APPENDIX

Mathematical formulas [1,2]

II

In the following formulas, m and n are integers, μ and ν can have any value.

Binomial theorem.

$$(1 \pm x)^n = \sum_{m=0}^{n} (\pm 1)^m \binom{n}{m} x^m = \sum_{m=0}^{n} \frac{(\pm 1)^m n!}{m!(n-m)!} x^m, \text{ any } x, \ n \text{ positive integer} \tag{1}$$

$$(1 \pm x)^\nu = \sum_{m=0}^{\infty} \frac{(\pm 1)^m \Gamma(\nu+1)}{m! \Gamma(\nu - m + 1)} x^m, \ 0 \le |x| < 1, \ \nu \neq -1, -2, -3, \ \ldots \tag{2}$$

$$(1 \pm x)^{-1} = \sum_{m=0}^{\infty} (\mp 1)^m x^m, \ \ 0 \le |x| < 1 \tag{3}$$

$$(1 \pm x)^{-2} = \sum_{m=0}^{\infty} (\mp 1)^m \ (m+1) x^m, \ \ 0 \le |x| < 1 \tag{4}$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{m=0}^{\infty} \frac{(2m)!}{(m!)^2} \left(\frac{x}{2}\right)^{2m}, 0 \le |x| < 1 \tag{5}$$

Gamma function.

$$\Gamma(n) = n!/n = (n-1)! \tag{6}$$

$$\Gamma(n+1) = n\Gamma(n) = n! \tag{7}$$

$$\Gamma(\nu+1) = \int_0^\infty x^\nu e^{-x} dx \tag{8}$$

$$\Gamma(1/2) = \sqrt{\pi} \tag{9}$$

$$\Gamma(n+1/2) = \frac{\sqrt{\pi}\Gamma(2n)}{2^{2n-1}\Gamma(n)} = \frac{\sqrt{\pi}(2n)!}{2^{2n} \ n!} \tag{10}$$

$$\Gamma(-n+1/2) = \frac{(-1)^n \pi}{\Gamma(n+1/2)} = \frac{(-1)^n \sqrt{\pi} 2^{2n-1} \Gamma(n)}{\Gamma(2n)} = \frac{(-1)^n \sqrt{\pi} 2^{2n} n!}{(2n)!} \tag{11}$$

Pochhammer symbol.

$$(\mu)_v = \frac{\Gamma(\mu+v)}{\Gamma(\mu)} \tag{12}$$

Hyperbolic formulas.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{13}$$

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} \tag{14}$$

$$e^x = \cosh x + \sinh x \tag{15}$$

$$e^{-x} = \cosh x - \sinh x \tag{16}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{17}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{18}$$

$$\cosh jx = \cos x \tag{19}$$

$$\sinh jx = j \sin x \tag{20}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \tag{21}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} \tag{22}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \tag{23}$$

$$\coth x = \frac{\cosh x}{\sinh x} \tag{24}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{25}$$

$$\cosh^2 x + \sinh^2 y = \cosh(x + y)\cosh(x - y) \tag{26}$$

$$\cosh(x \pm jy) = \cosh x \cosh jy \pm \sinh x \sinh jy = \cosh x \cos y \pm j \sinh x \sin y \tag{27}$$

$$\sinh(x \pm jy) = \sinh x \cosh jy \pm \cosh x \sinh jy = \sinh x \cos y \pm j \cosh x \sin y \tag{28}$$

$$\cosh^2 x = \frac{1 + \cosh 2x}{2} \tag{29}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2} \tag{30}$$

$$\frac{d}{dx} \cosh x = \sinh x \tag{31}$$

$$\frac{d}{dx} \sinh x = \cosh x \tag{32}$$

Trigonometric formulas.

