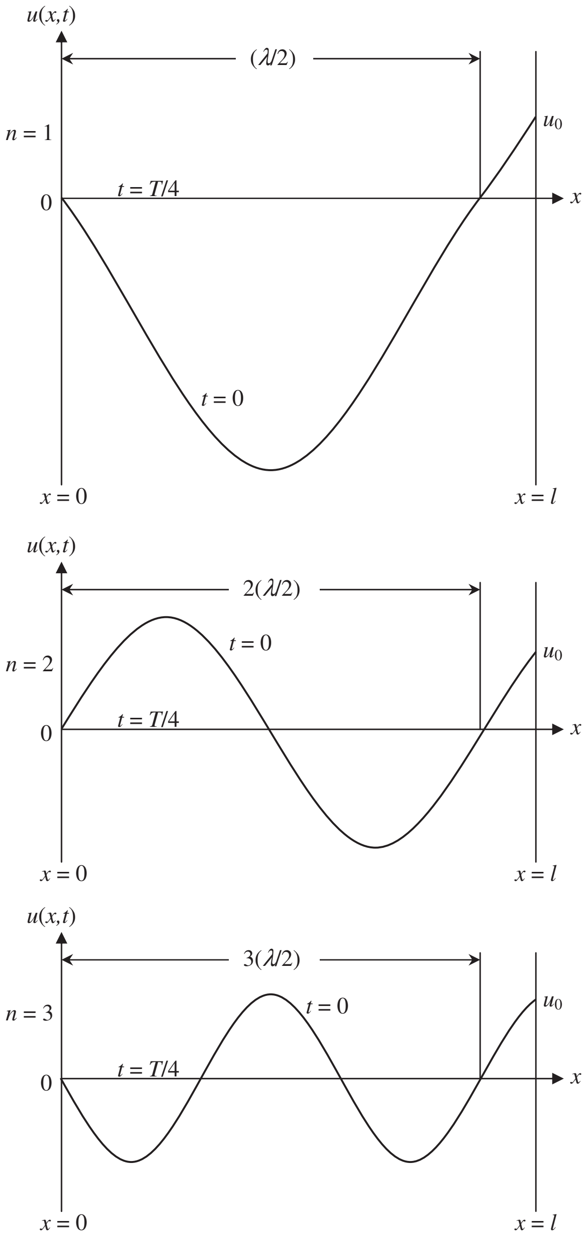

FIG. 2.7 Variation of the particle velocity $u(x,t)$ for $t = 0$, as a function of the distance along the tube of Fig. 2.4 for three frequencies, i.e., for three wavelengths.

At $x = l$, the rms particle velocity is u_0, and at $x = 0$, the particle velocity is zero. The period $T = 1/f$.

The constant of integration in Eq. (2.68), resulting from the integration of Eq. (2.4a), must be independent of x because we integrated with respect to x. The constant then represents an increment to the ambient pressure of the entire medium through which the wave is passing. Such an increment does not exist in our tube, so that in Eq. (2.68) we have set the constant of integration equal to zero. Integration of Eq. (2.68), after we have replaced $\tilde{u}(x)$ by its value from Eq. (2.64), yields

$$\tilde{p}(x) = j\rho_0 c\tilde{u}_0 \frac{\cos kx}{\sin kl} \tag{2.69}$$

or

$$p(x,t) = j\rho_0 c u_0 e^{j\omega t} \frac{\cos kx}{\sin kl}. \tag{2.70}$$

This result could alternatively have been obtained by setting $Z_T = \infty$ in Eq. (2.58). The pressure $\tilde{p}$ will be zero at those points of the tube where $kx = n\pi + \pi/2$ (where n is an integer or zero),

$$x|_{\tilde{p}=0} = \frac{\lambda}{2}\left(n + \frac{1}{2}\right). \tag{2.71}$$

The pressure will equal zero at one or more planes in the tube whenever l is greater than $\lambda/4$. Some examples are shown in Fig. 2.8. Here again, quantity n is equal to an approximate number of half wavelengths in the tube.

Refer once more to Fig. 2.7 which is drawn for $t = 0$. The instantaneous particle velocity is at its maximum (as a function of time). By comparison, in Fig. 2.8 at $t = 0$, the instantaneous sound pressure is zero. At a later time $t = T/4 = 1/(4f)$, the instantaneous particle velocity has become zero and the instantaneous sound pressure has reached its maximum. Equations (2.64) and (2.69) say that whenever kx is a small number the sound pressure leads by one-fourth period behind the particle velocity. At some other places in the tube, for example when x lies between $\lambda/4$ and $\lambda/2$, the sound pressure lags the particle velocity by one-fourth period.

To see the relation between p and u more clearly, refer to Fig. 2.7 and Fig. 2.8, for the case of $n = 2$. In Fig. 2.7, the particle motion is to the right whenever u is positive and to the left when it is negative. Hence, at $x = \lambda/2$, the particles on either side are moving toward each other, so that one-fourth period later the sound pressure will have built up to a maximum, as can be seen from Fig. 2.8. At the $2\lambda/2$ point, the particles are moving apart, so that the pressure is dropping to below barometric as can be seen from Fig. 2.8.

Fig. 2.7 and Fig. 2.8 also reveal that, wherever along the tube the magnitude of the velocity is zero, the magnitude of the pressure is a maximum, and vice versa. Hence, for maximum pressure, Eq. (2.67) applies.

Specific acoustic impedance. It still remains for us to solve for the specific acoustic impedance Z_s, at any plane x, in the tube. Taking the ratio of Eq. (2.69) to Eq. (2.64) or setting $Z_T = \infty$ in Eq. (2.60) yields

$$Z_s = \frac{\tilde{p}(l')}{-\tilde{u}(l')} = -j\rho_0 c \cot kl' = jX_s, \tag{2.72}$$

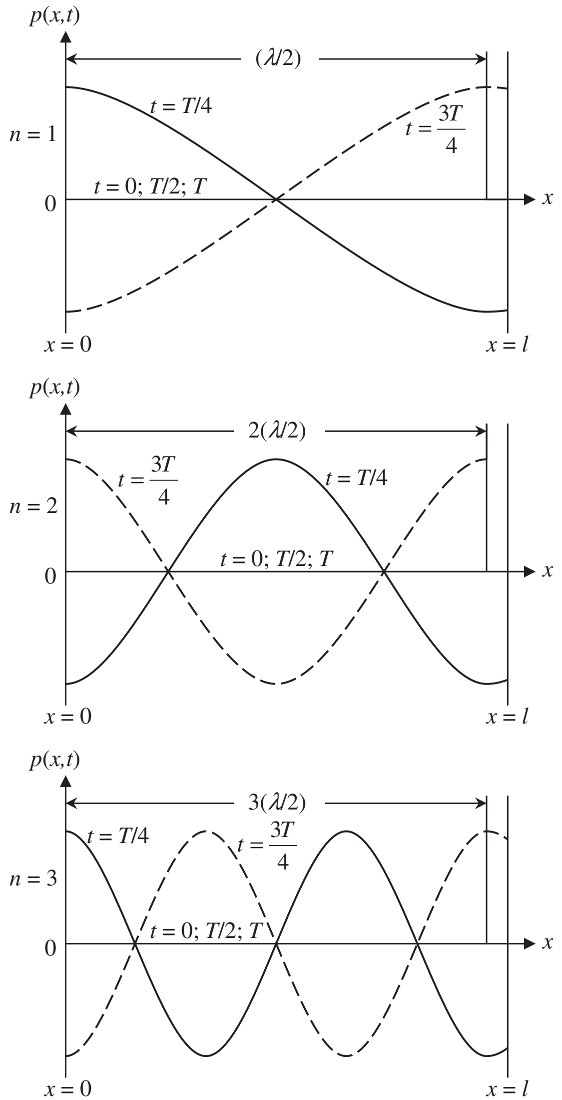

FIG. 2.8 Variation of the sound pressure *p* (*x*,*t*) as a function of the distance along the tube for three frequencies, i.e., for three wavelengths.

