CHAPTER

State variable analysis of circuits 14

CHAPTER OUTLINE

14.1 A BRIEF HISTORY

Two types of circuit simulation technique evolved during the 1960s and early 1970s. One was the nodal method, as used by CANCER [1] (Computer Analysis of Non-Linear Circuits Excluding Radiation), for example, which was the forerunner of SPICE (Simulation Program with Integrated Circuit Emphasis), pioneered by University of California-Berkeley. The other was the state variable approach, as used by SCEPTRE [2] (System for Circuit Evaluation and Prediction of Transient Radiation Effects), developed by IBM. In a nodal analysis, the node voltages and branch currents in the circuit are calculated for every frequency step. By contrast, in a state variable analysis, a set of frequency-dependent transfer functions are derived between the various sources and outputs within the circuit. From this single analysis, both frequency domain and time domain responses can be obtained. Hence, when the state variable method first appeared, it was hailed as the future of circuit simulation [3]. However, as the number of elements in integrated circuits increased during 1970s onwards, the state variable method proved too unwieldy and fell into disuse. Almost every circuit simulation tool available today uses some form of nodal analysis.

Acoustics: Sound Fields and Transducers. DOI: 10.1016/B978-0-12-391421-7.00014-2

14.2 WHAT IS STATE VARIABLE ANALYSIS?

The idea is to express the required information relating to the circuit or system response in terms of a first-order differential matrix equation. If the state vector is properly defined, it is always possible to write the equation in the proper form for linear systems. The state variable (or state vector if there is more than one) is that whose linearly independent parameters can describe the present state or any possible future state of the system. The state vector **x** satisfies the standard state equation [4]

$$\dot{\mathbf{x}} = \mathbf{A}\cdot\mathbf{x} + \mathbf{B}\cdot\mathbf{u} \tag{14.1}$$

where **x** is the state vector and **u** is the input vector. The output equation is

$$\mathbf{y} = \mathbf{C}\cdot\mathbf{x} + \mathbf{D}\cdot\mathbf{u} \tag{14.2}$$

The most convenient choice of state vector is one having parameters that describe the energy stored by different elements of the system. In electrical circuits, the parameters are normally chosen to be the capacitor voltages and inductor currents, thus allowing independently specifiable initial conditions [5]. Also, the output vector is normally chosen to be the voltages at the various nodes in the circuit. In this text, the input vector will contain the current sources and voltage sources. Equations (14.1) and (14.2) can be solved in general terms so that any system obeying the state equation can be described by a state vector at any instant in time. The most flexible method for solving Eqs. (14.1) and (14.2) is the Faddeev–Leverrier algorithm which enables either numerical or symbolic computation.

14.3 WHY USE STATE VARIABLE ANALYSIS?

Acoustical models have a relatively small numbers of elements, so that in most cases the state variable method can be applied without computational problems. There are a number of motivations for using such a method:

- The transfer function can be part of a DSP-based real-time model of the system for monitoring parameters such as voice-coil temperature or displacement so that they can be dynamically limited.
- The model can be used as a basis for response equalization.
- The poles and zeros of the system can be mapped in order to investigate its stability or sensitivity to component tolerances.
- The advent of symbolic handling in mathematical computer tools enables an algebraic transfer function to be generated in terms of the circuit element labels (e.g., L1, C2, R4, etc.). This enables us to examine the dependency of the poles and zeros on individual circuit elements.
- If we have a previously optimized transfer function, such as a Chebyshev or Butterworth polynomial, we can even go a stage further and equate the coefficients of the circuit transfer function polynomial with those of the optimized polynomial in order to obtain a set of simultaneous equations which can then be solved for each circuit element value. Hence an optimum circuit design can be obtained in one operation without any further iteration. This approach has been used with considerable success to create loudspeaker design "look-up" tables known as loudspeaker alignments (see Secs. 7.6 and 7.12), although these used to be worked out by hand [6,7].

14.4 WHAT ARE THE RESTRICTIONS?

In state variable analysis, there are some topological restrictions as follows

A loop must not contain capacitors only. A small resistance must be added to the path and this is known as "de-Q-ing", as shown in Fig. 14.1.

A node must not be the junction of inductors only. A large resistance must also join the node and is usually connected in parallel with one of the inductors, as shown in Fig. 14.1.

The previous two restrictions apply to all state variable programs. One, which is unique to the simple node voltage method described here, affects current-controlled (voltage or current) sources. An ideal current-controlled source has, by definition, zero input resistance. Unfortunately, this would result in the voltage at both input nodes being equal and therefore no longer independent of each other. Hence, a small resistance must be included across the input terminals so that the source effectively becomes a voltage-controlled one where the input current is simply the input voltage divided by the added resistance. The implementation of this is described in Sec. 14.12.

It should be noted that, in the case of a symbolically computed transfer function, the resistors that have been added due to the above restrictions can be set to zero in the final transfer function and will therefore have no influence over the final result.

14.5 SOME BASIC CIRCUIT THEORY

We will define a *branch* as the path through a single circuit *element* and a *node* as the point where two or more branches are connected. Then a *loop* is a set of branches connected end to end which form

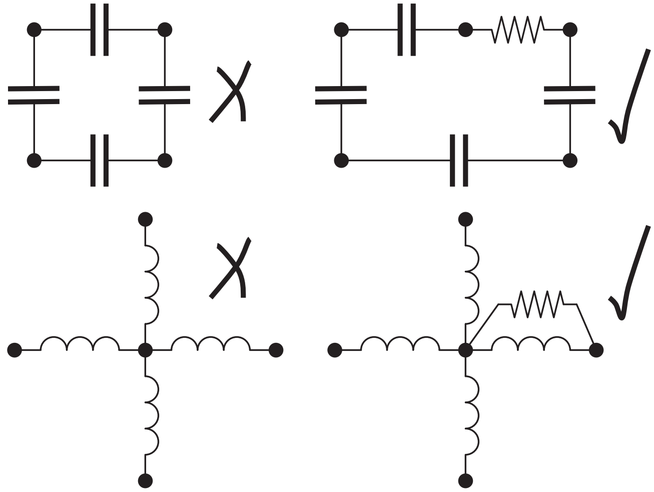

FIG. 14.1 "De-Q-ing" of all-capacitive loops and all-inductive nodes.

a closed circular path. Circuit *elements* considered here will be limited to resistors, capacitors, inductors, and voltage or current sources, together with their mechanical and acoustic analogies. In the case of sources, however, we will also consider some special types known as *controlled* sources, which each contain two isolated branches, one for the input, or *controlling* parameter, and one for the output, or *controlled* parameter. Firstly, though we need to state three key governing rules relating to circuits in general:

Kirchhoff's current law (KCL) states that the sum of all the currents flowing into any node is equal to the sum of the currents flowing out of that node.
Kirchhoff's voltage law (KVL) states that the sum of the voltages around any loop must be zero.
Ohm's law states that the current i flowing through any *passive* branch is equal to the voltage e across the branch divided by the impedance Z of the circuit element in that branch, that is, $i = e/Z$.

By a passive branch, we mean one that does not contain any sources, but is limited to resistors, capacitors or inductors so that

$$Z = \begin{cases} R, \text{ where R is the resistance} \\ 1/(j\omega C), \text{ where } C \text{ is the capacitance} \\ j\omega L, \text{ where } L \text{ is the inductance} \end{cases} \tag{14.3}$$

and ω is the angular frequency. From here on we will use the shorthand

$$s = j\omega = j2\pi f \tag{14.4}$$

In a time-domain analysis

$$s = \frac{d}{dt} \tag{14.5}$$

The three rules listed above will enable us to evaluate any circuit.

14.6 GRAPH THEORY

An electrical circuit in its most elementary form can be represented as a *graph*. This abstraction enables us to explore the properties of a circuit and to derive some very useful relationships. Fig. 14.2

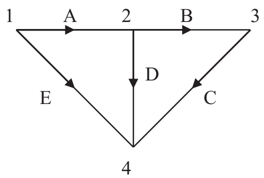

FIG. 14.2 Graph.

Table 14.1 Currents in graph

Node	A	B	C	D	E
1	+1	0	0	0	+1
2	−1	+1	0	+1	0
3	0	−1	+1	0	0
4	0	0	−1	−1	−1

shows an example of a graph, where the nodes are labeled 1 to 4 and the branches are labeled A to E. The arrows show the directions of current flow.

Next, we tabulate the graph using +1 to denote the node from which the current in each branch flows and −1 to denote the node to which it flows (see Table 14.1). If we make node 4 the reference, we can delete the last row and rewrite the table as a matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \tag{14.6}$$

Kirchhoff's current law can be expressed in terms of **A** as

$$\mathbf{A}\cdot\mathbf{i} = 0 \tag{14.7}$$

where **i** is a column vector of the currents in each branch. This represents the equations

$$\begin{aligned} i_A + i_E &= 0 \\ i_B - i_A + i_D &= 0 \\ i_C - i_B &= 0. \end{aligned} \tag{14.8}$$

Also, if we take the transpose of **A**

$$\mathbf{A}^t = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{14.9}$$

so that each row represents a branch and the column entries show which nodes each branch spans, we can write

$$\mathbf{e} = \mathbf{A}^t\cdot\mathbf{v} \tag{14.10}$$

which represents the equations

$$\begin{aligned} e_A &= v_1 - v_2 \\ e_B &= v_2 - v_3 \\ e_C &= v_3 \\ e_D &= v_2 \\ e_E &= v_1 \end{aligned} \tag{14.11}$$

where e_A to e_E are the branch voltages and v_1 to v_3 are the node voltages relative to the reference node (node 4). Equations (14.7) and (14.10) will be used later for automation.