Definitions

$$e^{\pm jx} = \sum_{k=0}^{\infty} \frac{(\pm jx)^k}{k!} \tag{33}$$

$$e^{jx} = \cos x + j \sin x \tag{34}$$

$$e^{-jx} = \cos x - j \sin x \tag{35}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \tag{36}$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2} \tag{37}$$

$$\cos jx = \cosh x \tag{38}$$

$$\sin jx = j \sinh x \tag{39}$$

$$\sec x = \frac{1}{\cos x} \tag{40}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} \tag{41}$$

$$\tan x = \frac{\sin x}{\cos x} = \sum_{k=0}^{\infty} \frac{2x}{\left(k + \frac{1}{2}\right)^2 \pi^2 - x^2} \tag{42}$$

$$\cot x = \frac{\cos x}{\sin x} = \sum_{k=0}^{\infty} \frac{(2 - \delta_{\mathrm{ok}})x}{x^2 - k^2\pi^2} \tag{43}$$

Formulas

$$\cos^2 x + \sin^2 x = 1 \tag{44}$$

$$\cos^2 x - \sin^2 y = \cos(x + y)\cos(x - y) \tag{45}$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \tag{46}$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \tag{47}$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \tag{48}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} \tag{49}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \tag{50}$$

$$\sin nx = \sin x \sum_{k=\frac{1-n}{2}}^{\frac{n-1}{2}} \cos(2kx) \tag{51}$$

$$\arctan x - \arctan y = \arctan\left(\frac{x - y}{1 + xy}\right) \tag{52}$$

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right) \tag{53}$$

Derivatives

$$\frac{d}{dx} \cos x = -\sin x \tag{54}$$

$$\frac{d}{dx} \sin x = \cos x \tag{55}$$

Integrals

$$\int_0^{2\pi} \cos(m\phi)\cos(n\phi)d\phi = \begin{cases} 0, & m \neq n \\ (1+\delta_{n0})\pi, & m = n \end{cases} \tag{56}$$

$$\int_0^{2\pi} \sin(m\phi)\sin(n\phi)d\phi = \begin{cases} 0, & m \neq n \\ (1-\delta_{n0})\pi, & m = n \end{cases} \tag{57}$$

Oblique-angled triangle (Fig. AII.1).

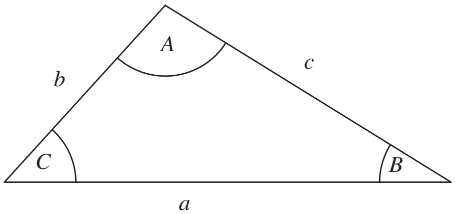

FIG. AII.1 Oblique-angled triangle.

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{58}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{59}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{60}$$

Legendre functions.

Definitions

$$P_n(z) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dz^n}(1-z^2)^n \text{ Rodrigues' formula} \tag{61}$$

$$P_n(z) = \frac{1}{2^n}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{(-1)^k\,(2n-2k)!}{k!\,(n-k)!(n-2k)!}z^{n-2k} \tag{62}$$

$$P_v^m(z) = (-1)^m(1-z^2)^{m/2}\frac{d^m}{dz^m}P_v(z) \tag{63}$$

Recursion formulas

$$(n+1)P_{n+1}(z) + nP_{n-1}(z) = (2n+1)zP_n(z) \tag{64}$$

$$n(n+1)(P_{n+1}(z) - P_{n-1}(z)) = (2n+1)(z^2-1)\frac{d}{dz}P_n(z) \tag{65}$$

Integrals

$$\int_0^{\pi} P_m(\cos\theta)P_n(\cos\theta)\sin\theta\, d\theta = \frac{2\delta_{mn}}{2n+1} \tag{66}$$

$$\int_0^{\delta} P_m(\cos\theta)\sin\theta\, d\theta = \frac{\delta^2}{2}, \qquad \delta \to 0 \tag{67}$$

$$\int_0^{\alpha} P_n(\cos\theta)\cos\theta\sin\theta\, d\theta = \begin{cases} (\sin^2\alpha)/2, & n = 0 \\ (1-\cos^3\alpha)/3, & n = 1 \\ -\sin^2\alpha\,\dfrac{P_n(\cos\alpha) + \cot\alpha\, P_n^1(\cos\alpha)}{(n-1)(n+2)}, & n \geq 2 \end{cases} \tag{68}$$

$$\int_0^{\alpha} P_n(\cos\theta)\sin\theta\, d\theta = \sin\alpha\, P_n^{-1}(\cos\alpha) \tag{69}$$