At $x = l$, the rms particle velocity is u_0, and at $x = 0$, it is zero. The period T equals $1/f$.

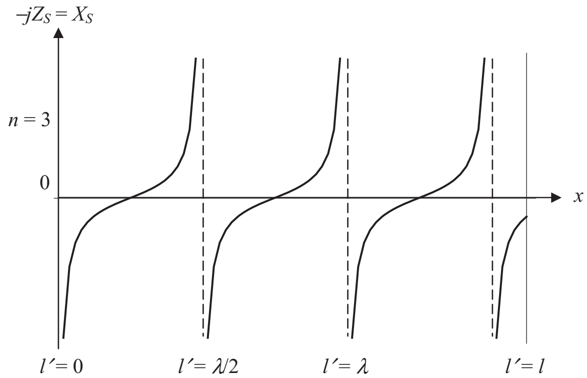

FIG. 2.9 The specific acoustic reactance (p_{rms}/u_{rms}) along the tube of Fig. 2.4 for a particular frequency, i.e., a particular wavelength where 3(λ/2) is a little less than the tube length *l*.

For this case, the number of zeros is three, and the number of poles is four.

where X_s is the reactance, and where we have set

$$x = l'. \tag{2.73}$$

That is, l' is the distance between any plane x in Fig. 2.4 and the rigid end of the tube at 0. The $-j$ indicates that at low frequencies where cot $kl' \approx 1/kl'$ the particle velocity leads the pressure in time by 90° and the reactance X_s is negative. At all frequencies the impedance is reactive and either leads or lags the pressure by exactly 90° depending, respectively, on whether X_s is negative or positive. The reactance X_s varies as shown in Fig. 2.9. If the value of kl' is small, we may approximate the cotangent by the first two terms of a series

$$\cot kl' \approx \frac{1}{kl'} - \frac{kl'}{3}. \tag{2.74}$$

This approximation is valid whenever the product of frequency times the distance from the rigid end of the tube to the point of measurement is very small. If the second term is very small, then it may be neglected with respect to the first.

Let us see how small the ratio of the distance l' to the wavelength λ must be if the second term of Eq. (2.74) is to be 3% or less of the first term. That is, let us solve for l'/λ from

$$\frac{2\pi l'}{3\lambda} \le 0.03\frac{\lambda}{2\pi l'}, \tag{2.75}$$

which gives us

$$\frac{l'}{\lambda} \le 0.05. \tag{2.76}$$

In other words, if cot kl' is to be replaced within an accuracy of 3% by the first term of its series expansion, l' must be less than one-twentieth wavelength in magnitude.

Assuming $l' < \lambda/20$, Eq. (2.72) becomes

$$Z_s = jX_s = -j\frac{\rho_0 c}{kl'} = \frac{1}{j\omega(l'/\rho_0 c^2)} \equiv \frac{1}{j\omega C_s} \text{ rayls.} \tag{2.77}$$

Hence, the specific acoustic impedance of a short length of tube can be represented as a "capacitance" called specific acoustic compliance, of magnitude $C_s = l'/\rho_0 c^2$. Note also that $C_s = l'/\gamma P_0$, because of Eq. (2.19).

The acoustic impedance is of the same type, except that an area factor appears so that

$$Z_A = \frac{\widetilde{p}}{S\widetilde{u}} = \frac{1}{j\omega(V/\rho_0 c^2)} \equiv \frac{1}{j\omega C_A} \text{ N}\cdot\text{s/m}^5, \tag{2.78}$$

where $V = l'S$ is the volume and S is the area of cross section of the tube. C_A is called the *acoustic compliance* and equals $V/\rho_0 c^2$. Note also that $C_A = V/\gamma P_0$, from Eq. (2.19).

Example 2.3. A cylindrical tube is to be used in an acoustic device as an impedance element. (a) The impedance desired is that of a compliance. What length should it have to yield a reactance of 1.4×10^3 rayls at an angular frequency of 1000 rad/s? (b) What is the relative magnitude of the first and second terms of Eq. (2.74) for this case?

Solution. The reactance of such a tube is

$$(a) \qquad X_s = 1.4 \times 10^3 = \frac{\gamma P_0}{\omega l'} = \frac{1.4 \times 10^5}{10^3 l'}.$$

Hence, $l' = 0.1$ m.

$$(b) \qquad \frac{kl'}{3} \div \frac{1}{kl'} = \frac{k^2 l'^2}{3} = \frac{\omega^2 l'^2}{3c^2} = \frac{10^6 \times 10^{-2}}{(3)(344.8)^2} = 0.028.$$

Hence, the second term is about 3% of the first term.

2.5 SOLUTION OF WAVE EQUATION FOR AIR IN A TUBE FILLED WITH ABSORBENT MATERIAL

Ducts and tubes are often filled with absorbent material in order to minimize standing waves, such as in transmission-line loudspeaker enclosures or exhaust-pipe mufflers, for example. Let us now modify the one-dimensional wave equation in rectangular coordinates, Eq. (2.34), taking into account the thermal and viscous losses in the material

$$\left(P_0 \frac{\partial^2}{\partial x^2} - j\omega R_f + \omega^2 \rho_0\right)\tilde{p} = 0, \tag{2.79}$$

where in the steady state we have let $\partial^2/\partial t^2 = -\omega^2$ and R_f is the specific flow resistance *per unit length* of the absorptive material in rayls/m. For simplicity we are assuming that the resistance is constant for all frequencies. A more comprehensive treatment of sound in absorbent materials will be given in Sec. 6.6. Notice too that we have omitted the specific heat ratio γ because we are assuming that the heat conduction within the material is such that the pressure fluctuations are isothermal. We define a complex density by

$$\boldsymbol{\rho} = \rho_0 + \frac{R_f}{j\omega}, \tag{2.80}$$

so that the wave equation simplifies to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{\boldsymbol{c}^2}\right)\tilde{p} = 0, \tag{2.81}$$

where

$$\boldsymbol{c} = \sqrt{\frac{P_0}{\boldsymbol{\rho}}}. \tag{2.82}$$

Hence the solution is

$$\tilde{p}(x) = \tilde{p}_+ e^{-j\boldsymbol{k}x} + \tilde{p}_- e^{j\boldsymbol{k}x}, \tag{2.83}$$

where the complex wave number is given by

$$\boldsymbol{k} = \frac{\omega}{\boldsymbol{c}} = \omega\sqrt{\frac{\boldsymbol{\rho}}{P_0}}, \tag{2.84}$$

and the characteristic impedance of the tube is

$$Z_s = \sqrt{\boldsymbol{\rho} P_0}. \tag{2.85}$$

In general, viscous or flow losses are dynamic and therefore associated with a change in the density of the medium whereas thermal conduction is static and therefore associated with a change in the bulk modulus. Viscous and thermal losses also occur in narrow unfilled tubes and these will be treated in some detail in Secs. 3.22 and 3.23.

