Chapter 3

Acoustics of speech and hearing

The production of speech by the vocal tract and the reception of sound by the cochlea, have a common mathematical framework in that they have both been represented as inhomogeneous transmission lines. While these representations may be improved upon, they capture the essence of the underlying physical mechanisims of production and reception processes. In this chapter we shall outline the underlying theory required to represent both these system. Then in the following chapter (4) we shall concentratate on production, and then in Chapter 7, develop models of the inner ear.

3.1 Equivalent Circuit for the Lossy Cylindrical Pipe

Consider the length dx of lossy cylindrical pipe of area A shown in Fig. 3.1a. Assume plane wave transmission so that the sound pressure and volume velocity are spatially dependent only upon x. Because of its mass, the air in the pipe exhibits an inertance which opposes acceleration. Because of its compressibility the volume of air exhibits a compliance. Assuming that the tube is smooth and hard-walled, energy losses can occur at the wall through viscous friction and heat conduction. Viscous losses are proportional to the square of the particle velocity, and heat conduction losses are proportional to the square of the sound pressure.

The characteristics of sound propagation in such a tube are easily described by drawing upon elementary electrical theory and some well known results for one-dimensional waves on transmission lines. Consider sound pressure analogous to the voltage and volume velocity analogous to the current in an electrical line. Sound pressure and volume velocity for plane wave propagation in the uniform tube satisfy the same wave equation as do voltage and current on a uniform transmission line. A dx length of lossy electrical line is illustrated in Fig. 3.1b. To develop the analogy let us write the relations for the

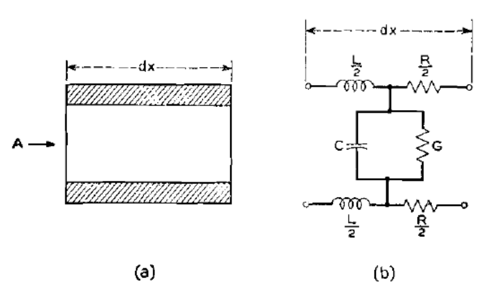

Figure 3.1: Incremental length of lossy cylindrical pipe. (a) acoustic representation; (b) electrical equivalent for a one-dimensional wave

electrical line. The per-unitlength inductance, capacitance, series resistance and shunt conductance are $L(x)$, $C(x)$, $R(x)$, and $G(x)$ respectively. Assuming sinusoidal time dependence for voltage and current, ($i(x,t) = I(x,\omega)e^{j\omega t}$ and $e(x,t) = E(x,\omega)e^{j\omega t}$), the differential current loss and voltage drop across the dx length of line are

$$dE = -\mathcal{Z}(x,s)Idx \quad \text{and} \quad dI = -\mathcal{Y}(x,s)Edx, \tag{3.1}$$

where $\mathcal{Y}(x,s) = G(x) + sC(x)$ and $\mathcal{Z}(x,s) = R(x) + sL(x)$, with complex *Laplace frequency* given by $s = \sigma + j\omega$.

Radian frequency ω is appropriate when taking the *Fourier transform* of say a voltage or a current, pressure, displacement, etc., however the Laplace frequency is the appropriate variable when taking the *Laplace transform* of causal functions,[1] such as an impedance $\mathcal{Z}(s)$ or an admittance $\mathcal{Y}(s)$, which due to their inherent causality must be *analytic* functions of complex frequency s, containing poles and zeros, or other singularities in the complex plane (e.g., branch cuts). More on this topic will be found in Sec. **??**. When the Laplace frequency s is used typically the right half s plane ($\sigma > 0$). It will be used with the Laplace transform (indicated as $\leftrightarrow$) when the frequency dependence indicates any causal function, such as the impulse response of a capacitor $\frac{1}{sC} \leftrightarrow \frac{1}{C}\Delta(t) = \int_{\omega=-\infty}^{\infty} \frac{1}{Cs}e^{-st}ds$, where $\Delta(t)$ is the Heaviside *unit step function*, zero for $t < 0$, 1 for $t > 0$ and undefined at $t = 0$. A short review s Laplace in Sec. ??

In matrix form these equations may be written

$$\frac{d}{dx}\begin{bmatrix} E(x,\omega) \\ I(x,\omega) \end{bmatrix} = -\begin{bmatrix} 0 & \mathcal{Z}(x,s) \\ \mathcal{Y}(x,s) & 0 \end{bmatrix}\begin{bmatrix} E(x,\omega) \\ I(x,\omega) \end{bmatrix}. \tag{3.2}$$

This is the 2×2 matrix form of the *Webester Horn equation* [Webster, 1919], which allows for the variation of the series *per-unit-length impedance* $\mathcal{Z}(x,s)$ and shunt *per-unit-length admittance* $\mathcal{Y}(x,s)$, as functions both the Horn area $A(x)$, and when losses are considered, the Laplace frequency s.

3.1.1 The homogeneous transmission line

Shown in Fig. 3.1b is the case of an elementary *homogeneous* piece of transmission line, where $\mathcal{Z}(s)$ and $\mathcal{Y}(s)$ are independent of position x, the equations for the voltage E and current I reduce to the wave equation

$$\frac{d^2E}{dx^2} - \mathcal{Z}\mathcal{Y}\,E = 0 \quad \text{and} \quad \frac{d^2I}{dx^2} - \mathcal{Z}\mathcal{Y}\,I = 0, \tag{3.3}$$

having solutions

$$\begin{bmatrix} E(x,\omega) \\ I(x,\omega) \end{bmatrix} = \begin{bmatrix} E_\omega^+ & E_\omega^- \\ I_\omega^+ & I_\omega^- \end{bmatrix}\begin{bmatrix} e^{-\gamma x} \\ e^{+\gamma x} \end{bmatrix} \tag{3.4}$$

where $\gamma(s) \equiv +\sqrt{\mathcal{Z}\mathcal{Y}} = \alpha(s) + j\beta(s)$ is the so-called *propagation constant* (the sign determines the wave direction), and the frequency-dependent *integration constants* E_ω^+, E_ω^-, I_ω^+ and I_ω^- are frequency dependent integration factors, determined by the 2-port terminal *boundary conditions*. Again, solutions Eq. 3.4 do *not* apply to the Webster Horn equation (Eq. 3.2), where a very different solution approach must be used due to the inhomogeneous *constituent relations* $\mathcal{Z}(x,s)$ and $\mathcal{Y}(x,s)$.

Transmission ($\top$-matrix) ABCD representations Specifically, as shown in Fig. 3.1 for a *finite* piece of homogeneous line, having length l, with sending-end (left) voltage and current E_1 and I_1, the receiving-end (right) voltage and current E_2 and I_2 are given by[2]

$$\begin{bmatrix} E_1(x,\omega) \\ I_1(x,\omega) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E_2(x+l,\omega) \\ I_2(x+l,\omega) \end{bmatrix}, \tag{3.5}$$

[1] A *causal function* is zero for $t < 0^-$.

[2] The transmission element C should not be confused with the acoustic compliance "C".