$$\begin{aligned} &\int_{\alpha}^{\pi} P_n(\cos\theta)P_m(\cos\theta)\,\sin\theta\, d\theta \\ &= \begin{cases} \dfrac{\sin\alpha(P_n(\cos\alpha)P_m'(\cos\alpha) - P_m(\cos\alpha)P_n'(\cos\alpha))}{m(m+1)-n(n+1)}, & m \neq n \\[2ex] \dfrac{1 + (P_m(\cos\alpha))^2\cos\alpha + 2\sum\limits_{j=1}^{m-1} P_j(\cos\alpha)\Big(P_j(\cos\alpha)\cos\alpha - P_{j+1}(\cos\alpha)\Big)}{2m+1}, & m = n \end{cases} \end{aligned} \tag{70}$$

Bessel functions.

Definitions

$$J_v(z) = \left(\frac{z}{2}\right)^v \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+v)!}\left(\frac{z}{2}\right)^{2m}, \text{ Bessel function} \tag{71}$$

$$Y_v(z) = \frac{\cos v\pi\, J_v(z) - J_{-v}(z)}{\sin v\pi}, \text{ Neumann function} \tag{72}$$

$$\begin{aligned} Y_n(z) = {} & \frac{2}{\pi}\ln\left(\frac{z}{2}\right)J_n(z) - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(m-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} \\ & -\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(\psi(k+1)+\psi(k+n+1))}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k+n} \end{aligned} \tag{73}$$

$$H_v^{(1)}(z) = J_v(z) + jY_v(z), \text{ Hankel function} \tag{74}$$

$$H_v^{(2)}(z) = J_v(z) - jY_v(z), \text{ Hankel function} \tag{75}$$

Integral representations

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{j(z\,\sin\phi - n\phi)}d\phi \tag{76}$$

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(z\,\sin\phi - n\phi)\,d\phi \tag{77}$$

$$J_v(z) = \frac{2}{\Gamma\left(v+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(\frac{z}{2}\right)^v\int_0^{\frac{\pi}{2}}(\sin\phi)^{2v}\cos(z\cos\phi)\,d\phi \tag{78}$$

Relations

$$J_{-n}(z) = (-1)^n J_n(z) \tag{79}$$

$$Y_{-n}(z) = (-1)^n Y_n(z) \tag{80}$$

$$Y_{-n-\frac{1}{2}}(z) = (-1)^n J_{n+\frac{1}{2}}(z) \tag{81}$$

$$Y_{n+\frac{1}{2}}(z) = (-1)^{n-1} J_{-n-\frac{1}{2}}(z) \tag{82}$$

Recursion formulas

$$Z_{v-1}(az) + Z_{v+1}(az) = \frac{2v}{az}Z_v(az) \tag{83}$$

$$Z_{v-1}(az) - Z_{v+1}(az) = \frac{2}{a}\cdot\frac{d}{dz}Z_v(az) \tag{84}$$

where Z can represent J, Y, $H^{(1)}$ or $H^{(2)}$.

Limiting forms for small arguments

$$J_v(z)|_{z\to 0} = \frac{1}{\Gamma(v+1)}\left(\frac{z}{2}\right)^v, \qquad v \neq -1, -2, -3, \cdots \tag{85}$$

$$Y_0(z)|_{z\to 0} = \frac{2}{\pi}\ln z \tag{86}$$

$$Y_v(z)|_{z\to 0} = \frac{\Gamma(v)}{\pi}\left(\frac{z}{2}\right)^v, \qquad v \neq 0 \tag{87}$$