2.6 FREELY TRAVELING PLANE WAVE

Sound pressure. If the rigid termination of Fig. 2.4 is replaced by a perfectly absorbing termination, a backward-traveling wave will not occur. Hence, Eq. (2.46) becomes

$$\tilde{p}(x) = \tilde{p}_+ e^{-jkx}, \tag{2.86}$$

where $\tilde{p}_+$ is the complex amplitude of the wave. This equation also applies to a plane wave traveling in free space.

Particle velocity. From Eq. (2.4a) in the steady state, we have

$$\tilde{u}(x) = -\frac{1}{jk\rho_0 c}\frac{\partial}{\partial x}\tilde{p}(x). \tag{2.87}$$

Hence,

$$\tilde{u}(x) = \frac{\tilde{p}_+}{\rho_0 c} e^{-jkx} = \frac{\tilde{p}(x)}{\rho_0 c}. \tag{2.88}$$

The particle velocity and the sound pressure are in phase. This is mathematical proof of the statement made in connection with the qualitative discussion of the wave propagated from a vibrating wall in Chapter 1 and Fig. 1.1.

Specific acoustic impedance. The specific acoustic impedance is

$$Z_s = \frac{\tilde{p}(x)}{\tilde{u}(x)} = \rho_0 c \text{ rayls}. \tag{2.89}$$

This equation says that in a plane freely traveling wave the specific acoustic impedance is purely resistive and is equal to the product of the average density of the gas and the speed of sound. This particular quantity is generally called the *characteristic impedance of the gas* because its magnitude depends on the properties of the gas alone. It is a quantity that is analogous to the surge impedance of an infinite electrical line. For air at 22°C and a barometric pressure of 10^5 Pa, its magnitude is 407 rayls.

2.7 FREELY TRAVELING CYLINDRICAL WAVE

Sound pressure. A solution to the cylindrical wave equation (2.23) is

$$\tilde{p}(w) = \tilde{p}_+ H_0^{(2)}(kw) + \tilde{p}_- H_0^{(1)}(kw), \tag{2.90}$$

where $\tilde{p}_+$ is the amplitude of the sound pressure in the outgoing wave at unit distance from the axis of symmetry and $\tilde{p}_-$ is the same for the reflected wave. $H_0{}^{(1)}(x)$ and $H_0{}^{(2)}(x)$ are Hankel functions defined by

$$H_0^{(1)}(x) = J_0(x) + jY_0(x), \tag{2.91}$$

$$H_0^{(2)}(x) = J_0(x) - jY_0(x), \tag{2.92}$$

where $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kind respectively, as plotted in Fig. 2.10. The "2" in parentheses denotes an outgoing cylindrical wave and the "1" denotes an incoming one. In the far field

$$\tilde{p}(w)\big|_{w\to\infty} = \sqrt{\frac{2}{\pi k w}}\left(\tilde{p}_+ e^{-j(kw-\pi/4)} + \tilde{p}_- e^{j(kw-\pi/4)}\right). \tag{2.93}$$

We can see from Fig. 2.10 that cylindrical waves, which are essentially two-dimensional due to the lack of axial dependency, differ from plane ones in two respects: Firstly the radial wavelength is longer nearer the axis of symmetry than in the far field. Secondly they decay in amplitude as they spread out, adopting an inverse square-root law in the far field. The latter makes sense when we consider that the area of the wave front is proportional to the radial distance w. The radiated power is the intensity

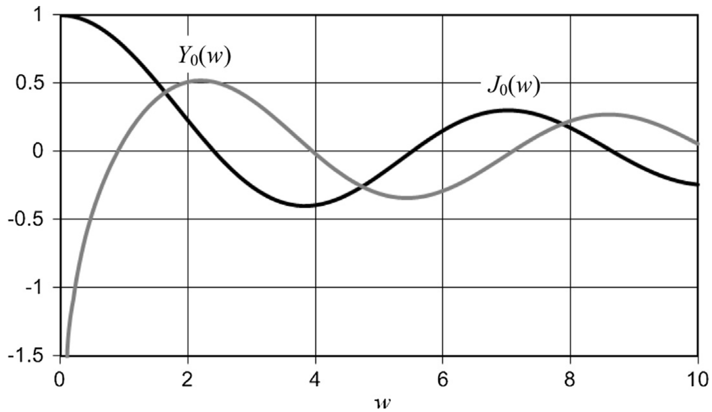

FIG. 2.10 Bessel functions of the first (black curve) and second (gray curve) kind.

multiplied by the area, where the intensity is given by Eq. (1.12). The intensity, in turn, is proportional to the square of the pressure and therefore inversely proportional to the radial distance. Hence the power remains constant. The same kind of wave deformation can be seen if you drop a pebble in a pond. Note the singularity in the $Y(x)$ function when $x = 0$. If there are no reflecting surfaces in the medium, only the first term of Eq. (2.90) is needed, i.e.,

$$\tilde{p}(w) = \tilde{p}_+ H_0^{(2)}(kw). \tag{2.94}$$

Particle velocity. With the aid of Eq. (2.4b), we solve for the particle velocity in the w direction:

$$\begin{aligned} \tilde{u}(w) &= -\frac{1}{jk\rho_0 c}\frac{\partial}{\partial w}\tilde{p}(w) \\ &= -j\frac{\tilde{p}_+}{\rho_0 c}H_1^{(2)}(kw). \end{aligned} \tag{2.95}$$

In the far field

$$\tilde{u}(w) = -j\frac{\tilde{p}_+}{\rho_0 c}\sqrt{\frac{2}{\pi k w}}e^{-j(kw-3\pi/4)} = \frac{\tilde{p}_+}{\rho_0 c}\sqrt{\frac{2}{\pi k w}}e^{-j(kw-\pi/4)}. \tag{2.96}$$

Specific acoustic impedance. The specific acoustic impedance is found from Eq. (2.94) divided by Eq. (2.95):

$$Z_s = \frac{\tilde{p}(w)}{\tilde{u}(w)} = j\rho_0 c\frac{H_0^{(2)}(kw)}{H_1^{(2)}(kw)} \text{ rayls.} \tag{2.97}$$

Plots of the magnitude and phase angle of the impedance as a function of kw are given in Fig. 2.11 and Fig. 2.12 respectively.