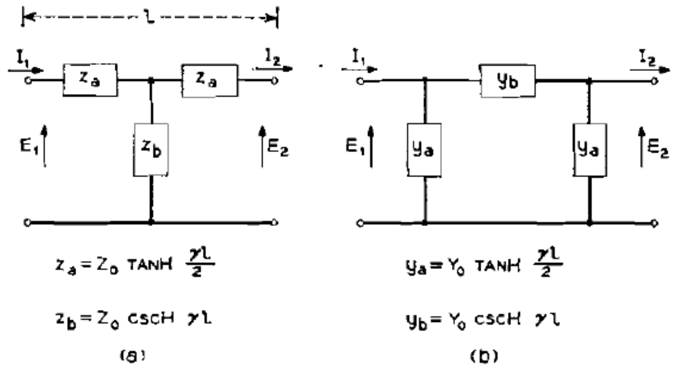

Figure 3.2: Equivalent 2-port networks for a length l of uniform transmission line. (a) T-section; (b) Π-section. These represent two alternative ways of specifying the impedances. For example the pipe representation of Fig. 3.1b is a T-section with $z_a = sL/2$, $z_c = R/2$ and $z_b = 1/(G + sC)$. in (a) z_c and (b) y_c need to be added.

where

$$A(s) \equiv \left.\frac{E_1}{E_1}\right|_{I_2=0}, \quad B(s) \equiv \left.\frac{E_1}{I_2}\right|_{E_1=0}, \quad C(s) \equiv \left.\frac{I_1}{E_2}\right|_{I_2=0} \quad D(s) \equiv \left.\frac{I_1}{I_2}\right|_{E_2=0}.$$

The inverse Laplace transforms of the $\top$-matrix parameters $a(t) \leftrightarrow A(s)$, $b(t) \leftrightarrow B(s)$, $c(t) \leftrightarrow C(s)$ and $d(t) \leftrightarrow D(s)$ must be analytic functions of Laplace frequency s in the right-half plane because they must represent causal functions (e.g., $a(t < 0^-) = 0$). This follows from the fact the the voltages and currents of the circuit must all obey causal relationships. Furthermore, the determinite of the $\top$-Matrix must be unity, that is $\Delta_\top \equiv AB - CD = 1$. This follows from the requirement that the passive physical system being modeled is reciprocal.

Given our example in Fig. 3.1a of a stub of transmission line of length l,

$$\begin{bmatrix} E_1(x,\omega) \\ I_1(x,\omega) \end{bmatrix} = \begin{bmatrix} \cosh(\gamma l) & Z_0 \sinh(\gamma l) \\ Y_0 \cosh(\gamma l) & \sinh(\gamma l) \end{bmatrix} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix}, \tag{3.6}$$

where

$$Z_0 = 1/Y_0 \equiv \sqrt{\mathcal{Z}/\mathcal{Y}} \tag{3.7}$$

is the *characteristic impedance* (Y_0 is the *characteristic admittance*) of the line and

$$\gamma \equiv +\sqrt{\mathcal{Z}\mathcal{Y}} \tag{3.8}$$

is the wave's *propagation function*. This relationship follows from Eq. 3.4, the solution of the wave equation.

One recalls from conventional circuit theory the lossless case corresponds to $\gamma = \sqrt{\mathcal{Z}\mathcal{Y}} = j\beta = s\sqrt{LC}$, and $Z_0 = \sqrt{L/C}$. The hyperbolic functions then reduce to circular functions which are purely reactive. Notice, too, for small loss conditions, (that is, $R \ll \omega L$ and $G \ll \omega C$) the attenuation and phase constants are approximately[3]

$$\begin{aligned} \alpha &\approx \frac{Y_0 R}{2} + \frac{Z_0 G}{2} \\ \beta &\approx \frac{\omega}{c}, \end{aligned} \tag{3.9}$$

[3] In the case of losses, the signs of α and β for the forward and reverse traveling waves must be determined such that the solution is both causal and stable. This involves the careful choice of the branch cut resulting from the square root in Eq. 3.8.

	$[\mathbb{Z}]$	$[\top]$	T-section (Fig. 3.2a)
$[\mathbb{Z}]$	$\begin{matrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{matrix}$	$\begin{matrix} A/C & \Delta_\top/C \\ 1/C & D/C \end{matrix}$	$\begin{matrix} z_a + z_c & z_b \\ z_b & z_a + z_c \end{matrix}$
$[\top]$	$\begin{matrix} z_{11}/z_{21} & \Delta_{\mathbb{Z}}/z_{21} \\ 1/z_{21} & z_{22}/z_{21} \end{matrix}$	$\begin{matrix} A & B \\ C & D \end{matrix}$	$\begin{matrix} (z_a + z_c)/z_b & z_a^2/z_b + 2z_a \\ 1/z_b & (z_a + z_c)/z_b \end{matrix}$

Table 3.1: Table of transformations between $\mathbb{Z}$-matrix (Eq. 3.10) and $\top$-matrix (Eq. 3.5) representations. Note that the sign of I_2 in the $\top$-matrix is switched for the $\mathbb{Z}$-matrix (i.e., $I_2' = -I_2$).
On the right-most column the forms are defined in terms of the T-secion parameters of Fig. 3.2a. $\Delta_\top$ and $\Delta_{\mathbb{Z}}$ represent the determinates of $[T]$ and $[\mathbb{Z}]$.

with wave velocity (i.e., speed of sound) $c \equiv 1/\sqrt{LC}$.

The $\top$-matrix is useful for calculations since it is designed to be cascaded, to build a network. However it is frequenly more convient to define a system in terms of the *impedance matrix* $\mathbb{Z}(s)$. We need to know how to transfrom between the $\top$ and $\mathbb{Z}$ matrix forms.

Impedance matrix representation The general 2-port *impedance matrix* $\mathbb{Z}(s)$ representation is defined as

$$\begin{bmatrix} E_1(x,\omega) \\ E_2(x,\omega) \end{bmatrix} = \mathbb{Z}(s) \begin{bmatrix} I_1 \\ I_2' \end{bmatrix} \equiv \begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix} \begin{bmatrix} I_1 \\ I'_2 \end{bmatrix} \tag{3.10}$$

with $I'_2 \equiv -I_2$.

The T-section format for the l length of line is shown in Fig. 3.2a, along with the Π-section format in Fig. 3.2b. These forms (T and Π sections) are the most schematic way of specifying a transmission line in terms of its physical parameters. From the T-section parameters (e.g., z_a, z_b) the $\top$-matrix and $\mathbb{Z}$-matrix parameters may be easily derived.

This relationship between the T-section parameters (Fig. 3.2a), the $\top$-matrix parameters $A(s)$, $B(s)$, $C(s)$ and $D(s)$ and the $\mathbb{Z}$-matrix elements z_{ij}, etc., are summarized in Table 3.1. By inspection of Eq. 3.10 and Fig. 3.2a,

$$z_{11} \equiv \left.\frac{E_1}{I_1}\right|_{I'_2=0} = z_a + z_b \qquad z_{12} \equiv \left.\frac{E_1}{I'_2}\right|_{I_1=0} = z_b$$
$$z_{21} \equiv \left.\frac{E_2}{I_1}\right|_{I'_2=0} = z_b \qquad z_{22} \equiv \left.\frac{E_2}{I'_2}\right|_{I_1=0} = z_a + z_c.$$

A system is *reciprocal* when $z_{12} = z_{21}$, or in terms of the $\top$-matrix representation, when $\Delta_\top \equiv AC - BD$. If a network is identical when run in reverse (if the system is invariant to having the ports flipped), the networks is called *symmetrical*, leading to $z_{11} = z_{22}$, or equivalently $A = D$.