14.7 WORKED EXAMPLE NO. 1: LOUDSPEAKER IN AN ENCLOSURE WITH A BASS-REFLEX PORT

In order to show how the various matrices are constructed by a computer program, we will familiarize ourselves with the state variable method by means of an example, which we will follow through by hand. The equivalent electrical circuit for a loudspeaker in a bass-reflex enclosure is shown in Fig. 14.3, using the admittance analogy for the mechanical and acoustical circuits, which are all referred to the electrical domain. The circuit elements are given by

$$R_1 = R_g + R_E \tag{14.12}$$

$$R_2 = \frac{(Bl)^2}{R_{MS}} \tag{14.13}$$

$$R_3 = \left(\frac{Bl}{S_D}\right)^2 \frac{1}{R_{AL}} \tag{14.14}$$

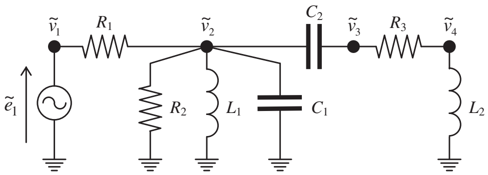

FIG. 14.3 Equivalent electrical circuit of loudspeaker in bass-reflex enclosure.

$$C_1 = \frac{M_{MS}}{(Bl)^2} \tag{14.15}$$

$$C_2 = \left(\frac{S_D}{Bl}\right)^2 M_{AT} \tag{14.16}$$

$$L_1 = (Bl)^2 C_{MS} \tag{14.17}$$

$$L_2 = \left(\frac{Bl}{S_D}\right)^2 C_{AB} \tag{14.18}$$

$$\tilde{v}_1 = sBl\tilde{x} \tag{14.19}$$

$$\tilde{v}_4 = \frac{Bl}{S_D}\tilde{U}_B \tag{14.20}$$

$$\tilde{p}(r) = s\,\frac{\rho_0 \tilde{U}_B}{4\pi r} \tag{14.21}$$

and from Part XXII the parameters are:

$\tilde{e}_1 = \tilde{e}_g$, the input voltage in V.
R_E is resistance of voice coil in Ω.
B is steady air-gap flux density in T.
l is length of wire in meters on the voice-coil winding.
$\tilde{x}$ is voice-coil displacement in m.
a is radius of diaphragm in m.
M_{MS} is mass of the diaphragm and voice coil, including radiation mass on both sides, in kg.
C_{MS} is total mechanical compliance of the suspensions in m/N.
R_{MS} is mechanical resistance of the suspensions in N·s/m.
S_D is effective area of diaphragm in m^2.
M_{AT} is acoustic mass of air in the port in kg/m^4, including end corrections.
C_{AB} is acoustic compliance of the box in m^5/N.
R_{AL} is combined acoustic resistance of air in port, box interior, and leakage in N·s/m^5.
$\tilde{U}_B$ is net volume velocity in m^3/s.
$\tilde{p}(r)$ is on-axis pressure in N/m^2 at a distance r (in m) from the diaphragm.
ρ_0 is ambient density of air in kg/m^3.

Because it is a low-frequency model, the coil inductance L_E and radiation resistance R_{AR} have been ignored. The values $\tilde{v}_1$, $\tilde{v}_2$, $\tilde{v}_3$, and $\tilde{v}_4$ are the voltages at nodes 1, 2, 3, and 4 respectively with respect to ground (node 0). Now we can create the *net list*, shown in Table 14.2, which completely describes the

Table 14.2 Net list for worked example No. 1

Element	From node	To node
$\tilde{e}_1$	1	0
R_1	1	2
R_2	2	0
R_3	3	4
C_1	2	0
C_2	2	3
L_1	2	0
L_2	4	0

circuit. From the net list we can create what is generally known as the **A** matrix [8], which is a mathematical representation of the circuit connectivity:

$$\mathbf{A} = \begin{array}{c} \begin{array}{cccccccc} \tilde{e}_1 & R_1 & R_2 & R_3 & C_1 & C_2 & L_1 & L_2 \end{array} \text{ node} \\ \left[\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \tag{14.22}$$

Note that we have omitted the row for the reference node zero, which is redundant in the computations that follow. This **A** matrix can then be partitioned into four matrices $\mathbf{A}_S$, $\mathbf{A}_R$, $\mathbf{A}_C$, and $\mathbf{A}_L$ representing the connectivity of the sources, resistors, capacitors, and inductors respectively:

$$\mathbf{A}_S = \begin{array}{c} \tilde{e}_1 \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right] \end{array}, \mathbf{A}_R = \begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{array}\right] \end{array}, \mathbf{A}_C = \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{array}\right] \end{array}, \mathbf{A}_L = \begin{array}{c} \begin{array}{cc} L_1 & L_2 \end{array} \text{ node} \\ \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \tag{14.23}$$

We will show how to use these matrices in due course.

Now let us return to the circuit of Fig. 14.3. The state variables are the *capacitor voltages* denoted by $\tilde{V}_{C1}$, $\tilde{V}_{C2}$ and *inductor currents* denoted by $\tilde{I}_{L1}$, $\tilde{I}_{L2}$, using the upper case. The capacitor currents and inductor voltages are denoted by $\tilde{i}_{C1}$, $\tilde{i}_{C2}$ and $\tilde{v}_{L1}$, $\tilde{v}_{L2}$ respectively, using the lower case to indicate that they are not state variables. The voltage source current is $\tilde{i}_{e1}$ and the node voltages are $\tilde{v}_1$, $\tilde{v}_2$, $\tilde{v}_3$, and $\tilde{v}_4$. The first step of the analysis is to apply KCL at each of the nodes:

At node 1

$$\tilde{i}_{e1} = \frac{\tilde{v}_2 - \tilde{v}_1}{R_1} \tag{14.24}$$

At node 2

$$\frac{\tilde{v}_1 - \tilde{v}_2}{R_1} = \frac{\tilde{v}_2}{R_2} + \tilde{I}_{L1} + \tilde{i}_{C1} + \tilde{i}_{C2} \tag{14.25}$$

At node 3

$$\tilde{i}_{C2} = \frac{\tilde{v}_3 - \tilde{v}_4}{R_3} \tag{14.26}$$

At node 4

$$\frac{\tilde{v}_3 - \tilde{v}_4}{R_3} = \tilde{I}_{L2} \tag{14.27}$$

For the voltage source

$$\tilde{v}_1 = \tilde{e}_1 \tag{14.28}$$

Next we apply KVL to the capacitors and inductors:
For C_1

$$\tilde{v}_2 = \tilde{V}_{C1} \tag{14.29}$$

For C_2

$$\tilde{v}_2 - \tilde{v}_3 = \tilde{V}_{C2} \tag{14.30}$$

For L_1

$$\tilde{v}_2 = \tilde{v}_{L1} \tag{14.31}$$

For L_2

$$\tilde{v}_4 = \tilde{v}_{L2} \tag{14.32}$$

It should be noted that the number of independent equations must equal the number of independent variables. In this case, there are four state variables $\tilde{V}_{C1}, \tilde{V}_{C2}, \tilde{I}_{L1}$, and $\tilde{I}_{L2}$, and hence four KVL equations related to them. However, although there are four node voltages and hence four KCL equations, one of them ($\tilde{v}_1$) is equal to the input voltage [see Eq. (14.28)] and is therefore not independent. Hence we have had to introduce an extra variable, namely, the input current $\tilde{i}_{e1}$. We now rearrange Eqs. (14.24) to (14.32) into the following set of simultaneous equations:

$$\begin{aligned}
\frac{\tilde{v}_1}{R_1} - \frac{\tilde{v}_2}{R_1} + \tilde{i}_{e1} &= 0 \\
-\frac{\tilde{v}_1}{R_1} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\tilde{v}_2 + \tilde{i}_{C1} + \tilde{i}_{C2} &= -\tilde{I}_{L1} \\
\frac{\tilde{v}_3}{\tilde{R}_3} - \frac{\tilde{v}_4}{R_3} - \tilde{i}_{C2} &= 0 \\
-\frac{\tilde{v}_3}{R_3} + \frac{\tilde{v}_4}{R_3} &= -\tilde{I}_{L2} \\
\tilde{v}_1 &= \tilde{e}_1 \\
\tilde{v}_2 &= \tilde{V}_{C1} \\
\tilde{v}_2 - \tilde{v}_3 &= \tilde{V}_{C2} \\
\tilde{v}_2 - \tilde{v}_{L1} &= 0 \\
\tilde{v}_4 - \tilde{v}_{L2} &= 0
\end{aligned} \tag{14.33}$$

where all sources and state variables are shown on the right hand side and the remaining unknown parameters are shown on the left. These equations can be written in matrix form as follows

$$\mathbf{M}\cdot\mathbf{v} = \mathbf{N}\cdot\mathbf{w} \tag{14.34}$$

where

$$\mathbf{M} = \begin{array}{c}
\begin{array}{ccccccccc} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{v}_4 & \tilde{i}_{e1} & \tilde{i}_{C1} & \tilde{i}_{C2} & \tilde{v}_{L1} & \tilde{v}_{L2} \end{array} \\
\begin{bmatrix}
\frac{1}{R_1} & -\frac{1}{R_1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{R_3} & -\frac{1}{R_3} & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -\frac{1}{R_3} & \frac{1}{R_3} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\end{array}, \quad \mathbf{v} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{v}_4 \\ \tilde{i}_{e1} \\ \tilde{i}_{C1} \\ \tilde{i}_{C2} \\ \tilde{v}_{L1} \\ \tilde{v}_{L2} \end{bmatrix} \tag{14.35}$$

$$\mathbf{N} = \begin{array}{c} \begin{array}{ccccc} \tilde{V}_{C1} & \tilde{V}_{C2} & \tilde{I}_{L1} & \tilde{I}_{L2} & \tilde{e}_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \mathbf{w} = \begin{bmatrix} \tilde{V}_{C1} \\ \tilde{V}_{C2} \\ \tilde{I}_{L1} \\ \tilde{I}_{L2} \\ \tilde{e}_1 \end{bmatrix} \tag{14.36}$$