Forms for large arguments

$$\begin{aligned} J_v(z) = \sqrt{\frac{2}{\pi z}}\Bigg[&\cos\left(z - v\frac{\pi}{2} - \frac{\pi}{4}\right)\sum_{k=0}^{\infty}\frac{(-1)^{k+v}\Gamma\left(2k - v + \frac{1}{2}\right)\Gamma\left(2k + v + \frac{1}{2}\right)}{\pi(2k)!(2z)^{2k}} \\ &- \sin\left(z - v\frac{\pi}{2} - \frac{\pi}{4}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k+v}\Gamma\left(2k - v - \frac{1}{2}\right)\Gamma\left(2k + v - \frac{1}{2}\right)}{\pi(2k-1)!(2z)^{2k-1}}\Bigg] \end{aligned} \tag{88}$$

$$\begin{aligned} Y_v(z) = \sqrt{\frac{2}{\pi z}}\Bigg[&\sin\left(z - v\frac{\pi}{2} - \frac{\pi}{4}\right)\sum_{k=0}^{\infty}\frac{(-1)^{k+v}\Gamma\left(2k - v + \frac{1}{2}\right)\Gamma\left(2k + v + \frac{1}{2}\right)}{\pi(2k)!(2z)^{2k}} \\ &+ \cos\left(z - v\frac{\pi}{2} - \frac{\pi}{4}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k+v}\Gamma\left(2k - v - \frac{1}{2}\right)\Gamma\left(2k + v - \frac{1}{2}\right)}{\pi(2k-1)!(2z)^{2k-1}}\Bigg] \end{aligned} \tag{89}$$

Asymptotic forms for very large arguments

$$J_v(z)|_{z\to\infty} = \sqrt{\frac{2}{\pi z}}\cos\left(z - v\frac{\pi}{2} - \frac{\pi}{4}\right) \tag{90}$$

$$Y_v(z)|_{z\to\infty} = \sqrt{\frac{2}{\pi z}}\sin\left(z - v\frac{\pi}{2} - \frac{\pi}{4}\right) \tag{91}$$

$$H_v^{(1)}(z)\Big|_{z\to\infty} = \sqrt{\frac{2}{\pi z}}\,e^{j\left(z - v\frac{\pi}{2} - \frac{\pi}{4}\right)} \tag{92}$$

$$H_{\nu}^{(2)}(z)\Big|_{z\to\infty} = \sqrt{\frac{2}{\pi z}}\,e^{-j\left(z-\nu\frac{\pi}{2}-\frac{\pi}{4}\right)} \tag{93}$$

Integrals

$$\int_0^a J_0(bx)dx = a\left\{J_0(ab)+\frac{\pi}{2}\left(J_1(ab)\mathbf{H}_0(ab)-J_0(ab)\mathbf{H}_1(ab)\right)\right\} \tag{94}$$

$$\int_0^a J_0(bx)xdx = \frac{a}{b}J_1(ab) \tag{95}$$

$$\int_0^a \left(1-\frac{x^2}{a^2}\right)^{\mu+\frac{1}{2}} J_0(bx)xdx = \frac{a^2}{2}\,\Gamma\left(\mu+\frac{3}{2}\right)\left(\frac{2}{ab}\right)^{\mu+\frac{3}{2}}\; J_{\mu+\frac{3}{2}}(ab),\ \text{Sonine's Integral} \tag{96}$$

$$\int_a^\infty \left(\frac{x^2}{a^2}-1\right)^{m+\frac{1}{2}} J_0(bx)xdx = \frac{a^2}{2}\Gamma\left(m+\frac{3}{2}\right)\left(\frac{2}{ab}\right)^{m+\frac{3}{2}}\; J_{-m-\frac{3}{2}}(ab),\ \text{Pritchard's Integral} \tag{97}$$

$$\int_0^\infty \left(1-\frac{x^2}{a^2}\right)^{m+\frac{1}{2}} J_0(bx)xdx = \frac{a^2}{2}\,\Gamma\left(m+\frac{3}{2}\right)\left(\frac{2}{ab}\right)^{m+\frac{3}{2}}\; H^{(2)}_{m+\frac{3}{2}}(ab) \tag{98}$$

$$\begin{aligned}\int_0^a \left(1-\frac{x^2}{a^2}\right)^{\mu+\frac{1}{2}} J_\nu(bx)xdx &= \sqrt{\pi}\;\frac{a^2}{2}\left(\frac{ab}{4}\right)^{\nu}\frac{\Gamma\left(\mu+\frac{3}{2}\right)}{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}+\mu+\frac{5}{2}\right)}\\ &\quad\times {}_1F_2\left(\frac{\nu}{2}+1;\ \frac{\nu}{2}+\mu+\frac{5}{2},\nu+1;-\frac{a^2b^2}{4}\right)\end{aligned} \tag{99}$$