It is important to note that there are cases where a physical system does not have an impedance representation, for example, when $C = 0$. The $\top$-matrix form, on the otherhand, always exists. Thus the prefered form must be for the $\top$-matrix form.

3.2 Acoustic transmission lines

Having reviewed the relations for the uniform, lossy electrical line, we may next interpret plane wave propagation in a uniform, lossy pipe (Fig. 3.1a) in analogous terms. Since the sound pressure $p(t) \leftrightarrow P(\omega)$ is considered analogous to voltage and the acoustic volume velocity, $u(t) \leftrightarrow U(\omega)$ analogous to current, the lossy, one dimensional, sinusoidal sound propagation is described by Eq. 3.3. The propagation constant is complex (that is, the velocity of propagation is in effect complex) and therefore the wave attenuates as it travels. In a smooth hard-walled tube the viscous and heat conduction losses can be represented, in effect, by an I^2R loss and an E^2G loss, respectively. The inertance of the air mass is analogous to the electrical inductance, and the compliance of the air volume is analogous to the electrical capacity. We can draw these parallels quantitatively.[4]

3.2.1 The Acoustic Mass "L"

The mass of air contained in the dx length of pipe in Fig. 3.1a is $\rho_0 A dx$, where ρ_0 is the air density. The *excess pressure* $p(x,t)$ accelerating this mass is (i.e., Newton's law):

$$dp = \rho_0 dx \frac{dv}{dt} = \rho_0 \frac{dx}{A} \cdot \frac{du(x,t)}{dt},$$

where $v(x,t)$ is particle velocity and $u(x,t)$ is the *volume velocity*, define as the area times the normal component of the particle velocity, namely

$$u(x,t) \equiv Av. \tag{3.11}$$

If we take the *Fourier Transform*[5] of these relations [e.g., $p(x,t) \leftrightarrow P(x,\omega)$ and $u(x,t) \leftrightarrow U(x,\omega)$] we obtain frequency domain relationships

$$\frac{d}{dx} P(x,\omega) = sL_a U(x,\omega), \tag{3.12}$$

where

$$L_a = \rho_0 / A \tag{3.13}$$

is the *per-unit-length acoustic inertance* .

It is our convention to use radian frequency ω for non-causal functions, having a Fourier transfrom, and complex radian frequency $s = \sigma + j\omega$ for *causal functions* of frequency having a *Laplace transform* (e.g., impedance sL_a defined via Eq. 3.12). We shall keep track of causality with this notational difference in frequency dependence (i.e., ω vs. s).

3.2.2 The Acoustic "C"

The analogous acoustic capacitance, or compliance, arises from the compressibility of the volume $\mathcal{V}$ of air contained in the dx length of tube shown in Fig. 3.1a. The elemental air volume $\mathcal{V} = Adx$ experiences compressions and expansions following the adiabatic gas law

$$p\mathcal{V}^\eta = \text{constant},$$

where p is the total pressure (the static + excess), $\mathcal{V}$ is the volume of the gas, and η is the adiabatic constant defined as the ratio of specific heat at constant pressure c_p to that at constant volume c_v.[6]

After taking the log and differentiating with respect to time

$$\frac{1}{p}\frac{dp}{dt} = -\frac{\eta}{\mathcal{V}}\frac{d\mathcal{V}}{dt}.$$

The diminution of the original air volume, owing to compression caused by an increase in pressure, must equal the volume current into the compliance; that is,

$$u = -\frac{d\mathcal{V}}{dt},$$

[4] The first-time reader might wish to omit the following detailed sections and jump to the summary Eq. 3.46 in Section 3.2.5.

[5] The *Fourier Transform relations* are defined as $P(\omega) \equiv \int_{-\infty}^{\infty} p(t) e^{-j\omega t} dt \;\; \leftrightarrow \;\; p(t) \equiv \int_{-\infty}^{\infty} P(\omega) e^{j\omega t} d\omega / 2\pi$.

[6] For diatomic air at normal conditions, $\eta = c_p/c_v = 1.4$.

and

$$\frac{1}{p}\frac{dp}{dt} = \frac{\eta}{\mathcal{V}}u.$$

For sinusoidal time dependence $P = P_0 + P(x,\omega)e^{j\omega t}$, where P_0 is the quiescent (ambient) static pressure, large[7] compared with the *excess pressure* $P(x,\omega)$.[8] The volume flow into the compliance of volume Adx is therefore

$$\frac{d}{dx}U(x,\omega) = s\frac{A}{P_0\eta}P(x,\omega). \tag{3.14}$$

The *per-unit-length acoustic compliance* is defined via Ohm's Law as the ratio of the volume velocity over the excess pressure

$$sC_a \equiv \frac{U(x,\omega)}{P(x,\omega)},$$

with

$$C_a = \frac{A}{P_0\eta} = \frac{A}{\rho_0 c^2}. \tag{3.15}$$

This last relation follows from the derivation of the acoustic wave equation Eq. 3.3 and the propagation constant Eq. 3.8, the speed of sound, which is given by [Morse, 1948]

$$c = \sqrt{\frac{P_0\eta}{\rho_0}}. \tag{3.16}$$

Thus $P_0\eta = \rho_0 c^2$.

3.2.3 The Acoustic "R"

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The acoustic R represents the power dissipated in viscous friction at the tube wall, a loss proportional to $|U|^2$ [Ingard, 1953]. The history of this analysis goes back to Stokes, Helmholtz and Kirchhoff, and Rayleigh [Keefe, 1984]. The easiest way to put in evidence this equivalent surface resistance is to consider the situation shown in Fig. **??**. Imagine that the tube wall is a plane surface, large in extent, and moving sinusoidally in the x-direction with harmonic velocity $U(x,y,\omega)$. The air particles proximate to the wall experience a force owing to the viscosity μ of the medium. The power expended per-unit-area in dragging the air with the plate is the loss to be determined.

Consider a layer of air dy thick and of unit area normal to the y axis, The net force on the layer is

$$\mu\left[\left(\frac{\partial u}{\partial y}\right)_{y+dy} - \left(\frac{\partial u}{\partial y}\right)_{y}\right] = \rho_0 dy\frac{\partial u}{\partial t},$$

where $u(x,y,\omega t)$ is the harmonic particle velocity in the x-direction. The diffusion equation specifying the air particle velocity as a function of the distance above the wall is then [Hildebrand, 1948]

$$\frac{\partial^2 u}{\partial y^2} = \frac{\rho_0}{\mu}\frac{\partial u}{\partial t}. \tag{3.17}$$

Taking a Fourier Transform we obtain the harmonic time dependence

$$\frac{d^2 U}{dy^2} = s\frac{\rho_0}{\mu}U = k_v^2 U, \tag{3.18}$$

[7] Under standard conditions $P_0 = 10^5$ [Pa].