Each row of **M** and each column of **N** must have at least one entry. Now let us partition the matrix **M** as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \tag{14.37}$$

where

$$\mathbf{M}_{11} = \begin{array}{c} \begin{array}{cccc} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{v}_4 \end{array} \\ \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 & 0 \\ 0 & 0 & \frac{1}{R_3} & -\frac{1}{R_3} \\ 0 & 0 & -\frac{1}{R_3} & \frac{1}{R_3} \end{bmatrix} \end{array} \quad \mathbf{M}_{12} = \begin{array}{c} \begin{array}{ccccc} \tilde{i}_{e1} & \tilde{i}_{C1} & \tilde{i}_{C2} & \tilde{v}_{L1} & \tilde{v}_{L2} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$
$$\mathbf{M}_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{M}_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{14.38}$$

On inspection of Eqs. (14.23) and (14.38), it becomes apparent that

$$\mathbf{M}_{12} = [\mathbf{A}_S \quad \mathbf{A}_C \quad 0] \tag{14.39}$$

and

$$\mathbf{M}_{21} = \begin{bmatrix} \mathbf{A}_S^t \\ \mathbf{A}_C^t \\ \mathbf{A}_L^t \end{bmatrix} \tag{14.40}$$

because they simply represent the nodes across which the voltage sources, capacitors, and inductors are connected. For example, $\tilde{i}_{C2}$ in column 3 of $\mathbf{M}_{12}$ leaves node 2 (hence the "1" in row 2) and enters node 3 (hence the "-1" in row 2). Also, matrix $\mathbf{M}_{22}$ just contains a negative unity matrix representing the inductors as follows:

$$\mathbf{M}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \tag{14.41}$$

Finally, matrix $\mathbf{M}_{11}$ for the resistors is a little more complicated. We recall that it was constructed using KCL, which can be expressed in terms of the $\mathbf{A}$ matrix, using Eq. (14.7) as follows

$$\mathbf{A}_R \cdot \mathbf{i}_R = 0 \tag{14.42}$$

where $\mathbf{i}_R$ is a column vector of the resistor currents. However, we wish to replace the resistor currents with the node voltages as in Eq. (14.38). First, we use the relationship of Eq. (14.10):

$$\mathbf{e}_R = \mathbf{A}_R^t \cdot \mathbf{v} \tag{14.43}$$

in order to express the resistor voltages in terms of node voltages. Multiplying this by an admittance matrix

$$\mathbf{Y}_R = \begin{bmatrix} 1/R_1 & 0 & 0 \\ 0 & 1/R_2 & 0 \\ 0 & 0 & 1/R_3 \end{bmatrix} \tag{14.44}$$

gives us the resistor currents

$$\mathbf{i}_R = \mathbf{Y}_R \cdot \mathbf{e}_R = \mathbf{Y}_R \cdot \mathbf{A}_R^t \cdot \mathbf{v} \tag{14.45}$$

Finally, inserting Eq. (14.45) into Eq. (14.42) yields the node voltages

$$\mathbf{A}_R \cdot \mathbf{Y}_R \cdot \mathbf{A}_R^t \cdot \mathbf{v} = 0 \tag{14.46}$$

which gives

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1/R_1 & 0 & 0 \\ 0 & 1/R_2 & 0 \\ 0 & 0 & 1/R_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{v}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{14.47}$$

or

$$\begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 & 0 \\ 0 & 0 & \frac{1}{R_3} & -\frac{1}{R_3} \\ 0 & 0 & -\frac{1}{R_3} & \frac{1}{R_3} \end{bmatrix} \cdot \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{v}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{14.48}$$

Therefore

$$\mathbf{M}_{11} = \mathbf{A}_R \cdot \mathbf{Y}_R \cdot \mathbf{A}_R^t \tag{14.49}$$

Similarly, we can partition $\mathbf{N}$ as follows:

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix} \tag{14.50}$$

where

$$\begin{array}{c} \begin{array}{cc} \mathbf{N}_{11} = \begin{array}{c} \begin{array}{cccc} \tilde{V}_{C1} & \tilde{V}_{C2} & \tilde{I}_{L1} & \tilde{I}_{L2} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array}, & \mathbf{N}_{12} = \begin{array}{c} \tilde{e}_1 \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \end{array} \\ \mathbf{N}_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_{22} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \tag{14.51}$$

On inspection of Eqs. (14.23) and (14.51), it becomes apparent that

$$\mathbf{N}_{11} = [0 - \mathbf{A}_L] \tag{14.52}$$

We note that from Eq. (14.36),

$$\mathbf{w} = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \tag{14.53}$$

where $\mathbf{x}$ is the state variable vector in Eqs. (14.1) and (14.2) given by

$$\mathbf{x} = \begin{bmatrix} \tilde{V}_{C1} \\ \tilde{V}_{C2} \\ \tilde{I}_{L1} \\ \tilde{I}_{L2} \end{bmatrix} \tag{14.54}$$

and $\mathbf{u}$ is the input vector in Eqs. (14.1) and (14.2) given by

$$\mathbf{u} = \begin{bmatrix} \tilde{e}_1 \end{bmatrix} \tag{14.55}$$

We note also from Eq. (14.35),

$$\mathbf{v} = \begin{bmatrix} \mathbf{y} \\ s \cdot \mathbf{E} \cdot \mathbf{x} \end{bmatrix} \tag{14.56}$$

where $\mathbf{y}$ is the output vector in Eq. (14.2) containing the node voltages

$$\mathbf{y} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{v}_4 \\ \tilde{i}_{e1} \end{bmatrix} \tag{14.57}$$

and $\mathbf{E}$ is a diagonal matrix containing the capacitor and inductor values, which relates the capacitor currents and inductor voltages in $\mathbf{v}$ to the state variables (or capacitor voltages and inductor currents) in $\mathbf{w}$ using ohms law as follows:

$$\begin{bmatrix} \tilde{i}_{C1} \\ \tilde{i}_{C2} \\ \tilde{v}_{L1} \\ \tilde{v}_{L2} \end{bmatrix} = s \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & L_2 \end{bmatrix} \cdot \begin{bmatrix} \tilde{V}_{C1} \\ \tilde{V}_{C2} \\ \tilde{I}_{L1} \\ \tilde{I}_{L2} \end{bmatrix} = s \cdot \mathbf{E} \cdot \mathbf{x} \tag{14.58}$$

We now rewrite Eq. (14.34) as

$$\mathbf{v} = \mathbf{M}^{-1} \cdot \mathbf{N} \cdot \mathbf{w} \tag{14.59}$$

or

$$\begin{bmatrix}\mathbf{y}\\ s\cdot\mathbf{E}\cdot\mathbf{x}\end{bmatrix}=\begin{array}{c}\begin{array}{ccccc}\tilde{V}_{C1} & \tilde{V}_{C2} & \tilde{I}_{L1} & \tilde{I}_{L2} & \tilde{e}_1\end{array}\\ \begin{bmatrix}0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & 0 & 0\\ 1 & -1 & 0 & -R_3 & 0\\ \frac{1}{R_1} & 0 & 0 & 0 & -\frac{1}{R_1}\\ -\frac{1}{R_1}-\frac{1}{R_2} & 0 & -1 & -1 & \frac{1}{R_1}\\ 0 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & -R_3 & 0\end{bmatrix}\end{array}\cdot\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} \tag{14.60}$$