$$\begin{aligned}\int_0^a \left(\frac{x}{a}\right)^{\mu} J_\nu(bx)xdx &= \frac{a^2}{\nu+\mu+2}\left(\frac{ab}{2}\right)^{\nu}\frac{1}{\Gamma(\nu+1)}\\ &\quad\times {}_1F_2\left(\frac{\nu}{2}+\frac{\mu}{2}+1;\ \frac{\nu}{2}+\frac{\mu}{2}+2,\ \nu+1;-\frac{a^2b^2}{4}\right)\end{aligned} \tag{100}$$

$$\int_0^a J_k(\alpha_{km}x/a)J_k(\alpha_{kn}x/a)xdx = \begin{cases} 0, & \alpha_{km} n \, \alpha_{kn} \\ a^2 J_{k+1}^2(\alpha_{kn})/2 & \alpha_{km} = \alpha_{kn} \end{cases}, \quad J_k(\alpha_{km}) = 0 \tag{101}$$

$$\int_0^a J_k(\alpha_{km}x/a)J_k(\alpha_{kn}x/a)xdx$$
$$= \left\{ \begin{array}{ll} a^2 \dfrac{\alpha_m J_k(\alpha_{kn})J_{k+1}(\alpha_{km}) - \alpha_n J_k(\alpha_{km})J_{k+1}(\alpha_{kn})}{\alpha_{km}^2 - \alpha_{kn}^2}, & \alpha_{km} \neq \alpha_{kn} \\ a^2 \dfrac{J_k^2(\alpha_{kn}) - J_{k-1}(\alpha_{kn})J_{k+1}(\alpha_{kn})}{2}, & \alpha_{km} = \alpha_{kn} \end{array} \right., \quad J_k(\alpha_{kn}) \neq 0 \tag{102}$$

$$2\int_0^1 \frac{J_1^2(zx)}{x\sqrt{1-x^2}}\,dx = 1 - \frac{J_1(2z)}{z} \tag{103}$$

$$2\int_1^\infty \frac{J_1^2(zx)}{x\sqrt{1-x^2}}\,dx = \frac{\mathbf{H}_1(2z)}{z} \tag{104}$$

$$2\int_0^1 \frac{J_1^2(zx)}{x}\sqrt{1-x^2}dx = 1 - {}_1F_2\left(\frac{1}{2};\frac{3}{2},2;-z^2\right) \tag{105}$$

$$2\int_1^\infty \frac{J_1^2(zx)}{x}\sqrt{1-x^2}dx = \frac{4}{\pi}\left\{\frac{1}{z} + \frac{z}{3}\ {}_2F_3\left(1,1;\frac{3}{2},2,\frac{5}{2};-z^2\right)\right\} \tag{106}$$

Expansions

$$J_0\left(z\sqrt{1+t^2}\right) = 2\sum_{n=0}^\infty \frac{(-1)^n}{1+\delta_{n0}}\ J_{2n}(z)J_{2n}(zt),\ \text{Gegenbauer summation theorem} \tag{107}$$

$$J_0\left(z\sqrt{1-t^2}\right)2\sum_{n=0}^\infty \frac{J_n(z)}{n!}\left(\frac{z}{2}\right)^n t^{2n},\ \text{Lommel addition theorem} \tag{108}$$

$$J_\nu(az)J_\mu(bz) = \frac{\left(\frac{1}{2}\,az\right)^\nu \left(\frac{1}{2}\,bz\right)^\mu}{\Gamma(\mu+1)} \sum_{n=0}^\infty \frac{(-1)^n\left(\frac{1}{2}\,az\right)^{2n}}{n!\Gamma(n+\nu+1)}\ {}_2F_1\left(-n,-n-\nu;\mu+1;\frac{b^2}{a^2}\right) \tag{109}$$

$$e^{\pm jx \cos\theta} = J_0(x) + \sum_{k=1}^{\infty} j^{\pm k} \cos(k\theta) J_k(x) \tag{110}$$

Wronskian

$$J_v(z)Y_{v+1}(z) - J_{v+1}(z)Y_v(z) = -\frac{2}{\pi z} \tag{111}$$

Hyperbolic Bessel functions.