[8] The *excess pressure* is the small (e.g. less than 10 Pascals) acoustic time-varying component riding on the ambient static pressure.

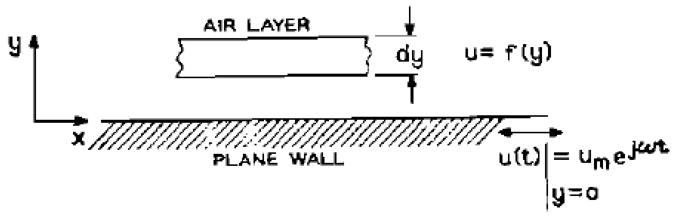

Figure 3.3: Relations illustrating viscous loss at the wall of a smooth tube

where $k_v = \sqrt{s\rho_0/\mu} = (1+j)\sqrt{\omega\rho_0/2\mu}$,[9] The harmonic velocity distribution is

$$U(x,y,\omega) = U(x,y=0,\omega)e^{-k_v y} = U(x,y=0,\omega)\ \cdot\ e^{-\sqrt{\omega\rho_0/2\mu}\,y}\ \cdot\ e^{-j\sqrt{\omega\rho_0/2\mu}\,y} \tag{3.20}$$

The distance required for the particle velocity to diminish to $1/e$ of its value at the driven wall is often called the *boundary-layer thickness* and is

$$\delta_v = \sqrt{2\mu/\omega\rho_0}. \tag{3.21}$$

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In air at a frequency of 100Hz, for example, $\delta_v \approx 0.2$ [mm].[10]

The viscous drag, per-unit area, on the plane wall is

$$F = -\mu \left.\frac{\partial U}{\partial y}\right|_{y=0} = \mu k_v U(y=0),$$

or

$$F = U\sqrt{s\mu\rho_0}. \tag{3.22}$$

Notice that this force has a real part and a positive reactive part. The latter acts to increase the apparent acoustic L. The average power dissipated per-unit surface area in this drag is

$$\bar{P} = \frac{1}{2}|F||U_m|\cos\theta = \frac{1}{2}|U_m|^2 R_s, \tag{3.23}$$

where $R_s = \sqrt{\omega\rho_0\mu/2}$ is the *per-unit-area surface resistance* and $\theta = \pi/2$ is the phase angle between F and U_m (e.g., 45°). For a length l of the acoustic tube, the inner surface area is Sl, where S is the circumference. Therefore, the average power dissipated per-unit length of the tube is $\bar{P}S = \frac{1}{2}u_m^2 SR$ or in terms of the acoustic volume velocity

$$\bar{P}S = \frac{1}{2}U_m^2 R_a,$$

where

$$R_a = \frac{S}{A^2}\sqrt{\omega\rho_0\mu/2}, \tag{3.24}$$

and A is the cross-sectional area of the tube. R_a is then the *per-unitlength acoustic resistance* for the analogy shown in Fig. 3.1.

[9] Note that this involves the *"usual 1/2 derivative"* formula, resulting from $\sqrt{s}$ [Lighthill, 1978, page 21, Eq. 77], which may be implimented by a time-convolution with the operator (and its Laplace transform)

$$\frac{\Delta(t)}{\sqrt{\pi t}} \star \frac{d}{dt} \leftrightarrow \frac{s}{\sqrt{s}} = \sqrt{s}. \tag{3.19}$$

[10] Some prefer the alternate definition of a more general *analytic boundary-layer thickness* $\delta_v = \sqrt{\mu/s\rho_0}$, without the factor of 2.

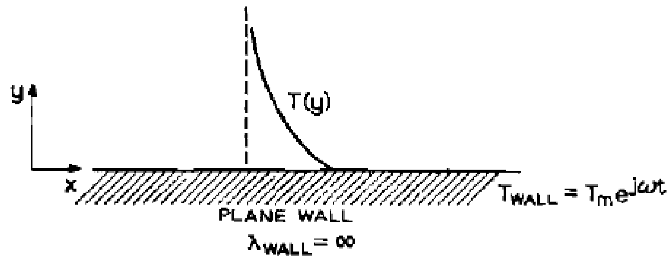

Figure 3.4: Relations illustrating heat conduction at the wall of a tube

As previously mentioned, the reactive part of the viscous drag contributes to the acoustic inductance per-unit-length. In fact, for the same area and surface relations applied above, the acoustic inductance obtained in the foregoing section should be increased by the frequency dependent factor $\frac{A^2}{S}\sqrt{\mu\rho_0/2\omega}$, or

$$L_a \approx \frac{\rho_0}{A}\left(1 + \frac{S}{A}\sqrt{\frac{\mu}{2\rho_0\omega}}\right). \tag{3.25}$$

Thus, the viscous boundary layer increases the apparent acoustic mass and slightly diminishes the cross-sectional area. For vocal tract analysis, the viscous boundary layer is typically so thin that the second term in Eq. 3.25 is negligible. For example, for a circular cross-section of 9 cm^2, the second term at a frequency of 500Hz is about (0.006) ρ_0/A (i.e., factor of 0.6%), and even smaller at higher frequencies.

The 2-port mass

Each acoustic element may also be written as a 2-port transmission matrix. For the acoustic mass element "L" this is equivilent to chosing the elemental dx to be a finite length, say l meters long. In this case we may describe the mass as the following 2-port transmission matrix notation

$$\begin{bmatrix} P_1(x,\omega) \\ U_1(x,\omega) \end{bmatrix} = \begin{bmatrix} 1 & R_a + sM_a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_2(x,\omega) \\ U_2(x,\omega) \end{bmatrix}, \tag{3.26}$$

The port variables on the left $[P_1, U_1]$ represent the input with volume velocity U_1 defined into the port, while the variables on the right $[P_2, U_2]$ represent the output, with volume velocity U_2 out of the port. The acoustic mass is given by

$$M_a l \equiv L_a = \frac{l\rho_0}{A}\left(1 + \frac{S}{A}\sqrt{\frac{\mu}{2\rho_0\omega}}\right). \tag{3.27}$$

where l is the length of the tube, ρ_0 is the density of air and A is the tube area. As will be discussed in greather depth below, the length must be modified to account for spreading of the waves as it leaves the tube, or if one tube is connected to a second having a different area. This end correction is a function of the ratio of the two tube areas.

3.2.4 The Acoustic "G"

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The analogous shunt conductance provides a power loss proportional to the square of the local sound pressure. Such a loss arises from heat conduction at the walls of the tube. The *per-unit-length conductance* can be deduced in a manner similar to that for the viscous loss. As before, it is easier to treat a simpler situation and extend the result to the vocal tube.