Referring to Fig. 14.3, we can see from this matrix how the node voltages are made up. For example, row 1 gives us $\tilde{v}_1 = \tilde{e}_1$ and row 4 gives us

$$\tilde{v}_4 = \tilde{V}_{C_1} - \tilde{V}_{C2} - R_3\tilde{I}_{L2}.$$

Similarly, the remaining rows give us the voltage source and capacitor currents and inductor voltages. By comparing Eq. (14.60) with the system of equations (14.1) and (14.2), we can partition the matrix thus:

$$\begin{bmatrix}\mathbf{y}\\ s\cdot\mathbf{E}\cdot\mathbf{x}\end{bmatrix} = \mathbf{M}^{-1}\cdot\mathbf{N}\cdot\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} = \begin{bmatrix}\mathbf{C} & \mathbf{D}\\ \mathbf{E}\cdot\mathbf{A} & \mathbf{E}\cdot\mathbf{B}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} \tag{14.61}$$

where

$$\mathbf{C}=\begin{array}{c}\begin{array}{cccc}\tilde{V}_{C1} & \tilde{V}_{C2} & \tilde{I}_{L1} & \tilde{I}_{L2}\end{array}\\ \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & -1 & 0 & -R_3\\ \frac{1}{R_1} & 0 & 0 & 0\end{bmatrix}\end{array},\quad \mathbf{D}=\begin{array}{c}\tilde{e}_1\\ \begin{bmatrix}1\\ 0\\ 0\\ 0\\ -\frac{1}{R_1}\end{bmatrix}\end{array}$$

$$\mathbf{E}\cdot\mathbf{A}=\begin{bmatrix}-\frac{1}{R_1}-\frac{1}{R_2} & 0 & -1 & -1\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 1 & -1 & 0 & -R_3\end{bmatrix},\mathbf{E}\cdot\mathbf{B}=\begin{bmatrix}\frac{1}{R_1}\\ 0\\ 0\\ 0\end{bmatrix} \tag{14.62}$$

Using

$$\mathbf{E}^{-1} = \begin{bmatrix} 1/C_1 & 0 & 0 & 0 \\ 0 & 1/C_2 & 0 & 0 \\ 0 & 0 & 1/L_1 & 0 \\ 0 & 0 & 0 & 1/L_2 \end{bmatrix} \tag{14.63}$$

gives

$$\mathbf{A} = \mathbf{E}^{-1}\cdot\mathbf{E}\cdot\mathbf{A} = \begin{bmatrix} -\dfrac{1}{C_1}\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}\right) & 0 & -1/C_1 & -1/C_1 \\ 0 & 0 & 0 & -1/C_2 \\ 1/L_1 & 0 & 0 & 0 \\ 1/L_2 & -1/L_2 & 0 & -R_3/L_2 \end{bmatrix}, \tag{14.64}$$

$$\mathbf{B} = \mathbf{E}^{-1}\cdot\mathbf{E}\cdot\mathbf{B} = \begin{bmatrix} \dfrac{1}{C_1R_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14.65}$$

Now we have furnished Eqs. (14.1) and (14.2) with **A**, **B**, **C**, and **D**, we can solve them using the Faddeev–Leverrier algorithm.

14.8 SOLUTION OF THE WORKED EXAMPLE USING THE FADDEEV–LEVERRIER ALGORITHM [9]

The transfer function $f_{i,j}$ between the j^{th} (voltage or current) source i^{th} node voltage is a rational polynomial of the form

$$f_{i,j} = \frac{\sum_{n=1}^{N} Q_n(i,j)s^{N-n}}{s^N + \sum_{n=1}^{N} P_n s^{N-n}} \tag{14.66}$$

except when the i^{th} output voltage node is also that of the j^{th} voltage source such that $\mathbf{D}(i,j) = 1$, in which case we add s^N to the numerator of Eq. (14.66) so that it becomes unity. The polynomial coefficients are calculated using the following procedure:

$$
\begin{aligned}
\mathbf{F}_1 &= \mathbf{I}, \\
\mathbf{G}_1 &= \mathbf{A}, \\
P_1 &= -Tr(\mathbf{G}_1), \\
\mathbf{F}_n &= \mathbf{G}_{n-1} + P_{n-1}.\mathbf{I}, \\
\mathbf{G}_n &= \mathbf{A}\cdot\mathbf{F}_n, \\
P_n &= -\frac{Tr(\mathbf{G}_n)}{n} \\
Q_n(i,j) &= [\mathbf{C}\cdot\mathbf{F}_n\cdot\mathbf{B} + \mathbf{D}\cdot P_n]_{i,j}
\end{aligned}
\tag{14.67}
$$

where $Tr(\ )$ denotes the trace of the matrix and $\mathbf{I}$ is the identity matrix. In our worked example of Fig. 14.3, we obtain for node 4:

$$f_{4,1} = \frac{\tilde{v}_4}{\tilde{e}_1} = \frac{Q_1(4,1)s^3}{s^4 + P_1 s^3 + P_2 s^2 + P_3 s + P_4} \tag{14.68}$$

where

$$P_1 = \frac{R_1 + R_2}{R_1 R_2 C_1} + \frac{R_3}{L_2} \tag{14.69}$$

$$P_2 = \frac{1}{L_1 C_1} + \frac{1}{L_2 C_1} + \frac{1}{L_2 C_2} + \frac{R_1 + R_2}{R_1 R_2 C_1}\frac{R_3}{L_2} \tag{14.70}$$

$$P_3 = \frac{R_1 + R_2}{R_1 R_2 C_1 C_2 L_2} + \frac{1}{L_1 C_1}\frac{R_3}{L_2} \tag{14.71}$$

$$P_4 = \frac{1}{L_1 C_1 L_2 C_2} \tag{14.72}$$

$$Q_1(4,1) = \frac{1}{R_1 C_1} = \frac{(Bl)^2}{R_E(M_{MD} + S_D^2 M_{AR})} \tag{14.73}$$

from Eq. (14.15). For node 2 we have

$$f_{2,1} = \frac{\tilde{v}_2}{\tilde{e}_1} = \frac{Q_1(2,1)s^3 + Q_2(2,1)s^2 + Q_3(2,1)s}{s^4 + P_1 s^3 + P_2 s^2 + P_3 s + P_4} \tag{14.74}$$

where the denominator coefficients are the same as for $f_{4,1}$, but the numerator coefficients are

$$Q_1(2,1) = \frac{1}{R_1 C_1} \tag{14.75}$$

$$Q_2(2,1) = \frac{1}{R_1C_1}\frac{R_3}{L_2} \tag{14.76}$$

$$Q_3(2,1) = \frac{1}{R_1C_1}\frac{1}{C_2L_2} \tag{14.77}$$

14.9 FAR-FIELD ON-AXIS PRESSURE

The far-field on-axis pressure is calculated from the voltage at node 4 by combining Eqs. (14.20) and (14.21):

$$\tilde{p}(r) = s\frac{\rho_0 S_D}{4\pi Blr}\tilde{v}_4 \tag{14.78}$$

Also, it is convenient to express the polynomial coefficients in terms of the Thiele–Small parameters which are the standard specifications for loudspeaker drive units:

$$\tilde{p}(r) = \frac{\tilde{e}_1 BlS_D\rho_0}{4\pi rR_E M_{MS}}\left(\frac{s^4}{s^4 + P_1s^3 + P_2s^2 + P_3s + P_4}\right) \tag{14.79}$$

where

$$P_1 = \frac{\omega_S}{Q_{TS}} + \frac{\omega_B}{Q_L} \tag{14.80}$$

$$P_2 = \left(1 + \frac{V_{AS}}{V_B}\right)\omega_S^2 + \omega_B^2 + \frac{\omega_S\omega_B}{Q_{TS}Q_L} \tag{14.81}$$

$$P_3 = \frac{\omega_S\omega_B^2}{Q_{TS}} + \frac{\omega_S^2\omega_B}{Q_L} \tag{14.82}$$

$$P_4 = \omega_S^2\omega_B^2 \tag{14.83}$$

where

ω_S is the angular suspension resonant-frequency in an infinite baffle given by

$$\omega_S = \frac{1}{\sqrt{L_1C_1}} = \frac{1}{\sqrt{M_{MS}C_{MS}}} \tag{14.84}$$

Q_{ES} is the electrical Q factor

$$Q_{ES} = \omega_S R_1 C_1 = \omega_S \frac{R_g + R_E}{(Bl)^2} M_{MS} \tag{14.85}$$

Q_{MS} is the mechanical Q factor

$$Q_{MS} = \omega_S R_2 C_1 = \omega_S \frac{1}{R_{MS}} M_{MS} \tag{14.86}$$

Q_{TS} is the total Q factor

$$Q_{TS} = \frac{Q_{ES} Q_{MS}}{Q_{ES} + Q_{MS}} \tag{14.87}$$

ω_B is the angular resonant-frequency of the box and port (including end corrections) given by

$$\omega_B = \frac{1}{\sqrt{L_2 C_2}} = \frac{1}{\sqrt{M_{AT} C_{AB}}} \tag{14.88}$$

Q_L is the acoustical Q factor due to box and port losses

$$Q_L = \omega_B \frac{L_2}{R_3} = \omega_B R_{AL} C_{AB} \tag{14.89}$$

V_B is the box volume which is related to the acoustic compliance by

$$V_B = \gamma P_0 C_{AB} \tag{14.90}$$

and V_{AS} is the suspension equivalent volume

$$V_{AS} = S_D^2 \gamma P_0 C_{MS} \tag{14.91}$$

There are just six Thiele–Small parameters which completely define a loudspeaker: R_E, Q_{ES}, Q_{MS}, f_S, S_D, and V_{AS}, where

$$Bl = S_D \sqrt{\frac{R_E \gamma P_0}{Q_{ES} \omega_S V_{AS}}} \tag{14.92}$$

14.10 WORKED EXAMPLE NO. 2: LOUDSPEAKER IN AN ENCLOSURE WITH A BASS-REFLEX PORT USING THE NORTON EQUIVALENT SOURCE

In order to illustrate the use of current sources in state-variable analysis, we will use *Norton's theorem* in order to convert the voltage source from the previous example into a current source, as shown in Fig. 14.4. It can be seen that a node is removed in the process.