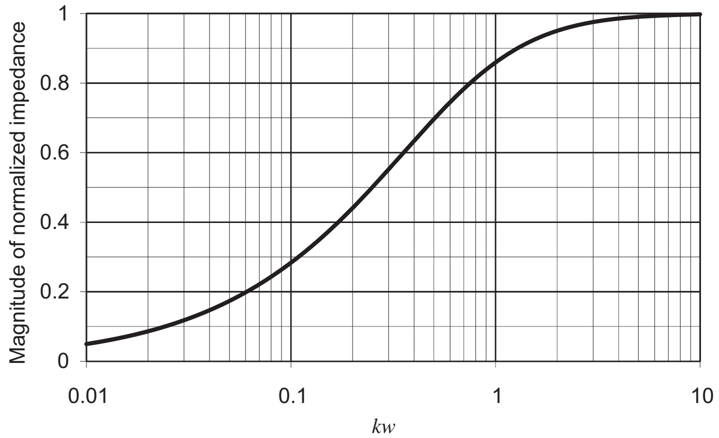

FIG. 2.11 Plot of the magnitude of the specific acoustic-impedance ratio $|Z_s|/(\rho_0 c)$ in a cylindrical freely traveling wave as a function of kw, where k is the wave-number equal to ω/c or $2\pi/\lambda$ and w is the distance from the axis of symmetry.

$|Z_s|$ is the magnitude of the ratio of pressure to particle velocity in a cylindrical free-traveling wave, and $\rho_0 c$ is the characteristic impedance of air.

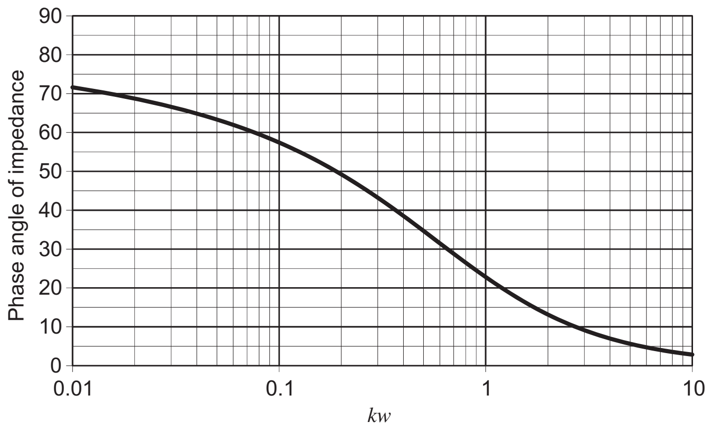

FIG. 2.12 Plot of the phase angle, in degrees, of the specific acoustic-impedance ratio $|Z_s|/\rho_0 c$ in a cylindrical wave as a function of kw, where k is the wave-number ω/c or $2\pi/\lambda$ and w is the distance from the axis of symmetry.

For large values of kw, that is, for large distances or for high frequencies, Eq. (2.97) becomes, approximately,

$$Z_s \approx \rho_0 c \text{ rayls.} \tag{2.98}$$

The impedance here is nearly purely resistive and approximately equal to the characteristic impedance for a plane freely traveling wave. In other words, the specific acoustic impedance a large distance from a cylindrical source in free space is nearly equal to that in a tube in which no reflections occur from the end opposite the source.

2.8 FREELY TRAVELING SPHERICAL WAVE

Sound pressure. A solution to the spherical wave equation (2.25) is

$$\tilde{p}(r) = \tilde{A}_{+}\frac{e^{-jkr}}{r} + \tilde{A}_{-}\frac{e^{jkr}}{r}, \tag{2.99}$$

where $\tilde{A}_{+}$ is the amplitude of the sound pressure in the outgoing wave at unit distance from the center of the sphere and $\tilde{A}_{-}$ is the same for the reflected wave. This equation can also be written in terms of *spherical* Hankel functions $h_0^{(1)}(x)$ and $h_0^{(2)}(x)$

$$\tilde{p}(r) = -jk\left(\tilde{A}_{+}h_0^{(2)}(kr) - \tilde{A}_{-}h_0^{(1)}(kr)\right), \tag{2.100}$$

which are also known as Hankel functions of fractional order, as defined by

$$h_0^{(1)}(x) = j_0(x) + jy_0(x), \tag{2.101}$$

$$h_0^{(2)}(x) = j_0(x) - jy_0(x), \tag{2.102}$$

$$j_0(x) = \frac{\sin x}{x}, \tag{2.103}$$

$$y_0(x) = -\frac{\cos x}{x}, \tag{2.104}$$

where $j_0(x)$ and $y_0(x)$ are spherical Bessel functions of the first and second kind respectively, as plotted in Fig. 2.13. The "2" in parentheses denotes an outgoing spherical wave and the "1" denotes an incoming one. These spherical Bessel functions are related to the cylindrical Bessel functions of half-integer order $J_{\frac{1}{2}}(x)$ and $Y_{\frac{1}{2}}(x)$ by

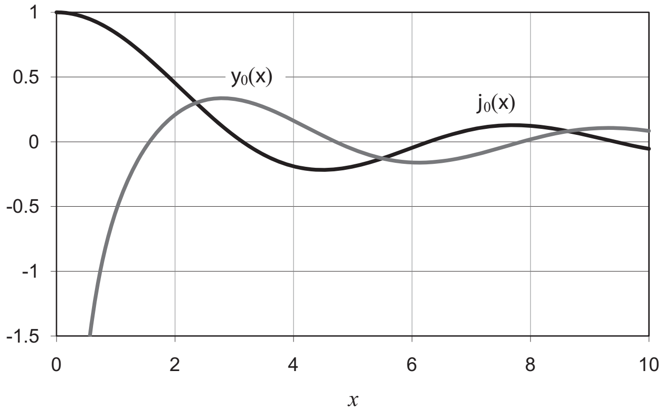

FIG. 2.13 Spherical Bessel functions of the first (black curve) and second (gray curve) kind.

$$j_0(x) = \sqrt{\frac{\pi}{2x}} J_{\frac{1}{2}}(x), \tag{2.105}$$

$$y_0(x) = \sqrt{\frac{\pi}{2x}} Y_{\frac{1}{2}}(x). \tag{2.106}$$

We can see that spherical waves differ from cylindrical ones in two respects: First, the radial wavelength remains constant as they progress, as is the case with plane waves. Second, although they decay in amplitude as they spread out, they adopt a direct inverse law in the far field. The latter makes sense when we consider that the area of the wave front is proportional to the square of the radial distance r. The radiated power is the intensity multiplied by the area, where the intensity is given by Eq. (1.12). The intensity, in turn, is proportional to the square of the pressure and therefore inversely proportional to the square of the radial distance. Hence the power remains constant.