Consider a highly conductive plane wall of large extent, such as shown in Fig. 3.4. The air above the boundary is essentially at constant pressure and has a coefficient of heat conduction λ and a specific heat c_p. Suppose the wall is given an oscillating temperature $T|_{x,y=0} = T_m(y=0)e^{j\omega t}$. The vertical temperature distribution produced in the air is described by the diffusion equation[Hildebrand, 1948].

$$\frac{\partial^2 T}{\partial y^2} = \frac{c_p\rho_0}{\lambda}\frac{\partial T}{\partial t},$$

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which in the frequency domain is

$$\frac{\partial^2 T_m}{\partial y^2} = j\omega\frac{c_p\rho_0}{\lambda}T_m. \tag{3.28}$$

The solution is therefore $T = T_m(y=0,\omega)e^{-k_h y + j\omega t}$, where

$$k_h = \sqrt{\frac{sc_p\rho_0}{\lambda}} \tag{3.29}$$

which is the same form as the velocity distribution due to viscosity. In a similar fashion, the boundary layer depth for temperature is $\delta_h = \sqrt{2\lambda/\omega c_p\rho_0}$ (or complex $\delta_h = \sqrt{\lambda/sc_p\rho_0}$), thus $k_h = (1+j)/\delta_h$.

Now consider more nearly the situation for the sound wave. Imagine an acoustic pressure wave moving parallel to the conducting boundary, that is, in the x-direction. We wish to determine the temperature distribution above the wall produced by the sound wave. The conducting wall is assumed to be maintained at some quiescent temperature and permitted no variation, that is, $\lambda_{wall} = \infty$. If the sound wavelength is long compared to the boundary extent under consideration, the harmonic pressure variation above the wall may be considered as $P = P_0 + p$, where P_0 is the quiescent atmospheric pressure and $p = p_m e^{j\omega t}$ is the pressure variation. (That is, the spatial variation of p with x is assumed small.) The gas laws prescribe

$$PV^\eta = \text{constant} \quad \text{and} \quad PV = RT \ \ \text{(for unit mass)}.$$

Taking differentials gives

$$\frac{d\mathcal{V}}{\mathcal{V}} = -\frac{1}{\eta}\frac{dP}{P} \quad \text{and} \quad \frac{dP}{P} + \frac{dV}{V} = \frac{dT}{T} \tag{3.30}$$

Combining the equations yields

$$\frac{dP}{P}\left(1 - \frac{1}{\eta}\right) = \frac{dT}{T}, \tag{3.31}$$

where

$$dP = p = p_m e^{j\omega t}$$
$$dT = \tau = \tau_m e^{j\omega t},$$

so from (Eq. 3.31)

$$\tau_m = \frac{T_0}{P_0}\left(\frac{\eta - 1}{\eta}\right)p_m \tag{3.32}$$

At the wall, $y = 0$ and $\tau(0) = 0$ (because $\lambda_{wall} = \infty$). Far from the wall (i.e., for y large), $|\tau(y)| = \tau_m$ as given in (Eq. 3.32). Using the result of (Eq. 3.29), the temperature distribution can be constructed as

$$\tau(y,t) = \left[1 - e^{-k_h y}\right]\tau_m e^{j\omega t},$$

or

$$\tau(y,t) = \frac{P_0}{T_0}\left(\frac{\eta - 1}{\eta}\right)\left[1 - e^{-k_h y}\right]p_m e^{j\omega t}. \tag{3.33}$$

This equation is the phasor at frequency ω, thus the ratio $\tau/p_m e^{j\omega t}$ is an admittance, and therefore an analytic function of s in the right-half plane.

Now consider the power dissipation at the wall corresponding to this situation. A long wavelength sound has been assumed so that the acoustic pressure variations above the boundary can be considered $p = p_m e^{j\omega t}$, and the spatial dependence of pressure neglected. Because of the temperature distribution above the boundary, however, the particle velocity will be nonuniform, and will have a component in the y-direction. The average power flow per-unit surface area into the boundary is $p\bar{u}_{y0}{}^t$, where u_{y0} is the velocity component in the y direction on the boundary. To examine this quantity, u_y is needed. Needs fixing

Conservation of mass in the y-direction requires

$$\rho_0 \frac{\partial u_y}{\partial y} = -\frac{\partial \rho_0}{\partial t}. \tag{3.34}$$

Also, for a constant mass of gas $d\rho_0/\rho_0 = -dV/V$ which with the second equation in (Eq. 3.30) requires

$$\frac{dP}{P} - \frac{d\rho_0}{\rho_0} = \frac{dT}{T}. \tag{3.35}$$

Therefore,

$$\frac{\partial u_y}{\partial y} = \left(\frac{1}{T_0}\frac{\partial \tau}{\partial t} - \frac{1}{P_0}\frac{\partial p}{\partial t}\right), \tag{3.36}$$

and

$$u_y = \int \frac{\partial u_y}{\partial y} \cdot dy$$

$$y_y = \frac{j\omega p}{P_0}\left\{\frac{\eta - 1}{\eta}\left(y + \frac{e^{-k_y y}}{ky}\right) - y\right\}. \tag{3.37}$$

And,

$$u_{y_0} = p\frac{\omega}{c}\frac{\eta - 1}{\rho_0 c}\frac{j}{1+j}\delta_h. \tag{3.38}$$

The equivalent energy flow into the wall is therefore

$$W_h = p\bar{u}_{y_0}{}^t = \frac{\omega}{c}\frac{\eta-1}{\rho_0 c}\delta_h \frac{1}{\sqrt{2}}\frac{1}{T}\int_0^T P_m^2 \cos\left(\omega t + \frac{\pi}{4}\right)\cos\omega t \cdot dt$$

$$W_h = \frac{1}{4}\frac{\omega}{c}\frac{\eta-1}{\rho_0 c}\delta_h p_m^2 = \frac{1}{2}G_\alpha p_m^2, \tag{3.39}$$

where G_α is an equivalent conductance per-unit wall area and is equal

$$G_\alpha = \frac{1}{2}\frac{\omega}{c}\frac{\eta-1}{\rho_0 c}\sqrt{\frac{2\lambda}{\omega c_p \rho_0}}. \tag{3.40}$$

The equivalent conductance per-unit-length of tube owing to heat conduction is therefore

$$G_\alpha = S\frac{\eta-1}{\rho_0 c^2}\sqrt{\frac{\lambda\omega}{2c_p\rho_0}}, \tag{3.41}$$

where S is the tube circumference. To reiterate, both the heat conduction loss G; and the viscous loss R_α are applicable to a smooth, rigid tube. The vocal tract is neither, so that in practice these losses might be expected to be somewhat higher. In addition, the mechanical impedance of the yielding wall includes a mass reactance and a conductance which contribute to the shunt element of the equivalent circuit. The effect of the wall reactance upon the tuning of the vocal resonances is generally small, particularly for open articulations. The contribution of wall conductance to tract damping is more important. Both of these effects are estimated in a later section.