Proof. Suppose we connect a resistor R_L between the output terminal and ground. In the case of the voltage source, the load current is $\tilde{e}/(R+R_L)$. Therefore, the output voltage is $\tilde{e}_0 = \tilde{e}R_L/(R+R_L)$. In the case of the current source, the output voltage is $\tilde{e}_0 = \tilde{i}(R//R_L) = \tilde{i}RR_L/(R+R_L)$. Hence, it can be seen that the two output voltages are equal when $\tilde{i} = \tilde{e}/R$.

Thus the circuit of Fig. 14.3 can be redrawn as shown in Fig. 14.5. The revised net list is shown in Table 14.3, and the **A** matrix becomes

$$\begin{array}{cccccccccc} & \frac{\tilde{e}_1}{R_1} & R_1 & R_2 & R_3 & C_1 & C_2 & L_1 & L_2 & \text{node} \\ \mathbf{A} = & \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 1 \\ -1 \end{array} & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 1 \\ -1 \\ 0 \end{array} & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] & \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \end{array} \tag{14.93}$$

which can be partitioned into four matrices: $\mathbf{A}_S$, $\mathbf{A}_R$, $\mathbf{A}_C$, and $\mathbf{A}_L$ representing the connectivity of the sources, resistors, capacitors, and inductors respectively:

$$\begin{array}{c} \frac{\tilde{e}_1}{R_1} \\ \mathbf{A}_S = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} R_1 \; R_2 \; R_3 \\ \mathbf{A}_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \end{array}, \begin{array}{c} C_1 \; C_2 \\ \mathbf{A}_C = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \end{array}, \begin{array}{cc} L_1 \; L_2 & \text{node} \\ \mathbf{A}_L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \end{array} \tag{14.94}$$

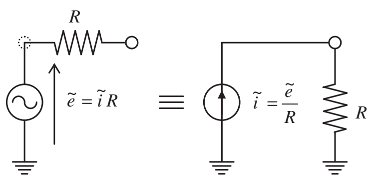

FIG. 14.4 Norton's theorem.

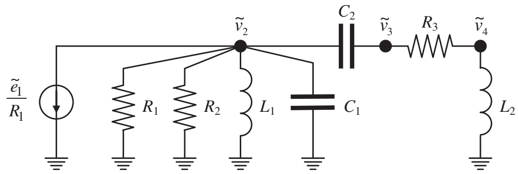

FIG. 14.5 Equivalent electrical circuit of loudspeaker in bass-reflex enclosure using Norton equivalent source.

Now let us return to the circuit of Fig. 14.5. As before, the first step of the analysis is to apply KCL at each of the nodes:

At node 2

$$\frac{\tilde{e}_1}{R_1} + \frac{\tilde{v}_2}{R_1} + \frac{\tilde{v}_2}{R_2} + \tilde{I}_{L1} + \tilde{i}_{C1} + \tilde{i}_{C2} = 0 \tag{14.95}$$

At node 3

$$\tilde{i}_{C2} = \frac{\tilde{v}_3 - \tilde{v}_4}{R_3} \tag{14.96}$$

At node 4

$$\frac{\tilde{v}_3 - \tilde{v}_4}{R_3} = \tilde{I}_{L2} \tag{14.97}$$

Next we apply KVL to the capacitors and inductors:
For C_1

$$\tilde{v}_2 = \tilde{V}_{C1} \tag{14.98}$$

For C_2

$$\tilde{v}_2 - \tilde{v}_3 = \tilde{V}_{C2} \tag{14.99}$$

Table 14.3 Net list for worked example No. 2

Element	From node	To node
$\tilde{e}_1/R_1$	2	0
R_1	2	0
R_2	2	0
R_3	3	4
C_1	2	0
C_2	2	3
L_1	2	0
L_2	4	0

For L_1

$$\tilde{v}_2 = \tilde{v}_{L1} \tag{14.100}$$

For L_2

$$\tilde{v}_4 = \tilde{v}_{L2} \tag{14.101}$$

We now rearrange Eqs. (14.95) to (14.101) into the following set of simultaneous equations:

$$\begin{aligned}
\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\tilde{v}_2 + \tilde{i}_{C1} + \tilde{i}_{C2} &= \frac{\tilde{e}_1}{R_1} - \tilde{I}_{L1} \\
\frac{\tilde{v}_3}{R_3} - \frac{\tilde{v}_4}{R_3} - \tilde{i}_{C2} &= 0 \\
-\frac{\tilde{v}_3}{R_3} + \frac{\tilde{v}_4}{R_3} &= -\tilde{I}_{L2} \\
\tilde{v}_2 &= \tilde{V}_{C1} \\
\tilde{v}_2 - \tilde{v}_3 &= \tilde{V}_{C2} \\
\tilde{v}_2 - \tilde{v}_{L1} &= 0 \\
\tilde{v}_4 - \tilde{v}_{L2} &= 0
\end{aligned} \tag{14.102}$$

where all sources and state variables are shown on the right hand side and the remaining unknown parameters are shown on the left. These equations can be written in matrix form as follows:

$$\mathbf{M}\cdot\mathbf{v} = \mathbf{N}\cdot\mathbf{w} \tag{14.103}$$

where

$$\mathbf{M} = \begin{array}{c}
\begin{array}{ccccccc} \tilde{v}_2 & \tilde{v}_3 & \tilde{v}_4 & \tilde{i}_{C1} & \tilde{i}_{C2} & \tilde{v}_{L1} & \tilde{v}_{L2} \end{array} \\
\begin{bmatrix}
\frac{1}{R_1}+\frac{1}{R_2} & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & \frac{1}{R_3} & -\frac{1}{R_3} & 0 & -1 & 0 & 0 \\
0 & -\frac{1}{R_3} & \frac{1}{R_3} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1
\end{bmatrix}
\end{array}, \quad \mathbf{v} = \begin{bmatrix} \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{v}_4 \\ \tilde{i}_{C1} \\ \tilde{i}_{C2} \\ \tilde{v}_{L1} \\ \tilde{v}_{L2} \end{bmatrix} \tag{14.104}$$

$$\mathbf{N} = \begin{array}{c} \begin{array}{ccccc} \tilde{V}_{C1} & \tilde{V}_{C2} & \tilde{I}_{L1} & \tilde{I}_{L2} & \tilde{e}_1/R_1 \end{array} \\ \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \mathbf{w} = \begin{bmatrix} \tilde{V}_{C1} \\ \tilde{V}_{C2} \\ \tilde{I}_{L1} \\ \tilde{I}_{L2} \\ \tilde{e}_1/R_1 \end{bmatrix} \tag{14.105}$$

If we delete the first and middle rows and columns of matrix **M** in Eq. (14.35), which represent the voltage source node voltage and current respectively, it can be seen that it is the same matrix **M** as in Eq. (14.104). Likewise, if we delete the first and middle rows of matrix **N** in Eq. (14.36), it can be seen that it is the same matrix **N** as in Eq. (14.105). Exactly the same method as before is used to solve Eq. (14.103), resulting in the same expressions for the node voltages.

14.11 WORKED EXAMPLE NO. 3: LOUDSPEAKER IN AN ENCLOSURE WITH A BASS-REFLEX PORT USING A TRANSFORMER AND GYRATOR

The equivalent circuit shown in Fig. 14.6 is essentially the same as that shown in Fig. 14.3, except that the mechanical and acoustical sections are in their own respective domains and the acoustical section is shown using the impedance analogy instead of the electrical one. A transformer separates the mechanical domain from the electrical one, while the gyrator performs the dual functions of separating the acoustical domain from the mechanical one as well as providing the transition from admittance to impedance analogies. The elements are now given by

$$t_1 = Bl \tag{14.106}$$

$$g_{1P} = g_{1S} = S_D \tag{14.107}$$

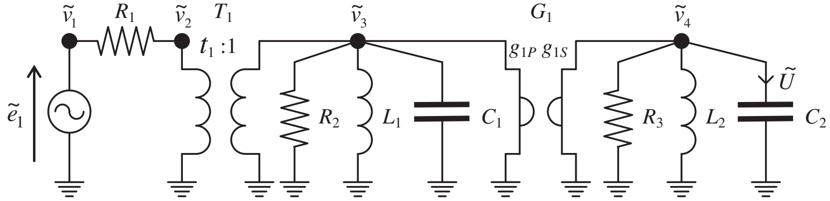

FIG. 14.6 Equivalent electrical circuit of loudspeaker in bass-reflex enclosure using transformer and gyrator.