If there are no reflecting surfaces in the medium, only the first term of this equation is needed, i.e.,

$$\tilde{p}(r) = \tilde{A}_+ \frac{e^{-jkr}}{r}. \tag{2.107}$$

Particle velocity. With the aid of Eq. (2.4b), solve for the particle velocity in the r direction:

$$\begin{aligned} \tilde{u}(r) &= \frac{1}{-jk\rho_0 c} \frac{\partial}{\partial r} \tilde{p}(r) \\ &= \frac{\tilde{A}_+}{\rho_0 c} \left(1 + \frac{1}{jkr}\right) \frac{e^{-jkr}}{r}. \end{aligned} \tag{2.108}$$

Specific acoustic impedance. The specific acoustic impedance is found from Eq. (2.107) divided by Eq. (2.108),

$$Z_s = \frac{\tilde{p}(r)}{\tilde{u}(r)} = \rho_0 c \frac{jkr}{1 + jkr} = \frac{\rho_0 ckr}{\sqrt{1 + k^2 r^2}} \,/90^\circ - \tan^{-1} kr \; rayls. \tag{2.109}$$

Plots of the magnitude and phase angle of the impedance as a function of kr are given in Fig. 2.14 and Fig. 2.15 respectively.

For large values of kr, that is, for large distances or for high frequencies, this equation becomes, approximately,

$$Z_s \approx \rho_0 c \text{ rayls}. \tag{2.110}$$

The impedance here is nearly purely resistive and approximately equal to the characteristic impedance for a plane freely traveling wave. In other words, the specific acoustic impedance a large distance from a spherical source in free space is nearly equal to that in a tube in which no reflections occur from the end opposite the source.

The important steady-state relations derived in this chapter are summarized in Table 2.2.

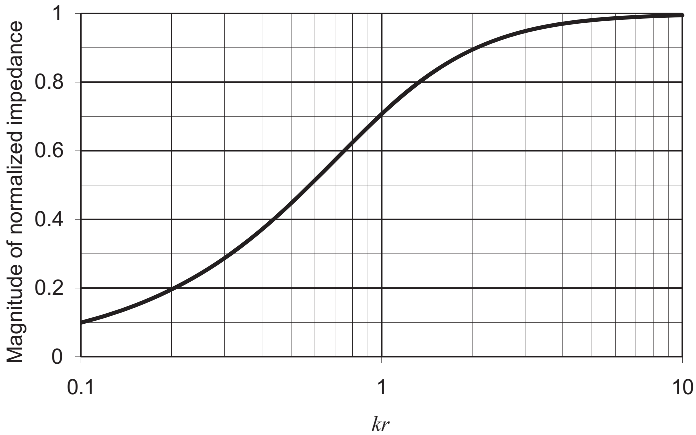

FIG. 2.14 Plot of the magnitude of the specific acoustic-impedance ratio $|Z_s|/(\rho_0 c)$ in a spherical freely traveling wave as a function of *kr*, where *k* is the wave-number equal to ω/c or $2\pi/\lambda$ and *r* is the distance from the center of the spherical source.

$|Z_s|$ is the magnitude of ratio of pressure to particle velocity in a spherical free-traveling wave, and $\rho_0 c$ is the characteristic impedance of air.

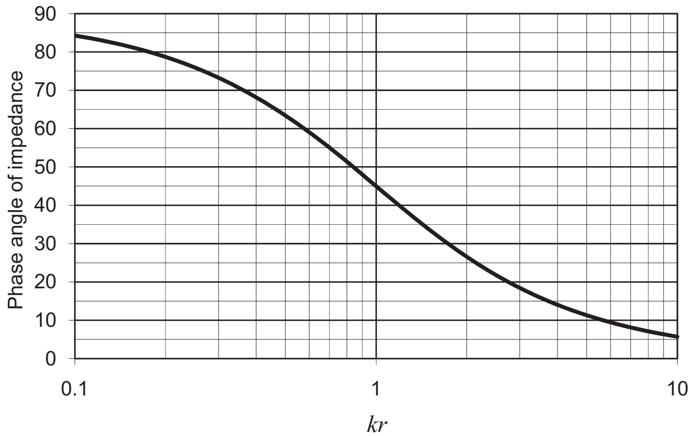

FIG. 2.15 Plot of the phase angle, in degrees, of the specific acoustic-impedance ratio $|Z_s|/\rho_0 c$ in a spherical wave as a function of *kr*, where *k* is the wave-number ω/c or $2\pi/\lambda$ and *r* is the distance from the center of the spherical source.

Table 2.2 General and Steady-state Relations for Small-signal Sound Propagation in Gases

Name	General equation	Steady-state equation
Wave equation in p or u	$\frac{\partial^2(\,)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2(\,)}{\partial t^2}$	$\frac{\partial^2(\,)}{\partial x^2} = -\frac{\omega^2}{c^2}(\,)$
	$\nabla^2(\,) = \frac{1}{c^2}\frac{\partial^2(\,)}{\partial t^2}$	$\nabla^2(\,) = -\frac{\omega^2}{c^2}(\,)$
	$\frac{\partial^2(pr)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2(pr)}{\partial t^2}$	$\nabla^2(pr) = -\frac{\omega^2}{c^2}(pr)$
Equation of motion	$\frac{\partial p}{\partial x} = -\rho_0\frac{\partial u}{\partial t}$	$u = \frac{-1}{j\omega\rho_0}\frac{\partial p}{\partial x}$
	$\text{grad}\, p = -\rho_0\frac{\partial \mathbf{q}}{\partial t}$	$p = -j\omega\rho_0\int u\,dx$
		$\text{grad}\, p = -j\omega\rho_0\mathbf{q}$
Displacement	$\xi = \int u\,dt$	$\xi = \frac{u}{j\omega}$
	$\xi = \int q\,dt$	$\xi = \frac{\mathbf{q}}{j\omega}$
Incremental density	$\rho = \frac{\rho_0}{\gamma P_0}p = \frac{p}{c^2}$	$\rho = \frac{\rho_0}{\gamma P_0}p = \frac{p}{c^2}$
	$\frac{\partial \rho}{\partial t} = -\rho_0\frac{\partial u}{\partial x}$	$\rho = -\frac{\rho_0}{j\omega}\frac{\partial u}{\partial x}$
Incremental temperature	$\Delta T = \frac{T_0}{P_0}\frac{\gamma-1}{\gamma}p$	$\Delta T = \frac{T_0}{P_0}\frac{\gamma-1}{\gamma}p$

PART V: SOLUTIONS OF THE HELMHOLTZ WAVE EQUATION IN THREE DIMENSIONS

2.9 RECTANGULAR COORDINATES

In the steady state, Eq. (2.20b) for the three-dimensional wave equation in rectangular coordinates can be written

$$(\nabla^2 + k^2)\tilde{p}(x,y,z) = 0, \tag{2.111}$$

where the Laplace operator is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.112}$$

and $k = \omega/c = 2\pi/\lambda$. Let the solution to Eq. (2.111) be of the form

$$\tilde{p}(x,y,z) = \tilde{p}_0 X(x)Y(y)Z(z). \tag{2.113}$$

Substituting this in Eq. (2.111) and dividing through by $X(x)Y(y)Z(z)$ yields

$$\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+k_x^2\right)+\left(\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+k_y^2\right)+\left(\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}+k_z^2\right) = 0, \tag{2.114}$$

where

$$k^2 = k_x^2+k_y^2+k_z^2. \tag{2.115}$$

For example, in the case of a plane wave with a direction of travel in the zx plane at an angle θ to the z axis, we have $k_z = k\cos\theta$, $k_x = k\sin\theta$, and $k_y = 0$. The first bracketed term of Eq. (2.114) depends upon x only, while the second term depends upon y only and the third term z only. However, they must all add up to zero which means that either they all have constant values, the combination of which is zero, or they are all zero. We shall assume the latter, in which case Eq. (2.114) can be separated into three equations for each ordinate as follows.