The 2-port compliance

The acoustic compliance element "$G_a + sC_a$" is equivilent to chosing the elemental dx to be a finite l meters long. In this case we may describe the mass as the following 2-port transmission matrix notation

$$\begin{bmatrix} P_1(x,\omega) \\ U_1(x,\omega) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ G_a + sC_a & 1 \end{bmatrix} \begin{bmatrix} P_2(x,\omega) \\ U_2(x,\omega) \end{bmatrix}, \tag{3.42}$$

The port variables on the left $[P_1, U_1]$ represent the input with volume velocity U_1 defined into the port, while the variables on the right $[P_2, U_2]$ represent the output, with volume velocity U_2 out of the port. The acoustic compliance is given by

$$C \equiv lC_a = \frac{Al}{\eta P_0} \tag{3.43}$$

where A is the tube area.

3.2.5 Summary of the Analogous Acoustic Elements

In matrix form the acoustic equations may be written in a manner similar to the electrical case, but in terms of the pressure P playing the role of a force, and the volume velocity U playing the role of the flow

$$\frac{d}{dx} \begin{bmatrix} P(x,\omega) \\ U(x,\omega) \end{bmatrix} = - \begin{bmatrix} 0 & \mathcal{Z}(x,s) \\ \mathcal{Y}(x,s) & 0 \end{bmatrix} \begin{bmatrix} P(x,\omega) \\ U(x,\omega) \end{bmatrix}. \tag{3.44}$$

with

$$\mathcal{Z} \equiv R_a + sL_z \qquad \text{and} \qquad \mathcal{Y} \equiv G_a + sC_a. \tag{3.45}$$

The per-unit-length analogous constants for the uniform pipe are

$$\begin{array}{ll} L_a = \frac{\rho_0}{A}, & C_a = \frac{A}{\rho_0 c^2}, \\ R_a = \frac{S}{A^2}\sqrt{\frac{\omega\rho_0\mu}{2}}, & G_a = S\frac{\eta-1}{\rho_0 c^2}\sqrt{\frac{\lambda\omega}{2c_p\rho_0}}, \end{array} \tag{3.46}$$

where A is tube area, S is tube circumference, ρ_0 is air density, c is sound velocity, μ is viscosity coefficient, A is coefficient of heat conduction, η is the adiabatic constant, and c_p is the specific heat of air at constant pressure.[11]

Having set down these quantities, it is possible to approximate the nonuniform vocal tract with as many right circular tube sections as desired by cascading the transmission matrix of each tube. The transmission characteristics can be determined either from calculations on equivalent network sections such as shown in Fig. 3.3, or from electrical circuit simulations of the clements.

We will presently apply the results of this section to some simplified analyses of the vocal tract. Before doing so, however, it is desirable to establish several fundamental relations for sound radiation from the mouth and for certain characteristics of the sources of vocal excitation.

[11] Summarizing the constants:

$$\begin{aligned} P_0 &= 10^5 \text{ Pa} \\ \rho_0 &= 114 \text{ kgm/m}^3 \text{ (moist air at body temperature 37deg C)} \\ c &= \sqrt{\gamma P_0/\rho_0} = 343 \text{ m/sec (moist air at body temperature, 37deg C)} \\ \eta &= c_p/c_v = 1.4 \\ c_p &= 0.24 \text{ cal/gm-degree(Odeg C, 1 atmos)transform to MKS units} \\ \mu &= 1.86 \times 10^{-4} \text{ dyne-sec/cm}^2\text{(20C, 0.76 m.Hg)transform to MKS units} \\ \lambda &= 0.055 \times 10^{-3} \text{ cal/gm-sec-deg(0deg C)transform to MKS units} \end{aligned}$$

3.3 Acoustic Horns

In this section we move from uniform tubes of constant crossection to the topic of horns, which have an area that is changing in the direction of wave propagatation. This is a very important topic to communication acoustics. First when the area changes, the impedance also must change. Thus horns are used to transform the acoustic impedance from one end to the other end of the horn. Second horns are a special case on inhomogeneous media, a very important topic to speech production and hearing. To deal with these issues we must start from the basic equations of acoustics in 3 dimensions. The equation of a horn is then typically an approximation which reduces the 2 or 3 dimentional wave propagation to a function of the axial variable. The methods used to do this are reviewed next.

In three dimensions the basic acoustic equations are based on two laws, Newton's second law of conservation of momentum

$$\nabla P = -s\rho_0 \mathbf{U} \quad [\mathtt{eq:gradP}] \tag{3.47}$$

and Hooke's Law for the adiabatic compressibility of air

$$\nabla \cdot \mathbf{U} = -\frac{s}{\eta P_0} P, \quad [\mathtt{eq:divU}] \tag{3.48}$$

where $P(\omega)$ is the pressure, $\mathbf{U}(\omega)$ is the vector particle velocity, with s the Laplace frequency $s = \sigma + j\omega$, ρ_0 the density of air, $\eta \equiv c_p/c_v$, P_0 is the static pressure of air. We refer to the ratio of pressure to particle velocity as the *specific acoustic impedance* in [Rayls], and the pressure over a volume velocity is the *acoustic impedance* in [*acoustic ohms*].

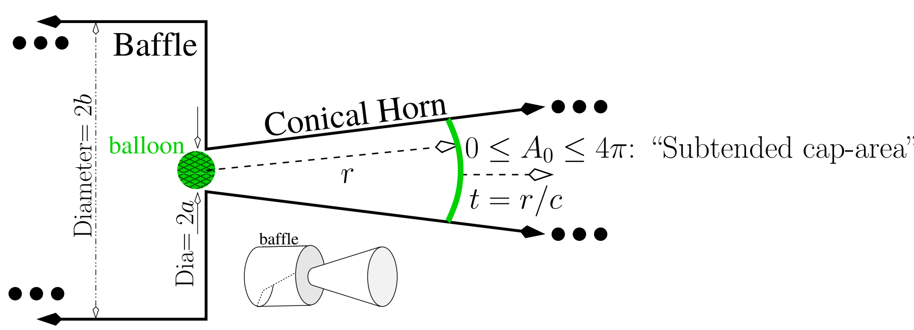

Figure 3.5: *Experimental setup showing a large pipe on the left terminating the wall containing a small hole with a balloon, shown in green. At time $t = 0$ the ballon is pricked and a pressure pulse is created. The baffel on the left is meant to represent an ∞ long tube having a very large radius compared to the horn input diameter $2a$, such that the acoustic admittance looking to the left ($A/\rho_0 c$ with $A \to \infty$), is very large compared to that looking into the horn, $Y_+(a, s)$ (Eq. 3.60). At time $t = b/c$ the outbound pressure pulse $\delta(t - b/c)/r$ has reached radius b.* [fig:Exp-horn]

3.3.1 Spherical acoustics

The conical horn is a special case with an exact solution due to its cylindrical symmetry. In spherical coordinates the pressure is given by Eq. 3.47

$$\nabla_r P \equiv \frac{dP}{dr} = -s\rho_0 U_r \tag{3.49}$$

and the velocity is determined by Eq. 3.48

$$\nabla_r \cdot \mathbf{U} \equiv \frac{1}{r^2}\frac{d}{dr} r^2 U_r = -\frac{s}{\eta P_0} P, \tag{3.50}$$

where the subscript r indicates the radial component of the pressure gradient and velocity divergence.