$$R_1 = R_E \tag{14.108}$$

$$R_2 = \frac{1}{R_{MS}} \tag{14.109}$$

$$R_3 = R_{AL} \tag{14.110}$$

$$C_1 = M_{MD} + S_D^2 M_{AR} \tag{14.111}$$

$$C_2 = C_{AB} \tag{14.112}$$

$$L_1 = C_{MS} \tag{14.113}$$

$$L_2 = M_{AP} \tag{14.114}$$

$$\tilde{v}_1 = \tilde{e}_1 \tag{14.115}$$

$$\tilde{v}_2 = sBl\tilde{x} \tag{14.116}$$

$$\tilde{v}_3 = \tilde{u} = s\tilde{x} \tag{14.117}$$

$$\tilde{v}_4 = \tilde{p} = \frac{1}{sC_2}\tilde{U} \tag{14.118}$$

$$\tilde{p}(r) = s\frac{\rho_0 \tilde{U}}{2\pi\, r} \tag{14.119}$$

Of course, the above quantities, except for R_1, are no longer electrical, but it is convenient for the purpose of the following analysis to keep them as electrical terms. In the computer program, it is not important what units are used, so long as they are consistent. The net list can be written as shown in Table 14.4, from which the following **A** matrix is created:

$$\mathbf{A} = \begin{array}{c} \begin{array}{cccccccccccc} \tilde{e}_1 & R_1 & R_2 & R_3 & T_{1P} & T_{1S} & G_{1P} & G_{1S} & C_1 & C_2 & L_1 & L_2 \end{array} \text{ node} \\ \left[\begin{array}{cccccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \tag{14.120}$$

which can be partitioned into six matrices: $\mathbf{A}_S$, $\mathbf{A}_R$, $\mathbf{A}_T$, $\mathbf{A}_G$, $\mathbf{A}_C$, and $\mathbf{A}_L$ representing the connectivity of the sources, resistors, transformers, gyrators, capacitors, and inductors respectively:

$$\mathbf{A}_S = \begin{array}{c} \tilde{e}_1 \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}, \mathbf{A}_R = \begin{array}{c} \begin{matrix} R_1 & R_2 & R_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \mathbf{A}_T = \begin{array}{c} \begin{matrix} T_{1P} & T_{1S} \end{matrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}, \mathbf{A}_G = \begin{array}{c} \begin{matrix} G_{1P} & G_{1S} \end{matrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \mathbf{A}_C = \begin{array}{c} \begin{matrix} C_1 & C_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \mathbf{A}_L = \begin{array}{c} \begin{matrix} L_1 & L_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \tag{14.121}$$

Now let us return to the circuit of Fig. 14.6. As before, the first step of the analysis is to apply KCL at each of the nodes. The primary and secondary currents of the transformer T_1 are denoted by $\tilde{i}_{T1P}$ and $\tilde{i}_{T1S}$ respectively. Likewise, the primary and secondary currents of the gyrator G_1 are denoted by $\tilde{i}_{G1P}$ and $\tilde{i}_{G1S}$ respectively.

At node 1

$$\tilde{i}_{e1} = \frac{\tilde{v}_2 - \tilde{v}_1}{R_1} \tag{14.122}$$

At node 2

$$\frac{\tilde{v}_1 - \tilde{v}_2}{R_1} = \tilde{i}_{T1P} \tag{14.123}$$

Table 14.4 Net list for worked example No. 3

Element	From node	To node
$\tilde{e}_1$	1	0
R_1	1	2
R_2	3	0
R_3	4	0
T_{1P}	2	0
T_{1S}	3	0
G_{1P}	3	0
G_{1S}	4	0
C_1	3	0
C_2	4	0
L_1	3	0
L_2	4	0

At node 3

$$-\tilde{i}_{T1S} = \frac{\tilde{v}_3}{R_2} + \tilde{I}_{L1} + \tilde{i}_{C1} + \tilde{i}_{G1P} \tag{14.124}$$

At node 4

$$-\tilde{i}_{G1S} = \frac{\tilde{v}_4}{R_3} + \tilde{I}_{L2} + \tilde{i}_{C2} \tag{14.125}$$

For the transformer T_1, we have the following pair of equations that define the voltage and current cross-coupling:

$$\tilde{v}_2 = t_1 \tilde{v}_3 \tag{14.126}$$

$$\tilde{i}_{T1S} = -t_1 \tilde{i}_{T1P} \tag{14.127}$$

For the gyrator G_1, we have the following pair of equations that define the forward and reverse coupling via the mutual conductances g_{1P} and g_{1S} respectively:

$$-\tilde{i}_{G1S} = g_{1P} \tilde{v}_3 \tag{14.128}$$

$$\tilde{i}_{G1P} = g_{1S} \tilde{v}_4 \tag{14.129}$$

For the voltage source

$$\tilde{v}_1 = \tilde{e}_1. \tag{14.130}$$

Next we apply KVL to the capacitors and inductors:

For C_1

$$\tilde{v}_3 = \tilde{V}_{C1} \tag{14.131}$$

For C_2

$$\tilde{v}_4 = \tilde{V}_{C2} \tag{14.132}$$

For L_1

$$\tilde{v}_3 = \tilde{v}_{L1} \tag{14.133}$$

For L_2

$$\tilde{v}_4 = \tilde{v}_{L2} \tag{14.134}$$

We now rearrange Eqs. (14.122) to (14.134) into the following set of simultaneous equations:

$$
\begin{aligned}
&\frac{1}{R_1}\tilde{v}_1 - \frac{1}{R_1}\tilde{v}_2 + \tilde{i}_{e1} = 0 \\
&-\frac{1}{R_1}\tilde{v}_1 + \frac{1}{R_1}\tilde{v}_2 + \tilde{i}_{T1P} = 0 \\
&\frac{1}{R_2}\tilde{v}_3 + \tilde{i}_{T1S} + \tilde{i}_{G1P} + \tilde{i}_{C1} = -\tilde{I}_{L1} \\
&\frac{1}{R_3}\tilde{v}_4 + \tilde{i}_{G1S} + \tilde{i}_{C2} = -\tilde{I}_{L1} \\
&\tilde{v}_2 - t_1\tilde{v}_3 = 0 \\
&t_1\tilde{i}_{T1P} + \tilde{i}_{T1S} = 0 \\
&\tilde{v}_3 + \frac{1}{g_{1P}}\tilde{i}_{G1S} = 0 \\
&\tilde{v}_4 - \frac{1}{g_{1S}}i_{G1P} = 0 \\
&\tilde{v}_1 = \tilde{e}_1 \\
&\tilde{v}_3 = \tilde{V}_{C1} \\
&\tilde{v}_4 = \tilde{V}_{C2} \\
&\tilde{v}_3 - \tilde{v}_{L1} = 0 \\
&\tilde{v}_4 - \tilde{v}_{L2} = 0
\end{aligned}
\tag{14.135}
$$

where all sources and state variables are shown on the right hand side and the remaining unknown parameters are shown on the left. These equations can be written in matrix form as follows:

$$\mathbf{M}\cdot\mathbf{v} = \mathbf{N}\cdot\mathbf{w} \tag{14.136}$$

where

$$\mathbf{M} = \begin{array}{c} \begin{matrix} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{v}_4 & \tilde{i}_{T1P} & \tilde{i}_{T1S} & \tilde{i}_{G1P} & \tilde{i}_{G1S} & \tilde{i}_{e1} & \tilde{i}_{C1} & \tilde{i}_{C2} & \tilde{v}_{L1} & \tilde{v}_{L2} \end{matrix} \\ \begin{bmatrix} R_1^{-1} & -R_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -R_1^{-1} & R_1^{-1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_2^{-1} & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_3^{-1} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & g_{1P} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -g_{1S} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}, \quad \mathbf{v} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{v}_4 \\ \tilde{i}_{e1} \\ \tilde{i}_{T1P} \\ \tilde{i}_{T1S} \\ \tilde{i}_{G1P} \\ \tilde{i}_{G1S} \\ \tilde{i}_{C1} \\ \tilde{i}_{C2} \\ \tilde{v}_{L1} \\ \tilde{v}_{L2} \end{bmatrix} \tag{14.137}$$

$$\mathbf{N} = \begin{array}{c} \begin{matrix} \tilde{V}_{C1} & \tilde{V}_{C2} & \tilde{I}_{L1} & \tilde{I}_{L2} & \tilde{e}_1 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad \mathbf{w} = \begin{bmatrix} \tilde{V}_{C1} \\ \tilde{V}_{C2} \\ \tilde{I}_{L1} \\ \tilde{I}_{L2} \\ \tilde{e}_1 \end{bmatrix} \tag{14.138}$$

Now let us partition the matrix **M** as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{31} & \mathbf{M}_{33} \end{bmatrix} \tag{14.139}$$

where

$$\mathbf{M}_{11} = \begin{array}{c} \begin{array}{cccc} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{v}_4 \end{array} \\ \begin{bmatrix} R_1^{-1} & -R_1^{-1} & 0 & 0 \\ -R_1^{-1} & R_1^{-1} & 0 & 0 \\ 0 & 0 & R_2^{-1} & 0 \\ 0 & 0 & 0 & R_3^{-1} \end{bmatrix} \end{array} \mathbf{M}_{12} = \begin{array}{c} \begin{array}{cc} \tilde{i}_{T1P} & \tilde{i}_{T1S} \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array} \mathbf{M}_{13} = \begin{array}{c} \begin{array}{ccccccc} \tilde{i}_{G1P} & \tilde{i}_{G1S} & \tilde{i}_{e1} & \tilde{i}_{C1} & \tilde{i}_{C2} & \tilde{v}_{L1} & \tilde{v}_{L2} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{M}_{21} = \begin{bmatrix} 0 & 1 & -t_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{M}_{22} = \begin{bmatrix} 0 & 0 \\ t_1 & 1 \end{bmatrix} \mathbf{M}_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{14.140}$$

$$\mathbf{M}_{31} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{M}_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{M}_{33} = \begin{bmatrix} 0 & g_{1P} & 0 & 0 & 0 & 0 & 0 \\ -g_{1S} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