***The plane wave equation in* x.**

$$\left(\frac{\partial^2}{\partial x^2}+k_x^2\right)X = 0. \tag{2.116}$$

***The plane wave equation in* y.**

$$\left(\frac{\partial^2}{\partial y^2}+k_y^2\right)Y = 0. \tag{2.117}$$

***The plane wave equation in* z.**

$$\left(\frac{\partial^2}{\partial z^2}+k_z^2\right)Z = 0. \tag{2.118}$$

The solutions to Eqs. (2.116), (2.117), and (2.118) are

$$X(x) = X_+e^{-jk_xx}+X_-e^{jk_xx},$$

$$Y(y) = Y_+e^{-jk_yy}+Y_-e^{jk_yy}, \text{ and}$$

$$Z(z) = Z_+e^{-jk_zz}+Z_-e^{jk_zz}$$

respectively so that the solution to Eq. (2.111) is

$$\tilde{p}(x,y,z) = \tilde{p}_+e^{-j(k_xx+k_yy+k_zz)}+\tilde{p}_-e^{j(k_xx+k_yy+k_zz)}. \tag{2.119}$$

2.10 CYLINDRICAL COORDINATES

In problems where there is axial symmetry, cylindrical coordinates are often useful, as shown in Fig. 2.16. We shall use these for planar circular radiators. In the xy plane of the rectangular coordinate

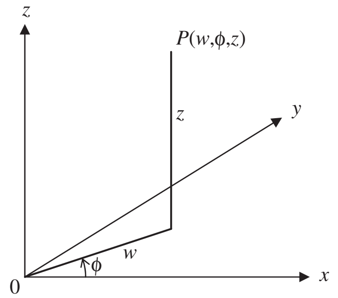

FIG. 2.16 Cylindrical coordinates.

system, the x and y ordinates are replaced by polar ordinates w and ϕ where the radial ordinate w is given by

$$w = \sqrt{x^2 + y^2} \tag{2.120}$$

and the azimuthal ordinate ϕ is given by

$$\phi = \arctan(y/x). \tag{2.121}$$

Conversely

$$x = w \cos \phi, \tag{2.122}$$

$$y = w \sin \phi. \tag{2.123}$$

The rectangular z ordinate simply becomes the axial cylindrical ordinate. The three-dimensional wave equation in cylindrical coordinates is

$$(\nabla^2 + k^2)\tilde{p}(w, \phi, z) = 0, \tag{2.124}$$

where the Laplace operator is given by

$$\nabla^2 = \frac{\partial^2}{\partial w^2} + \frac{1}{w}\frac{\partial}{\partial w} + \frac{1}{w^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}, \tag{2.125}$$

which is often written in the following short form:

$$\nabla^2 = \frac{1}{w}\frac{\partial}{\partial w}\left(w\frac{\partial}{\partial w}\right) + \frac{1}{w^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}. \tag{2.126}$$

Let the solution to Eq. (2.124) be of the form

$$\tilde{p}(w, \phi, z) = \sum_{n=0}^{\infty} \tilde{p}_n W_n(w)\Phi_n(\phi)Z(z). \tag{2.127}$$

Substituting this in Eq. (2.124), multiplying through by w^2, and dividing through by $W_n(w)\Phi(\phi)Z(z)$ yields

$$\frac{w^2}{W_n}\frac{\partial^2 W_n}{\partial w^2} + \frac{w}{W_n}\frac{\partial W_n}{\partial w} + k_w^2 w^2 = -\frac{1}{\Phi_n}\frac{\partial^2 \Phi_n}{\partial \phi^2} - \frac{w^2}{Z}\frac{\partial^2 Z}{\partial z^2} - k_z^2 w^2, \tag{2.128}$$

where

$$k^2 = k_w^2 + k_z^2. \tag{2.129}$$

If both sides of Eq. (2.128) are equated to a constant of separation n^2, then Eq. (2.128) can then be separated into three equations for each ordinate as follows.

***The radial equation in* w.**

$$\left(\frac{\partial^2}{\partial w^2} + \frac{1}{w}\frac{\partial}{\partial w} + k_w^2 - \frac{n^2}{w^2}\right) W_n(w) = 0. \tag{2.130}$$

The solution to this equation is of the form

$$W_n(w) = W_{n+} H_n^{(2)}(k_w w) + W_{n-} H_n^{(1)}(k_w w), \tag{2.131}$$

where $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are Hankel functions defined by

$$H_n^{(1)}(x) = J_n(x) + jY_n(x), \tag{2.132}$$

$$H_n^{(2)}(x) = J_n(x) - jY_n(x), \tag{2.133}$$

where $J_n(x)$ and $Y_n(x)$ are Bessel functions of the first and second kind respectively, as plotted in Fig. 2.17 and Fig. 2.18. The "2" in parentheses denotes an outgoing cylindrical wave and the "1" denotes an incoming one.

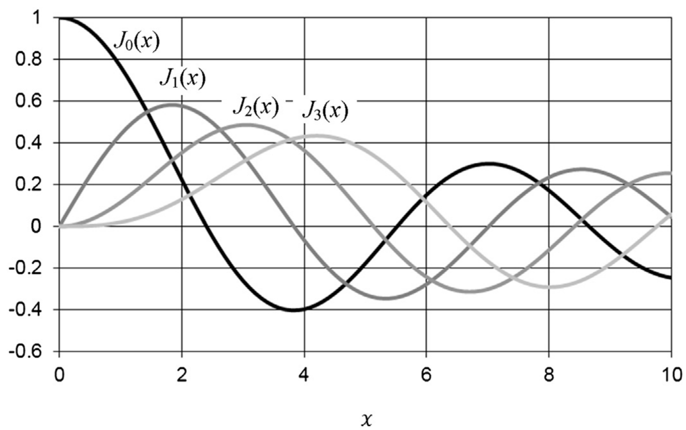

FIG. 2.17 Bessel functions of the first kind.