If we define the *per-unit length series acoustic impedance*

$$\mathcal{Z}(r,s) = s\frac{\rho_0}{A(r)} \tag{3.51}$$

and the *per-unit length shunt acoustic admittance*

$$\mathcal{Y}(r,s) = s\frac{A(r)}{\eta P_0}, \tag{3.52}$$

with the horn's *subtended cap area* as $A(r) = A_0 r^2$ (A_0 is the fraction of subtended cap area relative to 4π, as shown in Fig. 3.5), then the transformed spherical equations may be written in *Webster form* (i.e. see Eq. 3.2) in terms of the pressure P volume velocity $V = A(r)U_r$ as

$$\frac{d}{dr}\begin{bmatrix} P(r,\omega) \\ V(r,\omega) \end{bmatrix} = -\begin{bmatrix} 0 & \mathcal{Z}(r,s) \\ \mathcal{Y}(r,s) & 0 \end{bmatrix}\begin{bmatrix} P(r,\omega) \\ V(r,\omega) \end{bmatrix}. \quad [\texttt{eq:Webster2}] \tag{3.53}$$

This is a scaled form of the conical horn (it is a horn because the *per-unit-length impedance* $\mathcal{Z}$ and admittance $\mathcal{Y}$ are a function of r), with an angle that subtends 4π [sr] (steradians). Expressing the solution in terms of the volume velocity after scaling $A(r)$, we obtain the traditional conical horn equation Eq. 4.67 [Salmon, 1946a,b, Morse, 1948, p. 271].

Thus the spherical wave solution may be expressed as a Webster (conical) horn equation. If one reduces this to a second-order equation in pressure, the classic Webster horn equation results [Webster, 1919, Salmon, 1946a,b, Morse, 1948, Kinsler and Frey, 1962, Leach, 1996, Pierce, 1981].

The functions $\mathcal{Z}$ and $\mathcal{Y}$ define the *acoustic characteristic impedance (resistance)* that depends on the radius

$$Z_0(r) \equiv \sqrt{\frac{\mathcal{Z}}{\mathcal{Y}}} = \frac{\sqrt{\rho_0 \eta P_0}}{A(r)} = \frac{\rho_0 c}{A(r)} = \frac{1}{Y_0(r)} \quad [\texttt{eq:Z0}] \tag{3.54}$$

and a wave *propagation function*, closely related to the sound speed by

$$\gamma \equiv \sqrt{\mathcal{Z}\mathcal{Y}} = s/c \tag{3.55}$$

that for all horns, is independent of the axial coordinate r [Morse, 1948].

Equation 3.53 may be reexpressed as a single equation in pressure as

$$\frac{d^2}{dr^2} rP = \gamma^2 rP. \tag{3.56}$$

Following d'Alembert (1747), the solution to this equation in spherical coordinates, corresponding to a spherical symmetry, is given by the sum of an outbound $\mathcal{P}^+(r,\omega)$ and an inbound $\mathcal{P}^-(r,\omega)$ wave [Salmon, 1946a,b, Morse, 1948, Pierce, 1981, Leach, 1996]

$$P(r,\omega) = \mathcal{P}^+_\omega \frac{e^{-sr/c}}{r} + \mathcal{P}^-_\omega \frac{e^{+sr/c}}{r} \quad [\texttt{eq:dAlembert}] \tag{3.57}$$

where $\mathcal{P}^+_\omega(a) \leftrightarrow p^+(t)$ and $\mathcal{P}^-_\omega(a) \leftrightarrow p^-(t)$ are the Fourier "source strengths" and a is the radius corresponding to the source location (the radius the waves are launched from). When $\mathcal{P}^+_\omega(a) = 1$ and $\mathcal{P}^-_\omega(a) = 0$, the resulting outbound pressure wave is a Dirac delta function

$$p(r,t) = \frac{\delta(t - r/c)}{r} \leftrightarrow P(r,\omega) = \frac{e^{-sr/c}}{r}. \quad [\texttt{eq:p+}] \tag{3.58}$$

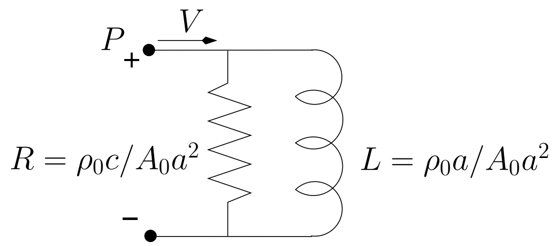

Figure 3.6: *Equivalent circuit of the* radiation acoustic impedance *of a conical horn for Eq. 3.60. Parameter A_0 is a number between 0 and 4π determined by the horn's* subtended cap area, *as defined by Fig. 3.5, while a is the horn's throat radius. The resistance R and the reactance sL are equal at frequency given by $ka = 1$, or $\omega_c = c/a$. This makes the connection between resonant scattering and the reactive component of the mass. This mass is also called the* spreading inertance. [fig:Zrad]

3.3.2 Particle velocity and the radiation impedance

Substituting the expression for the pressure (Eq. 3.57) into Eq. 3.47 results in an expression for the radial component of the particle velocity

$$U_r = -\frac{1}{\mathcal{Z}}\frac{\partial}{\partial r}\left[\mathcal{P}^+_\omega \frac{e^{-sr/c}}{r} + \mathcal{P}^-_\omega \frac{e^{+sr/c}}{r}\right] = Y^+\mathcal{P}^+ - Y^-\mathcal{P}^-, \quad [\mathtt{eq:Ur}] \tag{3.59}$$

corresponding to out and inbound velocity waves $\mathcal{U}^\pm_r = Y^\pm \mathcal{P}^\pm$. From the above definition the *acoustic radiation admittances* at $r = a$ for outbound (Y^+) and inbound (Y^-) waves are

$$Y^\pm(a, s) \equiv \frac{A_0 a^2}{\sqrt{\rho_0 \eta P_0}} \pm \frac{A_0 a^2}{s\rho_0 a}. \quad [\mathtt{eq:Ypm}] \tag{3.60}$$

Here the first, real term which is the characteristic admittance $Y_0(a)$, corresponds to radiated (absorbed) energy, while the second, reactive complex term, to the stored kinetic energy (wave momentum). As depicted in Fig. 3.6, the sum of two admittances represent the parallel combination of impedances, in this case the characteristic resistance $Z_0 \equiv \rho_0 c/A_0 a^2$ and the acoustic mass $\pm\rho_0 a/A_0 a^2$ [Bauer, 1944, Salmon, 1946a,b, Morse, 1948]. The minus sign on the inbound wave U^-_r has been chosen to correspond to the direction of mass flow so that the real part of the radiation admittance remains positive for waves of both directions. It is obvious that a converging wave and a diverging wave cannot have the same impedance. This shows up as the converging wave having a negative mass, as indicated by the sign of Y^- in /EqYpm, corresponding to the inbound wave.

Setting the expression for the admittance to zero gives the frequencies of the poles $s_\pm(r) = \pm c/a$, which are related to the size of the acoustic wavelength relative to the size of the sphere. This is the same as $ka = 1$ where $k = 2\pi/\lambda$ is the wave number and λ is the wavelength. One may conclude that the reactance is related to sphereical *resonant scattering*.