On inspection of Eqs. (14.121) and (14.140), it becomes apparent that

$$\mathbf{M}_{12} = \mathbf{A}_T \tag{14.141}$$

$$\mathbf{M}_{13} = [\mathbf{A}_G \quad \mathbf{A}_S \quad \mathbf{A}_C \quad 0] \tag{14.142}$$

and

$$\mathbf{M}_{31} = \begin{bmatrix} \mathbf{A}_G^t \\ \mathbf{A}_S^t \\ \mathbf{A}_C^t \\ \mathbf{A}_L^t \end{bmatrix} \tag{14.143}$$

because they simply represent the nodes across which the transformers, gyrators, voltage sources, capacitors, and inductors are connected. Also, matrix $\mathbf{M}_{33}$ contains a negative unity matrix representing the inductors plus some off-diagonal terms for the forward and reverse mutual conductances of the gyrator. The matrices $\mathbf{M}_{21}$ and $\mathbf{M}_{22}$ represent the voltage and current transfer characteristics respectively of the transformer. We note that

$$\mathbf{A}_T^t = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{14.144}$$

The first row of $\mathbf{M}_{21}$ is obtained by multiplying the second row of $\mathbf{A}_T^t$ (which represents the secondary voltage) by $-t$ and then adding it to the first row (which represents the primary voltage). This gives us the voltage relationship of Eq. (14.126). The second row of $\mathbf{M}_{21}$ is simply null. If there is more than one transformer, the same process is repeated for each successive pair of rows. The first row of $\mathbf{M}_{22}$ is null with the second row giving the current transfer ratio. Matrices $\mathbf{M}_{23}$ and $\mathbf{M}_{32}$ are both null. Finally, matrix $\mathbf{M}_{11}$ is given by

$$\mathbf{M}_{11} = \mathbf{A}_R \cdot \mathbf{Y}_R \cdot \mathbf{A}_R^t \tag{14.145}$$

as before. Also, matrix $\mathbf{N}$ is constructed in exactly the same way as before and the same method is used to solve Eq. (14.136).

14.12 WORKED EXAMPLE NO. 4: LOUDSPEAKER IN AN ENCLOSURE WITH A BASS-REFLEX PORT USING CONTROLLED SOURCES

The equivalent circuit shown in Fig. 14.7 is essentially the same as that shown in Fig. 14.6, except for two modifications which, as we shall see, do not affect its operation. First, the transformer has been replaced by the combination of a current-controlled current source (CC_1) in the forward direction and a voltage-controlled voltage source (VV_1) in the reverse direction such that the current source represents Eq. (14.127) and the voltage source Eq. (14.126). Second, the gyrator has been replaced by two voltage-controlled current sources: one in the forward direction (VC_1) representing Eq. (14.128) and the other in the reverse direction representing Eq. (14.129). Because the nodal method used here prevents the connection of a short-circuit between two nodes, a small value resistor (R_4) is connected between the current-sensing terminals of the current-controlled current source CC_1. In the case of a symbolic transfer function, R_4 can be set to zero in the final solution.

Although it is simpler to use transformers and gyrators directly, the purpose of this circuit is illustrative since it is quite common to encounter acoustical systems with active components, for example loudspeakers with current or motional feedback, where the amplifier can be represented as

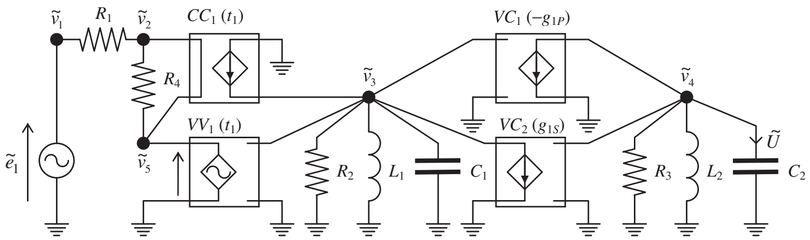

FIG. 14.7 Equivalent electrical circuit of loudspeaker in bass-reflex enclosure using controlled sources.

a controlled source. After using Norton's theorem to replace the voltage source $\tilde{e}_1$ with a current source $\tilde{e}_1/R_1$, the new net list can now be written as shown in Table 14.5, from which the following $\mathbf{A}$ matrix is created:

$$\mathbf{A} = \begin{array}{cc} \begin{array}{ccccccccccccccccc} \tilde{e}_1 & R_1 & R_2 & R_3 & R_4 & VV_{1P} & VV_{1S} & CC_{1P} & CC_{1S} & VC_{1P} & VC_{1S} & VC_{2P} & VC_{2S} & C_1 & C_2 & L_1 & L_2 \end{array} & \text{node} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array} \tag{14.146}$$

which can be partitioned into six matrices: $\mathbf{A}_S$, $\mathbf{A}_R$, $\mathbf{A}_\mathrm{T}$, $\mathbf{A}_\mathrm{G}$, $\mathbf{A}_C$, and $\mathbf{A}_L$ representing the connectivity of the sources, resistors, transformers, gyrators, capacitors, and inductors respectively:

Table 14.5 Net list for worked example No. 4

Element	From node	To node
$\tilde{e}_1$	1	0
R_1	1	2
R_2	3	0
R_3	4	0
R_4	2	5
VV_{1P}	3	0
VV_{1S}	5	0
CC_{1P}	2	5
CC_{1S}	0	3
VC_{1P}	3	0
VC_{1S}	4	0
VC_{2P}	4	0
VC_{2S}	3	0
C_1	3	0
C_2	4	0
L_1	3	0
L_2	4	0

$$\mathbf{A}_S = \begin{array}{c} \frac{\tilde{e}_1}{R_1} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}, \mathbf{A}_R = \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array}, \mathbf{A}_{VVP} = \begin{array}{c} VV_{1P} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array}, \mathbf{A}_{VVS} = \begin{array}{c} VV_{1S} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{array}, \mathbf{A}_{CCP} = \begin{array}{c} CC_{1P} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \end{array},$$

$$\mathbf{A}_{CCS} = \begin{array}{c} CC_{1S} \\ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \end{array}, \mathbf{A}_{VCP} = \begin{array}{c} \begin{array}{cc} VC_{1P} & VC_{2P} \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}, \mathbf{A}_{VCS} = \begin{array}{c} \begin{array}{cc} VC_{1S} & VC_{2S} \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \mathbf{A}_C = \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}, \mathbf{A}_L = \begin{array}{c} \begin{array}{cc} L_1 & L_2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array} \tag{14.147}$$

Now let us return to the circuit of Fig. 14.7. As before, the first step of the analysis is to apply KCL at each of the nodes. The secondary current of the voltage-controlled voltage source VV_1 is denoted by $\tilde{i}_{VV1S}$. Similarly, the secondary current of the current-controlled current source CC_1 is denoted by $\tilde{i}_{CC1S}$. Likewise, the secondary currents of the voltage-controlled current sources VC_1 and VC_2 are denoted by $\tilde{i}_{VC1S}$ and $\tilde{i}_{VC2S}$ respectively.

At node 1

$$\tilde{i}_{e1} = \frac{\tilde{v}_2 - \tilde{v}_1}{R_1} \tag{14.148}$$

At node 2

$$\frac{\tilde{v}_1 - \tilde{v}_2}{R_1} = \frac{\tilde{v}_2 - \tilde{v}_5}{R_4} \tag{14.149}$$

At node 3

$$\tilde{i}_{CC1S} = \frac{\tilde{v}_3}{R_2} + \tilde{I}_{L1} + \tilde{i}_{C1} + \tilde{i}_{VC2S} \tag{14.150}$$

At node 4

$$\tilde{i}_{VC1S} + \frac{\tilde{v}_4}{R_3} + \tilde{I}_{L2} + \tilde{i}_{C2} = 0 \tag{14.151}$$

At node 5

$$\frac{\tilde{v}_2 - \tilde{v}_5}{R_4} = \tilde{i}_{VV1S} \tag{14.152}$$

For the current-controlled current source CC_1, we have the following cross-coupled current relationship:

$$\tilde{i}_{CC1S} = t_1 \frac{\tilde{v}_2 - \tilde{v}_5}{R_4} \tag{14.153}$$

For the voltage-controlled voltage source VV_1, we have the following cross-coupled voltage relationship:

$$\tilde{v}_5 = t_1 \tilde{v}_3 \tag{14.154}$$

For the voltage-controlled current sources VC_1 and VC_2, we have the following pair of equations that define the forward and reverse coupling via the mutual conductances g_{1P} and g_{1S} respectively:

$$\tilde{i}_{VC1S} = g_{1P} \tilde{v}_3 \tag{14.155}$$

$$\tilde{i}_{VC2S} = g_{1S} \tilde{v}_4 \tag{14.156}$$

For the voltage source

$$\tilde{v}_1 = \tilde{e}_1 \tag{14.157}$$

Next we apply KVL to the capacitors and inductors:

For C_1

$$\tilde{v}_2 = \tilde{V}_{C1} \tag{14.158}$$

For C_2

$$\tilde{v}_3 = \tilde{V}_{C2} \tag{14.159}$$

For L_1

$$\tilde{v}_2 = \tilde{v}_{L1} \tag{14.160}$$

For L_2

$$\tilde{v}_3 = \tilde{v}_{L2} \tag{14.161}$$

We now rearrange Eqs. (14.148) to (14.161) into the following set of simultaneous equations:

$$\begin{aligned}
&\frac{1}{R_1}\tilde{v}_1 - \frac{1}{R_1}\tilde{v}_2 + \tilde{i}_{e1} = 0 \\
&-\frac{1}{R_1}\tilde{v}_1 + \left(\frac{1}{R_1}+\frac{1}{R_4}\right)\tilde{v}_2 - \frac{1}{R_4}\tilde{v}_5 = 0 \\
&\frac{1}{R_2}\tilde{v}_3 - \tilde{i}_{CC1S} + \tilde{i}_{VC2S} + \tilde{i}_{C1} = -\tilde{I}_{L1} \\
&\frac{1}{R_3}\tilde{v}_4 + \tilde{i}_{VC1S} + \tilde{i}_{C2} = -\tilde{I}_{L2} \\
&-\frac{1}{R_4}\tilde{v}_2 + \frac{1}{R_4}\tilde{v}_5 + \tilde{i}_{VV1S} = 0 \\
&\tilde{v}_5 - t_1\tilde{v}_3 = 0 \\
&-\frac{t_1}{R_4}\tilde{v}_2 + \frac{t_1}{R_4}\tilde{v}_5 + i_{CC1S} = 0 \\
&\tilde{v}_3 - \frac{1}{g_{1P}}\tilde{i}_{VC1S} = 0 \\
&\tilde{v}_4 - \frac{1}{g_{1S}}\tilde{i}_{VC2S} = 0 \\
&\tilde{v}_1 = \tilde{e}_1 \\
&\tilde{v}_3 = \tilde{V}_{C1} \\
&\tilde{v}_4 = \tilde{V}_{C2} \\
&\tilde{v}_3 - \tilde{v}_{L1} = 0 \\
&\tilde{v}_4 - \tilde{v}_{L2} = 0
\end{aligned} \tag{14.162}$$

where all sources and state variables are shown on the right hand side and the remaining unknown parameters are shown on the left. These equations can be written in matrix form as follows:

$$\mathbf{M}\cdot\mathbf{v} = \mathbf{N}\cdot\mathbf{w} \tag{14.163}$$

where

$$
\mathbf{M}=\begin{array}{c}
\begin{matrix} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{v}_4 & \tilde{v}_5 & \tilde{i}_{VV1S} & \tilde{i}_{CC1S} & \tilde{i}_{VC1S} & \tilde{i}_{VC2S} & \tilde{i}_{e1} & \tilde{i}_{C1} & \tilde{i}_{C2} & \tilde{v}_{L1} & \tilde{v}_{L2} \end{matrix} \\
\begin{bmatrix}
R_1^{-1} & -R_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-R_1^{-1} & R_1^{-1}+R_2^{-1} & 0 & 0 & -R_4^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & R_2^{-1} & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & R_3^{-1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -R_4^{-1} & 0 & 0 & R_4^{-1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -t_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & t_1^{-1}R_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -g_{1P}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -g_{1S}^{-1} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
\end{array},\ \mathbf{v}=\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{v}_4 \\ \tilde{v}_5 \\ \tilde{i}_{VV1S} \\ \tilde{i}_{CC1S} \\ \tilde{i}_{VC1S} \\ \tilde{i}_{VC2S} \\ \tilde{i}_{e1} \\ \tilde{i}_{C1} \\ \tilde{i}_{C2} \\ \tilde{v}_{L1} \\ \tilde{v}_{L2} \end{bmatrix}
\tag{14.164}
$$

$$
\mathbf{N}=\begin{array}{c}
\begin{matrix} \tilde{V}_{C1} & \tilde{V}_{C2} & \tilde{I}_{L1} & \tilde{I}_{L2} & \tilde{e}_1 \end{matrix} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array},\ \mathbf{w}=\begin{bmatrix} \tilde{V}_{C1} \\ \tilde{V}_{C2} \\ \tilde{I}_{L1} \\ \tilde{I}_{L2} \\ \tilde{e}_1 \end{bmatrix}
\tag{14.165}
$$

Now let us partition the matrix **M** as follows:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{31} & \mathbf{M}_{33} \end{bmatrix} \tag{14.166}$$

where

$$\mathbf{M}_{11} = \begin{array}{c} \begin{array}{ccccc} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{v}_4 & \tilde{v}_5 \end{array} \\ \begin{bmatrix} R_1^{-1} & -R_1^{-1} & 0 & 0 & 0 \\ -R_1^{-1} & R_1^{-1}+R_2^{-1} & 0 & 0 & -R_4^{-1} \\ 0 & 0 & R_2^{-1} & 0 & 0 \\ 0 & 0 & 0 & R_3^{-1} & 0 \\ 0 & -R_4^{-1} & 0 & 0 & R_4^{-1} \end{bmatrix} \end{array} \mathbf{M}_{12} = \begin{array}{c} \begin{array}{cc} \tilde{i}_{VV1S} & \tilde{i}_{CC1S} \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array} \mathbf{M}_{13} = \begin{array}{c} \begin{array}{ccccccc} \tilde{i}_{VC1S} & \tilde{i}_{VC2S} & \tilde{i}_{e1} & \tilde{i}_{C1} & \tilde{i}_{C2} & \tilde{v}_{L1} & \tilde{v}_{L2} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{M}_{21} = \begin{bmatrix} 0 & 0 & -t_1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{M}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & t_1^{-1}R_4 \end{bmatrix} \quad \mathbf{M}_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_{31} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{M}_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{M}_{33} = \begin{bmatrix} g_{1P}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{1S}^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{14.167}$$

On inspection of Eqs. (14.147) and (14.167), it becomes apparent that

$$\mathbf{M}_{12} = [\mathbf{A}_{VVS}\,\mathbf{A}_{CCS}] \tag{14.168}$$

$$\mathbf{M}_{13} = [\mathbf{A}_{VCS}\,\mathbf{A}_{S}\,\mathbf{A}_{C}\;0] \tag{14.169}$$

and

$$\mathbf{M}_{31} = \begin{bmatrix} \mathbf{A}_{VCP}^t \\ \mathbf{A}_S^t \\ \mathbf{A}_C^t \\ \mathbf{A}_L^t \end{bmatrix} \tag{14.170}$$

because they simply represent the nodes across which the controlled sources, voltage sources, capacitors, and inductors are connected. Also, matrix $\mathbf{M}_{33}$ contains a negative unity matrix representing the inductors plus some diagonal terms for the mutual conductances of the voltage-controlled current sources. The first row of $\mathbf{M}_{21}$ represents the transfer characteristic of the voltage-controlled voltage source from Eq. (14.154):

$$\mathbf{M}_{21} = \begin{bmatrix} \mathbf{A}_{VVS} - t_1 \mathbf{A}_{VVP}^t \\ -\mathbf{A}_{CCP}^t \end{bmatrix} \tag{14.171}$$

The second row of $\mathbf{M}_{21}$ in combination with the second row of $\mathbf{M}_{22}$ gives the current transfer characteristic of the current-controlled current source from Eq. (14.153), where the parameter $t_1^{-1}R_4$ in $\mathbf{M}_{22}$ may be regarded as an equivalent mutual conductance between the input voltage and output current. The first row of $\mathbf{M}_{22}$ is null as are also the matrices $\mathbf{M}_{23}$ and $\mathbf{M}_{32}$. Finally, matrix $\mathbf{M}_{11}$ is given by

$$\mathbf{M}_{11} = \mathbf{A}_R \cdot \mathbf{Y}_R \cdot \mathbf{A}_R^t \tag{14.172}$$

as before. Also, matrix $\mathbf{N}$ is constructed in exactly the same way as before and the same method is used to solve Eq. (14.163).

14.13 GYRATOR COMPRISING TWO CURRENT-CONTROLLED VOLTAGE SOURCES

As a footnote, the only controlled source which has not been considered in the above worked examples is the current-controlled voltage source. A gyrator is shown in Fig. 14.8a. In Sec. 14.12, the gyrator used in Sec. 14.11 was replaced with a pair of voltage-controlled current sources as shown in Fig. 14.8b. Alternatively, a gyrator can be replaced by a pair of current-controlled voltage sources as shown in Fig. 14.8c. In each case, the governing equations are

$$\tilde{i}_S = g_P \tilde{v}_P \tag{14.173}$$

$$\tilde{i}_P = g_S \tilde{v}_S \tag{14.174}$$

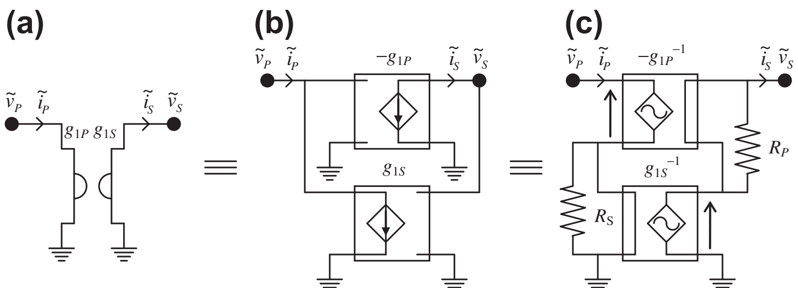

FIG. 14.8 The gyrator (a) can be replaced by a pair of voltage-controlled current sources (b) or a pair of current-controlled voltage sources (c).

The configuration of Fig. 14.8c can be formulated in matrix system of equations in exactly the same way as a voltage-controlled voltage source (see Sec. 14.12) where the voltage gains of the forward and reverse controlled sources are $-(R_P g_P)^{-1}$ and $(R_S g_S)^{-1}$ respectively.

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