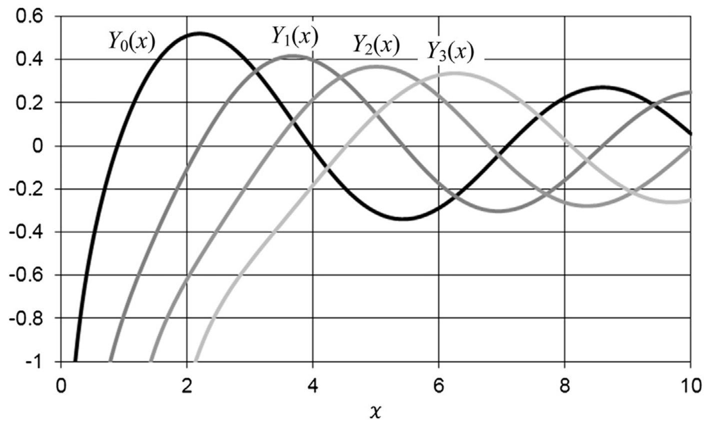

FIG. 2.18 Bessel functions of the second kind.

The azimuthal equation in ϕ.

$$\left(\frac{\partial^2}{\partial\phi^2} + n^2\right)\Phi_n(\phi) = 0. \tag{2.134}$$

The solution to this equation is of the form

$$\Phi_n(\phi) = A_n\cos(n\phi) + B_n\sin(n\phi). \tag{2.135}$$

It can be seen that the integer n denotes the n^{th} harmonic of the azimuthal modes of vibration where $\phi = 2\pi$ represents a full rotation about the z axis. The values of A_n and B_n depend on where the nodes and antinodes lie on the circumference. For example, setting $B_n = 0$ would place the nodes at $\phi = 0$, π, and 2π.

The axial equation in z.

$$\left(\frac{\partial^2}{\partial z^2} + k_z^2\right)Z(z) = 0. \tag{2.136}$$

The solution to this plane wave equation is of the form

$$Z(z) = Z_+e^{-jk_zz} + Z_-e^{jk_zz}, \tag{2.137}$$

where the $+$ sign denotes a forward traveling wave and the $-$ sign a reverse one. From Eq. (2.129) we observe that

$$k_z = \begin{cases} \sqrt{k^2 - k_w^2}, & k \geq k_w \cdot \\ -j\sqrt{k_w^2 - k^2}, & k < k_w \end{cases}, \tag{2.138}$$

Hence for $k < k_w$ the forward traveling term becomes an *evanescent* decaying one. Evanescent waves typically occur close to sound sources in the form of non-propagating standing waves.

2.11 SPHERICAL COORDINATES

So far, we have only considered the one-dimensional spherical wave equation and its solution. In many problems where there are spherical surfaces but no axial or rotational symmetry, it is necessary to use spherical coordinates as shown in Fig. 2.19. The x, y, and z ordinates are replaced by spherical ordinates r, ϕ, and θ, where the radial ordinate r is given by

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{2.139}$$

the inclination angle θ is given by

$$\theta = \operatorname{arccot}\left(z/\sqrt{x^2 + y^2}\right), \tag{2.140}$$

and the azimuth angle ϕ is given by

$$\phi = \arctan(y/x). \tag{2.141}$$

Conversely

$$x = r\sin\theta\cos\phi, \tag{2.142}$$

$$y = r\sin\theta\sin\phi, \tag{2.143}$$

$$z = r\cos\theta. \tag{2.144}$$

The three-dimensional wave equation in spherical coordinates is

$$(\nabla^2 + k^2)\tilde{p}(r, \theta, \phi) = 0, \tag{2.145}$$

where the Laplace operator is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \tan\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}, \tag{2.146}$$

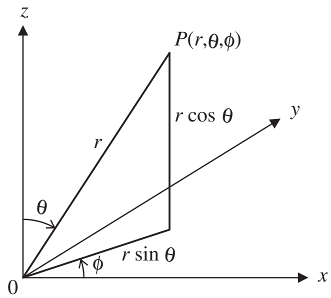

FIG. 2.19 Spherical coordinates.

which is often written in the following short form:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{2.147}$$

Let the solution to Eq. (2.145) be of the form

$$\tilde{p}(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\tilde{p}_{mn}R_n(r)Z_{mn}(\theta)\Phi_m(\phi). \tag{2.148}$$

Substituting this in Eq. (2.145), multiplying through by r^2, and dividing through by $R_n(r)\,Z_{mn}(\theta)\Phi_n(\phi)$ yields

$$\frac{r^2}{R_n}\frac{\partial^2 R_n}{\partial r^2} + \frac{2r}{R_n}\frac{\partial R_n}{\partial r} + k^2r^2 = -\frac{1}{Z_{mn}}\frac{\partial^2 Z_{mn}}{\partial\theta^2} - \frac{1}{Z_{mn}\tan\theta}\frac{\partial Z_{mn}}{\partial\theta} - \frac{1}{\Phi_m \sin^2\theta}\frac{\partial^2\Phi_m}{\partial\phi^2}. \tag{2.149}$$

If both sides of Eq. (2.149) are equated to a constant of separation $n(n+1)$, then Eq. (2.149) can then be separated into three equations for each ordinate as follows.

***The radial equation in* r.** After equating the left hand side of Eq. (2.149) to $n(n+1)$, we have

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2 - \frac{n(n+1)}{r^2}\right)R_n(r) = 0. \tag{2.150}$$

The solution to this equation is of the form

$$R_n(r) = R_{n+}h_n^{(2)}(kr) + R_{n-}h_n^{(1)}(kr), \tag{2.151}$$

where $h_n^{(1)}(x)$ and $h_n^{(2)}(x)$ are spherical Hankel functions, which are also known as Hankel functions of fractional order, as defined by

$$h_n^{(1)}(x) = j_n(x) + jy_n(x), \tag{2.152}$$

$$h_n^{(2)}(x) = j_n(x) - jy_n(x), \tag{2.153}$$

where $j_n(x)$ and $y_n(x)$ are spherical Bessel functions of the first and second kind respectively, as plotted in Fig. 2.20 and Fig. 2.21. The "2" in parentheses denotes an outgoing spherical wave and the "1" denotes an incoming one. These spherical Bessel functions are related to the cylindrical Bessel functions of integer order $J_{n+\frac{1}{2}}(x)$ and $Y_{n+\frac{1}{2}}(x)$ by

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x), \tag{2.154}$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}}Y_{n+\frac{1}{2}}(x). \tag{2.155}$$

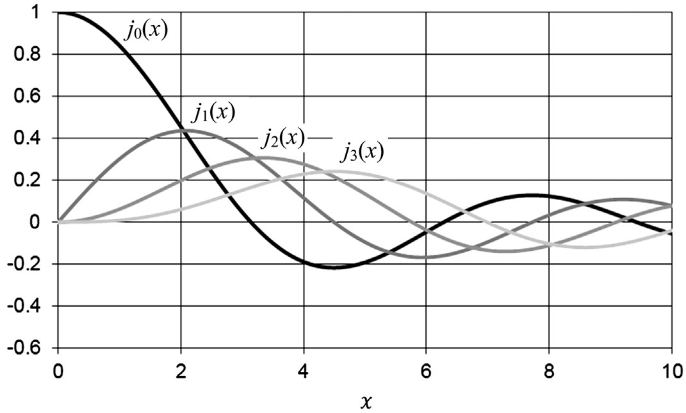

FIG. 2.20 Spherical Bessel functions of the first kind.