This reactive term is well known in loud speaker design, as it explains why loudspeakers cannot radiate energy at low frequencies. This shunt reactance in the radiation impedance limits the power radiated for frequencies greater than the cutoff frequency $s_\pm$. This is the same for an electrical antenna smaller than the wavelength (e.g., $ka < 1$).

In the time domain, an inverse Laplace transform of the two admittances is

$$y_\pm(a, t) = Y_0\delta(t) \pm \frac{A_0 a}{\rho_0}\Delta(t) \leftrightarrow Y^\pm(a, s), \quad [\mathtt{eq:ypm}] \tag{3.61}$$

where $\delta(t)$ is the Dirac impulse function and $\Delta(t)$ is its first integral, the unit step function. Since the velocity is the product of the admittance and the pressure, the time domain admittance $y^+(a, s)$ represents the radial particle velocity corresponding to an outbound wave pressure wave impulse, launched from a spherical radiator having radius a (e.g., Eq. 3.58).

In the case of a uniform tube, there is no mass reactance and the forward traveling pressure wave is simply $p^+\delta(t-x/c)$, with a corresponding velocity wave $v^+ = Z_0 p^+\delta(t-x/c)$.

The velocity wave, found by convolution of the pressure impulse Eq. 3.58 with the admittance $y_+(t)$, consists of a delta function plus a outbound constant velocity, that decreases with the radius r. The spherical wave on the other hand has a mass reactance (i.e. $s\rho_0 r$) in the wave radiation impedance. The outbound step function, being the inverse transform of the imaginary part of the admittance, represents the reactive storage of energy. This reactive part is the part not radiated, but corresponds to the velocity component that is delayed, relative to the pressure component, representing stored energy in the acoustic field.

The relations between pressure and velocity $[P, U_r]$ and the out and inbound pressure $[\mathcal{P}^+, \mathcal{P}^-]$, given at a, by Eq. 3.57 and Eq. 3.59, may be summarized in matrix form as, evaluated at the input $(r = a)$

$$\begin{bmatrix} P(a,\omega) \\ U_r(a,\omega) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ Y^+(a,s) & -Y^-(a,s) \end{bmatrix} \begin{bmatrix} \mathcal{P}^+(a,\omega) \\ \mathcal{P}^-(a,\omega) \end{bmatrix} \quad \text{[eq : ImpedanceWaves]} \tag{3.62}$$

with $\mathcal{P}^+$ being the outbound and $\mathcal{P}^-$ being the inbound d'Alembert (1746) waves for a spherical radiator of radius a, where $\mathcal{P}^\pm(a,\omega) = \mathcal{P}^\pm_\omega(a)e^{\mp sa/c}/a$ and the radiation admittances $Y^\pm(a,s)$ are given by Eq. 3.60.

3.3.3 Wave variables

A second order linear differential equation may be transformed into another set of basis functions. The natural basis set is the d'Alembert waves of Eq. 3.57, but other transformations are possible. In the following we shall transform Eq. 4.67 using *wave variables* [Fettweis, 1986].

Wave variables are linearly related to the pressure and velocity (Eq. 3.62) via the relations $P = P^+ + P^-$ and $U_r = U^+ - U^-$. In order that this linear transformation be unique, we further require that the wave variable ratios to be constrained such that they are equal to the characteristic impedance, which in general may depend on the positional coordinate, and if losses are considered, on the Laplace frequency s

$$\frac{P^\pm}{U_r^\pm} = Z_0(r,s) \equiv \sqrt{\frac{\mathcal{Z}(r,s)}{\mathcal{Y}(r,s)}}. \quad \text{[eq : z0]} \tag{3.63}$$

Defining the ratio of wave variables in this way leads to a unique linear transformation between *impedance variables* (i.e., the usual pressure and velocity variables $[P, U_r]$, the ratio of which define the impedance) and wave variables $[P^+, P^-]$. Eq. 3.63 makes them similar to localized plane waves. Wave variables (i.e., the localized plane waves) are special in that they characterize causal wave-fronts.

The uniqueness of the relations is proved by writing them in matrix form

$$\begin{bmatrix} P(r,\omega) \\ U_r(r,\omega) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ Y_0 & -Y_0 \end{bmatrix} \begin{bmatrix} P^+(r,\omega) \\ P^-(r,\omega) \end{bmatrix}, \quad \text{[eq : Wavepm1]} \tag{3.64}$$

and noting that the determinant (i.e., -2$y_0(r) = -2\sqrt{\mathcal{Y}/\mathcal{Z}}$) is non-zero. In homogeneous media, wave variables identically reduce to plane waves.

From the inverse of Eq. 3.65 we may determine the wave variables given $\mathcal{P}^\pm$ as

$$\begin{bmatrix} P^+(r,\omega) \\ P^-(r,\omega) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & +Z_0 \\ 1 & -Z_0 \end{bmatrix} \begin{bmatrix} P(r,\omega) \\ U_r(r,\omega) \end{bmatrix}, \quad \text{[eq : Wavepm1]} \tag{3.65}$$

We view this transformation as a differential form of Weyl's famous integral expansion of the spherical wave in terms of plane waves. If we cascade Eq. 3.62 with Eq. 3.65 we may directly write the wave variables in terms of the d'Alembert solutions $\mathcal{P}^\pm$

s plane-wave
57. Talk to

$$\begin{bmatrix} P^+(r,\omega) \\ P^-(r,\omega) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & +Z_0 \\ 1 & -Z_0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ Y^+(a,s) & -Y^-(a,s) \end{bmatrix} \begin{bmatrix} \mathcal{P}^+(a,\omega) \\ \mathcal{P}^-(a,\omega) \end{bmatrix}, \quad \text{[eq : Wavepm2]} \tag{3.66}$$

This transformation represents a linear mapping from d'Alembert waves to that obey Eq. 3.63, similar to plane waves. The price for this much simpler impedance relationship (e.g., it is real when the medium is lossless), is that both types of wave variables are required to expand each of the d'Alembert waves.

As a specific example, when there is only an outbound wave (e.g., an infinite line driven by an impulse source), $\mathcal{P}_\omega^+ = 1$, $\mathcal{P}_\omega^- = 0$ and $\mathcal{P}^+ = e^{-sr/c}/r \leftrightarrow \delta(t - r/c)/r$

$$P^{\pm}(r, \omega) = \frac{1 \pm Z_0 Y^+}{2} e^{-sr/c} \quad [\texttt{eq:Wavepm3}] \tag{3.67}$$

thus

$$P^{+}(r, \omega) = \left(1 + \frac{c}{2sa}\right) \frac{e^{-sr/c}}{r} \quad [\texttt{eq:Wavepm4}] \tag{3.68}$$

and

$$P^{-}(r, \omega) = -\frac{c}{2sa} \frac{e^{-sr/c}}{r} \quad [\texttt{eq:Wavepm5}] \tag{3.69}$$

resulting in a reflectance of

$$R(s) = -\frac{c}{s2a + c} \tag{3.70}$$

which is -1 for frequencies below resonance, and goes to zero as 1 over frequency, above.