Definitions

$$I_v(z) = j^{-v} J_v(jz), \text{ Hyperbolic Bessel function} \tag{112}$$

$$I_v(z) = \left(\frac{z}{2}\right)^v \sum_{m=0}^{\infty} \frac{1}{m!(m+v)!} \left(\frac{z}{2}\right)^{2m} \tag{113}$$

$$K_v(z) = \frac{\pi}{2} j^{v+1} H_v^{(1)}(jz), \text{ Hyperbolic Neumann function} \tag{114}$$

$$K_v(z) = \frac{\pi}{2}(-j)^{v+1} H_v^{(2)}(-jz) \tag{115}$$

$$\begin{aligned} K_n(z) = {} & (-1)^{n+1} \ln\left(\frac{z}{2}\right) I_n(z) + \frac{1}{2} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} \\ & + \frac{(-1)^n}{2} \sum_{k=0}^{\infty} \frac{\psi(k+1) + \psi(k+n+1)}{k!(k+n)!} \left(\frac{z}{2}\right)^{2k+n} \end{aligned} \tag{116}$$

Recursion formulas

$$I_{v-1}(z) - I_{v+1}(z) = \frac{2v}{z} I_v(z) \tag{117}$$

$$I_{v-1}(z) + I_{v+1}(z) = 2 \frac{d}{dz} I_v(z) \tag{118}$$

$$K_{v-1}(z) - K_{v+1}(z) = -\frac{2v}{z} K_v(z) \tag{119}$$

$$K_{v-1}(z) + K_{v+1}(z) = -2 \frac{d}{dz} K_v(z) \tag{120}$$

Limiting forms for small arguments

$$I_v(z)|_{z \to 0} = \frac{1}{\Gamma(v+1)} \left(\frac{z}{2}\right)^v, \qquad v \neq -1, -2, -3, \cdots \tag{121}$$

$$K_0(z)|_{z\to 0} = -\ln z \tag{122}$$

$$K_\nu(z)|_{z\to 0} = \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu \tag{123}$$

Wronskian

$$I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = \frac{1}{z} \tag{124}$$

Struve function.

Definitions

$$\mathbf{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1}\sum_{m=0}^{\infty}\frac{1}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{2m}, \text{ Struve function} \tag{125}$$

Integral representation

$$\mathbf{H}_\nu(z) = \frac{2}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu\int_0^{\frac{\pi}{2}}(\sin\phi)^{2\nu}\sin(z\cos\phi)d\phi \tag{126}$$

Recursion formulas

$$\mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = \frac{2\nu}{z}\,\mathbf{H}_\nu(z) + \frac{\left(\frac{1}{2}\,z\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)} \tag{127}$$

$$\mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\frac{d}{dz}\,\mathbf{H}_\nu(z) - \frac{\left(\frac{1}{2}\,z\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)} \tag{128}$$

Integrals

$$\int_0^a \mathbf{H}_0(bx)dx = a^2\,\frac{b}{\pi}\,{}_2F_3\left(1,1;\frac{3}{2},\frac{3}{2},2;-\,\frac{a^2b^2}{4}\right) \tag{129}$$

Spherical Bessel functions.

Definitions

$$j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z), \text{ Spherical Bessel function} \tag{130}$$

$$y_n(z) = \sqrt{\frac{\pi}{2z}}\, Y_{n+\frac{1}{2}}(z), \text{ Spherical Neumann function} \tag{131}$$

$$h_n^{(1)}(z) = j_n(z) + j y_n(z), \text{ Spherical Hankel function} \tag{132}$$

$$h_n^{(2)}(z) = j_n(z) - j y_n(z), \text{ Spherical Hankel function} \tag{133}$$

$$j_n(z) = (-1)^n z^n \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z}, \text{ Rayleigh's formula} \tag{134}$$

