Frequency-response shapes for loudspeakers [1]

I

In the following high-pass filter functions $G(s)$, the order of the function is denoted by N and the frequency at which the magnitude of the response is $1/\sqrt{2}$ (that is, 3 dB below the pass-band level) is denoted by $\omega_{3\mathrm{dB}}$.

Synchronous.

$$G(s) = \left(\frac{s}{s+\omega_0}\right)^N$$

where

$$\omega_0 = \omega_{3\mathrm{dB}}\sqrt{2^{1/\mathrm{N}}-1}$$

Bessel. The Bessel polynomials are generated from the following power series in s:

$$B_n(s) = \sum_{k=0}^{n} a_k s^k$$

where $s = j\omega$ and

$$a_k = \frac{(2n-k)!}{2^{n-k}\,k!(n-k)!}$$

Also, we define a frequency scaling factor γ such that

$$|B_n(j\gamma)| = a_0\sqrt{2} = \frac{2n!}{2^{n-1/2}n!}$$

Bessel polynomials

1st-order: $B_1(s) = s+1$
2nd-order: $B_2(s) = s^2+3s+3$
3rd-order: $B_3(s) = s^3+6s^2+15s+15$
4th-order: $B_4(s) = s^4+10s^3+45s^2+105s+105$
5th-order: $B_5(s) = s^5+15s^4+105s^3+420s^2+945s+945$
6th-order: $B_6(s) = s^6+21s^5+210s^4+1260s^3+4725s^2\ 10395s+10395$

If the real parts of the roots or poles are α_1, α_2,…α_N and the imaginary parts of the roots or poles are β_1, β_2,…β_N, then

$$\omega_n = \frac{\gamma}{\sqrt{\alpha_n^2 + \beta_n^2}}$$

$$Q_n = \frac{\sqrt{\alpha_n^2 + \beta_n^2}}{2\alpha_n}$$

Odd order

$$G(s) = \frac{s^N}{\left(s^2 + \frac{\omega_1}{Q_1}s + \omega_1^2\right)\left(s^2 + \frac{\omega_2}{Q_2}s + \omega_2^2\right)\cdots\left(s^2 + \frac{\omega_{(N-1)/2}}{Q_{(N-1)/2}}s + \omega_{(N-1)/2}^2\right)\left(s + \omega_{(N+1)/2}\right)}$$

Even order

$$G(s) = \frac{s^N}{\left(s^2 + \frac{\omega_1}{Q_1}s + \omega_1^2\right)\left(s^2 + \frac{\omega_2}{Q_2}s + \omega_2^2\right)\cdots\left(s^2 + \frac{\omega_{N/2}}{Q_{N/2}}s + \omega_{N/2}^2\right)}$$

Butterworth.
Odd order

$$G(s) = \frac{s^N}{\left(s^2 + \frac{\omega_{3\mathrm{dB}}}{Q_1}s + \omega_{3\mathrm{dB}}^2\right)\left(s^2 + \frac{\omega_{3\mathrm{dB}}}{Q_2}s + \omega_{3\mathrm{dB}}^2\right)\cdots\left(s^2 + \frac{\omega_{3\mathrm{dB}}}{Q_{(N-1)/2}}s + \omega_{3\mathrm{dB}}^2\right)\left(s + \omega_{3\mathrm{dB}}\right)}$$

The poles lie on a circle of radius $\omega_{3\mathrm{dB}}$, each at an angle of θ_n to the real axis, where

$$Q_n = \frac{1}{2\cos\theta_n}$$

and

$$\theta_n = \pm\frac{n}{N}\pi, \qquad n = 1, 2, \cdots (N-1)/2$$

Even order

$$G(s) = \frac{s^N}{\left(s^2 + \frac{\omega_{3\mathrm{dB}}}{Q_1}s + \omega_{3\mathrm{dB}}^2\right)\left(s^2 + \frac{\omega_{3\mathrm{dB}}}{Q_2}s + \omega_{3\mathrm{dB}}^2\right)\cdots\left(s^2 + \frac{\omega_{3\mathrm{dB}}}{Q_{N/2}}s + \omega_{3\mathrm{dB}}^2\right)}$$

where

$$\theta_n = \pm\,\frac{2n-1}{2N}\pi, \qquad n = 1, 2, \cdots N/2$$

Chebyshev. ***Chebyshev polynomials***

1st-order: $C_1(\Omega) = \Omega$
2nd-order: $C_2(\Omega) = 2\Omega^2 - 1$
3rd-order: $C_3(\Omega) = 4\Omega^3 - 3\Omega$
4th-order: $C_4(\Omega) = 8\Omega^4 - 8\Omega^2 + 1$
5th-order: $C_5(\Omega) = 16\Omega^5 - 20\Omega^3 + 5\Omega$
6th-order: $C_6(\Omega) = 32\Omega^6 - 48\Omega^4 + 18\Omega^2 - 1$

where

$$\Omega = \frac{\omega}{\omega_P}$$

and ω_P is the pass-band limit which is defined as the frequency at which the final 0 dB crossing occurs before roll-off. Let R be the maximum permitted magnitude of the pass band ripples in dB. Thus we define a ripple factor by

$$\varepsilon = \sqrt{10^{0.1\,R} - 1}$$

Also, we define a frequency scaling factor γ by

$$\gamma = \frac{\omega_{3\mathrm{dB}}}{\omega_{\mathrm{p}}} = \cosh\left(\frac{1}{N}\operatorname{arccosh}\left(\frac{1}{\varepsilon}\right)\right)$$

and then find the roots of the polynomial

$$1 + \varepsilon^2 C_N^2$$

so that the imaginary parts of the roots α_1, α_2,…α_N give the real parts of the poles and the real parts of the roots β_1, β_2,…β_N give the imaginary parts of the poles. Then

$$\omega_n = \frac{\gamma}{\sqrt{\alpha_n^2 + \beta_n^2}}$$

$$Q_n = \frac{\sqrt{\alpha_n^2 + \beta_n^2}}{2\alpha_n}$$

Odd order

$$G(s) = \frac{s^N}{\left(s^2 + \frac{\omega_1}{Q_1}s + \omega_1^2\right)\left(s^2 + \frac{\omega_2}{Q_2}s + \omega_2^2\right)\cdots\left(s^2 + \frac{\omega_{(N-1)/2}}{Q_{(N-1)/2}}s + \omega_{(N-1)/2}^2\right)\left(s + \omega_{(N+1)/2}\right)}$$

Even order

$$G(s) = \frac{s^N}{\left(s^2 + \frac{\omega_1}{Q_1}s + \omega_1^2\right)\left(s^2 + \frac{\omega_2}{Q_2}s + \omega_2^2\right)\cdots\left(s^2 + \frac{\omega_{N/2}}{Q_{N/2}}s + \omega_{N/2}^2\right)}$$

References

[1] Zverev AI. Handbook of Filter Synthesis. New Jersey: Wiley; 1967.