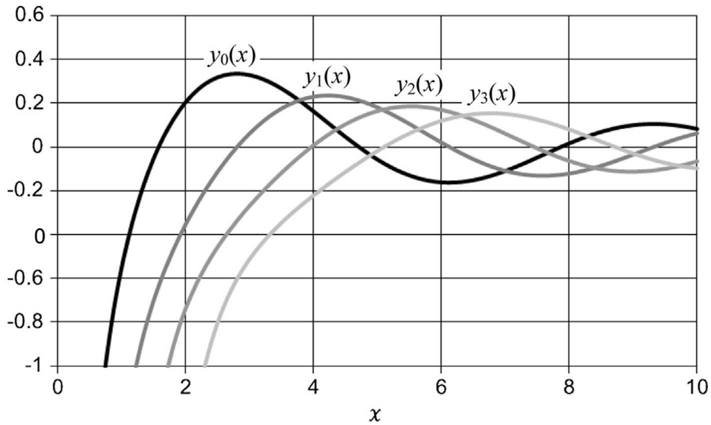

FIG. 2.21 Spherical Bessel functions of the second kind.

The inclination equation in θ. After equating the right hand side of Eq. (2.149) to $n(n+1)$, we have

$$\frac{1}{Z_{mn}}\frac{\partial^2 Z_{mn}}{\partial\theta^2}+\frac{1}{Z_{mn}\tan\theta}\frac{\partial Z_{mn}}{\partial\theta}+n+1 = -\frac{1}{\Phi_m \sin^2\theta}\frac{\partial^2\Phi_m}{\partial\phi^2}. \tag{2.156}$$

Equating the left hand side of Eq. (2.156) to another constant of separation $m^2/\sin^2\theta$ yields

$$\left(\frac{\partial^2}{\partial\theta^2}+\frac{1}{\tan\theta}\frac{\partial}{\partial\theta}+n(n+1)-\frac{m^2}{\sin^2\theta}\right)Z_{mn}(\theta) = 0. \tag{2.157}$$

After substituting $z=\cos\theta$, the inclination equation becomes

$$\left[\left(1-z^2\right)\frac{\partial^2}{\partial z^2}-2z\frac{\partial}{\partial z}+n(n+1)-\frac{m^2}{1-z^2}\right)\right]Z_{mn}(z) = 0. \tag{2.158}$$

The solution to this equation is of the form

$$Z_{mn}(z) = \Theta_{mn} P_n^m(z) \tag{2.159}$$

or

$$Z_{mn}(\theta) = \Theta_{mn} P_n^m(\cos\theta), \tag{2.160}$$

where $P_n^m(\cos\theta)$ is the *associated* Legendre function. In the case of axisymmetry, where $m = 0$, it reduces to the Legendre function (or Legendre polynomial) denoted by $P_n(\cos\theta)$, as plotted in Fig. 2.22.

The azimuth equation in $\boldsymbol{\phi}$***.*** Equating the right hand side of Eq. (2.156) to the constant of separation $m^2/\sin^2\theta$ yields

$$\left(\frac{\partial^2}{\partial\phi^2} + m^2\right)\Phi_m(\phi) = 0. \tag{2.161}$$

The solution to this equation is of the form

$$\Phi_m(\phi) = A_m\cos(m\phi) + B_m\sin(m\phi). \tag{2.162}$$

It can be seen that the integer m denotes the m^{th} harmonic of the azimuthal modes of vibration where $\phi = 2\pi$ represents a full rotation about the z axis. The values of A_m and B_m depend on where the nodes and antinodes lie on the circumference. For example, setting $B_m = 0$ would place the nodes at $\phi = 0$, π, and 2π. Now the complete solution to Eq. (2.145) may be written as

$$\begin{aligned}\tilde{p}(r,\theta,\phi) = &\sum_{n=0}^{\infty}\sum_{m=0}^{n}\tilde{p}_{mn}\left(R_{n+}h_n^{(1)}(kr) + R_{n-}h_n^{(2)}(kr)\right)\\ &\times P_n^m(\cos\theta)(A_m\cos(m\phi) + B_m\sin(m\phi)).\end{aligned} \tag{2.163}$$

which in the case of axial symmetry ($m = 0$) simplifies to

$$\tilde{p}(r,\theta) = \sum_{n=0}^{\infty}\tilde{p}_n\left(R_{n+}h_n^{(1)}(kr) + R_{n-}h_n^{(2)}(kr)\right)P_n(\cos\theta). \tag{2.164}$$

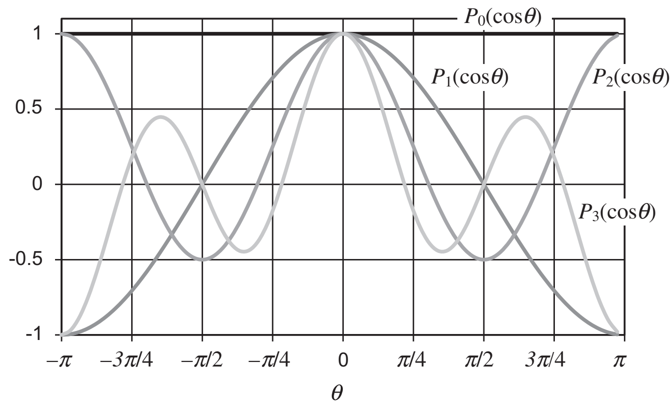

FIG. 2.22 Legendre functions.

Notes

[1] Nonvector derivations of the wave equation are given in Rayleigh, Theory of Sound, Vol. 2, pp. 1–15, (Dover, 1945); P.M. Morse, Vibration and Sound, 2nd ed. (Acoustical Society of America, New York, 1981), pp. 217–225; L.E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, Fundamentals of Acoustics, 4th ed. (John Wiley & Sons, Inc., New York, 2000), pp. 113–213; and other places.

[2] A vector derivation of the wave equation is given in two papers that must be read together: W.J. Cunningham, Application of Vector Analysis to the Wave Equation, J Acoust Soc Am 1950; 22:61 and R.V.L. Hartley, Note on Application of Vector Analysis to the Wave Equation, J Acoust Soc Am 1950;22:511.

[3] If a mass of gas is chosen so that its weight in grams is equal to its molecular weight (known to chemists as the gram-molecular weight, or the mole), then the volume of this mass at 0°C and 0.76 m Hg is the same for all gases and equals 0.02242 m^3. Then $R = 8.314$ watt-sec per degree centigrade per gram-molecular weight. If the mass of gas chosen is n times its molecular weight, then $R = 8.314\ n$.

[4] Beranek See LL. Acoustic Measurements. New York: Acoustical Society of America; 1988. p. 49.

[5] Serway RA, Jewett JW. Principles of Physics: A Calculus-Based Text. 4th ed. Calif: Thomson Brooks/Cole, Belmont; 2006. ISBN 053449143X, p. 550.

[6] Webster AG. Acoustical Impedance, and the Theory of Horns and of the Phonograph. Proc Natl Acad Sci USA 1919;5:275–82.

[7] For the type of source we have assumed and no dissipation, this case breaks down for $kl = n\pi$.