$$y_n(z) = -(-1)^n z^n \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\cos z}{z}, \text{ Rayleigh's formula} \tag{135}$$

$$j_0(z) = \frac{\sin z}{z} \tag{136}$$

$$j_1(z) = -z\,\frac{1}{z}\frac{d}{dz}\frac{\sin z}{z} = \frac{\sin z}{z^2} - \frac{\cos z}{z} \tag{137}$$

$$j_2(z) = z^2\,\frac{1}{z}\frac{d}{dz}\left(\frac{1}{z}\frac{d}{dz}\frac{\sin z}{z}\right) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{2}{z^2}\cos z \tag{138}$$

$$y_0(z) = -\frac{\cos z}{z} \tag{139}$$

$$y_1(z) = z\frac{1}{z}\frac{d}{dz}\frac{\cos z}{z} = -\frac{\cos z}{z^2} - \frac{\sin z}{z} \tag{140}$$

$$y_2(z) = -z^2\,\frac{1}{z}\frac{d}{dz}\left(\frac{1}{z}\frac{d}{dz}\frac{\cos z}{z}\right) = \left(-\frac{3}{z^3} + \frac{1}{z}\right)\cos z - \frac{3}{z^2}\sin z \tag{141}$$

Recursion formulas

$$f_{n-1}(z) + f_{n+1}(z) = \frac{2n+1}{z} f_n(z) \tag{142}$$

$$n f_{n-1}(z) - (n+1) f_{n+1}(z) = (2n+1)\frac{d}{dz} f_n(z) \tag{143}$$

where f can represent j, y, $h^{(1)}$ or $h^{(2)}$.

Limiting forms for small arguments

$$j_v(z)|_{z\to 0} = \frac{\sqrt{\pi}}{2\Gamma\left(v+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^v, \qquad v \neq -1, -2, -3, \cdots \tag{144}$$

$$y_v(z)|_{z\to 0} = -\frac{\Gamma\left(v+\frac{1}{2}\right)}{2\sqrt{\pi}}\left(\frac{2}{z}\right)^{v+1} \tag{145}$$

Asymptotic forms for large arguments

$$j_v(z)|_{z\to\infty} = \frac{\sin\left(z - v\frac{\pi}{2}\right)}{z} \tag{146}$$

$$y_v(z)|_{z\to\infty} = -\frac{\cos\left(z - v\frac{\pi}{2}\right)}{z} \tag{147}$$

$$h_v^{(1)}(z)\Big|_{z\to\infty} = -j\frac{e^{j\left(z-v\frac{\pi}{2}\right)}}{z} \tag{148}$$

$$h_v^{(2)}(z)\Big|_{z\to\infty} = j\frac{e^{-j\left(z-v\frac{\pi}{2}\right)}}{z} \tag{149}$$

Hypergeometric function.

$$\begin{aligned} {}_pF_q(a_1, a_2, \cdots a_p; b_1, b_2, \cdots b_q; z) &= \frac{\Gamma(b_1)\Gamma(b_2)\cdots(b_q)}{\Gamma(a_1)\Gamma(a_2)\cdots(a_p)} \\ &\times \sum_{n=0}^{\infty}\frac{\Gamma(n+a_1)\Gamma(n+a_2)\cdots(n+a_p)}{\Gamma(n+b_1)\Gamma(n+b_2)\cdots(n+b_q)}\frac{z^n}{n!} \end{aligned} \tag{150}$$

Dirac delta function.

$$\delta(z-a) = \begin{cases} 0, & z \neq a \\ \text{Indeterminate}, & z = a \end{cases} \tag{151}$$

$$\delta(z) = \frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\Bigg|_{a\to 0} \tag{152}$$

$$\int_{-\infty}^{\infty}\delta(z)dz = 1 \tag{153}$$

$$\int_{-\infty}^{\infty} F(z)\delta(z-a)dz = F(a) \tag{154}$$

Miscellaneous functions.

$$\delta_{mn}\begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}, \text{Kronecker delta function} \tag{155}$$

$$(\mu)_v = \frac{\Gamma(\mu+v)}{\Gamma(\mu)}, \text{ Pochhammer symbol} \tag{156}$$

References

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